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Generalized θ-$\Phi_{\aleph}$-Contractions and θ-$\mathscr{F}_{\aleph}$-Expansion of Darbo-Type and Their Applications

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Abstract. In this work, we present the concept of θ-$\Phi_{\aleph}$-contraction, θ-$\Phi_{\aleph}$-Suzuki contraction, θ-$\Phi_{\aleph}$-Kannan type contraction, and θ-$\mathscr{F}_{\aleph}$-expansion, and establish some novel fixed point theorems in the light of Banach space. In order to verify our results, we construct some examples. Furthermore, we use our results to check the existence of a solution to differential equations.

1. Introduction and Preliminaries

Among the most flexible tools in mathematics, differential equations provide the basis for real-world dynamic system comprehension and modeling. In fields ranging from physics, engineering, biology, and economics, they are indispensable since they clarify how quantities vary in regard to

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one another. From simulating the spread of diseases to predicting planetary motion, differential equations provide a basis for converting real-world phenomena into mathematical language. Their flexibility is in their ability to capture the essence of linear or nonlinear, slow or fast change.

Second-order differential equations are special among the several varieties of differential equations in that they can characterize systems in which acceleration, or the second derivative of a parameter, is absolutely vital. These equations are important for characterizing processes including forces, oscillations, and energy transformations. They are fundamental for mechanical vibrations—that is, for the swing of a pendulum, the bounce of a spring, or the swaying of a skyscraper in the breeze. Using components like capacitors and inductors, electrical engineering circuits mimic the flow of energy between electric and magnetic fields. Second-order equations help one to better grasp predator-prey dynamics—that is, interactions between species produce changes in population size.

Second-order differential equations are interesting in that they expose intricate features such as stability, damping, and resonance. They could clarify, for example, why a bridge might fall in particular wind conditions or how a tuning fork generates a single, continuous sound. Apart from helping us to forecast the behavior of systems, these equations clarify how to maximize or regulate them. Second-order differential equations find many useful applications in design of automobile shock absorbers, tuning of musical instruments, and heart rate determination. By exposing latent patterns and relationships, their answers can help us better grasp our built and natural surroundings.

We give a brief overview with definitions, fundamental ideas, and necessary conclusions to reach completeness. The following notations and ideas will be applied across this work:

1. **Notations for Sets:**

- $\mathbb{R}$: The set of real numbers.
- $\mathbb{R}_+$: The interval $[0, \infty)$.
- $\mathscr{N}$: The set of positive integers.
- $\overline{Q}$: Set $Q's$ closure.
- $\overline{co}Q$: Set $Q's$ convex hull closure.

2. **Banach Space and Related Concepts:**

- $\mathscr{E}$: A Banach space.
- $\varpi_{\mathscr{E}}$: The collection of all bounded subsets for $\mathscr{E}$, defined as:

$$\varpi_{\mathscr{E}} = \{\mathcal{D} \neq \emptyset : \mathcal{D} \text{ represent a bounded subset for } \mathscr{E}\}.$$

- $\ker \aleph$: A function $\aleph : \varpi_{\mathscr{E}} \to [0, \infty)$,with kernel defined as:

$$\ker \aleph = \{\mathcal{D} \in \varpi_{\mathscr{E}} : \aleph(\mathcal{D}) = 0\}.$$

- $\wp_{\mathscr{E}}$: The subfamily of $\varpi_{\mathscr{E}}$ comprising of all relatively compact sets.

- $\mho$: The set of all convex, bounded, non-empty and closed subsets of $\mathscr{E}$, defined as:

$$\mho = \{\mathcal{D} : \mathcal{D} \neq \emptyset, \text{ convex, closed, and bounded subset of } \mathscr{E}\}.$$

Numerous scholars have shown a keen interest in fixed point theory. One of the two primary outgrowths of this concept is the Banach contraction principle, which was the initial major finding about contraction maps on metric spaces. In the second branch, the emphasis is on operators with discontinuity in convex and compact subsets of a Banach space, which has two noteworthy outcomes in this domain: first, Brouwer's fixed point theorem, which asserts that , Every continuous map from the closed unit ball of $\mathbb{R}^n$ into itself must have a fixed point, and second, its infinite dimensional extension. The fixed point theorem of Schauder declares that ,every continuous and compact mapping on convex, bounded, and closed subset of a Banach space. Compactness is essential in both the theorems. In order to handle such obstacle, one of the effective way is to employ the concept of Measure of noncompactness ($\mathscr{MNC}$). Measures of noncompactness is a mathematical concept that is widely used in nonlinear functional analysis. This concept has proven to be very useful in various fields, including the theory of operators, the normed spaces geometry, the functional differential and integral equations theory.

In certain situations, it might not always produce a contraction operator but rather another kind of operator, such as an expansive or non-expansive one. Among the most fascinating areas of fixed point theory research is the study of expansive mappings. The idea of expanding mapping was first presented by Gillespie *et al.* [9] and established fixed point result for such expansion. There are many authors, who consider expansions and their fixed point problems [13,15,20]. Górnicki [10] generalized the concept of expansions to $\mathscr{F}$-expansion. Zada *et al.* [23] developed the notion of $\mathscr{F}_{\aleph}$-expansion in the non-compact circumstances and demonstrated fixed point results under such expansion in Banach space. The axiomatic definition for $\mathscr{MNC}$ is as follows:

Definition 1.1. *[1] We say that a mapping* $\aleph : \varpi_{\mathscr{E}} \to [0,\infty)$ *represent* $\mathscr{MNC}$ *in* $\mathscr{E}$ *when* $\forall\ \mathcal{D}, \mathcal{D}_1, \mathcal{D}_2 \in \varpi_{\mathscr{E}}$ *it fulfills the below axioms:*

(i) *The family* $\ker \aleph = \mathcal{D} \in \varpi_{\mathscr{E}} : \varpi(\mathcal{D}) = 0$ *is a nonempty with* $\ker \aleph \subseteq \wp_{\mathscr{E}}$;

(ii) *(Monotonic)* $\mathcal{D}_1 \subset \mathcal{D}_2 \Rightarrow \aleph(\mathcal{D}_1) \leq \aleph(\mathcal{D}_2)$;

(iii) *(Invariant under closure)* $\aleph(\overline{\mathcal{D}}) = \aleph(\mathcal{D})$;

(iv) *(Invariant under convex hull)* $\aleph(\overline{co}\mathcal{D}) = \aleph(\mathcal{D})$;

(v) $\aleph(\eta\mathcal{D}_1 + (1-\eta)\mathcal{D}_2) \leq \eta\aleph(\mathcal{D}_1) + (1-\eta)\aleph(\mathcal{D}_2),\ \forall \eta \in [0,1]$;

(vi) *(Generalized Cantor's intersection theorem) If* $\{\mathcal{D}_n\}$ *denotes a sequence for closed sets within* $\varpi_{\mathscr{E}}$ *in a manner that* $\mathcal{D}_{n+1} \subset \mathcal{D}_n$, $\forall n \in \mathscr{N}$ *with* $\lim\limits_{s\to+\infty} \aleph(\mathcal{D}_n) = 0$, *thus* $\mathcal{D}_\infty = \bigcap_{n=1}^{+\infty} \mathcal{D}_n \neq \emptyset$.

