

Special Anisotropic Conformal Changes of Conic Pseudo-Finsler Surfaces**Nabil L. Youssef¹, Ebtsam H. Taha¹, A. A. Kotb², S. G. Elgendi^{3,*}**¹*Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt*²*Department of Mathematics, Faculty of Science, Damietta University, Damietta, Egypt*³*Department of Mathematics, Faculty of Science, Islamic University of Madinah, Madinah, Kingdom of Saudi Arabia***Corresponding author: salah.ali@fsc.bu.edu.eg, salahelgendi@yahoo.com*

Abstract. This study presents many special anisotropic conformal changes of a conic pseudo-Finsler surface (M, F) , such as C-anisotropic and horizontal C-anisotropic conformal transformations, which reduce to C-conformal when the conformal factor is solely position-dependent. Furthermore, we present vertical C-anisotropic conformal changes and demonstrate that they are characterized by the property of (M, F) being Riemannian. Additionally, we examine the anisotropic conformal transformation that fulfils the ϕT -condition, the horizontal ϕT -condition, and the vertical ϕT -condition. The first two conditions reduce to the σT -condition when the conformal factor relies solely on a positional variable. We demonstrate that, under the vertical ϕT -condition change, every Landsberg surface is Berwaldian. Thus, the vertical ϕT -condition is equivalent to the T -condition. Furthermore, we examine the scenario when the anisotropic conformal factor becomes the main scalar of the non-Riemannian surface (M, F) . We present an example of a Finslerian Schwarzschild-de Sitter solution having Finslerian spherical symmetry and apply our results to it.

1. INTRODUCTION

A conformal transformation entails the scaling of a Finsler metric by a smooth positive function known as the conformal factor. A transformation that relies solely on the manifold coordinates (position) is referred to as an isotropic conformal transformation [5, 9, 12, 13, 17, 18]. Knebelman initially introduced the theory of isotropic conformal transformation of Finsler spaces in 1929 [12]. M. Hashiguchi [9] subsequently introduced a particular transformation termed a C-conformal change, which serves as an essential tool in the study of Finsler geometry. This transformation facilitates the examination of invariant qualities and the classification of special Finsler spaces. It is an extension of the concept of concurrent vector fields. Concurrent vector fields have been

Received: Jun. 25, 2025.

2020 *Mathematics Subject Classification.* 53B40, 53C60.

Key words and phrases. anisotropic conformal change; conic pseudo-Finsler surface; modified Berwald frame; C-conformal; C-anisotropic conformal; anisotropic ϕT -condition.

extensively studied in the Riemannian geometry. In 1950, Tachibana [20] expanded this concept to Finsler geometry and identified the spaces in which such vector fields exist. In 2019, N. Youssef et al. presented a general idea known as a semi-concurrent vector field in [24]. The Cartan tensor is a fundamental non-Riemannian quantity in Finsler geometry, invariant under isotropic conformal changes, encouraging some geometers to define and explore the C -conformal transformation. In Finsler surfaces [1–3, 15, 21], the C -conformal transformation is characterized by F being a Riemannian metric.

The T -tensor is a crucial object in Finsler geometry which holds significant relevance. The σT -condition is particularly important in determining the preservation of features such as the Landsberg or Berwald character under conformal transformations. Elgendi, in [5], examined the relationship between the T -condition and the σT -condition, demonstrating their equivalence in Finsler surfaces. Furthermore, the Landsberg unicorn's problem is closely related to both T -condition and σT -condition [5–8].

On the other hand, when the conformal factor is dependent upon both direction (tangent or velocity vectors) and position, the transformation is referred to as an anisotropic conformal transformation [10, 11, 22, 23]. Anisotropic conformal transformations of Finsler spaces vary in direction, as opposed to uniform scaling in conformal transformations. The direction-dependent provides more comprehensive metric deformations, essential for describing directional phenomena such as relativistic spacetime. The investigation of anisotropic transformations enhances the examination of Finsler spaces.

We introduce and investigate the concepts of C -anisotropic and $\bar{C}$ -anisotropic conformal changes of a pseudo-Finsler surface in §3 due to the fact that the Cartan tensor does not remain invariant under an anisotropic conformal transformation of a conic pseudo-Finsler surface [22]. In particular, we explore under what conditions the anisotropic conformal transformation of a conic pseudo-Finsler surface is C -anisotropic and $\bar{C}$ -anisotropic conformal change, and identify necessary and sufficient conditions such that the property of C -anisotropic is preserved. When this property is preserved, we determine the conditions under which F is projectively flat.

Based on the dependence of the anisotropic conformal factor on position and directional arguments, we introduce and examine the horizontal C -anisotropic, horizontal $\bar{C}$ -anisotropic, vertical C -anisotropic and vertical $\bar{C}$ -anisotropic conformal changes. Also, we demonstrated that the conformal transformation between two non-Riemannian metrics is horizontal C -anisotropic and horizontal $\bar{C}$ -anisotropic if and only if the geodesic spray is invariant (Theorem 3.1). We show that the conformal changes of C , $\bar{C}$, horizontal C , and horizontal $\bar{C}$ -anisotropic are the same when the conformal factor is just a function of x only, but the vertical C -anisotropic change is not. Additionally, the vertical C -anisotropic conformal change is characterized by the property that the Finsler surface is Riemannian.

We are motivated to introduce the ϕT -condition, horizontal ϕT -condition and vertical ϕT -condition in light of the fact that the T -tensor is not invariant under an anisotropic conformal

transformation of a conic pseudo-Finsler surface (with conformal factor $\phi(x, y)$) [22, 23]. The anisotropic ϕT -, horizontal ϕT -, and vertical ϕT -conditions are investigated and characterized in §4. Given an anisotropic conformal transformation that satisfies the anisotropic ϕT -condition in (M, F) and $(M, \bar{F})$, we determine the conditions under which the Landsberg metric is Berwaldian, as shown in Theorem 4.1. In §5, we assume that F represents a non-Riemannian metric, with the anisotropic conformal factor being the main scalar $\mathcal{I}$ of F . This specific case for the conformal factor yields interesting results.

In §6, an example of a Finslerian sphere is provided by an anisotropic conformal transformation of a Riemannian sphere. We examine the conditions under which this transformation meets the criteria for C-anisotropic, $\bar{C}$ -anisotropic, horizontal C-anisotropic and the ϕT -condition. This example represents a Finslerian Schwarzschild-de Sitter solution that possesses the symmetry of a Finslerian sphere. This is because, the Finslerian analogue of Birkhoff's theorem states that a Finslerian gravitational field with the symmetry of a "Finslerian sphere" in a vacuum must be static [14]. Also, the Einstein field equations including a non-zero cosmological constant have been modified to the Finslerian framework as presented in [16]. This modification yields a Finslerian Schwarzschild-de Sitter solution that possesses the symmetry of a Finslerian sphere. Finally, we end our work by some concluding remarks along with a table which summarizes all special anisotropic conformal transformations we studied in §7.

2. NOTATION AND PRELIMINARIES

Let M be an n -dimensional smooth manifold. The tangent bundle of M is denoted by (TM, π_M, M) , where TM is the disjoint union of all tangent vectors at each point in base manifold M and $\pi_M : TM \rightarrow M$ denote the canonical projection from TM onto M . Define $TM_0 = TM \setminus (0)$ as the tangent bundle with the zero section removed. Then (TM_0, π_M, M) is sub-bundle of non zero vectors. On the base manifold M , we use local coordinates (x^i) . In the tangent bundle TM , the coordinates are given by (x^i, y^i) , where x^i are the base coordinates and y^i represent the components of the tangent vectors.

A function $f \in C^\infty(TM_0)$ is said to be positively homogeneous of degree r in the directional argument y and denoted by $h(r)$ if $f(x, \lambda y) = \lambda^r f(x, y)$, $\forall \lambda > 0$.

A conic sub-bundle of TM is a non-empty open subset $\mathcal{A}$ of TM_0 which satisfies $\pi(\mathcal{A}) = M$ and is invariant under scaling of tangent vectors by positive real numbers.

Definition 2.1. A conic pseudo-Finsler metric on M is a smooth function $F : \mathcal{A} \rightarrow \mathbb{R}$ which satisfies the following conditions:

- (i): $F(x, y)$ is $h(1)$: $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda \in \mathbb{R}^+$,
- (ii): For each point of $\mathcal{A}$, the Finsler metric tensor $g_{ij}(x, y) = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j F^2(x, y)$, is non-degenerate.

The pair (M, F) is called a conic pseudo-Finsler manifold.

For a Finsler metric F , a unique nonlinear Cartan (Ehresmann) connection exists on $\mathcal{A} \subset TM$ that is both torsion-free and compatible with F . The nonlinear connection coefficients are determined

by

$$G_i^j = \frac{1}{4} \dot{\partial}_i \left[g^{jk} \left(y^m \partial_m \dot{\partial}_k F^2 - \partial_k F^2 \right) \right].$$

This nonlinear connection defines the horizontal derivatives $\delta_i := \partial_i - G_i^j \dot{\partial}_j$, where $\dot{\partial}_i := \frac{\partial}{\partial y^i}$.

The coefficients of the geodesic spray coefficients of F can be expressed as

$$G^i = \frac{1}{4} g^{ik} \left(y^m \partial_m \dot{\partial}_k F^2 - \partial_k F^2 \right).$$

It is clear that G^i are smooth $h(2)$ functions in $\mathcal{A}$, moreover, the geodesic spray can be defined globally on TM_0 by $S = y^i \partial_i - 2G^i \dot{\partial}_i$.

