

## VALUE DISTRIBUTION AND UNIQUENESS THEOREMS FOR DIFFERENCE OF ENTIRE AND MEROMORPHIC FUNCTIONS

SUBHAS S. BHOOSNURMATH\* AND SMITA R KABBUR

ABSTRACT. We investigate the value distribution and uniqueness problems of difference polynomials of entire and meromorphic functions.

### 1. INTRODUCTION AND MAIN RESULTS

Let  $f(z)$  be a meromorphic functions of finite order. We define difference operator as ,

$$\Delta_c f = f(z+c) - f(z), \quad \text{and} \quad \Delta_c^n f = \Delta_c^{n-1}(\Delta_c f), \quad n \geq 2,$$

where  $c$  is a non-zero constant. In particular, if  $c = 1$  we use the usual difference notation  $\Delta_c f = \Delta f$ .

Certain estimates involving the derivative  $f \mapsto f'$  of a meromorphic function play key roles in the construction and applications of classical Nevanlinna theory. Recently, there has been an increasing interest in studying difference equations in the complex plane. Halburd and Korhonen [2] established a version of Nevanlinna theory based on difference operator. Bergweiler and Langley [12] considered the value distribution of zeros of difference operators that can be viewed as discrete analogues of the zeros of  $f'(z)$ . Ishizaki and Yanagihara [13] developed a version of Wiman - Valiron theory for difference equations of entire functions of small growth. Growth estimates for the difference analogue of logarithmic derivative  $\frac{f(z+c)}{f(z)}$  were given by Halburd and Korhonen [1] . This result has potentially large number of applications in the study of difference equation. Many ideas and methods from the theory of differential equations are utilized to obtain information about meromorphic solutions of difference equations. The analogue of Clunie Lemma used to ensure that finite order meromorphic solutions of certain non-linear difference equations have large number of poles. All concepts of Nevanlinna theory related to ramification have natural difference analogue.

Let  $f$  be a transcendental entire function and  $n$  be a positive integer. Hayman [14] and Clunie [22] proved that  $f^n f'$  assumes every non-zero value  $a \in \mathbb{C}$  infinitely often.

Let  $f$  be a transcendental entire function. As for the value distribution of differential polynomial  $f^n(f-1)f'$ , Fang [15] showed that  $f^n(f-1)f'$  assumes every non-zero value  $a \in \mathbb{C}$  infinitely often for  $n \geq 4$ . We recall the following uniqueness theorem due to Lin and Yi [16, 17].

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## 2. PRELIMINARIES AND LEMMAS.

**Theorem A.** Let  $f$  and  $g$  be two non-constant (resp transcendental) entire functions and let  $n \geq 7$  be an integer. If  $f^n(f-1)f'$  and  $g^n(g-1)g'$  share 1 (resp  $z$ ) CM then  $f = g$ .

Recently, value distribution in difference analogue has become a subject of great interest. For analogue results in difference Laine and Yang [4] proved the following result.

**Theorem B.** Let  $f$  be a transcendental entire function of finite order and  $c$  be a non-zero complex constant. Then for  $n \geq 2$ ,  $f(z)^n f(z+c)$  assumes every non-zero value  $a \in \mathbb{C}$  infinitely often.

Following analogue results in difference are proved by J. Zhang [5].

**Theorem C.** Let  $f(z)$  be a transcendental entire function of finite order and  $\alpha(z)$  be a small function with respect to  $f(z)$ . Suppose that  $c$  is non-zero complex constant and  $n \geq 2$  is an integer then  $f(z)^n(f(z)-1)f(z+c)$  has infinitely many zeros.

**Theorem D.** Let  $f(z)$  and  $g(z)$  be a transcendental entire function of finite order and  $\alpha(z)$  be a small function with respect to both  $f(z)$  and  $g(z)$ . Suppose that  $c$  is non-zero complex constant and let  $n \geq 7$ . If  $f(z)^n(f(z)-1)f(z+c)$  and  $g(z)^n(g(z)-1)g(z+c)$  share  $\alpha(z)$  CM, then  $f(z) \equiv g(z)$ .

**In this paper we consider the difference analogue of  $f^n(f^m-1)f'$  and prove Theorem 2. For  $m = 1$  our result reduces to Theorem 4. Also proved Theorem 3 where we investigate the uniqueness problem when the difference polynomial share  $\alpha(z)$  with ignoring multiplicity.**

For a non-constant meromorphic function  $f$  and a set  $S \subset \mathbb{C} \cup \{\infty\}$ , we define  $E_f(S) = \bigcup_{a \in S} \{z \in \mathbb{C} \mid f(z) = a, \text{ counting multiplicities}\}$ . We say that two non-constant meromorphic functions  $f$  and  $g$  share  $a$  CM, if  $E_f(S) = E_g(S)$  and  $S = \{a\}$ .

In 1976, Gross [18] proved that there exists three finite sets  $S_j (j = 1, 2, 3)$  such that for any two non-constant entire functions  $f$  and  $g$ ,  $E_f(S_j) = E_g(S_j) (j = 1, 2, 3)$  imply  $f = g$ . In the same paper, Gross posed the following question:

**Question 1.** Can one find two (or possibly even one) finite sets  $S_j (j = 1, 2)$  such that any two entire functions  $f$  and  $g$  satisfying  $E_f(S_j) = E_g(S_j) (j = 1, 2)$  must be identical?