Employing the $\mathscr{MNC}$ concept, Darbo [8] presented a fixed point result, that establishes the existence of fixed points.The Darbo's fixed-point theorem in regards to a measure of noncompactness $\aleph$ could be expressed as:

Theorem 1.1. *[8] Let $\mathcal{D} \in \mho$ and $\Psi : \mathcal{D} \to \mathcal{D}$ is continuous mapping defined on every subset $\mathcal{D}_0$ of $\mho$ such that*

$$\aleph(\Psi(\mathcal{D}_0)) \leq k\aleph(\mathcal{D}_0),$$

where $\mathcal{D}_0 \subset \mathcal{D}$, $k \in [0, 1)$ with $\aleph$ as $\mathscr{MNC}$ as described on $\mathscr{E}$, as a result Ψ assures a fixed-point ($\mathcal{FP}$) in $\mho$.

Finally, we recall some essential functions, which we will use for the contraction of our main results.

Definition 1.2. *[16] Θ represent collection of functions $\theta : \mathbb{R} \longrightarrow \mathbb{R}$ where the following assumptions hold for it:*

- (Θ_1) *θ denotes a non-decreasing mapping with no discontinuity;*
- (Θ_2) *$\lim_{n\to\infty} \theta^n(k) = -\infty$, for every $0 < k$;*
- (Θ_3) *$k > \theta(k)$, for all $k \geq 0$.*

Definition 1.3. *[4] Φ is the set of mappings $\phi : (0, \infty) \to \mathbb{R}$ that fullfill the following conditions:*

- (Φ_1) *ϕ increases strictly;*
- (Φ_2) *For every $\{\mathcal{D}_n\}_{n\in\mathcal{N}}$ a sequence of positive numbers the $\lim_{n\to\infty} \mathcal{D}_n = 0$ iff $\lim_{n\to\infty} \phi(\mathcal{D}_n) = -\infty$;*
- (Φ_3) *ϕ is continuous in $(0, +\infty)$.*

Definition 1.4. *[18] Suppose $\mathscr{F} : (0, \infty) \longrightarrow \mathbb{R}$ denotes a mapping in sense that*

- (F_1) *$\mathscr{F}$ identified as an increasing;*
- (F_2) *$\lim_{n\to+\infty} \delta_n = 0 \Longleftrightarrow \lim_{n\to+\infty} \mathscr{F}(\delta_n) = -\infty$, for arbitrary any $\{\delta_n\} \subset (0, \infty)$;*
- (F_3) *It is possible to find out $y \in (0, 1)$ to the extent that $0 = \lim_{s\to 0^+} \delta^y \mathscr{F}(\delta)$.*

Symbolize with $\mathbb{F}$, the collection of such mappings as $\mathscr{F} : (0, \infty) \longrightarrow \mathbb{R}$ which fulfill the axioms $(F_1) - (F_2)$. Moreover with $\mathbb{S}$, those all mappings as $\tau : (0, \infty) \longrightarrow \mathbb{R}$ to the extent that $\lim_{t\to s^+} \inf \tau(k) > 0$, $\forall s \in [0, \infty)$.

The following lemma is necessary to determine the $\mathcal{FP}$ of expansion maps.

Lemma 1.1. *[10] For surjective mapping $\mathfrak{g} : \mathcal{M} \to \mathcal{M}$, $\exists$ a mapping as described $\mathfrak{g}^* : \mathcal{M} \to \mathcal{M}$ such that $\mathfrak{g} \circ \mathfrak{g}^* : \mathcal{M} \to \mathcal{M}$ is an identity mapping.*

The aim of our work is to utilize definition (1.2) and (1.3) and establish Darbo type results.

2. Generalized Darbo type Contractions with Fixed Point Theorems

Here we present, θ-$\Phi_\aleph$ contraction, θ-$\Phi_\aleph$-Suzuki contractions, and θ-$\Phi_\aleph$-Kannan type contraction of Darbo type and $\mathcal{FP}$ theorems to the mappings holding such contraction conditions in Banach space endowed by $\mathscr{MNC}$. First we need to know what θ-$\Phi_\aleph$ contraction is:

Definition 2.1. *For $\mathcal{D} \in \mho$, the map $\Psi_c : \mathcal{D} \to \mathcal{D}$ will be identified as θ-$\Phi_\aleph$ contraction when $\exists$ $\phi \in \Phi$ along $\theta \in \Theta$ in a manner where*

$$\aleph(\mathcal{D}_1) > 0 \Rightarrow \theta(\aleph(\Psi_c(\mathcal{D}_1))) \leq \phi(\theta(\Delta(\mathcal{D}_1, \mathcal{D}_2))), \tag{2.1}$$

while $\mathcal{D}_1$ and $\mathcal{D}_2$ denote subsets for $\mathcal{D}$, $\aleph(\mathcal{D}_1)$, $\aleph(\Psi_c(\mathcal{D}_1))$, $\aleph(\Psi_c(\mathcal{D}_2)) > 0$, $\aleph$ represent an $\mathscr{MNC}$ described in $\mathscr{E}$ and

$$\Delta(\mathcal{D}_1, \mathcal{D}_2) = \max\Big\{\aleph(\mathcal{D}_1), \aleph(\Psi_c(\mathcal{D}_1)), \aleph(\Psi_c(\mathcal{D}_2)), \frac{1}{2}\aleph(\Psi_c(\mathcal{D}_1) \cup \Psi_c(\mathcal{D}_2))\Big\}.$$

In the view of the above θ-$\Phi_{\aleph}$-contraction, our findings are as follows.