In the case of positive definite of Finsler surface, Berwald introduced a global frame in [3] called Berwald frame. Given that the angular metric h_{ij} has a matrix representation (h_{ij}) with rank one, we can determine a unique vector field $m_i(x, y)$ and sign $\varepsilon = \pm 1$, where

$$h_{ij} = \varepsilon m_i m_j.$$

As, $g_{ij} = \ell_i \ell_j + h_{ij}$, where $\ell_i = \dot{\partial}_i F$ and $\ell^i = \frac{y^i}{F}$. Then, the metric tensor can be expressed as

$$g_{ij} = \ell_i \ell_j + \varepsilon m_i m_j. \quad (2.1)$$

We denote $g = \det(g_{ij})$ and we have the following relations:

$$\ell^i \ell_i = 1 \quad \ell^i m_i = \ell_i m^i = 0, \quad m^i m_i = \varepsilon. \quad (2.2)$$

Hence, (ℓ^i, m^i) is orthonormal frame, called modified Berwald frame introduced by Báscó and Matsumoto [2] which is suitable for both positive definite and non-positive definite surfaces.

The main scalar $\mathcal{I}(x, y)$ is one of the most significant quantities in Finsler surfaces, it is an $h(0)$ smooth function defined from the Cartan tensor [3] by,

$$FC_{ijk} = \mathcal{I} m_i m_j m_k. \quad (2.3)$$

For a smooth $f \in C^\infty(TM_0)$ we define the v-scalar derivatives $(f_{;1}, f_{;2})$ and h-scalar derivatives $(f_{,1}, f_{,2})$ in (M, F) as follows:

$$F \dot{\partial}_i f = f_{;1} \ell_i + f_{;2} m_i, \quad \delta_i f = f_{,1} \ell_i + f_{,2} m_i, \quad (2.4)$$

where

$$f_{;1} = y^i \dot{\partial}_i f, \quad f_{;2} = \varepsilon F(\dot{\partial}_i f) m^i, \quad f_{,1} = (\delta_i f) \ell^i, \quad f_{,2} = \varepsilon (\delta_i f) m^i.$$

Specifically, if f is $h(r)$, then $f_{;1} = rf$, the commutation formulas [3] are given by:

$$f_{;1,2} - f_{;2,1} = -Rf_{;2}, \quad (2.5)$$

$$f_{,1,2} - f_{,2,1} = f_{,2}, \quad (2.6)$$

$$f_{;2,2} - f_{,2,2} = -\varepsilon(f_{,1} + \mathcal{I} f_{;2} + \mathcal{I}_{,1} f_{;2}), \quad (2.7)$$

where R is said to be the Gauss curvature or the h -scalar curvature.

Definition 2.2. [22] *The anisotropic conformal change of a conic pseudo-Finsler metric F is defined by*

$$F \mapsto \bar{F}(x, y) = e^{\phi(x, y)} F(x, y), \quad F^2(\dot{\partial}_i \dot{\partial}_j \phi + (\dot{\partial}_i \phi)(\dot{\partial}_j \phi)) m^i m^j + \varepsilon = \varepsilon + \sigma - (\phi_{;2})^2 \neq 0, \quad (2.8)$$

$$\sigma = \phi_{;2;2} + \varepsilon \mathcal{I} \phi_{;2} + 2(\phi_{;2})^2, \quad (2.9)$$

given that the conformal factor $\phi(x, y)$ is a smooth $h(0)$ -function on $\mathcal{A}$. In this case, F is said to be anisotropic conformally changed to $\bar{F}$. In addition, we say that the anisotropic conformal transformation is proper if the conformal factor is a non-isotropic and non-homothetic function which means $\phi_{;2} \neq 0$.

Now, we define the v -scalar derivatives $(f_{;a}, f_{;b})$ and h -scalar derivatives $(f_{,a}, f_{,b})$ in $(M, \bar{F})$ for f by:

$$\bar{F} \dot{\partial}_i f = f_{;a} \bar{\ell}_i + f_{;b} \bar{m}_i, \quad \bar{\delta}_i f = f_{,a} \bar{\ell}_i + f_{,b} \bar{m}_i,$$

where

$$f_{;a} = y^i \dot{\partial}_i f, \quad f_{;b} = \varepsilon \bar{F}(\dot{\partial}_i f) \bar{m}^i, \quad f_{,a} = (\bar{\delta}_i f) \bar{\ell}^i, \quad f_{,b} = \varepsilon (\bar{\delta}_i f) \bar{m}^i.$$

In [22], we discuss the anisotropic conformal change on a conic pseudo-Finsler surface F equipped with modified Berwald frame and determined how this change affects the components of the Berwald frame, the main scalar and the geodesic spray coefficients of $\bar{F}$ as follows:

$$\bar{\ell}_i = e^{\phi} [\ell_i + \phi_{;2} m_i], \quad \bar{\ell}^i = e^{-\phi} \ell^i, \quad \bar{m}_i = e^{\phi} \sqrt{\frac{\varepsilon}{\rho}} m_i, \quad \bar{m}^i = e^{-\phi} \sqrt{\varepsilon \rho} [m^i - \varepsilon \phi_{;2} \ell^i]. \quad (2.10)$$

$$\bar{\mathcal{I}} = \sqrt{\varepsilon \rho} [\mathcal{I} + 2\varepsilon \phi_{;2} - \frac{\varepsilon \rho_{;2}}{2\rho}]. \quad (2.11)$$

$$\bar{G}^i = G^i + Q m^i + P \ell^i, \quad (2.12)$$

where

$$\rho = \frac{1}{\sigma + \varepsilon - (\phi_{;2})^2}. \quad (2.13)$$

$$2Q = \varepsilon \rho F^2 (\phi_{;2} \phi_{;1} + \phi_{;1;2} - 2\phi_{;2}), \quad (2.14)$$

$$2P = -\rho F^2 \phi_{;2} (\phi_{;2} \phi_{;1} + \phi_{;1;2} - 2\phi_{;2}) + F^2 \phi_{;1}. \quad (2.15)$$

Furthermore, from (2.14) and (2.15), we get

$$2\varepsilon \phi_{;2} Q + 2P = F^2 \phi_{;1}. \quad (2.16)$$

Lemma 2.1. *Let (M, F) be Finsler surface. Each smooth function f is on M satisfies $m^i \partial_i f = 0$ is constant.*

Proof. Since $f \in C^\infty(M)$, then we have $\delta_i f = \partial_i f$ and $\delta_i f = f_{;1} \ell_i + f_{;2} m_i$. Then, $m^i \partial_i f = \varepsilon f_{;2}$. Moreover, if $m^i \partial_i f = 0$, we obtain $f_{;1} = f_{;2} = f_{;2} = 0$. Thereby, using the commutation formula (2.7), we get $f_{;1} = 0$. Consequently, f is a constant function. $\square$

3. C-ANISOTROPIC AND HORIZONTAL C-ANISOTROPIC CHANGE

Hashiguchi [9] introduced the concept of C-conformal transformation in Finsler geometry as a special form of a conformal transformation. This class is a subset of conformal transformations, distinguished by specific conditions on the Cartan tensor and the conformal factor. In Finsler surfaces, the C-conformal transformation is characterized by the fact that F is a Riemannian metric. More precisely,

Definition 3.1. [9] *The (isotropic) conformal change $\bar{F}(x, y) = e^{\theta(x)}F(x, y)$ is said to be C-conformal if it is not a homothetic change and satisfies*

$$C_{jk}^i \partial_i \theta = 0.$$

Lemma 3.1. *The conformal change of a two-dimensional conic pseudo-Finsler metric F defined as $\bar{F} = e^\theta F$ is C-anisotropic conformal if and only if F is Riemannian.*

Proof. Let $\bar{F} = e^\theta F$ be a C-conformal transformation, then

$$0 = C_{jk}^i \partial_i \theta = (\partial_i \theta) \frac{\mathcal{I}}{F} m^i m_j m_k,$$

which is equivalent to either $\mathcal{I} = 0$ (F is Riemannian) or $m^i \partial_i \theta = 0$. Since θ is a function of x only, then, by Lemma 2.1, θ is constant which is a contradiction. $\square$

It should be noted that, in the case of a conformal transformation, $\bar{C}_{jk}^i = C_{jk}^i$ as the conformal factor depends only on the position only. However, in the case of the anisotropic conformal transformation (2.8), we have $\bar{C}_{jk}^i \neq C_{jk}^i$ as the conformal factor depends on both x and y . This inspires us to introduce some special anisotropic conformal transformations, namely, C-anisotropic conformal and horizontal C-anisotropic conformal transformations. We also discuss under what conditions these transformations are preserved.