Many authors have worked on it and got related results. We recall the following result by Li and Yang [19].

**Theorem E.** Let  $m \geq 2$ ,  $n > 2m+6$  with  $n$  and  $n-m$  having no common factors. Let  $a$  and  $b$  be two non-zero constants such that the equation  $\omega^n + a\omega^{n-m} + b = 0$  has no multiple roots. Let  $S = \{\omega \mid \omega^n + a\omega^{n-m} + b = 0\}$ . Then for any two non-constant meromorphic functions  $f$  and  $g$ , the conditions  $E_f(S) = E_g(S)$  and  $E_f\{\infty\} = E_g\{\infty\}$  imply  $f = g$ .

J Zhang [5] considered the difference analogue of this result and proved the following result.

**Theorem F.** Let  $m \geq 2$ ,  $n \geq 2m+4$  with  $n$  and  $n-m$  having no common factors. Let  $a$  and  $b$  be two non-zero constants such that the equation  $\omega^n + a\omega^{n-m} + b = 0$  has no multiple roots. Let  $S = \{\omega \mid \omega^n + a\omega^{n-m} + b = 0\}$ . Suppose that  $f$  is a non-constant meromorphic function of finite order. Then  $E_{f(z)}(S) = E_{f(z+c)}(S)$

and  $E_{f(z)}\{\infty\} = E_{f(z+c)}\{\infty\}$  imply  $f(z) = f(z+c)$ .

In 1998, Frank - Reinders [20] proved following result. Let the polynomial  $P$  be defined as,

$$P(\omega) = \frac{(n-1)(n-2)}{2}\omega^n - n(n-2)\omega^{n-1} + \frac{n(n-1)}{2}\omega^{n-2} - c,$$

where  $n(\geq 3)$  is an integer and  $c(\neq 0, 1)$  is a constant.

**Theorem G.** Let  $S = \{\omega | P(\omega) = 0\}$ , where  $P(\omega)$  is as defined above and  $n \geq 11$  be an integer. Then for any two non-constant meromorphic functions  $f$  and  $g$  the condition  $E_{f(z)}(S) = E_{g(z)}(S)$  implies  $f = g$ .

In this section, **we consider the difference analogue of this result and prove Theorem 4. The techniques used here greatly improves the condition on  $n$  from ' $n \geq 11$ ' to ' $n \geq 8$ '.**

For a non-constant meromorphic function  $f$  and a set  $S \subset \mathbb{C} \cup \{\infty\}$ , we define  $E_f(S) = \bigcup_{a \in S} \{z \in f(z) - a = 0, \text{ counting multiplicities}\}$ . We say that two non-constant meromorphic functions  $f$  and  $g$  share a CM, if  $E_f(S) = E_g(S)$  and  $S = \{a\}$ .

In 1976, Gross [18] proved that there exists three finite sets  $S_j (j = 1, 2, 3)$  such that for any two non-constant entire functions  $f$  and  $g$ ,  $E_f(S_j) = E_g(S_j) (j = 1, 2, 3)$  imply  $f = g$ . In the same paper, Gross posed the following question:

**Question 1.** Can one find two (or possibly even one) finite sets  $S_j (j = 1, 2)$  such that any two entire functions  $f$  and  $g$  satisfying  $E_f(S_j) = E_g(S_j) (j = 1, 2)$  must be identical?

Many authors have worked on it and got related results. We recall the following result by Li and Yang [19].

**Theorem H.** Let  $m \geq 2$ ,  $n > 2m+6$  with  $n$  and  $n-m$  having no common factors. Let  $a$  and  $b$  be two non-zero constants such that the equation  $\omega^n + a\omega^{n-m} + b = 0$  has no multiple roots. Let  $S = \{\omega | \omega^n + a\omega^{n-m} + b = 0\}$ . Then for any two non-constant meromorphic functions  $f$  and  $g$ , the conditions  $E_f(S) = E_g(S)$  and  $E_f\{\infty\} = E_g\{\infty\}$  imply  $f = g$ .

J Zhang [5] considered the difference analogue of this result and proved the following result.

**Theorem I.** Let  $m \geq 2$ ,  $n \geq 2m+4$  with  $n$  and  $n-m$  having no common factors. Let  $a$  and  $b$  be two non-zero constants such that the equation  $\omega^n + a\omega^{n-m} + b = 0$  has no multiple roots. Let  $S = \{\omega | \omega^n + a\omega^{n-m} + b = 0\}$ . Suppose that  $f$  is a non-constant meromorphic function of finite order. Then  $E_{f(z)}(S) = E_{f(z+c)}(S)$  and  $E_{f(z)}\{\infty\} = E_{f(z+c)}\{\infty\}$  imply  $f(z) = f(z+c)$ .

In 1998, Frank - Reinders [20] proved following result. Let the polynomial  $P$  be defined as,

$$P(\omega) = \frac{(n-1)(n-2)}{2}\omega^n - n(n-2)\omega^{n-1} + \frac{n(n-1)}{2}\omega^{n-2} - c,$$

where  $n(\geq 3)$  is an integer and  $c(\neq 0, 1)$  is a constant.