Theorem 2.1. *Let $\mathcal{D} \in \mho$. When $\Psi_c : \mathcal{D} \to \mathcal{D}$ represent θ-$\Phi_{\aleph}$ contraction with being continuous. It results to a $\mathcal{FP}$ for Ψ_c in $\mathcal{D}$*

Proof. Let us formulate a sequence $\{\mathcal{D}_n\}_{n=0}^{\infty}$ in a way that

$$\mathcal{D}_0 = \mathcal{D} \text{ and } \mathcal{D}_n = \overline{co}(\Psi_c \mathcal{D}_{n-1}), \ \forall n \in \mathscr{N}. \tag{2.2}$$

We must show that $\mathcal{D}_{n+1} \subset \mathcal{D}_n$ and $\Psi_c\mathcal{D}_n \subset \mathcal{D}_n$, $\forall n \in \mathscr{N}$. Now, the 1st inclusion is accomplished through induction. When $n = 1$, so with the help of (2.2), we come up with $\mathcal{D}_0 = \mathcal{D}$ and $\mathcal{D}_1 = \overline{co}(\Psi_c\mathcal{D}_0) \subset \mathcal{D}_0$. Next, for $n > 1$, we assume that $\mathcal{D}_n \subset \mathcal{D}_{n-1}$, then $\overline{co}(\Psi_c(\mathcal{D}_n)) \subset \overline{co}(\Psi_c(\mathcal{D}_{n-1}))$, Using equation (2.2). We obtained the first inclusion

$$\mathcal{D}_{n+1} \subset \mathcal{D}_n. \tag{2.3}$$

With the help of above inclusion (2.3), we are able to derive the second inclusion as:

$$\Psi_c\mathcal{D}_n \subset \overline{co}(\Psi_c\mathcal{D}_n) = \mathcal{D}_{n+1} \subset \mathcal{D}_n. \tag{2.4}$$

We will now examine two scenarios pertaining to $\aleph$. When an integer m which is non-negative can be identified in a way that $\aleph(\mathcal{D}_m) = 0$, then $\mathcal{D}_m$ is compact. However $\Psi_c : \mathcal{D}_m \to \mathcal{D}_m$, so by Schauder's Theorem, Ψ_c ensure a $\mathcal{FP}$ in $\mathcal{D}_m \subset \mathcal{D}$. In place of this, if we consider $\aleph(\mathcal{D}_n) > 0$, $\forall n \in \mathscr{N}$. After that, we'll need to provide testimony affirming that $\mathcal{D}_\infty \subset \mathcal{D}_n \in \mho$. Initially, we should verify that $\aleph(\mathcal{D}_n) \to 0$ as $n \to +\infty$. Based on inclusion (2.3), it can be written $\aleph(\mathcal{D}_{n+1}) < \aleph(\mathcal{D}_n)$, that is $\{\aleph(\mathcal{D}_n)\}$ is non-increasing sequence which converges to $s \in \mathbb{R}$ along $s \geq 0$. Furthermore, since $\aleph(\mathcal{D}_n) \in (0, \infty)$ and $s \in [0, \infty)$, Using contraction condition (2.1) with $\mathcal{D}_1 = \mathcal{D}_n$ and $\mathcal{D}_2 = \mathcal{D}_{n+1}$, we write

$$\begin{aligned} \theta(\aleph(\mathcal{D}_{n+1})) &= \theta(\aleph(\overline{co}(\Psi_c(\mathcal{D}_n)))) \\ &= \theta(\aleph(\Psi_c(\mathcal{D}_n))) \\ &\leq \phi(\theta(\Delta(\mathcal{D}_n, \mathcal{D}_{n+1}))), \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} \Delta(\mathcal{D}_n, \mathcal{D}_{n+1}) &= \max\Big\{\aleph(\mathcal{D}_n), \aleph(\Psi_c(\mathcal{D}_n)), \aleph(\Psi_c(\mathcal{D}_{n+1})), \frac{1}{2}\aleph(\Psi_c(\mathcal{D}_n) \cup \Psi_c(\mathcal{D}_{n+1}))\Big\} \\ &\leq \max\Big\{\aleph(\mathcal{D}_n), \aleph(\mathcal{D}_n), \aleph(\mathcal{D}_{n+1}), \frac{1}{2}\aleph(\mathcal{D}_n \cup \mathcal{D}_{n+1})\Big\} \\ &= \max\Big\{\aleph(\mathcal{D}_n), \aleph(\mathcal{D}_n), \aleph(\mathcal{D}_{n+1}), \frac{1}{2}\aleph(\mathcal{D}_n)\Big\} \\ &= \aleph(\mathcal{D}_n). \end{aligned}$$

Thus from inequality (2.5), we obtain that

$$\theta(\aleph(\mathcal{D}_{n+1})) \leq \phi(\theta(\aleph(\mathcal{D}_n))). \tag{2.6}$$

Thus

$$\begin{aligned}\theta(\aleph(\mathcal{D}_{n+1})) &\leq \phi(\theta(\aleph(\mathcal{D}_n)))\\ &\leq \phi^2(\theta(\aleph(\mathcal{D}_{n-1})))\\ &\leq \phi^3(\theta(\aleph(\mathcal{D}_{n-2})))\\ &\vdots\\ &\leq \phi^{n+1}(\theta(\aleph(\mathcal{D}_{n_0}))).\end{aligned}$$

Taking $n \to \infty$ and applying (Θ_1), we have $\lim_{n\to+\infty} \theta(\aleph(\mathcal{D}_{n+1})) = -\infty$ and utilizing property (F_2), It is possible for us to say $\lim_{n\to+\infty} \aleph(\mathcal{D}_{n+1}) = 0$. Therefore with the help of definition (vi)-(1.1), $\emptyset \neq \bigcap_{n=1}^{+\infty} \mathcal{D}_n = \mathcal{D}_\infty$ and $\Psi_c\mathcal{D}_\infty \subset \mathcal{D}_\infty$ as $\Psi_c\mathcal{D}_n \subset \mathcal{D}_n$. Also, since $\mathcal{D}_\infty \subset \mathcal{D}_n$ for all $n \in \mathcal{N}$, so via definition (ii)-1.1, $\aleph(\mathcal{D}_\infty) \leq \aleph(\mathcal{D}_n)$, $\forall n \in \mathcal{N}$. Thus $\aleph(\mathcal{D}_\infty) = 0$, which means $\mathcal{D}_\infty \in \ker \aleph$, and hence $\mathcal{D}_\infty$ yields to be bounded. But $\mathcal{D}_\infty$ represent closed in a sense that $\mathcal{D}_\infty$ is compact. Consequently, Schauder's Theorem ensures a $\mathcal{FP}$ for Ψ_c in $\mathcal{D}_\infty \subset \mathcal{D}$. □

To illustrate the Theorem 2.1, the below example is constructed.