Definition 3.2. *The proper anisotropic conformal change $\bar{F} = e^\phi F$ is said to be C-anisotropic conformal if it satisfies $C_{jk}^i \partial_i \phi = 0$. Similarly, the anisotropic conformal change $\bar{F} = e^\phi F$ is said to be $\bar{C}$ -anisotropic conformal if it satisfies $\bar{C}_{jk}^i \partial_i \phi = 0$.*

Proposition 3.1. *Let F be conformally anisotropic changed to $\bar{F}$ by (2.8). Then, we have:*

- (i): *The anisotropic conformal transformation is C-anisotropic conformal if and only if either F is a Riemannian metric or $m^i \partial_i \phi = 0$.*
- (ii): *The anisotropic conformal transformation is $\bar{C}$ -anisotropic conformal if and only if either $\mathcal{I} = \frac{\varepsilon}{2}(\ln \rho - 4\phi)_{,2}$ ($\bar{F}$ is a Riemannian metric) or $m^i \partial_i \phi - \varepsilon \phi_{,2} \ell^i \partial_i \phi = 0$.*
- (iii): *Assume that (2.8) is an anisotropic conformal transformation between two non-Riemannian metrics. The property of C-anisotropic conformal is invariant if and only if $\ell^i \partial_i \phi = 0$.*

Proof.

- (i): *The anisotropic conformal transformation is C-anisotropic conformal if and only if*

$$0 = C_{jk}^i \partial_i \phi = \mathcal{I} m^i m_j m_k \partial_i \phi.$$

From which we have $\mathcal{I} m^i \partial_i \phi = 0$. That is, the anisotropic conformal transformation (2.8) is C-anisotropic conformal if and only if either F is a Riemannian metric or $m^i \partial_i \phi = 0$.

(ii): The anisotropic conformal transformation is $\bar{C}$ -anisotropic conformal if and only if

$$0 = \bar{C}_{jk}^i \partial_i \phi = \bar{\mathcal{I}} (m^i m_j m_k - \varepsilon \phi_{,2} \ell^i m_j m_k) \partial_i \phi.$$

Therefore, the anisotropic conformal transformation (2.8) is $\bar{C}$ -anisotropic conformal if and only if either $\bar{\mathcal{I}} = 0$ or $m^i \partial_i \phi - \varepsilon \phi_{,2} \ell^i \partial_i \phi = 0$. By making use of (2.11), (2.8) is $\bar{C}$ -anisotropic conformal if and only if either $\mathcal{I} = \frac{\varepsilon}{2} [\frac{\rho_{,2}}{\rho} - 4\phi_{,2}] = \frac{\varepsilon}{2} (\ln \rho - 4\phi)_{,2}$ or $m^i \partial_i \phi - \varepsilon \phi_{,2} \ell^i \partial_i \phi = 0$.

(iii): Consider the anisotropic conformal change (2.8) between non-Riemannian metrics F and $\bar{F}$. Now, we have:

($\implies$) If the property of C-anisotropic conformal is invariant, that is, our anisotropic conformal transformation is both C-anisotropic and $\bar{C}$ -anisotropic transformation, then we have $m^i \partial_i \phi = 0$ and $m^i \partial_i \phi - \varepsilon \phi_{,2} \ell^i \partial_i \phi = 0$. Since $\phi_{,2} \neq 0$, we conclude $\ell^i \partial_i \phi = 0$.

($\impliedby$) If $\ell^i \partial_i \phi = 0$ and the anisotropic conformal change (2.8) is C-anisotropic change (i.e., $m^i \partial_i \phi = 0$, by (i) above), then, since $\phi_{,2} \neq 0$, we have $m^i \partial_i \phi - \varepsilon \phi_{,2} \ell^i \partial_i \phi = 0$. That is, (2.8) is $\bar{C}$ -anisotropic conformal change.

□

Remark 3.1. A Finsler surface (M, F) is projectively flat if and only if it satisfies the Hamel's equation ($\partial_1 \partial_2 F = \partial_2 \partial_1 F$) [4, Eqn. (2.70)]. The geodesic coefficient of the two-dimensional metric F is written as $2G^i = y^r (\partial_r F) \ell^i + \frac{F^2 (\partial_2 \partial_1 F - \partial_1 \partial_2 F)}{h} m^i$, where $h = \frac{\varepsilon}{\ell^1 m^2 - \ell^1 m^1}$ [15, Eqn. (1.4)]. Consequently, the Finsler surface (M, F) is projectively flat if and only if $G^k m_k = 0$.

Proposition 3.2. Assuming that $\bar{F} = e^\phi F$ is an anisotropic conformal transformation between two non-Riemannian metrics, we have the following:

- (i): If F is a locally Minkowski metric and the anisotropic conformal transformation (2.8) is $\bar{C}$ -anisotropic conformal change, then the two sprays S and $\bar{S}$ are projectively equivalent if and only if $\phi_{,2,1} = 0$ in some coordinate system.
- (ii): If F is projectively flat metric, then the anisotropic conformal transformation (2.8) is $\bar{C}$ -anisotropic if and only if $\phi_{,2} = \phi_{,1} \left[\phi_{,1} - \frac{G^k}{F^2} \ell_k \right]$. Moreover, the anisotropic conformal transformation (2.8) is C-anisotropic if and only if $\phi_{,2} = -\frac{\phi_{,2}}{F^2} G^k m_k$.
- (iii): If the property of C-anisotropic conformal is preserved, then F is projectively flat if and only if ϕ is a first integral of the geodesic spray S .

Proof. For a Finsler surface equipped with modified Berwald frame, we have, by [22, Eqn. (4.8)], the following equalities:

$$\left. \begin{aligned} \text{(a)} \quad F^2 \ell^k \partial_k \phi &= F^2 \phi_{,1} + 2G^k \phi_{,2} m_k \\ \text{(b)} \quad F m^k \partial_k \phi &= \varepsilon F \phi_{,2} + G_k^i \phi_{,2} m^k m_i \end{aligned} \right\} \quad (3.1)$$

This leads to the equivalence:

$$m^i \partial_i \phi - \varepsilon \phi_{,2} \ell^i \partial_i \phi = 0 \iff \phi_{,2} = \phi_{,2} [\phi_{,1} + \frac{2\phi_{,2}}{F^2} G^k m_k - \frac{\varepsilon}{F} G_k^i m^k m_i], \quad (3.2)$$

$$m^i \partial_i \phi = 0 \iff \phi_{,2} = -\frac{\varepsilon \phi_{,2}}{F} G_k^i m^k m_i, \quad (3.3)$$

$$\ell^i \partial_i \phi = 0 \iff \phi_{,1} = -\frac{2}{F^2} \phi_{,2} G^k m_k. \quad (3.4)$$

(i): Let F be a locally Minkowski metric F (i.e., $G^i = 0$ in some coordinate system [18, Definition 5.3]) and (2.8) be $\bar{C}$ -anisotropic. By Proposition 3.1 **(ii)** and (3.2), we have

$$\phi_{,2} = \phi_{,2} \phi_{,1}. \quad (3.5)$$

Using the commutation formula (2.6) and (2.14), along with (3.5), we get

$$Q = \frac{1}{2} \varepsilon \rho F^2 (\phi_{,2} \phi_{,1} + \phi_{,1,2} - 2\phi_{,2}) = \frac{1}{2} \varepsilon \rho F^2 (\phi_{,2} \phi_{,1} + \phi_{,2,1} - \phi_{,2}) = \frac{1}{2} \varepsilon \rho F^2 \phi_{,2,1}.$$

Consequently, the two sprays S and $\bar{S}$ are projectively equivalent if and only if $\phi_{,2,1} = 0$ in some coordinate system.

(ii): Assuming that F is projectively flat, by Remark 3.1, we have

$$G^k m_k = 0. \quad (3.6)$$

Differentiating (3.6) with respect to y^i , where $F \partial_j m_i = -(\ell_i - \varepsilon \mathcal{I} m_i) m_j$, we get $0 = G_i^k m_k + \frac{G_k^i}{F} (-\ell_k + \varepsilon \mathcal{I} m_k) m_i$. Using (3.6), we obtain

$$G_i^k m_k m^i = \frac{\varepsilon G^k}{F} \ell_k. \quad (3.7)$$

Substituting (3.6) and (3.7) into (3.2), then by Proposition 3.1 **(ii)**, the anisotropic conformal transformation (2.8) is $\bar{C}$ -anisotropic conformal if and only if $\phi_{,2} = \phi_{,2} [\phi_{,1} - \frac{G^k}{F^2} \ell_k]$. On the other hand, plugging (3.7) into (3.3) leads to the anisotropic conformal transformation (2.8) is C -anisotropic if and only if $\phi_{,2} = -\frac{\phi_{,2}}{F^2} G^k \ell_k$.

(iii): Let the property of C -anisotropic conformal be preserved under the anisotropic conformal change (2.8), from Proposition 3.1 **(iii)** and (3.4), we have $\phi_{,1} = -\frac{2}{F^2} \phi_{,2} G^k m_k$. Since $\phi_{,2} \neq 0$, then, by Remark 3.1, the metric F is projectively flat ($G^k m_k = 0$) if and only if the anisotropic conformal factor is a first integral of the geodesic spray S (i.e. $\phi_{,1} = 0$). $\square$

Remark 3.2. **(i):** Let $\bar{F} = e^\phi F$ be non-Riemannian with ϕ is horizontally constant (i.e. $\delta_i \phi = 0$), then the anisotropic conformal transformation (2.8) is $\bar{C}$ -anisotropic if and only if $2\phi_{,2} G^k m_k = \varepsilon F G_k^i m^k m_i$, by Proposition 3.1 **(ii)** and (3.2).

(ii): When the conformal factor in (2.8) is a function of positional argument, both C-anisotropic and $\bar{C}$ -anisotropic conformal transformations reduce to the well-known C-conformal transformation.