**Theorem J.** Let  $S = \{\omega | P(\omega) = 0\}$ , where  $P(\omega)$  is as defined above and  $n \geq 11$  be an integer. Then for any two non-constant meromorphic functions  $f$  and  $g$  the condition  $E_{f(z)}(S) = E_{g(z)}(S)$  implies  $f = g$ .

In this section, **we consider the difference analogue of this result and**

prove **Theorem 4**. The techniques used here greatly improves the condition on  $n$  from ' $n \geq 11$ ' to ' $n \geq 8$ '.

3. LEMMAS.

In order to prove our results, we need following lemmas.

**Lemma 1.** [12]. Let  $f(z)$  be a meromorphic function of finite order  $\rho$  and let  $c$  be a fixed non-zero complex constant. Then for each  $\epsilon > 0$ , we have

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = O(r^{\rho-1+\epsilon}).$$

**Lemma 2.** [6]. Let  $f(z)$  be a meromorphic function of finite order  $\rho$  and let  $c$  be a fixed non-zero complex constant. Then for each  $\epsilon > 0$ , we have

$$T(r, f(z+c)) = T(r, f) + O(r^{\rho-1+\epsilon}).$$

It is evident that  $S(r, f(z+c)) = S(r, f)$ .

**Lemma 3.** [7]. Let  $f(z)$  be a meromorphic function of finite order  $\rho$  and let  $c$  be a fixed non-zero complex constant. Then

$$N\left(r, \frac{1}{f(z+c)}\right) \leq N\left(r, \frac{1}{f}\right) + S(r, f),$$

$$N(r, f(z+c)) \leq N(r, f) + S(r, f),$$

$$\bar{N}\left(r, \frac{1}{f(z+c)}\right) \leq \bar{N}\left(r, \frac{1}{f}\right) + S(r, f),$$

$$\bar{N}(r, f(z+c)) \leq \bar{N}(r, f) + S(r, f),$$

outside of possible exceptional set with finite logarithmic measure.

**Lemma 4.** Let  $f(z)$  be a transcendental entire function of finite order  $\rho$ . Let  $F = f(z)^n(f(z)^m - 1)f(z+c)$ . Then

$$(1) \quad T(r, F) = (n+m+1)T(r, f) + S(r, f)$$

From this lemma it is clear that  $S(r, F) = S(r, f)$  and similarly  $S(r, G) = S(r, f)$

**Proof.** Since  $f$  is entire function of finite order, we deduce from Lemma 1 and the standard Valiron Mohon'ko theorem that,

$$\begin{aligned} (n+m+1)T(r, f(z)) &= T(r, f(z)^{n+1}(f(z)^m - 1)) + S(r, f) \\ &= m(r, f(z)^{n+1}(f(z)^m - 1)) + S(r, f) \end{aligned}$$

$$\begin{aligned} \text{or, } (n+m+1)T(r, f(z)) &\leq m\left(r, \frac{f(z)^{n+1}(f(z)^m - 1)}{f(z)^n(f(z)^m - 1)f(z+c)}\right) + m(r, F) + S(r, f) \\ &\leq m\left(r, \frac{f(z)}{f(z+c)}\right) + m(r, F) + S(r, f) \\ &\leq T(r, F) + S(r, f) \end{aligned}$$

Therefore, we have

$$(2) \quad (n+m+1)T(r, f(z)) \leq T(r, F) + S(r, f)$$

On the other hand, by Lemma 2 and the fact that  $f$  is a transcendental entire function of finite order, we get

$$\begin{aligned} T(r, F) &\leq T(r, f(z)^n(f(z)^m - 1)) + T(r, f(z+c)) + S(r, f) \\ &= (n+m)T(r, f(z)) + T(r, f(z+c)) + S(r, f) \\ &\leq (n+m+1)T(r, f(z)) + S(r, f) \end{aligned}$$

$$(3) \quad \text{i.e., } T(r, F) \leq (n+m+1)T(r, f(z)) + S(r, f)$$

Thus (3.1) follows from (3.2) and (3.3).

**Lemma 5.** Let  $f(z)$  and  $g(z)$  be a meromorphic function of finite order. If  $n \geq m+6$ ,  $n, m$  are positive integers and

$$(4) \quad f(z)^n(f(z)^m - 1)f(z+c) = g(z)^n(g(z)^m - 1)g(z+c)$$

then  $f = tg$ , where  $t^m = 1$ .

**Proof.** Let  $h(z) = \frac{f(z)}{g(z)}$ . If  $h(z)^{n+m}h(z+c) \neq 1$ , then from (3.4) we have

$$\begin{aligned} g(z)^n h(z)^n (g(z)^m h(z)^m - 1) g(z+c) h(z+c) &= g(z)^n (g(z)^m - 1) g(z+c) \\ h(z)^n (g(z)^m h(z)^m - 1) h(z+c) &= g(z)^m - 1 \\ h(z)^{n+m} h(z+c) g(z)^m - h(z)^n h(z+c) - g(z)^m + 1 &= 0 \\ g(z)^m (h(z)^{n+m} h(z+c) - 1) &= h(z)^n h(z+c) - 1 \end{aligned}$$

$$(5) \quad \text{or, } g(z)^m = \frac{h(z)^n h(z+c) - 1}{h(z)^{n+m} h(z+c) - 1}$$