Example 2.1. *Suppose $\mathcal{D} = [-8, 9]$ represent a subset to the Banach space $\mathbb{R}$. So evidently $\mathcal{D} \in \mho$. Next, define $\phi : [1, \infty) \longrightarrow \mathbb{R}$, $\Psi_c : \mathcal{D} \to \mathcal{D}$, $\theta : \mathbb{R} \longrightarrow \mathbb{R}$, as $\phi(k) = \ln k + k$, $\Psi_c(k) = 1 - k$, and $\theta(k) = \sqrt[3]{k} - \frac{1}{2}(k \geq 1)$ respectively. It is obvious to justify that Ψ_c is continuous, $\theta \in \Theta$ and $\phi \in \Phi$. Further, $\mathcal{MNC}$ is described as $\aleph : \mathcal{B}(\mathcal{E}) \to [0, \infty)$ by*

$$\aleph(\mathcal{D}) = \sup_{k,u\in\mathcal{D}} |k - u| = diam(\mathcal{D}).$$

Next, suppose $\mathcal{D}_1 = [2, 3]$ with $\mathcal{D}_2 = [2, 9]$ represent subsets for $\mathcal{D}$. So $\aleph(\mathcal{D}_1) = \aleph(\Psi_c(\mathcal{D}_1)) = 1$, $\aleph(\Psi_c(\mathcal{D}_2)) = 7$, $\aleph(\Psi_c(\mathcal{D}_1) \cup \Psi_c(\mathcal{D}_2)) = 8$, and hence

$$\begin{aligned}\Delta(\mathcal{D}_1, \mathcal{D}_2) &= \max\left\{\aleph(\mathcal{D}_1), \aleph(\Psi_c(\mathcal{D}_1)), \aleph(\Psi_c(\mathcal{D}_2)), \frac{1}{2}\aleph(\Psi_c(\mathcal{D}_1) \cup \Psi_c(\mathcal{D}_2))\right\}\\ &= \max\{1, 1, 7, 4\} = 7.\end{aligned}$$

Thus from contraction (2.1), we write

$$\theta(\phi(\aleph(\mathcal{D}_0))) = \theta(\phi(1)) = 1 < 2.57591 = \theta(\phi(\Delta(\mathcal{D}_1, \mathcal{D}_2))).$$

That is θ-ϕ-weak contraction. As a result with the help of theorem 2.1, Ψ_c assures a $\mathcal{FP}$ $\frac{1}{2} \in \mathcal{D}$.

A number of crucial consequences can be deduced from Theorem 2.1. We introduce a number of useful corollaries that expand upon and encompass various well-known findings in the literature. Notably, When we choose $\theta(t) = e^t$ and $\phi(t) = e^{\phi(t)}$ we conclude the corollary below.

Corollary 2.1. *Suppose $\mathcal{D} \in \mho$. If the θ-ϕ map $\Psi_c : \mathcal{D} \to \mathcal{D}$ represent a map with ni discontinuity such that*

$$\aleph(\Psi_c(\mathcal{D}_1)) \leq \phi(\aleph(\mathcal{D}_1))), \tag{2.7}$$

if we choose $\phi(t) = t - \tau$, we conclude the following corollary.

Corollary 2.2. *Let $\mathcal{D} \in \mho$ with $\Psi_c : \mathcal{D} \to \mathcal{D}$ representing a map with no discontinuity such that*

$$\tau + \theta(\aleph(\Psi_c \mathcal{D})) \leq \theta(\Delta(\mathcal{D}_1, \mathcal{D}_2)),$$

where $\mathcal{D} \subseteq \mho$. Consequently, Ψ_c will assures a $\mathcal{FP}$ in $\mathcal{D}$.

Remark 2.1. *From the above corollary we can derive the Darbo's $\mathcal{FP}$ theorem when we choose $\theta(k) = \ln(k)$, for every $k > 0$.*

Now, this work introduces θ-$\Phi_{\aleph}$-Suzuki and θ-$\Phi_{\aleph}$-Kannan type contractions as below:

Definition 2.2. *Suppose $\mathcal{D} \in \mho$. The $\Psi_c : \mathcal{D} \to \mathcal{D}$ mapping is θ-$\Phi_{\aleph}$-Suzuki contraction when $\exists\ \phi \in \Phi$ and $\theta \in \Theta$ such that for any $\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{D}$, we have*

$$\frac{1}{2}\aleph(\Psi_c(\mathcal{D}_1)) < \aleph(\mathcal{D}_1) \Rightarrow \theta(\aleph(\Psi_c(\mathcal{D}_1))) \leq \phi(\theta(\Delta(\mathcal{D}_1, \mathcal{D}_2))),$$

where

$$\Delta(\mathcal{D}_1, \mathcal{D}_2) = \max\left\{\aleph(\mathcal{D}_1), \aleph(\Psi_c(\mathcal{D}_1)), \aleph(\Psi_c(\mathcal{D}_2)), \frac{1}{2}\aleph(\Psi_c(\mathcal{D}_1) \cup \Psi_c(\mathcal{D}_2))\right\}.$$

With the help of the above contraction the following theorem can be constructed, it is straightforward to prove it, so we skip the proof.

Theorem 2.2. *Assume $\mathcal{D} \in \mho$ and suppose $\Psi_c : \mathcal{D} \to \mathcal{D}$ represent a θ-$\Phi_{\aleph}$-Suzuki contraction, as a result Ψ_c will assures a unique $\mathcal{FP}$.*

Definition 2.3. *Assume $\mathcal{D} \in \mho$. Then $\Psi_c : \mathcal{D} \to \mathcal{D}$ will be θ-$\Phi_{\aleph}$-Kannan type contraction when $\exists$ $\theta \in \Theta$ with $\phi \in \Phi$ in a manner that*

$$\theta(\aleph(\Psi_c(\mathcal{D}_1))) \leq \phi\left(\theta\left(\frac{\aleph(\Psi_c(\mathcal{D}_1) + \Psi_c(\mathcal{D}_2))}{2}\right)\right). \tag{2.8}$$

In the view of the above θ-$\Phi_{\aleph}$-Kannan type contraction, we give the following result.

Theorem 2.3. *Assume $\mathcal{D} \in \mho$ and $\Psi_c : \mathcal{D} \to \mathcal{D}$ represent a map with no discontinuity. if we have $\theta \in \Theta$, $\phi \in \Phi$, and $\tau > 0$ such a way*

$$\theta(\aleph(\Psi_c(\mathcal{D}_1))) \leq \phi\left(\theta\left(\frac{\aleph(\Psi_c(\mathcal{D}_1) + \Psi_c(\mathcal{D}_2))}{2}\right)\right). \tag{2.9}$$

then Ψ_c has at least a $\mathcal{FP}$.