Definition 3.3. The anisotropic conformal change (2.8) is said to be horizontal C-anisotropic conformal if $(\delta_i \phi) C_{jk}^i = 0$. Similarly, it is said to be horizontal $\bar{C}$ -anisotropic conformal if $(\delta_i \phi) \bar{C}_{jk}^i = 0$.

Theorem 3.1. Let (M, F) be a Finsler surface and (2.8) be the anisotropic conformal transformation. Then we have the following:

- (i):** the anisotropic conformal transformation (2.8) is horizontal C-anisotropic conformal if and only if either F is Riemannian or $\phi_{,2} = 0$.
- (ii):** the anisotropic conformal transformation (2.8) is horizontal $\bar{C}$ -anisotropic conformal if and only if $\mathcal{I} = \frac{\varepsilon}{2}(\ln \rho - 4\phi)_{,2}$ or $\phi_{,2} = \phi_{,1}\phi_{,2}$.
- (iii):** The anisotropic conformal change (2.8) between non-Riemannian surfaces is a horizontal C-anisotropic and $\bar{C}$ -anisotropic if and only if the geodesic spray is invariant.

Proof. **(i):** The anisotropic conformal transformation (2.8) is horizontal C-anisotropic conformal means that

$$0 = (\delta_i \phi) C_{jk}^i = (\phi_{,1} \ell_i + \phi_{,2} m_i) \frac{\mathcal{I}}{F} m^i m_j m_k = \frac{\varepsilon \phi_{,2} \mathcal{I}}{F} m_j m_k,$$

which is equivalent that either F being Riemannian or $\phi_{,2} = 0$.

(ii): From (2.10) and (2.11), the anisotropic conformal transformation (2.8) is a horizontal $\bar{C}$ -anisotropic conformal if and only if

$$\begin{aligned} 0 &= (\delta_i \phi) \bar{C}_{jk}^i = \frac{1}{F} [(\phi_{,1} \ell_i + \phi_{,2} m_i) (\mathcal{I} + 2\varepsilon \phi_{,2} - \frac{\varepsilon \rho_{,2}}{2\rho}) (m^i m_j m_k - \varepsilon \phi_{,2} \ell^i m_j m_k)] \\ &= \frac{1}{F} [(\mathcal{I} + 2\varepsilon \phi_{,2} - \frac{\varepsilon \rho_{,2}}{2\rho}) (\varepsilon \phi_{,2} m_j m_k - \varepsilon \phi_{,1} \phi_{,2} m_j m_k)]. \end{aligned}$$

Hence, the anisotropic conformal transformation (2.8) is horizontal $\bar{C}$ -anisotropic conformal if and only if $\mathcal{I} = \frac{\varepsilon}{2}[\frac{\rho_{,2}}{\rho} - 4\phi_{,2}] = \frac{\varepsilon}{2}(\ln \rho - 4\phi)_{,2}$ or $\phi_{,2} = \phi_{,1}\phi_{,2}$.

(iii): ($\Leftarrow$) If the geodesic spray is invariant, then ϕ is horizontally constant [22, Theorem 4.11].

Consequently, the anisotropic conformal transformation (2.8) is horizontal C-anisotropic and $\bar{C}$ -anisotropic.

($\Rightarrow$) If the anisotropic conformal change (2.8) is a horizontal C-anisotropic and $\bar{C}$ -anisotropic. Also, F and $\bar{F}$ are non-Riemannian. Then, we get $\phi_{,2} = 0$, and $\phi_{,2} = \phi_{,2}\phi_{,1}$.

Therefore, $\phi_{,1} = \phi_{,2} = 0$ (as $\phi_{,2} \neq 0$) which implies that the geodesic spray is invariant. $\square$

Proposition 3.3. Let (M, F) (resp. $(M, \bar{F})$) be non-Riemannian surface and (2.8) be a horizontal C-anisotropic (resp. horizontal $\bar{C}$ -anisotropic). The geodesic spray is invariant if and only if the conformal factor is first integral of S (resp. either $\phi_{,2} = 0$ or $\phi_{,1} = 0$).

Proof. Let the conformal factor be first integral of S and (2.8) be horizontal C-anisotropic conformal, provided that F is non-Riemannian, then by Proposition 3.1, we get $\phi_{,1} = 0 = \phi_{,2}$ which is

equivalent to ϕ is constant horizontally which means the geodesic spray is invariant under the anisotropic change (2.8) by [22, Theorem 4.11]. In the case of $(M, \bar{F})$ being a non-Riemannian surface and (2.8) being horizontally $\bar{C}$ -anisotropic, we have $\phi_{,2} = \phi_{,2}\phi_{,1}$. Then the geodesic spray is invariant if and only if either $\phi_{,2} = 0$ or $\phi_{,1} = 0$. $\square$

Proposition 3.4. *Let $(M, \bar{F})$ be non-Riemannian surface and (2.8) be a horizontal $\bar{C}$ -anisotropic. The two geodesic sprays S and $\bar{S}$ are projectively equivalent if and only if $\phi_{,2,1} = 0$ (i.e. $\phi_{,2}$ is first integral of the geodesic spray S).*

Proof. The two geodesic sprays S and $\bar{S}$ are projectively equivalent if and only if $Q = 0$, by [22, Theorem 5.2]. Then from (2.14) and (2.6), we have

$$0 = \phi_{,2}\phi_{,1} + \phi_{,1,2} - 2\phi_{,2} = \phi_{,2}\phi_{,1} + \phi_{,2,1} - \phi_{,2} \quad (3.8)$$

Since (2.8) is a horizontal $\bar{C}$ -anisotropic conformal change, provided that $(M, \bar{F})$ is a non Riemannian metric, then from Proposition 3.1 (ii) and (3.8), the two geodesic sprays S and $\bar{S}$ are projectively equivalent if and only if $\phi_{,2,1} = 0$. $\square$

Remark 3.3. *Let (2.8) be the anisotropic conformal transformation between non-Riemannian metrics. We have the following:*

- (i): *From the commutation formula (2.6), the conditions that ϕ and $\phi_{,2}$ are first integral of the geodesic spray S is equivalent to that S and $\bar{S}$ are invariant under the anisotropic conformal change (2.8).*
- (ii): *The anisotropic conformal transformation (2.8) is C-anisotropic and horizontal C-anisotropic conformal if and only if $G_k^i m^k m_i = 0$, which follows from Proposition 3.1 (i), (3.3) and Theorem 3.1.*
- (iii): *The anisotropic conformal transformation (2.8) is $\bar{C}$ -anisotropic and horizontal $\bar{C}$ -anisotropic conformal if and only if $2\phi_{,2}G^k m_k = \varepsilon F G_k^i m^k m_i$.*

We have discussed the C-anisotropic conformal transformation $((\partial_i \phi)C_{jk}^i = 0)$ and horizontal C-anisotropic conformal transformation $((\delta_i \phi)C_{jk}^i = 0)$. These two types of anisotropic conformal transformation reduce to C-conformal transformation when the conformal factor ϕ is a function of x only. If we take the condition as $(\dot{\partial}_i \phi)C_{jk}^i = 0$ (what we call “vertical C anisotropic conformal”), then it does not reduce to the C-conformal transformation.

Definition 3.4. *The proper anisotropic conformal change (2.8) is said to be vertical C-anisotropic conformal change if $(\dot{\partial}_i \phi)C_{jk}^i = 0$. Similarly, (2.8) is said to be vertical $\bar{C}$ -anisotropic conformal change if $(\dot{\partial}_i \phi)\bar{C}_{jk}^i = 0$.*

Proposition 3.5. *Let (M, F) be a conic pseudo-Finsler surface. The anisotropic conformal change (2.8) is vertical C-anisotropic if and only if F is Riemannian. Moreover, the anisotropic conformal change (2.8) is vertical $\bar{C}$ -anisotropic if and only if $\bar{F}$ is Riemannian.*

Proof. Since, we have, by (2.2), (2.3) and (2.4),

$$(\dot{\partial}_i \phi)C_{jk}^i = \frac{I}{F} (\dot{\partial}_i \phi) m^i m_j m_k = \frac{I}{F^2} \varepsilon \phi_{,2} m_j m_k.$$

Consequently, the anisotropic conformal transformation (2.8) is vertical C -anisotropic if and only if F is Riemannian (as $\phi_{,2} \neq 0$). Secondly, the anisotropic conformal change (2.8) is vertical $\bar{C}$ -anisotropic if and only if

$$0 = (\dot{\partial}_i \phi) \bar{C}_{jk}^i = \phi_{,2} m_i \bar{I} (m^i m_j m_k - \varepsilon \phi_{,2} \ell^i m_j m_k) = \varepsilon \phi_{,2} \bar{I} m_j m_k.$$

As $\phi_{,2} \neq 0$, the anisotropic conformal transformation (2.8) is vertical $\bar{C}$ -anisotropic if and only if $\bar{F}$ is Riemannian. $\square$

Vector fields in differential geometry, especially concurrent vector fields, are essential for understanding manifold geometry. Concurrent vector fields play a key role in characterizing geometric properties in Riemannian and Finsler spaces. In 1950, Tachibana established the basis for investigating concurrent vector fields in Finsler spaces [20]. Then, Hashiguchi studied the effect of C -conformal change on Finsler metrics in 1976, linking them to concurrent vector fields [9]. The notion of a semi-concurrent vector field was introduced by N. Youssef and colleagues in 2019 [24].