If 1 is a Picard exceptional value of  $h(z)^{n+m}h(z+c)$ , applying the Nevanlinna second main theorem with Lemma 2, we get

$$\begin{aligned} T(r, h(z)^{n+m}h(z+c)) &\leq \bar{N}(r, h(z)^{n+m}h(z+c)) + \bar{N}\left(r, \frac{1}{h(z)^{n+m}h(z+c)}\right) \\ &\quad + \bar{N}\left(r, \frac{1}{h(z)^{n+m}h(z+c) - 1}\right) + S(r, h) \\ &\leq 2T(r, h(z)) + 2T(r, h(z+c)) + S(r, h) \end{aligned}$$

$$(6) \quad \text{or, } T(r, h(z)^{n+m}h(z+c)) \leq 4T(r, h(z)) + S(r, h)$$

On the other hand, combining the standard Valiron - Mohon'ko theorem with (3.6) and Lemma 2, we get

$$\begin{aligned} (n+m)T(r, h(z)) &= T(r, h(z)^{n+m}) + S(r, h) \\ &\leq T(r, h(z)^{n+m}h(z+c)) + T(r, h(z+c)) + S(r, h) \\ &\leq 5T(r, h(z)) + S(r, h) \end{aligned}$$

$$(7) \quad \text{or, } (n+m-5)T(r, h(z)) \leq S(r, h)$$

which contradicts the hypothesis that  $n \geq m+6$ .

Therefore 1 is not a Picard exceptional value of  $h(z)^{n+m}h(z+c)$ . Thus there exists  $z_0$  such that  $h(z_0)^{n+m}h(z_0+c) = 1$  then by (3.5), we have  $h(z_0)^n h(z_0+c) = 1$ . Hence  $h(z_0)^m = 1$ , and

$$(8) \quad \bar{N}\left(r, \frac{1}{h(z)^{n+m}h(z+c) - 1}\right) \leq \bar{N}\left(r, \frac{1}{h(z)^m - 1}\right) \leq mT(r, h) + O(1)$$

Denote,

$$(9) \quad H(z) = h(z)^{n+m}h(z+c)$$

We have  $T(r, H) \leq (n+m+1)T(r, h) + S(r, h)$ . Applying the second main theorem to  $H$  and using Lemma 2 and (3.8), we get

$$\begin{aligned} T(r, H) &\leq \bar{N}(r, H(z)) + \bar{N}\left(r, \frac{1}{H(z)}\right) + \bar{N}\left(r, \frac{1}{H(z)-1}\right) + S(r, H) \\ &\leq \bar{N}(r, H(z)) + \bar{N}\left(r, \frac{1}{H(z)}\right) + mT(r, H(z)) + S(r, h) \\ &\leq (m+4)T(r, h) + S(r, h) \end{aligned}$$

Therefore, we have

$$(10) \quad T(r, H(z)) \leq (m+4)T(r, h) + S(r, h).$$

On the other hand using (3.9) and (3.10) we have

$$\begin{aligned} (n+m)T(r, h) &= T(r, h(z)^{n+m}) + S(r, h) \\ &\leq T(r, H(z)) + T(r, h(z+c)) + S(r, h) \\ &\leq (m+5)T(r, h) + S(r, h) \end{aligned}$$

or,  $(n-5)T(r, h) \leq S(r, h)$ . which contradicts our hypothesis,  $n \geq m+6$ .

Therefore,  $h(z)^{n+m}h(z+c) \equiv 1$ , and  $h(z)^nh(z+c) \equiv 1$ . Thus  $h(z)^m \equiv 1$ .

Hence we get  $f(z) = tg(z)$ , where  $t^m = 1$ .

**Lemma 6.** [21]. Let  $F$  and  $G$  be two non-constant meromorphic function. If  $F$  and  $G$  share 1 CM, then one of the following cases hold:

$$i) \max \{T(r, F), T(r, G)\} \leq N_2\left(r, \frac{1}{F}\right) + N_2(r, F) + N_2(r, G) + \left(r, \frac{1}{G}\right) + S(r, F) + S(r, G),$$

$$ii) \quad FG \equiv 1,$$

$$iii) \quad F \equiv G,$$

where  $N_2\left(r, \frac{1}{F}\right)$  denoted the counting function of zeros of  $F$  such that simple zeros are counted once and multiple zeros are counted twice.

**Lemma 7.** [8]. Let  $F$  and  $G$  be two non-constant meromorphic function. Let  $F$  and  $G$  share 1 IM. Let,

$$(11) \quad H = \frac{F''}{F'} - 2\frac{F'}{F-1} - \frac{G''}{G'} + 2\frac{G'}{G-1}$$

If  $H \neq 0$ , then

$$\begin{aligned} T(r, F) + T(r, G) &\leq 2 \left[ N_2\left(r, \frac{1}{F}\right) + N_2(r, F) + N_2(r, G) + N_2\left(r, \frac{1}{G}\right) \right] \\ &\quad + 3 \left[ \bar{N}(r, F) + \bar{N}(r, G) + \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}\left(r, \frac{1}{G}\right) \right] \\ (12) \quad &+ S(r, F) + S(r, G) \end{aligned}$$

**Lemma 8.** [20]. Let

$$Q(\omega) = (n-1)^2(\omega^n - 1)(\omega^{n-2} - 1) - n(n-2)(\omega^{n-1} - 1)^2$$

be a polynomial of degree  $2n-2$  ( $n \geq 3$ ).