Proof. Let a sequence $\{\mathcal{D}_n\}_{n=0}^{\infty}$ in such a way that

$$\mathcal{D}_0 = \mathcal{D} \text{ and } \mathcal{D}_n = \overline{co}(\Psi_c \mathcal{D}_{n-1}), \ \forall n \in \mathcal{N}. \tag{2.10}$$

We need to show that $\mathcal{D}_{n+1} \subset \mathcal{D}_n$ and $\Psi_c \mathcal{D}_n \subset \mathcal{D}_n$, $\forall n \in \mathcal{N}$. Our proof of first inclusion is based on induction. When $n = 1$, then with the help of (2.10), we come up with $\mathcal{D}_0 = \mathcal{D}$ and $\mathcal{D}_1 = \overline{co}(\Psi_c \mathcal{D}_0) \subset \mathcal{D}_0$. Next, for $n > 1$ we assume that $\mathcal{D}_n \subset \mathcal{D}_{n-1}$, then $\overline{co}(\Psi_c(\mathcal{D}_n)) \subset \overline{co}(\Psi_c(\mathcal{D}_{n-1}))$, utilizing equation (2.10), our first inclusion is obtained

$$\mathcal{D}_{n+1} \subset \mathcal{D}_n. \tag{2.11}$$

The secondary inclusion has been acquired by the application of the inclusion (2.11),

$$\Psi_c \mathcal{D}_n \subset \overline{co}(\Psi_c \mathcal{D}_n) = \mathcal{D}_{n+1} \subset \mathcal{D}_n. \tag{2.12}$$

We will now examine two situations based on the values of $\aleph$. If a non-negative integer m can be identified such a way $\aleph(\mathcal{D}_m) = 0$, then $\mathcal{D}_m$ turns out to be compact. However $\Psi_c : \mathcal{D}_m \to \mathcal{D}_m$, so Schauder's Theorem ensures a $\mathcal{FP}$ for Ψ_c in $\mathcal{D}_m \subset \mathcal{D}$. In place of this, if we consider $\aleph(\mathcal{D}_n) > 0$, $\forall n \in \mathcal{N}$. So it must be justified that $\mathcal{D}_\infty \subset \mathcal{D}_n \in \mho$. Primarily we should verify $\aleph(\mathcal{D}_n) \to 0$ as $n \to +\infty$. Based on inclusion (2.11), we may express the following as a sequence $\{\aleph(\mathcal{D}_n)\}$, where $\mathcal{D}_{n+1} < \aleph(\mathcal{D}_n)$ is decreasing down and will eventually converge to $s \in \mathbb{R}$ with $s \geq 0$. Therefore $\aleph(\mathcal{D}_n) \in (0, \infty)$ along $s \in [0, \infty)$, thus by assumption on τ, $\lim\limits_{t \to s^+} \inf \tau(t) > 0$, implies to have $r > 0$ with $n_0 \in \mathcal{N}$ such a way $\tau(\aleph(\mathcal{D}_n)) \geq r$, $\forall n \geq n_0$. Using contraction condition (2.9) with $\mathcal{D}_1 = \mathcal{D}_n$ and $\mathcal{D}_2 = \mathcal{D}_{n+1}$, we write

$$\theta(\aleph(\mathcal{D}_{n+1})) = \theta(\aleph(\mathrm{co}(\Psi_c(\mathcal{D}_n)))) = \theta(\aleph(\Psi_c(\mathcal{D}_n))) \leq \phi\left(\theta\left(\frac{\aleph(\Psi_c(\mathcal{D}_n) + \Psi_c(\mathcal{D}_{n+1}))}{2}\right)\right).$$

Consequently,

$$\begin{aligned}
\theta(\aleph(\mathcal{D}_n)) \leq & \phi\left(\theta\left(\frac{\aleph(\Psi_c(\mathcal{D}_n) + \Psi_c(\mathcal{D}_{n+1}))}{2}\right)\right) \\
\leq & \phi \cdot \phi\left(\theta\left(\frac{\aleph(\Psi_c(\mathcal{D}_{n-1}) + \Psi_c(\mathcal{D}_n))}{2}\right)\right) \\
\leq & \phi^2 \cdot \phi\left(\theta\left(\frac{\aleph(\Psi_c(\mathcal{D}_{n-2}) + \Psi_c(\mathcal{D}_{n-1}))}{2}\right)\right) \\
& \vdots \\
\leq & \phi^{n+1}\left(\theta\left(\frac{\aleph(\Psi_c(\mathcal{D}_{n_0}) + \Psi_c(\mathcal{D}_1))}{2}\right)\right), \ \forall n > n_0.
\end{aligned}$$

Therefore

$$\lim_{n \to \infty} \theta(\aleph(\mathcal{D}_n)) = -\infty.$$

Since θ satisfies condition (Θ_2), it is clear that

$$\lim_{n \to \infty} \aleph(\mathcal{D}_n) = 0.$$

Therefore the definition 1.1(vi) implies, $\mathcal{D}_\infty = \bigcap_{n=1}^{+\infty} \mathcal{D}_n \neq \emptyset$ and $\Psi_c\mathcal{D}_\infty \subset \mathcal{D}_\infty$ as $\Psi_c\mathcal{D}_n \subset \mathcal{D}_n$. Also, since $\mathcal{D}_\infty \subset \mathcal{D}_n$ for every $n \in \mathcal{N}$, as a result of definition 1.1(ii), $\aleph(\mathcal{D}_\infty) \leq \aleph(\mathcal{D}_n)$, $\forall n \in \mathcal{N}$. Thus $\aleph(\mathcal{D}_\infty) = 0$, which is $\mathcal{D}_\infty \in \ker \aleph$, hence $\mathcal{D}_\infty$ comes to be bounded. But $\mathcal{D}_\infty$ is closed such that $\mathcal{D}_\infty$ is compact. Consequently, by Schauder's Theorem Ψ_c assures a $\mathcal{FP}$ in $\mathcal{D}_\infty \subset \mathcal{D}$. □

From the above theorem we deduce several pivotal results in particular, if we choose $\theta(k) = e^k$ with $\phi(k) = k^{2\alpha}$, we conclude the below corollary.

Corollary 2.3. *Suppose $\mathcal{D} \in \mho$. If the Kannan-type mapping $\Psi_c : \mathcal{D} \to \mathcal{D}$ has no discontinuity such that*

$$\aleph(\Psi_c(\mathcal{D}_1)) \leq \alpha(\aleph(\Psi_c(\mathcal{D}_1)) + \aleph(\Psi_c(\mathcal{D}_2))), \tag{2.13}$$

where $\alpha \in [0, \frac{1}{2})$, $\mathcal{D}_1$ and $\mathcal{D}_2$ are subsets of $\mathcal{D}$. As a result Ψ_c ensure a $\mathcal{FP}$ in $\mathcal{D}$.

By following these contractions, one can prove a number of findings that prove or expand upon a number of well-known theorems.

3. Generalized Darbo type Expansion with Fixed Point Theorems

We introduce θ-$\mathscr{F}_\aleph$ expansion and then provide $\mathcal{FP}$ results satisfying such expansion.

Definition 3.1. *The mapping $\Psi_c : \mathcal{D} \to \mathcal{D}$ represent θ-$\mathscr{F}_\aleph$-expanding if $\aleph(\mathcal{D}_0) > 0$, for all $\mathcal{D}_0 \subset \mathcal{D}$ and there exists $\tau \in \mathbb{S}$ with $\mathscr{F} \in \mathbb{F}$, and $\theta \in \Theta$ in a way that*

$$\mathscr{F}(\aleph(\Psi_c(\mathcal{D}_0))) \geq \theta(\mathscr{F}(\aleph(\mathcal{D}_0))) + \tau(\aleph(\mathcal{D}_0)), \tag{3.1}$$

The below result is presented in the context of the θ-$\mathscr{F}_\aleph$-expanding map.