Definition 3.5. [24] A vector field $X^i(x)$ on M is said to be semi-concurrent if it satisfies

$$X^i C_{ijk} = 0. \quad (3.9)$$

Remark 3.4. [15, Example 3.1.1.1] Let (M, F) be a conic pseudo-Finsler surface. The vector field $X^i(x) = A(x, y)\ell^i + B(x, y)m^i$ in M is a function of position only if and only if $A_{,2} = B$ and $B_{,2} = -\varepsilon(A - IB)$. Similarly, a covariant vector field $w_i(x) = C(x, y)\ell_i + D(x, y)m_i$ is a function of x only if and only if $C_{,2} = D$ and $D_{,2} = -\varepsilon(C - ID)$. Also, we have if one of “ A and B ” or “ C and D ” vanishes then the other vanishes also.

We introduce an alternative proof to the known result that a two-dimensional Finsler manifold admitting a semi-concurrent vector field is Riemannian [24, Theorem 3.9 (a)].

Proposition 3.6. A conic pseudo-Finsler surface (M, F) admits semi-concurrent vector field if and only if F is Riemannian.

Proof. Assume that (M, F) admits semi-concurrent vector field $X^i(x) = A(x, y)\ell^i + B(x, y)m^i$, then we have

$$0 = X^i(x) C_{ijk} = \frac{I}{F} (A(x, y)\ell^i + B(x, y)m^i) m_i m_j m_k = \frac{\varepsilon I}{F} B(x, y) m_j m_k.$$

Which is equivalent to either F is Riemannian or $B(x, y) = 0$. From Remark 3.4, the later shows that X^i is zero vector, which is contradiction. Hence the result. $\square$

Now, we will determine the necessary and sufficient condition for the property of admitting a semi-concurrent vector field to be preserved under the anisotropic conformal change (2.8).

Proposition 3.7. Let (2.8) be the anisotropic conformal transformation of a Finsler space (M, F) which admits a semi-concurrent vector field. A necessary and sufficient condition for $(M, \bar{F})$ to admit a semi-concurrent vector field is that the function $(4\phi - \ln \rho)$ is isotropic.

Proof. Let (M, F) admits a semi-concurrent vector field, by Proposition 3.6, we get $\mathcal{I} = 0$. Also, $(M, \bar{F})$ admits a semi-concurrent vector field if and only if $\bar{F}$ is Riemannian, or equivalently $\bar{\mathcal{I}} = 0$, that is

$$\mathcal{I} + 2\varepsilon\phi_{;2} - \frac{\varepsilon\rho_{;2}}{2\rho} = 0. \quad (3.10)$$

Therefore, (3.10) becomes $2\varepsilon\phi_{;2} - \frac{\varepsilon\rho_{;2}}{2\rho} = 0$, which is $(4\phi - \ln \rho)_{;2} = 0$. Then, $(M, \bar{F})$ admits a semi-concurrent vector field is equivalent to $(4\phi - \ln \rho)$ is an isotropic function. $\square$

Remark 3.5. (i): From (2.13) and Proposition 3.7 a necessary and sufficient condition for $(M, \bar{F})$ to admit a semi-concurrent vector field is that $\phi_{;2;2;2} + 6\phi_{;2}\phi_{;2;2} + 4(\phi_{;2})^3 + 4\varepsilon\phi_{;2} = 0$, provided that, (M, F) admits a semi-concurrent vector field.

(ii): If ϕ is a function of position only, the property of admitting a semi-concurrent vector field is conserved. In other words, the property of admitting a semi-concurrent vector field is preserved if the anisotropic conformal change is reduced to an isotropic conformal change.

Corollary 3.1. Let (M, F) be a conic pseudo-Finsler surface and (2.8) be a proper anisotropic conformal transformation. If $\bar{g} = e^{4\phi}g$, then the Finsler metric $\bar{F}$ admits a semi-concurrent vector if and only if $\phi_{;2;2} = (\phi_{;2})^2$.

Proof. Since $\bar{g} = e^{4\phi}g$ is equivalent to $\sigma = (\phi_{;2})^2$, then from (2.9) we get

$$\mathcal{I} + \varepsilon\phi_{;2} = \frac{-\varepsilon\phi_{;2;2}}{\phi_{;2}}. \quad (3.11)$$

Also, by using (2.13) we have $\rho = \varepsilon$, $\rho_{;2} = 0$. Moreover, Remark 3.5 and (3.11) gives rise to the Finsler metric $\bar{F}$ admits a semi-concurrent vector if and only if

$$0 = \mathcal{I} + 2\varepsilon\phi_{;2} - \frac{\varepsilon\rho_{;2}}{2\rho} = \frac{-\varepsilon\phi_{;2;2}}{\phi_{;2}} + \varepsilon\phi_{;2} = \frac{\varepsilon(-\phi_{;2;2} + (\phi_{;2})^2)}{\phi_{;2}}.$$

Hence, $\bar{F}$ admits a semi-concurrent vector if and only if $\phi_{;2;2} = (\phi_{;2})^2$. $\square$

4. ANISOTROPIC ϕT -CONDITION

The T -tensor " $FT_{ijkh} = \mathcal{I}_{;2} m_i m_j m_k m_h$ " is one of the fundamental objects in Finsler geometry. The Finsler metric satisfies the T -condition if its T -tensor vanishes identically. The T -condition and σT -condition are equivalent in the case of Finsler surfaces [5, Theorem 3.5].

Definition 4.1. A Finsler space (M, F) satisfies the σT -condition if M admits a non-constant smooth function σ such that $(\partial_i \sigma) T^i_{jkr} = 0$.

We present an alternative proof of [5, Theorem 3.5]

Proposition 4.1. A two-dimensional Finsler metric satisfies σT -condition if and only if has vanishing T -tensor.

Proof. Let (M, F) satisfy σT -condition, that is, $(\partial_i \sigma) T^i_{jkr} = 0$. Then, by (2.4), (2.2) and the definition of T -tensor, we have

$$0 = \partial_i \sigma = (\delta_i \sigma) T^i_{jkr} = (\sigma_{,1} \ell_i + \sigma_{,2} m_i) \frac{\mathcal{I}_{,2}}{F} m^i m_j m_k m_r = \frac{\varepsilon \mathcal{I}_{,2} \sigma_{,2}}{F} m_j m_k m_r. \quad (4.1)$$

Consequently, the Finsler metric F satisfies σT -condition if and only if either F has vanishing T -tensor (i.e., $\mathcal{I}_{,2} = 0$) or $\sigma_{,2} = 0$. The later gives $\sigma_{,2} = \sigma_{,2} = 0$. Then, σ is a constant function, by Lemma 2.1, which contradicts σ is non-constant function. $\square$

From Proposition 4.1, any Landsberg surface satisfies the σT -condition is Berwaldian [5]. Under the anisotropic conformal transformation (2.8), we see that the conformal factor depends on both position and direction arguments and the transformation of the T -tensor is not invariant:

$$\bar{T}_{ijkh} = \frac{\varepsilon e^{3\phi}}{\rho} \left[T_{ijkh} + \frac{1}{2F\rho} \left(4\varepsilon \rho \phi_{,2,2} + \rho_{,2} (\mathcal{I} + 2\varepsilon \phi_{,2} + \frac{\varepsilon \rho_{,2}}{2\rho}) - \varepsilon \rho_{,2,2} \right) m_i m_j m_h m_k \right].$$

This leads us to define ϕT -condition, $\phi \bar{T}$ -condition, horizontal ϕT -condition and horizontal $\phi \bar{T}$ -condition and determine the conditions under which these properties coincide with the T -condition. In other words, we aim to find the conditions such that any Landsberg surface that satisfies these properties is a Berwald surface. In Finsler surfaces, the T -tensor components T^i_{jkr} are defined by $FT^i_{jkr} = \mathcal{I}_{,2} m^i m_j m_k m_r$. By using (2.10), under the anisotropic conformal transformation (2.8). We have

$$\bar{F} \bar{T}^i_{jkr} = \bar{\mathcal{I}}_{,b} \bar{m}^i \bar{m}_j \bar{m}_k \bar{m}_r = e^{2\phi} \sqrt{\frac{\varepsilon}{\rho}} \bar{\mathcal{I}}_{,2} (m^i - \varepsilon \phi_{,2} \ell^i) m_j m_k m_r \quad (4.2)$$