Then

$$Q(\omega) = (\omega - 1)^4(\omega - \beta_1)(\omega - \beta_2), \dots, (\omega - \beta_{2n-6})$$

where  $\beta_j \in \mathbb{C} \setminus \{0, 1\}$ .

#### 4. STATEMENT AND PROOF OF MAIN RESULT.

**Theorem 1.** Let  $f(z)$  be entire function of finite order and  $\alpha(z)$  be a small function with respect to  $f(z)$ . Suppose that  $c$  is a non-zero complex constant and  $n$  is an integer. If  $n \geq 2$ , then  $f(z)^n(f(z)^m - 1)f(z+c) - \alpha(z)$  has infinitely many zeros.

**Proof.** Let  $F(z) = f(z)^n(f(z)^m - 1)f(z+c)$ .

Contrary to the assumption, suppose  $f(z)^n(f(z)^m - 1)f(z+c) - \alpha(z)$  has finitely many zeros.

By second fundamental theorem, Lemma 3, we have

$$\begin{aligned} T(r, F) &\leq \bar{N}(r, F(z)) + \bar{N}\left(r, \frac{1}{F(z)}\right) + \bar{N}\left(r, \frac{1}{F(z)-1}\right) + S(r, F) \\ &\leq \bar{N}\left(r, \frac{1}{f(z)^n(f(z)^m - 1)f(z+c)}\right) + \bar{N}\left(r, \frac{1}{F(z) - \alpha(z)}\right) + S(r, f) \\ &\leq (m+2)T(r, f) + S(r, f) \end{aligned}$$

using Lemma 4 we get

$$(n-1)T(r, f) \leq S(r, f), \text{ which contradicts our assumption.}$$

**Theorem 2.** Let  $f(z)$  and  $g(z)$  be two transcendental entire function of finite order and  $\alpha(z)$  be a small function with respect to  $f(z)$  and  $g(z)$ . Suppose that  $c$  is a non-zero complex constant and let  $n \geq m+6$ . If  $f(z)^n(f(z)^m - 1)f(z+c)$  and  $g(z)^n(g(z)^m - 1)g(z+c)$  share  $\alpha(z)$  CM, then  $f(z) \equiv tg(z)$ , where  $t^m = 1$ .

**Proof.** Let

(13)

$$F(z) = \frac{f(z)^n(f(z)^m - 1)f(z+c)}{\alpha(z)} \quad \text{and} \quad G(z) = \frac{g(z)^n(g(z)^m - 1)g(z+c)}{\alpha(z)}$$

Then  $F(z)$  and  $G(z)$  share 1 CM, except the zeros and poles of  $\alpha(z)$ . By Lemma 4

$$(14) \quad T(r, F(z)) = (n+m+1)T(r, f) + S(r, f)$$

$$(15) \quad \text{and} \quad T(r, G(z)) = (n+m+1)T(r, g) + S(r, g)$$

Since  $f$  and  $g$  are transcendental entire functions,

$$N_2(r, F) = S(r, f), \quad \text{and} \quad N_2(r, G) = S(r, g)$$

By Lemma 3, we have

$$\begin{aligned} N_2\left(r, \frac{1}{F}\right) &\leq 2\bar{N}\left(r, \frac{1}{f^n}\right) + N\left(r, \frac{1}{f^m - 1}\right) + N\left(r, \frac{1}{f(z+c)}\right) + S(r, f) \\ &\leq 2\bar{N}\left(r, \frac{1}{f}\right) + mT(r, f) + T(r, f(z+c)) + S(r, f) \\ &\leq (m+3)T(r, f) + S(r, f) \end{aligned}$$

Therefore

$$(16) \quad N_2(r, F) + N_2\left(r, \frac{1}{F}\right) \leq (m+3)T(r, f) + S(r, f)$$

Similarly,

$$(17) \quad N_2(r, G) + N_2\left(r, \frac{1}{G}\right) \leq (m+3)T(r, g) + S(r, g)$$

By condition of the theorem, Suppose case (i) of Lemma 6 holds.

Substituting (4.4) and (4.5) in (i) of Lemma 6, we have

$$T(r, F) + T(r, G) \leq (2m+6)(T(r, f) + T(r, g)) + S(r, f) + S(r, g)$$

Using (4.2) and (4.3) we get

$$\begin{aligned} (n+m+1)\{T(r, f) + T(r, g)\} &\leq (2m+6)(T(r, f) + T(r, g)) + S(r, f) + S(r, g) \\ \text{or, } (n-m-5)\{T(r, f) + T(r, g)\} &\leq S(r, f) + S(r, g) \end{aligned}$$

which contradicts our assumption,  $n \geq m+6$ .

Hence by Lemma 6  $F(z)G(z) \equiv \alpha(z)^2$  or  $F(z) \equiv G(z)$ .