Theorem 3.1. *When the mapping $\Psi_c : \mathcal{D} \to \mathcal{D}$ represent θ-$\mathscr{F}_\aleph$-expanding, surjective with continuity. Then Ψ_c assures a $\mathcal{FP}$ in $\mathcal{D}$.*

Proof. Since $\Psi_c : \mathcal{D} \to \mathcal{D}$ is surjective, as a result by Lemma 1.1 one can find a mapping $\Psi_c^* : \mathcal{D} \to \mathcal{D}$ such that $\Psi_c \circ \Psi_c^*$ is the identity mapping on $\mathcal{D}$. Suppose $\mathcal{D}_1$ and $\mathcal{D}_2$ represent any subsets of $\mathcal{D}$ in a sense that $\mathcal{D}_2 = \Psi_c^*(\mathcal{D}_1)$. Suppose that $\aleph(\mathcal{D}_2) > 0$, so via (3.1), one can write

$$\mathscr{F}(\aleph(\Psi_c(\mathcal{D}_2))) \geq \theta(\mathscr{F}(\aleph(\mathcal{D}_2))) + \tau(\aleph(\mathcal{D}_2)). \tag{3.2}$$

Since $\Psi_c(\mathcal{D}_2) = \Psi_c(\Psi_c^*(\mathcal{D}_1)) = (\Psi_c \circ \Psi_c^*)(\mathcal{D}_1) = \mathcal{D}_1$, so that inequality (3.2) becomes

$$\mathscr{F}(\aleph(\mathcal{D}_1)) \geq \theta(\mathscr{F}(\aleph(\Psi_c^*(\mathcal{D}_1)))) + \tau(\aleph(\Psi_c^*(\mathcal{D}_1))). \tag{3.3}$$

However, if $\Psi_c^* u = u$, then $\Psi_c u = \Psi_c(\Psi_c^* u) = u$. Thus to investigate the $\mathcal{FP}$ of Ψ_c, it is enough to find it for the Ψ_c^*. For this, a sequence is described as $\{\mathcal{D}_n\}_{n=0}^{\infty}$ such a way

$$\mathcal{D}_0 = \mathcal{D} \text{ and } \mathcal{D}_n = \overline{co}(\Psi_c^*\mathcal{D}_{n-1}),\ \forall\ n \in \mathcal{N}. \tag{3.4}$$

It is obvious to show that $\mathcal{D}_{n+1} \subset \mathcal{D}_n$ and $\Psi_c^*\mathcal{D}_n \subset \mathcal{D}_n$. Further, if one take an integer m which is non-negative along $\aleph(\mathcal{D}_m) = 0$, then $\mathcal{D}_m$ is compact and Theorem Schauder, ensures the existence of $\mathcal{FP}$ of Ψ_c^* in $\mathcal{D}_m \subset \mathcal{D}$. Then as the sequence $\{\aleph(\mathcal{D}_n)\}$ is decreasing converging to non-negative real number s. Assume that $\aleph(\mathcal{D}_{n+1}) > 0$, then $0 < \aleph(\mathcal{D}_{n+1}) = \aleph(\overline{co}(\Psi_c^*(\mathcal{D}_n))) = \aleph(\Psi_c^*(\mathcal{D}_n))$, that is, $\aleph(\Psi_c^*(\mathcal{D}_n)) \in (0,\infty)$ and $s \in [0,\infty)$, so that $\lim_{t\to s^+}\inf \tau(\aleph(\Psi_c^*(\mathcal{D}_n))) > 0$, that is we found $0 < r$ and $n_0 \in \mathscr{N}$ along $\tau(\aleph(\Psi_c^*(\mathcal{D}_n))) \geq r$, for all $n \geq n_0$. Application of (3.3) with $\mathcal{D} = \mathcal{D}_n$, we write

$$\begin{aligned}\tau(\aleph(\Psi_c^*(\mathcal{D}_n))) + \theta(\mathscr{F}(\aleph(\mathcal{D}_{n+1}))) &= \tau(\aleph(\Psi_c^*(\mathcal{D}_n))) + \mathscr{F}(\aleph(\overline{co}(\Psi_c^*(\mathcal{D}_n))))\\ &= \tau(\aleph(\Psi_c^*(\mathcal{D}_n))) + \mathscr{F}(\aleph(\Psi_c^*(\mathcal{D}_n)))\\ &\leq \mathscr{F}(\aleph(\mathcal{D}_n)).\end{aligned}$$

Moving forward, we write

$$\theta(\mathscr{F}(\aleph(\mathcal{D}_{n+1}))) \leq \mathscr{F}(\aleph(\mathcal{D}_n)) - \tau(\aleph(\Psi_c^*(\mathcal{D}_n))) \leq \mathscr{F}(\aleph(\mathcal{D}_n)) - r.$$

Consequently,

$$\theta(\mathscr{F}(\aleph(\mathcal{D}_n))) \leq \mathscr{F}(\aleph(\mathcal{D}_{n-1})) - r.$$

By routine calculation, one can easily obtained that

$$\theta(\mathscr{F}(\aleph(\mathcal{D}_n))) \leq \mathscr{F}(\aleph(\mathcal{D}_{n_0})) - (n-n_0)r,\ \forall n > n_0.$$

Clearly $\lim_{n\to+\infty}\theta(\mathscr{F}(\aleph(\mathcal{D}_n))) = -\infty$, and utilizing property (F_2), one could say $\mathscr{F}(\lim_{n\to+\infty}\aleph(\mathcal{D}_n)) = 0$. So by Definition 1.1, $\mathcal{D}_\infty = \bigcap_{n=1}^{+\infty}\mathcal{D}_n \neq \emptyset$ and $\mathcal{D}_\infty \supset \Psi_c^*\mathcal{D}_\infty$ as $\mathcal{D}_n \supset \Psi_c^*\mathcal{D}_n$. Also, since $\mathcal{D}_\infty \subset \mathcal{D}_n$, $\forall n \in \mathscr{N}$, so by Definition 1.1(ii), $\aleph(\mathcal{D}_\infty) \leq \aleph(\mathcal{D}_n)$, for every $n \in \mathscr{N}$. Thus $\aleph(\mathcal{D}_\infty) = 0$ and hence $\mathcal{D}_\infty \in \ker\aleph$, that is $\mathcal{D}_\infty$ turns out to be bounded. But $\mathcal{D}_\infty$ is closed such that $\mathcal{D}_\infty$ is compact. Hence by Theorem Schauder, Ψ_c^* assures a $\mathcal{FP}$ in $\mathcal{D}_\infty \subset \mathcal{D}$. Therefore, Ψ_c ensures a $\mathcal{FP}$ in $\mathcal{D}_\infty \subset \mathcal{D}$. □

To further demonstrate the Theorem, the following example is developed (3.1).