Also, from [23, §2] we have

$$\bar{\mathcal{I}}_{,b} = \sqrt{\varepsilon \rho} \bar{\mathcal{I}}_{,2}, \quad (4.3)$$

$$\bar{\mathcal{I}}_{,a} = e^{-\phi} [\bar{\mathcal{I}}_{,1} - \frac{2\varepsilon}{F^2} Q \bar{\mathcal{I}}_{,2}], \quad (4.4)$$

$$\bar{\mathcal{I}}_{,b} = e^{-\phi} \sqrt{\varepsilon \rho} [\bar{\mathcal{I}}_{,2} - \phi_{,2} \bar{\mathcal{I}}_{,1} - \frac{\varepsilon}{F^2} (\varepsilon P + Q_{,2} - \varepsilon \mathcal{I} Q - 2\phi_{,2} Q) \bar{\mathcal{I}}_{,2}], \quad (4.5)$$

where

$$\bar{\mathcal{I}}_{,2} = \frac{\sqrt{\varepsilon \rho}}{2\rho} [\rho_{,2} (\mathcal{I} + 2\varepsilon \phi_{,2} + \frac{\varepsilon}{2} (\ln \rho)_{,2}) + 2\rho (\mathcal{I}_{,2} + 2\varepsilon \phi_{,2,2}) - \varepsilon \rho_{,2,2}]. \quad (4.6)$$

$$\bar{\mathcal{I}}_{,1} = \frac{\sqrt{\varepsilon \rho}}{2\rho} [\rho_{,1} (\mathcal{I} + 2\varepsilon \phi_{,2} + \frac{\varepsilon}{2} (\ln \rho)_{,2}) + 2\rho (\mathcal{I}_{,1} + 2\varepsilon \phi_{,2,1}) - \varepsilon \rho_{,2,1}]. \quad (4.7)$$

$$\bar{\mathcal{I}}_{,2} = \frac{\sqrt{\varepsilon \rho}}{2\rho} [\rho_{,2} (\mathcal{I} + 2\varepsilon \phi_{,2} + \frac{\varepsilon}{2} (\ln \rho)_{,2}) + 2\rho (\mathcal{I}_{,2} + 2\varepsilon \phi_{,2,2}) - \varepsilon \rho_{,2,2}]. \quad (4.8)$$

The Landsbergian property of $\bar{F}$ is characterized by $\bar{\mathcal{I}}_{,a} = 0$ and the Berwaldian property of $\bar{F}$ is characterized by $\bar{\mathcal{I}}_{,a} = 0 = \bar{\mathcal{I}}_{,b}$.

Definition 4.2. The anisotropic conformal change (2.8) is said to be satisfying the anisotropic ϕT -condition if $(\partial_i \phi) T^i_{jkr} = 0$. Similarly, the anisotropic conformal change (2.8) is said to be satisfying the anisotropic $\phi \bar{T}$ -condition if $(\partial_i \phi) \bar{T}^i_{jkr} = 0$.

Lemma 4.1. The anisotropic conformal change (2.8) satisfies the anisotropic ϕT -condition if and only if either $m^i \partial_i \phi = 0$ or F has vanishing T -tensor. Similarly, The anisotropic conformal change (2.8) satisfies the anisotropic $\phi \bar{T}$ -condition if and only if either $m^i \partial_i \phi = \varepsilon \phi_{;2} \ell^i \partial_i \phi$ or $\bar{F}$ has vanishing $\bar{T}$ -tensor.

Proof. The anisotropic conformal change (2.8) satisfies the anisotropic ϕT -condition, by definition, means that

$$0 = (\partial_i \phi) T^i_{jkr} = \frac{I_{;2}}{F} m_j m_k m_r (m^i \partial_i \phi).$$

Which is equivalent to either F has vanishing T -tensor or $m^i \partial_i \phi = 0$. Similarly, from (4.2), (2.8) satisfies the anisotropic $\phi \bar{T}$ -condition is equivalent to

$$0 = (\partial_i \phi) \bar{T}^i_{jkr} = e^\phi \sqrt{\frac{\varepsilon}{\rho}} \frac{\bar{I}_{;2}}{F} (m^i \partial_i \phi - \varepsilon \phi_{;2} \ell^i \partial_i \phi) m_j m_k m_r.$$

Thereby, the anisotropic $\phi \bar{T}$ -condition is satisfied if and only if either $m^i \partial_i \phi = \varepsilon \phi_{;2} \ell^i \partial_i \phi$ or $\bar{I}_{;2} = 0$. Then, either $\bar{F}$ has vanishing $\bar{T}$ -tensor, by [23, Remark 4.1], or $m^i \partial_i \phi = \varepsilon \phi_{;2} \ell^i \partial_i \phi$. $\square$

Theorem 4.1. Let $\bar{F} = e^\phi F$ be the anisotropic conformal transformation (2.8). Then we have:

- (i): If F is a Landsberg metric and (2.8) satisfies the anisotropic ϕT -condition, provided that, $m^i \partial_i \phi \neq 0$, then F is Berwaldian.
- (ii): If $(M, \bar{F})$ is a Landsberg surface and (2.8) satisfies the anisotropic $\phi \bar{T}$ -condition, provided that, $m^i \partial_i \phi \neq \varepsilon \phi_{;2} \ell^i \partial_i \phi$, then $(M, \bar{F})$ is Berwaldian. In other words, a non-Riemannian Landsberg surface $(M, \bar{F})$ is Berwaldian if the anisotropic $\phi \bar{T}$ -condition is satisfied, provided that, (2.8) does not $\bar{C}$ -anisotropic conformal change.

Proof. (i) and first part of (ii) Follows from Lemma 4.1 and the fact that any Landsberg surface with a vanishing T -tensor is Berwaldian. The proof of second part of (ii): As, (2.8) does not satisfy $\bar{C}$ -anisotropic conformal change and $(M, \bar{F})$ is a non-Riemannian surface, then from Proposition 3.1, we get

$$m^i \partial_i \phi \neq \varepsilon \phi_{;2} \ell^i \partial_i \phi. \quad (4.9)$$

Since (2.8) satisfies the anisotropic $\phi \bar{T}$ -condition, then from Lemma 4.1 and (4.9), we get $\bar{I}_{;2} = 0$, consequently, $(M, \bar{F})$ has vanishing $\bar{T}$ -tensor. Hence $(M, \bar{F})$ is Berwaldian. $\square$

Remark 4.1. Every C -anisotropic conformal transformation satisfies the ϕT -condition. Correspondingly, every $\bar{C}$ -anisotropic conformal transformation satisfies the $\phi \bar{T}$ -condition. However, the converse is not true in general, that is, not every anisotropic conformal transformation satisfies the ϕT -condition is C -anisotropic conformal transformation.

Definition 4.3. The anisotropic conformal transformation (2.8) is said to be satisfying the horizontal ϕT -condition if $(\delta_i \phi) T^i_{jkr} = 0$. Similarly, the anisotropic conformal transformation (2.8) is said to be satisfying the horizontal $\phi \bar{T}$ -condition if $(\delta_i \phi) \bar{T}^i_{jkr} = 0$.

Proposition 4.2. Let (M, F) be a conic pseudo-Finsler surface. Then we have the following:

- (i): The anisotropic conformal transformation (2.8) satisfies horizontal ϕT -condition if and only if either F has vanishing T -tensor or $\phi_{,2} = 0$.
- (ii): The anisotropic conformal transformation (2.8) satisfies horizontal $\phi \bar{T}$ -condition if and only if $\bar{F}$ has vanishing $\bar{T}$ -tensor or $\phi_{,2} = \phi_{,1} \phi_{,2}$.

Proof. (i): Assume that the anisotropic conformal transformation (2.8) satisfies horizontal ϕT -condition, then we have

$$0 = (\delta_i \phi) T^i_{jkr} = (\phi_{,1} \ell_i + \phi_{,2} m_i) \mathcal{I}_{,2} m^i m_j m_k m_r = \varepsilon \phi_{,2} \mathcal{I}_{,2} m_j m_k m_r,$$

which is equivalent to that either F has vanishing T -tensor or $\phi_{,2} = 0$.

- (ii): From (4.2), the anisotropic conformal transformation (2.8) satisfies horizontal $\phi \bar{T}$ -condition if and only if

$$\begin{aligned} 0 &= (\delta_i \phi) \bar{T}^i_{jkr} = [(\phi_{,1} \ell_i + \phi_{,2} m_i) \bar{\mathcal{I}}_{,2} (m^i m_j m_k m_r - \varepsilon \phi_{,2} \ell^i m_j m_k m_r)] \\ &= \bar{\mathcal{I}}_{,2} (\varepsilon \phi_{,2} m_j m_k m_r - \varepsilon \phi_{,1} \phi_{,2} m_j m_k m_r). \end{aligned}$$

Hence, the anisotropic conformal transformation (2.8) satisfies horizontal $\phi \bar{T}$ -condition if and only if $\bar{F}$ has vanishing $\bar{T}$ -tensor or $\phi_{,2} = \phi_{,1} \phi_{,2}$. □

Theorem 4.2. Assume that (2.8) is a vertical C-anisotropic conformal change with $\phi_{,2}$ being horizontally constant. Then the following are equivalent:

- (i): The Finsler metric $\bar{F}$ is Landsbergian.
- (ii): The anisotropic conformal change (2.8) satisfies the horizontal $\phi \bar{T}$ -condition.
- (iii): The Finsler metric $\bar{F}$ is Berwaldian or the two geodesic sprays S and $\bar{S}$ are projectively equivalent.

Proof. We assume that (2.8) is a vertical C-anisotropic change. By Proposition 3.5, F is a Riemannian metric and $\phi_{,2}$ is horizontally constant, by assumption. Then, according to [23, Corollary 2.5], we have that ρ and $\rho_{,2}$ are horizontally constant. This implies that

$$\mathcal{I} = \rho_{,1} = \rho_{,2} = \rho_{,2,1} = \rho_{,2,2} = 0. \quad (4.10)$$

(i) $\iff$ (ii) It follows by [23, Proposition 4.7] and Proposition 4.2 (ii).