Suppose,  $F(z)G(z) \equiv \alpha(z)^2$ .

$$\text{That is } f(z)^n (f(z)^m - 1) f(z+c) g(z)^n (g(z)^m - 1) g(z+c) = \alpha(z)^2$$

or,

$$f(z)^n (f(z)-1) (f(z)^{m-1} - \dots - 1) f(z+c) g(z)^n (g(z)-1) (g(z)^{m-1} - \dots - 1) g(z+c) = \alpha(z)^2$$

Then,

$$N\left(r, \frac{1}{f}\right) = S(r, f) \quad \text{and} \quad N\left(r, \frac{1}{f-1}\right) = S(r, f)$$

From this we have,

$$\delta(0, f) + \delta(1, f) + \delta(\infty, f) = 3, \text{ which is impossible.}$$

From this we conclude that,  $F(z) \equiv G(z)$ .

i. e.  $f(z)^n (f(z)^m - 1) f(z+c) \equiv g(z)^n (g(z)^m - 1) g(z+c)$

Using Lemma 5, we conclude that  $f(z) = tg(z)$ , where  $t^m = 1$ .

**Theorem 3.** Let  $f(z)$  and  $g(z)$  be two transcendental entire function of finite order, and  $\alpha(z)$  be a small function with respect to  $f(z)$  and  $g(z)$ . Suppose that  $c$  is a non-zero complex constant and let  $n \geq 4m+12$ . If  $f(z)^n (f(z)^m - 1) f(z+c)$  and  $g(z)^n (g(z)^m - 1) g(z+c)$  share  $\alpha(z)$  IM, then  $f(z) = tg(z)$ , where  $t^m = 1$ .

**Proof.** Let

$$(18) \quad F(z) = \frac{f(z)^n (f(z)^m - 1) f(z+c)}{\alpha(z)} \quad \text{and} \quad G(z) = \frac{g(z)^n (g(z)^m - 1) g(z+c)}{\alpha(z)}$$

Then  $F(z)$  and  $G(z)$  share 1 IM. Let  $H$  be as defined in Lemma 7.

Using Lemma 3, we have

$$\begin{aligned} \bar{N}\left(r, \frac{1}{F}\right) &\leq \bar{N}\left(r, \frac{1}{f^n}\right) + \bar{N}\left(r, \frac{1}{f^m - 1}\right) + \bar{N}\left(r, \frac{1}{f(z+c)}\right) + S(r, f) \\ (19) \quad &\leq (m+2)T(r, f) + S(r, f) \end{aligned}$$

Using (4.4), (4.5) and (4.7) in (4.10) of Lemma 7, we get

$$\begin{aligned} T(r, F) + T(r, G) &\leq 2 \left[ N_2 \left( r, \frac{1}{f} \right) + N_2 \left( r, \frac{1}{G} \right) \right] + 3 \left[ \overline{N} \left( r, \frac{1}{F} \right) + \overline{N} \left( r, \frac{1}{G} \right) \right] \\ &\quad + S(r, f) + S(r, g) \\ &\leq 2(m+3) [T(r, f) + T(r, g)] + 3(m+2) [T(r, f) + T(r, g)] \\ &\quad + S(r, f) + S(r, g) \\ &\leq (5m+12) [T(r, f) + T(r, g)] + S(r, f) + S(r, g) \end{aligned}$$

Using (4.2) and (4.3), we have

$$(20) \quad (n+4m-11)(T(r, f) + T(r, g)) \leq S(r, f) + S(r, g)$$

which contradicts the hypothesis that  $n \geq 4m+12$ .

Thus we get  $H \equiv 0$ .

Integrating  $H$  twice, we obtain

$$(21) \quad F = \frac{(b+1)G + (a-b-1)}{bG + (a-b)} \quad \text{and} \quad G = \frac{(a-b-1) - (a-b)F}{Fb - (b+1)}$$

In the following, we will prove that  $FG \equiv 1$  or  $F \equiv G$ .

**Case 1.**  $b \neq 0, -1$ . If  $a-b-1 \neq 0$ , then by (4.9) we have

$$(22) \quad \overline{N} \left( r, \frac{1}{F} \right) = \overline{N} \left( r, \frac{1}{G - \frac{a-b-1}{b+1}} \right)$$

By the Nevanlinna second main theorem and Lemma 4, we have

$$\begin{aligned} T(r, G) &\leq \overline{N}(r, G) + \overline{N} \left( r, \frac{1}{G} \right) + \overline{N} \left( r, \frac{1}{G - \frac{a-b-1}{b+1}} \right) + S(r, G) \\ &\leq \overline{N} \left( r, \frac{1}{G} \right) + \overline{N} \left( r, \frac{1}{F} \right) + S(r, g) \\ &\leq \overline{N} \left( r, \frac{1}{g(z)^n (g(z)^m - 1) g(z+c)} \right) + \overline{N} \left( r, \frac{1}{f(z)^n (f(z)^m - 1) f(z+c)} \right) + S(r, g) \\ &\leq (m+2)T(r, g) + (m+2)T(r, f) + S(r, g) \end{aligned}$$

Therefore we have,

$$(23) \quad (n+m+1)T(r, g) \leq (m+2)T(r, f) + (m+2)T(r, g) + S(r, g)$$

$$(24) \quad (n+m+1)T(r, f) \leq (m+2)T(r, g) + (m+2)T(r, f) + S(r, f)$$

From (4.11) and (4.12), we get

$$(n+m+1)(T(r, f) + T(r, g)) \leq (m+2)(T(r, f) + T(r, g)) + S(r, f) + S(r, g)$$

or,  $(n-1)(T(r, f) + T(r, g)) \leq S(r, f) + S(r, g)$

This contradicts the assumption that  $n \geq 4m+12$ .