Example 3.1. *Suppose $\mathcal{D} = [-8, 9] \subseteq \mathbb{R}$. Therefore, it is evident $\mathcal{D} \in \mho$. Moreover define $\Psi_c : \mathcal{D} \to \mathcal{D}$ with $\mathscr{F} : (0,\infty) \longrightarrow \mathbb{R}$ and $\tau : (0,\infty) \longrightarrow \mathbb{R}$ as $\Psi_c(k) = 1 - k$, $\theta(k) = \sqrt[3]{k} - \frac{1}{2}(k \geq 1)$, $\tau(k) = \ln\sqrt{k}$ with $\mathscr{F}(k) = \ln k + k$, correspondingly. It is simple to verify that $\mathscr{F} \in \mathbb{F}$ with $\tau \in \mathbb{S}$, $\theta \in \Theta$, and Ψ_c is continuous. Further, define an $\mathscr{MNC}$, $\aleph : \mathcal{B}(\mathscr{E}) \to [0,\infty)$ by*

$$\aleph(\mathcal{D}) = diam(\mathcal{D}) = \sup_{k,u\in\mathcal{D}}|k-u|.$$

Now, let $\mathcal{D}_0 = [0, 1]$ be subset of $\mathcal{D}$. As a result $\aleph(\mathcal{D}_0) = \aleph(\Psi_c(\mathcal{D}_0)) = 1$. Thus from expansion (3.1), *we write*

$$\theta(\mathscr{F}(\aleph(\mathcal{D}_0))) + \tau(\aleph(\mathcal{D}_0)) = \theta(\mathscr{F}(1)) + \tau(1) = 0.5 < 1 = \mathscr{F}(\aleph(\Psi_c(\mathcal{D}_0))).$$

That is, θ-$\mathscr{F}_\aleph$-weak expansion. As a result, Ψ_c has a $\mathcal{FP}$ $\frac{1}{2} \in \mathcal{D}$ according to Theorem (3.1).

Remark 3.1. *We can deduce many essential expansion from expansion* (3.1)*. We provide some preferable expansions that extend and covers various familiar theorems in the literature. Notably:*

(1) *If we choose $\mathscr{F}(k) = \ln k, 0 < k$, we deduce*

$$\aleph(\Psi_c(\mathcal{D})) \geq \aleph((\mathcal{D}))^\theta e^{\tau(\aleph(\mathcal{D}))},\ \textit{for all}\ \mathcal{D} \subset \mathcal{D};$$

(2) *If we take* $\mathscr{F}(k) = \ln k + k, k > 0$, *we deduce*

$$\aleph(\Psi_c(\mathcal{D})) \geq (\aleph(\mathcal{D}))^{\theta} e^{\tau(\aleph(\mathcal{D}))-\aleph(\Psi_c(\mathcal{D}))+\aleph(\mathcal{D})}, \text{ for all } \mathcal{D} \subset \mathcal{D};$$

(3) *If we choose* $\mathscr{F}(k) = \ln(k^2 + k), k > 0$, *we deduce*

$$\aleph(\Psi_c(\mathcal{D}))(\aleph(\Psi_c(\mathcal{D})) + 1) \geq (\aleph(\mathcal{D}))^{\theta}(\aleph(\mathcal{D}) + 1)e^{\tau(\aleph(\mathcal{D}))}, \text{ for all } \mathcal{D} \subset \mathcal{D};$$

(4) *If we choose* $\mathscr{F}(k) = \arctan\left(\frac{-1}{k}\right)$ *with* $k > 0$, *we deduce*

$$\aleph(\Psi_c(\mathcal{D})) \geq \frac{\aleph(\mathcal{D}) + \theta \tan\tau(\aleph(\mathcal{D}))}{\theta - \tan\tau(\aleph(\mathcal{D})) + \aleph(\mathcal{D})}, \text{ for all } \mathcal{D} \subset \mathcal{D}.$$

By applying these extensions, one can prove a number of findings that generalise and build upon a number of well-established theorems from the existing body of literature.

4. Applications

Here in this part, the study applies the result for the existence of solution to second order differential equations. Suppose $(\mathscr{E}, \|.\|)$ represent a Banach space, $\mathbf{B}(a, r)$ denotes a closed ball along r radius with center a, and $\mathbf{B}_r$ denotes the ball $\mathbf{B}(0, r)$. Our goal is to demonstrate sufficient conditions for the existence of solution to the differential equation:

$$\begin{cases} u''(t) = \psi(t, u(t)), \quad t \in [0,1], \\ u(0) = u_0, u(1) = u_1. \end{cases} \tag{4.1}$$

Where $\psi : [0,1] \times \mathbb{R} \to \mathbb{R}$ represent a function with no discontinuous. It is possible to express the previously mentioned problem as an integral equation:

$$u(t) = F(t) + \xi \int_0^1 \mathbb{G}_r(t,s)u(s)ds, \qquad t \in [0,1], \tag{4.2}$$

in which $F(t) = u_0 + t(u_1 - u_0)$ and $\mathbb{G}_r(t,s)$ is the Green's function, described by

$$\mathbb{G}_r(t,s) = \begin{cases} s(1-s) & 0 \leq s \leq t \leq t, \\ t(1-s) & 0 \leq t \leq s \leq 1. \end{cases} \tag{4.3}$$

We are now in the position to exhibit the existence result.

Theorem 4.1. *Suppose* $u, v \in \mathbf{B}_r$ *with* $\psi : [0,1] \times \mathbb{R} \to \mathbb{R}$ *represent a mapping with no discontinuous in the extent that*

$$\|u(t) - v(t))\| \leq \frac{1}{\xi} \sup_{t \in [0,1]} \sqrt{(\left|u(t) - v(t)\right|^2 - r)e^{-\tau}}, \tag{4.4}$$

and

$$\frac{\mathfrak{M}}{6} \leq r, \tag{4.5}$$

where $|u(t)-v(t)|>r$, $r\geq 0$, *and* $\sup_{t\in[0,1]}\left|u(s)\right|=\mathfrak{M}<\infty$. *Then there exists a solution of the second order differential equation* (4.2) *in* $\mathbf{B}_r$. *Consequently, there exists a solution of the differential equation of second order* (4.1) *in* $\mathbf{B}_r$.