(ii) $\implies$ (iii) Assume that (2.8) satisfies the horizontal $\phi \bar{T}$ -condition. By Proposition 4.2, we have $\bar{\mathcal{I}}_{,2} = 0$ or $\phi_{,2} = \phi_{,2} \phi_{,1}$. First, if $\bar{\mathcal{I}}_{,2} = 0$, substituting from (4.10) into (4.7) and (4.8), we get

$$\bar{\mathcal{I}}_{,2} = 0 = \bar{\mathcal{I}}_{,1} = \bar{\mathcal{I}}_{,2}. \quad (4.11)$$

Substituting (4.11) into (4.4) and (4.5), we find that $\bar{F}$ is Berwaldian. Secondly, if $\phi_{;2}\phi_{;1} = \phi_{;2}$ and $\phi_{;2}$ is horizontally constant, by assumption and using the commutation formula (2.6), we have

$$0 = \phi_{;2}\phi_{;1} + \phi_{;2,1} - \phi_{;2} = \phi_{;2}\phi_{;1} + \phi_{;1,2} - \phi_{;2} = Q. \quad (4.12)$$

(iii) $\implies$ (i) Let $\bar{F}$ be a Berwald metric then $\bar{F}$ Landsbergian. Also, if the two geodesic spray S and $\bar{S}$ are projectively equivalent, using (4.10), we get

$$Q = \bar{\mathcal{I}}_{;1} = 0. \quad (4.13)$$

Substituting (4.13) into (4.4), we find that $\bar{F}$ is Landsbergian. $\square$

Definition 4.4. The proper anisotropic conformal change (2.8) is said to be satisfying the vertical ϕT -condition if $(\dot{\partial}_i \phi) T^i_{jkr} = 0$. Similarly, (2.8) is said to be satisfying the vertical $\phi \bar{T}$ -condition if $(\dot{\partial}_i \phi) \bar{T}^i_{jkr} = 0$.

Lemma 4.2. Let (M, F) be a conic pseudo-Finsler surface. The anisotropic conformal change (2.8) satisfies the vertical ϕT -condition if and only if F has a vanishing T -tensor.

Proof. We have, by (2.2),

$$(\dot{\partial}_i \phi) T^i_{jkr} = \frac{\mathcal{I}_{;2}}{F} (\dot{\partial}_i \phi) m^i m_j m_k m_r = \frac{\varepsilon \mathcal{I}_{;2}}{F^2} \phi_{;2} m_j m_k m_r.$$

Consequently, the anisotropic conformal transformation (2.8) satisfies the vertical ϕT -condition if and only if F has a vanishing T -tensor. $\square$

According to Lemma 4.2, we have

Remark 4.2. (i): The anisotropic conformal transformation (2.8) satisfies the vertical $\phi \bar{T}$ -condition if and only if $\bar{F}$ has a vanishing $\bar{T}$ -tensor.
(ii): The ϕT -condition is equivalent to the T -condition.

From Remark 4.2 and the fact that any Landsberg surface satisfies the T -condition is Berwaldian, we have

Corollary 4.1. Let F be anisotropically conformal to $\bar{F}$ by (2.8). Then, we have the following:

- (i): If F is a Landsberg metric and (2.8) satisfies the vertical ϕT -condition, then, F is Berwaldian.
- (ii): If $\bar{F}$ is a Landsberg metric and (2.8) satisfies the vertical $\phi \bar{T}$ -condition, then, $\bar{F}$ is Berwaldian.

5. SPECIAL CHOICE OF THE CONFORMAL FACTOR

In this section, we assume that F is a non-Riemannian metric and the conformal factor of the anisotropic conformal transformation (2.8) is the main scalar $\mathcal{I}$ of F , that is,

$$\bar{F}(x, y) = e^{\mathcal{I}(x, y)} F(x, y). \quad (5.1)$$

From (2.9)-(2.15) then we have the following:

$$\sigma = I_{;2;2} + \varepsilon I I_{;2} + 2(I_{;2})^2, \quad (5.2)$$

$$2Q = \varepsilon \rho F^2 (I_{;2} I_{;1} + I_{;1;2} - 2I_{;2}), \quad (5.3)$$

$$2P = -\rho F^2 I_{;2} (I_{;2} I_{;1} + I_{;1;2} - 2I_{;2}) + F^2 I_{;1}. \quad (5.4)$$

Furthermore, from (5.3) and (5.4), we get

$$2\varepsilon I_{;2} Q + 2P = F^2 I_{;1}. \quad (5.5)$$

Proposition 5.1. *Under the anisotropic conformal change (5.1), we have the following:*

- (i): *If the conic pseudo-Finsler surface (M, F) has vanishing T-tensor, then the anisotropic conformal change (5.1) reduces to isotropic conformal change.*
- (ii): *If (M, F) is a Berwald surface, then the geodesic spray is invariant.*
- (iii): *If (M, F) is a Landsberg surface, then the geodesic spray of $\bar{F}$ has the coefficients*

$$\bar{G}^i = G^i - \varepsilon \rho F^2 I_{;2} m^i + \rho F^2 I_{;2} I_{;2} \ell^i.$$

Proof. (i): It follows from the fact that a Finsler F has a vanishing T-tensor if and only if the main scalar (conformal factor) is a function of x only.

(ii): Let (M, F) be a Berwald surface, then $I_{;1} = I_{;2} = 0$. Substituting into (5.3) and (5.4), we get $P = Q = 0$. Then, by (2.12), the geodesic spray is invariant under the anisotropic change.

(iii): Since F is Landsbergian, that is, $I_{;1} = 0$. Thus, (5.3) and (5.4) become $Q = -\varepsilon \rho F^2 I_{;2}$ and $P = \rho F^2 I_{;2} I_{;2}$. Hence, the result follows from (2.12). $\square$

Lemma 5.1. *Assume that the anisotropic conformal change (5.1) satisfies the C-anisotropic conformal change. Then, F is weakly Berwaldian if and only if $G_k^i m^k m_i = 0$.*

Proof. From Proposition 3.1 and (3.3), the anisotropic conformal transformation (5.1) satisfies the C-anisotropic conformal change if and only if $I_{;2} = -\frac{\varepsilon \phi_{;2}}{F} G_k^i m^k m_i$. Then F is weakly Berwald if and only if $G_k^i m^k m_i = 0$. $\square$

Let $\bar{F}$ be a non-Riemannian metric. From Proposition 3.1: the property of C-anisotropic conformal is invariant if and only if

$$\ell^i \partial_i I = 0 \iff I_{;1} = -\frac{2}{F^2} I_{;2} G^k m_k.$$

Lemma 5.2. *Under the anisotropic conformal change (5.1), if $\bar{F}$ is a non-Riemannian metric and the property of C-anisotropic conformal is invariant, then, the Finsler metric F is Landsbergian if and only if it is projectively flat metric.*

Theorem 5.1. *The Finsler surface (M, F) is Berwaldian if one of the following is satisfied:*

- (i): *(M, F) is a Landsberg surface and (5.1) is horizontal C-anisotropic conformal transformation.*

(ii): (M, F) is a Landsberg or weak Berwald surface and (5.1) is horizontal $\bar{C}$ -anisotropic conformal transformation.

Proof. **(i):** Since (M, F) is a Landsberg surface (i.e., $\mathcal{I}_1 = 0$) and (5.1) is a horizontal C -anisotropic conformal transformation, then from Proposition 3.1, we have $\mathcal{I}_2 = 0$. Thus, (M, F) is a Berwald surface.

(ii): From Proposition 3.1, the anisotropic conformal change (5.1) is horizontal $\bar{C}$ -anisotropic if and only if $\mathcal{I}_2 = \mathcal{I}_2 \mathcal{I}_1$. Then (M, F) is a Berwald surface if (M, F) is a Landsberg or weak Berwald surface. □

6. A FINSLERIAN SCHWARZSCHILD-DE SITTER SOLUTION

An anisotropic conformal transformation differs from a conformal transformation in which it can transform a Riemannian metric into a Finslerian metric [22, 23]. For instance, consider the Riemannian metric defined on the 2-dimensional sphere by:

$$F(\theta, \eta; y^\theta, y^\eta) = \sqrt{(y^\theta)^2 + \sin^2(\theta) (y^\eta)^2}.$$

By using the conformal factor

$$\phi = \ln \left(\frac{\sqrt{(1 - a^2 \sin^2 \theta) (y^\theta)^2 + \sin^2 \theta (y^\eta)^2} - a \sin^2 \theta y^\eta}{(1 - a^2 \sin^2 \theta) \sqrt{(y^\theta)^2 + \sin^2 \theta (y^\eta)^2}} \right),$$

we get the Finsler metric

$$\bar{F} = e^\phi F = \frac{\sqrt{(1 - a^2 \sin^2(\theta)) (y^\theta)^2 + \sin^2(\theta) (y^\eta)^2}}{1 - a^2 \sin^2(\theta)} - \frac{a \sin^2(\theta) y^\eta}{1 - a^2 \sin^2(\theta)}, \quad (6.1)$$

where $0 \leq a < 1$ is a constant known as the Finsler parameter. It is worth noting that $\bar{F}$ is a two dimensional Randers-Finsler space with constant positive flag curvature $\lambda = 1$ and is homotopy equivalent to the two dimensional sphere with volume 4π , when $a = 0$, $\bar{F} = F$. The Finslerian analogue of Birkhoff's theorem states that a Finslerian gravitational field with the symmetry of a "Finslerian sphere" in vacuum must be static [14]. Building on this, the Einstein field equations with a non-zero cosmological constant ($\Lambda \neq 0$) have been extended to the Finslerian framework in [16], resulting in a Finslerian Schwarzschild-de Sitter solution that possesses the symmetry of a "Finslerian sphere". An example of such a Finslerian sphere is given by the expression (6.1). We have the following:

- (i):** The Finsler metric F is a Riemannian metric of constant curvature and does not satisfy the Hamel's equations, therefore it is not locally projectively flat.
- (ii):** F is C -anisotropic conformal, horizontal C -anisotropic, vertical C -anisotropic and satisfies the ϕT -condition, as F is a Riemannian metric.