Thus  $a-b-1 = 0$ , then by (4.9), we have

$$(25) \quad F = \frac{(b+1)G}{bG+1}$$

Since  $F$  is an entire function, from (4.13) we have

$$\overline{N} \left( r, \frac{1}{G + \frac{1}{b}} \right) = 0.$$

Proceeding as above we deduce a contradiction.

**Case 2:**  $b = -1$ ,  $a \neq -1$ . Then by (4.9) we get

$$F = \frac{a}{(a+1) - G}, \quad \text{and} \quad \bar{N}\left(r, \frac{1}{G - (a+1)}\right) = \bar{N}(r, F) = 0$$

Similarly as in Case 1, we get contradiction.

Hence  $a = -1$ , Thus we get  $FG \equiv 1$ .

Proceeding as in theorem 2, we get contradiction.

**Case 3:** If  $b = 0$ ,  $a \neq 1$ . From (4.9) we have

$$F = \frac{G + (a-1)}{a} \quad \text{and} \quad \bar{N}\left(r, \frac{1}{F}\right) = \bar{N}\left(r, \frac{1}{G - (a+1)}\right)$$

Proceeding as in Case 1, we get a contradiction.

Thus  $a = 1$ , implies  $F = G$ .

Proceeding as in theorem 2 we get  $f = tg$ , where  $t^m = 1$ .

**Theorem 4.** Let  $n \geq 8$  be an integer and  $c(\neq 0, 1)$  is a constant such that the equation  $P(\omega) = 0$  has no multiple roots. Where

$$(26) \quad P(\omega) = \frac{(n-1)(n-2)}{2}\omega^n - n(n-2)\omega^{n-1} + \frac{n(n-1)}{2}\omega^{n-2} - c$$

Let  $S = \{\omega | P(\omega) = 0\}$ , Suppose that  $f$  is a non-constant meromorphic function of finite order then  $E_{f(z)}(S) = E_{f(z+c)}(S)$  and  $E_{f(z)}(\{\infty\}) = E_{f(z+c)}(\{\infty\})$  implies  $f(z) = f(z+c)$ .

**Proof.** From the condition of the theorem, we have  $E_{f(z)}(S) = E_{f(z+c)}(S)$

Then there exists a polynomial  $Q(z)$ , such that

$$(27) \quad \frac{P(z+c)}{P(z)} = e^{Q(z)}$$

and  $T(r, e^{Q(z)}) = m(r, e^{Q(z)}) = S(r, f)$

Rewriting (4.15), we have

$$P(z+c) = e^{Q(z)}P(z)$$

or,

$$\begin{aligned} & \frac{(n-1)(n-2)}{2}f(z+c)^n - n(n-2)f(z+c)^{n-1} + \frac{n(n-1)}{2}f(z+c)^{n-2} - c \\ &= e^{Q(z)} \left[ \frac{(n-1)(n-2)}{2}f(z)^n - n(n-2)f(z)^{n-1} + \frac{n(n-1)}{2}f(z)^{n-2} - c \right] \end{aligned}$$

or,

$$(28) \quad \begin{aligned} & f(z+c)^{n-2} \left[ \frac{(n-1)(n-2)}{2}f(z+c)^2 - n(n-2)f(z+c) + \frac{n(n-1)}{2} \right] \\ &= e^{Q(z)} \left[ \frac{(n-1)(n-2)}{2}f(z)^n - n(n-2)f(z)^{n-1} + \frac{n(n-1)}{2}f(z)^{n-2} - c + ce^{-Q(z)} \right] \end{aligned}$$

Let

$$F(z) = \frac{(n-1)(n-2)}{2}f(z)^n - n(n-2)f(z)^{n-1} + \frac{n(n-1)}{2}f(z)^{n-2},$$

or,

$$F(z) = f(z)^{n-2} \left[ \frac{(n-1)(n-2)}{2} f(z)^2 - n(n-2)f(z) + \frac{n(n-1)}{2} \right],$$

or,

$$(29) \quad F(z) = f(z)^{n-2}(f - \alpha_1)(f - \alpha_2)$$

where  $\alpha_1$ , and  $\alpha_2$  are roots of the equation,

$$\frac{(n-1)(n-2)}{2} f(z)^2 - n(n-2)f(z) + \frac{n(n-1)}{2} = 0.$$

Therefore (4.16) can be rewritten as,

$$(30) \quad \begin{aligned} f(z+c)^{n-2} \left[ \frac{(n-1)(n-2)}{2} f(z+c)^2 - n(n-2)f(z+c) + \frac{n(n-1)}{2} \right] \\ = e^{Q(z)} \left[ F(z) - (c - ce^{-Q(z)}) \right] \end{aligned}$$

Using standard Valiron Moho'nko theorem, we get

$$(31) \quad T(r, F) = nT(r, f) + S(r, f)$$

Applying second main theorem to  $F$  and using (4.17) and (4.18), we have

$$\begin{aligned} T(r, F) &\leq \bar{N}(r, F) + \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}\left(r, \frac{1}{F - (c - ce^{-Q(z)})}\right) + S(r, F) \\ &\leq \bar{N}(r, F) + \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}\left(r, \frac{1}{f(z+c)^{n-2}}\right) \\ &\quad + \bar{N}\left(r, \frac{1}{\frac{(n-1)(n-2)}{2} f(z+c)^2 - n(n-2)f(z+c) + \frac{n(n-1)}{2}}\right) + S(r, f) \end{aligned}$$

$$\begin{aligned} \text{or, } T(r, F) &\leq \bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f(z)^{n-2}}\right) + \bar{N}\left(r, \frac{1}{f - \alpha_1}\right) \\ &\quad + \bar{N}\left(r, \frac{1}{f - \alpha_2}\right) + \bar{N}\left(r, \frac{1}{f(z+c)^{n-2}}\right) \\ &\quad + \bar{N}\left(r, \frac{1}{\frac{(n-1)(n-2)}{2} f(z+c)^2 - n(n-2)f(z+c) + \frac{n(n-1)}{2}}\right) + S(r, f) \\ &\leq 4T(r, f) + 3T(r, f(z+c)) + S(r, f) \end{aligned}$$

By (4.19), we have

$$(32) \quad \begin{aligned} n T(r, f) &\leq 4T(r, f) + 3T(r, f(z+c)) + S(r, f) \\ \text{or, } (n-4)T(r, f(z)) &\leq 3T(r, f(z+c)) + S(r, f) \end{aligned}$$

$$(33) \quad \text{Similarly } (n-4)T(r, f(z+c)) \leq 3T(r, f(z)) + S(r, f)$$

Using (4.20) and (4.21), we have

$$\begin{aligned} (n-4)(T(r, f(z)) + T(r, f(z+c))) &\leq 3(T(r, f(z)) + T(r, f(z+c))) + S(r, f) \\ \text{or, } (n-7)(T(r, f(z)) + T(r, f(z+c))) &\leq S(r, f), \end{aligned}$$

which contradicts the assumption  $n \geq 8$ . Therefore  $e^{Q(z)} = 1$ .  
Hence from (4.15), we get  $P(z+c) = P(z)$ .

$$\begin{aligned} \text{or, } & \frac{(n-1)(n-2)}{2} f(z+c)^n - n(n-2)f(z+c)^{n-1} + \frac{n(n-1)}{2} f(z+c)^{n-2} - c \\ & = \frac{(n-1)(n-2)}{2} f(z)^n - n(n-2)f(z)^{n-1} + \frac{n(n-1)}{2} f(z)^{n-2} - c \\ \text{or, } & \frac{(n-1)(n-2)}{2} (f(z)^n - f(z+c)^n) - n(n-2) (f(z)^{n-1} - f(z+c)^{n-1}) \\ (34) \quad & + \frac{n(n-1)}{2} (f(z)^{n-2} - f(z+c)^{n-2}) \equiv 0. \end{aligned}$$

Taking

$$h(z) = \frac{f(z+c)}{f(z)},$$

we get

$$(35) \quad \frac{(n-1)(n-2)}{2} (h^n - 1) f(z)^2 - n(n-2) (h^{n-1} - 1) f(z) + \frac{n(n-1)}{2} (h^{n-2} - 1) = 0$$

Suppose  $h$  is not a constant. From (4.23), we have

$$(36) \quad \{(n-1)(n-2)(h^n - 1)f(z) - n(n-2)(h^{n-1} - 1)\}^2 = -n(n-2)Q(h)$$

where  $Q(h)$  is defined as in Lemma 8. Using (4.24) and Lemma 8, we get

$$(37) \quad \begin{aligned} & \{(n-1)(n-2)(h^n - 1)f(z) - n(n-2)(h^{n-1} - 1)\}^2 \\ & = -n(n-2)(h-1)^4(h-\beta_1)(h-\beta_2), \dots, (h-\beta_{2n-6}) \end{aligned}$$

From (4.24), all zeros of  $(h-\beta_j)$  have order at least 2. Applying second fundamental theorem to  $h$ , we get

$$\begin{aligned} (2n-8)T(r, h) & \leq \sum_{j=1}^{2n-6} \bar{N}\left(r, \frac{1}{h-\beta_j}\right) + S(r, h) \\ & \leq \frac{1}{2} \sum_{j=1}^{2n-6} \bar{N}\left(r, \frac{1}{h-\beta_j}\right) + S(r, h) \\ & \leq (n-3)T(r, h) + S(r, h) \end{aligned}$$

which contradicts the assumption,  $n \geq 8$ .

Thus  $h$  is a constant, From (5.3.23) we obtain  $h^n - 1 = 0$ . Therefore  $h = 1$ .  
Hence we conclude that  $f(z) \equiv f(z+c)$ .

**Theorem 5.** Let  $n \geq 7$  be an integer and  $c(\neq 0, 1)$  is a constant such that the equation  $P(\omega) = 0$  has no multiple roots, where  $P(\omega)$  is as defined in theorem 4. Let  $S = \{\omega | P(\omega) = 0\}$ , Suppose that  $f$  is a non-constant entire function of finite order then  $E_{f(z)}(S) = E_{f(z+c)}(S)$  implies  $f(z) = f(z+c)$ .

**Proof.**  $f$  and  $g$  are entire functions. Taking  $N(r, f) = N(r, g) = 0$ , in the above theorem, we obtain conclusion of this theorem.

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DEPARTMENT OF MATHEMATICS, KARNATAK UNIVERSITY, DHARWAD-580003, INDIA

\*CORRESPONDING AUTHOR