(iii): In a non-Riemannian Randers space $(M, \bar{F})$, the 1-form is not covariantly constant with respect to the Riemannian metric $\alpha = \frac{\sqrt{(1-a^2 \sin^2(\theta)) (y^\theta)^2 + \sin^2(\theta) (y^\eta)^2}}{1-a^2 \sin^2(\theta)}$. Hence, it is neither Berwaldian nor Landsbergian (by using [18, Theorem 3.2].)

In more details: the Riemannian metric a_{ij} and its inverse a^{ij} are given by

$$(a_{ij}) = \begin{pmatrix} \frac{1}{1-a^2 \sin^2 \theta} & 0 \\ 0 & \frac{\sin^2 \theta}{(1-a^2 \sin^2 \theta)^2} \end{pmatrix}, \quad (a^{ij}) = \begin{pmatrix} 1-a^2 \sin^2 \theta & 0 \\ 0 & \frac{(1-a^2 \sin^2 \theta)^2}{\sin^2 \theta} \end{pmatrix}.$$

The corresponding non-zero coefficients γ_{ij}^h of the Levi-Civita connection are given by:

$$\gamma_{\theta\theta}^\theta = \frac{a^2 \sin \theta \cos \theta}{(1-a^2 \sin^2 \theta)}, \quad \gamma_{\theta\eta}^\eta = \frac{\cos \theta (1+a^2 \sin^2 \theta)}{\sin \theta (1-a^2 \sin^2 \theta)}, \quad \gamma_{\eta\eta}^\theta = -\frac{\cos \theta \sin \theta (1+a^2 \sin^2 \theta)}{(1-a^2 \sin^2 \theta)^2}.$$

Since, the 1-form $\beta = b_i y^i$, with $b_\theta = 0$, $b_\eta = \frac{-a \sin \theta}{1-a^2 \sin^2 \theta}$, the covariant derivative of b_θ with respect to the Levi-Civita connection is given by:

$$\nabla_\eta b_\theta = \partial_\eta b_\theta - \gamma_{\theta\eta}^\theta b_\theta - \gamma_{\theta\eta}^\eta b_\eta = -\frac{-a \sin \theta}{(1-a^2 \sin^2 \theta)} \frac{\cos \theta (1+a^2 \sin^2 \theta)}{\sin \theta (1-a^2 \sin^2 \theta)} = \frac{a \cos \theta (1+a^2 \sin^2 \theta)}{(1-a^2 \sin^2 \theta)^2}.$$

Therefore, the 1-form is not covariantly constant with respect to the Riemannian metric α .

(iv): The Finsler metric $\bar{F}$ is of constant curvature $\lambda = 1$ and not locally projectively flat [19, Theorem 1.1].

(v): By using Mapple's package uploaded to

https://github.com/salahelgendi/Special_anisotropics_Section_6

we get $m^i \partial_i \phi - \varepsilon \phi_{,2} \ell^i \partial_i \phi \neq 0$ and $\phi_{,2} \neq \phi_{,2} \phi_{,1}$. According to Proposition 3.1 and Theorem 3.1, the anisotropic conformal change (6.1) is not $\bar{C}$ -anisotropic conformal and does not satisfy the horizontal $\bar{C}$ -anisotropic property, as $\bar{F}$ is not Riemannian. Also, (6.1) is not a vertical $\bar{C}$ -anisotropic conformal change.

(vii): As F is a Riemannian metric, it admits a semi-concurrent vector field. However, $\bar{F}$ is not (as it is a non-Riemannian metric).

7. CONCLUDING REMARKS

In conformal theory, special conformal transformations like C-conformal transformations have been studied. This transformation depends on the Cartan tensor (which is invariant under the conformal change) and the conformal factor (which only depends on position). The Cartan tensor is not invariant under anisotropic conformal transformations and the conformal factor depends on the position and direction. This motivated us to study C-anisotropic, $\bar{C}$ -anisotropic, horizontal C-anisotropic and horizontal $\bar{C}$ -anisotropic transformations.

Under the anisotropic conformal transformation $\bar{F} = e^\phi F$ defined by (2.8), the following points are to be highlighted:

(i): We have characterized C-anisotropic, $\bar{C}$ -anisotropic, horizontal C-anisotropic and horizontal $\bar{C}$ -anisotropic transformations.

- (ii):** We have determined the conditions under which the properties of C -anisotropic and horizontal C -anisotropic are preserved under the anisotropic conformal change (2.8).
- (iii):** Let (2.8) be the anisotropic conformal transformation between two non-Riemannian surfaces then we have:
- (a):** If the property of C -anisotropic conformal is preserved, then F is projectively flat if and only if ϕ is a first integral of the geodesic spray S .
 - (b):** If F is a flat metric and the anisotropic conformal transformation (2.8) is $\bar{C}$ -anisotropic conformal change, then the two sprays S and $\bar{S}$ are projectively equivalent if and only if $\phi_{,2}$ is a first integral of the geodesic spray S .
 - (c):** If (2.8) satisfies the horizontal C -anisotropic (or horizontal $\bar{C}$ -anisotropic), the geodesic spray is invariant if and only if the conformal factor is first integral of the geodesic spray S .
 - (d):** If (2.8) satisfies the horizontal $\bar{C}$ -anisotropic, the two geodesic sprays S and $\bar{S}$ are projectively equivalent if and only if $\phi_{,2}$ is a first integral of the geodesic spray S .
 - (e):** If the conformal factor is a function of x only, the C -anisotropic, $\bar{C}$ -anisotropic, horizontal C -anisotropic and horizontal $\bar{C}$ -anisotropic transformations are reduced to the well-known C -conformal transformation. However, the vertical C -anisotropic conformal transformation does not reduce to C -conformal and it characterizes the property of F to be a Riemannian metric.

Secondly, in conformal transformation, a smooth non-constant function σ on M that satisfies certain conditions with the T -tensor is called the σT -condition. Based on the fact that the T -tensor is not invariant under the anisotropic conformal transformation (2.8) and the conformal factor depend on position and direction arguments, we discuss under what condition (2.8) satisfies ϕT -condition, $\phi \bar{T}$ -condition, horizontal ϕT -condition and horizontal $\phi \bar{T}$ -condition. This is achieved through the existence of a function ϕ that satisfies certain conditions with the T and $\bar{T}$ -tensors.

- (i):** We have established the characterization of the anisotropic conformal transformation (2.8) satisfies the ϕT -condition, $\phi \bar{T}$ -condition, horizontal ϕT -condition and horizontal $\phi \bar{T}$ -condition.
- (ii):** Since any Landsberg metric satisfying the σT -condition is Berwaldian, we have found under what conditions a Landsberg metric F (resp. $\bar{F}$) that satisfies ϕT -condition or horizontal ϕT -condition (resp. $\phi \bar{T}$ -condition or and horizontal $\phi \bar{T}$ -condition) is Berwaldian.
- (iii):** Under a vertical C -anisotropic conformal change with horizontally constant $\phi_{,2}$, the Finsler metric $\bar{F}$ is Landsbergian if and only if the transformation satisfies the horizontal $\phi \bar{T}$ -condition, which is also equivalent to $\bar{F}$ being Berwaldian or the two geodesic sprays S and $\bar{S}$ are projectively equivalent.
- (iv):** The property that the anisotropic conformal transformation (2.8) satisfies the vertical ϕT -condition is equivalent to that it satisfies the T -condition.

(v): If the conformal factor depends on the position only, the ϕT -condition, $\phi \bar{T}$ -condition, horizontal ϕT -condition and horizontal $\phi \bar{T}$ -condition are simplified to the notion of σT -condition. However, the ϕT -condition does not simplify to the σT -condition and is equivalent to the T -condition.

Finally, we provide the following table which summarizes all special anisotropic conformal transformations that we studied.

Table: Equivalence of special anisotropic conformal changes

Special anisotropic change	equivalent conditions
C-anisotropic	F is Riemannian or $m^i \partial_i \phi = 0$
$\bar{C}$ -anisotropic	$\bar{F}$ is Riemannian or $m^i \partial_i \phi - \varepsilon \phi_{,2} \ell^i \partial_i \phi = 0$
horizontal C-anisotropic	F is Riemannian or $\phi_{,2} = 0$
horizontal $\bar{C}$ -anisotropic	$\bar{F}$ is Riemannian or $\phi_{,2} = \phi_{,2} \phi_{,1}$
vertical C-anisotropic	F is Riemannian
ϕT -condition	F has vanishing T -tensor or $m^i \partial_i \phi = 0$
$\phi \bar{T}$ -condition	$\bar{F}$ has vanishing $\bar{T}$ -tensor or $m^i \partial_i \phi = 0$
horizontal ϕT -condition	F has vanishing T -tensor or $\phi_{,2} = 0$
horizontal $\phi \bar{T}$ -condition	$\bar{F}$ has vanishing $\bar{T}$ -tensor or $\phi_{,2} = \phi_{,2} \phi_{,1}$
vertical ϕT	F is Riemannian

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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