Proof. Let $\mathbf{B}_r=\{u\in C([0,\mathfrak{T}],\mathbb{R}):\|u\|\leq r\}$. Then, $\mathbf{B}_r$ is a non-empty, bounded, convex, and closed subset of $(\mathscr{E},\|.\|)$. Define $\theta:\mathbb{R}\longrightarrow\mathbb{R}$ as $\theta(k)=\ln k$ and $\Phi:(0,\infty)\to\mathbb{R}$, with $\tau>0$, and the operator $\Psi_c:\mathbf{B}_r\to\mathbf{B}_r$ by

$$\Psi_c u(t)=F(t)+\xi\int_0^1\mathbb{G}_r(t,s)u(s)ds.$$

Moreover, define $\mathscr{M}\mathscr{N}\mathscr{C}$ $\aleph:\mathcal{B}(\mathscr{E})\to[0,\infty)$ by

$$\aleph\left(\Psi_c\mathcal{D}_1\right)=\sup_{t\in[0,1]}\left|\Psi_c u\left(t\right)-\Psi_c v\left(t\right)\right|^2,\text{ for all }\ u,v\in\mathcal{D}.$$

First, we will prove that $\Psi_c:\mathbf{B}_r\to\mathbf{B}_r$ is well-defined. Assume $u\in\mathbf{B}_r$ for a specific value of r. For every $t\in[0,1]$, it follows that

$$\begin{aligned}\left|\Psi_c u(t)\right|=&\left|F(t)+\xi\int_0^1\mathbb{G}_r(t,s)u(s)ds\right|\\ \leq&F(t)+\xi\int_0^1\mathbb{G}_r(t,s)\left|u(s)\right|ds\\ \leq&\sup_{s\in[0,1]}\left|u(s)\right|\int_0^1\mathbb{G}_r(t,s)ds\\ \leq&\mathfrak{M}\left[\frac{t^3}{6}-\frac{t^2}{2}+\frac{t}{2}\right]_0^1\\ =&\frac{\mathfrak{M}}{6}\\ \leq&r.\end{aligned}$$

That is $\left\|\Psi_c\left(u\right)\right\|\leq r$, $\forall u\in\mathbf{B}_r$, that means that $\Psi_c\left(u\right)\in\mathbf{B}_r$ and consequently $\Psi_c:\mathbf{B}_r\to\mathbf{B}_r$ is well-defined.

Next, we need to prove that $\Psi_c:\mathbf{B}_r\to\mathbf{B}_r$ is a contraction. Assume

$$\begin{aligned}\left|\Psi_c u\left(t\right)-\Psi_c v\left(t\right)\right| &= \left|F(t)+\xi\int_0^1\mathbb{G}_r(t,s)u(s)ds-F(t)-\xi\int_0^1\mathbb{G}_r(t,s)v(s)ds\right|\\ &\leq \xi\int_0^1\mathbb{G}_r(t,s)\left|u(s)-v(s)\right|ds\\ &\leq \sup_{t\in[0,1]}\left|u(t)-v(t)\right|\xi\int_0^1\mathbb{G}_r(t,s)ds\\ &\leq \sup_{t\in[0,1]}\sqrt{(\left|u(t)-v(t)\right|^2-r)e^{-\tau}}\int_0^1\mathbb{G}_r(t,s)ds\end{aligned}$$

$$= \left[\frac{t^3}{6} - \frac{t^2}{2} + \frac{t}{2}\right]_0^1 \sup_{t\in[0,1]} \sqrt{(\left|u(t) - v(t)\right|^2 - r)e^{-\tau}}$$
$$= \frac{1}{6}\sup_{t\in[0,1]} \sqrt{(\left|u(t) - v(t)\right|^2 - r)e^{-\tau}}.$$

Using the definition of $\mathscr{MNC}$, we get

$$\|\Psi_c u - \Psi_c v\|^2 \leq \left\{(\|u - v\|^2 - r)e^{-\tau}\right\},$$

or

$$\aleph\left(\Psi_c \mathcal{D}_1\right) \leq \left(\aleph\left(\mathcal{D}_1\right)\right)e^{-\tau}, \tag{4.6}$$

therefore implying that

$$\aleph\left(\Psi_c \mathcal{D}_1\right) \leq \Delta(\mathcal{D}_1, \mathcal{D}_2)e^{-\tau}, \tag{4.7}$$

where

$$\Delta(\mathcal{D}_1, \mathcal{D}_2) = \sup\left\{\aleph(\mathcal{D}_1), \aleph(\Psi_c(\mathcal{D}_1)), \aleph(\Psi_c(\mathcal{D}_2)), \frac{1}{2}\aleph(\Psi_c(\mathcal{D}_1) \cup \Psi_c(\mathcal{D}_2))\right\}.$$

We obtain by applying logarithms to inequality (4.7).

$$\ln(\aleph\left(\Psi_c \mathcal{D}_1\right)) \leq -\tau + \ln\left(\Delta(\mathcal{D}_1, \mathcal{D}_2)\right),$$

which implies that

$$\tau + \theta(\aleph\left(\Psi_c \mathcal{D}_1\right)) \leq \theta\left(\Delta(\mathcal{D}_1, \mathcal{D}_2)\right).$$

Ψ_c assures a $\mathcal{FP}$ in $\mathbf{B}_r$. For that reason, the corollary (2.2) ensure a solution in $\mathbf{B}_r$. □

5. Conclusion

Several new θ-$\Phi_{\aleph}$-contraction and θ-$\mathscr{F}_{\aleph}$-expansion have been introduced through measure of noncompactness. There are several existing results that can be derived from the $\mathcal{FP}$ results that were established in Banach space. For the validity of established results, we construct some examples, and also we have established the existence of a solution to the second order differential equation in order to ensure the accuracy of our results.

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References

[1] J. Banaś, On Measures of Noncompactness in Banach Spaces, Comment. Math. Univ. Carol. 21 (1980), 131–143. https://eudml.org/doc/17021.

[2] G. Darbo, Punti Uniti in Trasformazioni a Codominio Non Compatto, Rend. Sem. Mat. Univ. Padova 24 (1955), 84–92.

[3] V. Parvaneh, N. Hussain, M. Khorshidi, N. Mlaiki, H. Aydi, Fixed Point Results for Generalized $\mathcal{F}$-Contractions in Modular B-Metric Spaces with Applications, Mathematics 7 (2019), 887. https://doi.org/10.3390/math7100887.

[4] J. Garcia-Falset, K. Latrach, On Darbo-Sadovskii's Fixed Point Theorems Type for Abstract Measures of (weak) Noncompactness, Bull. Belg. Math. Soc. - Simon Stevin 22 (2015), 797–812. https://doi.org/10.36045/bbms/1450389249.

[5] S. Banach, Sur les Opérations dans les Ensembles Abstraits et Leur Application aux Équations Intégrales, Fundam. Math. 3 (1922), 133–181. https://doi.org/10.4064/fm-3-1-133-181.

[6] L.E.J. Brouwer, Über Abbildung von Mannigfaltigkeiten, Math. Ann. 71 (1911), 97–115. https://eudml.org/doc/158520.

[7] J. Chen, X. Tang, Generalizations of Darbo's Fixed Point Theorem via Simulation Functions with Application to Functional Integral Equations, J. Comput. Appl. Math. 296 (2016), 564–575. https://doi.org/10.1016/j.cam.2015.10.012.

[8] G. Darbo, Punti Uniti in Trasformazioni a Codominio Non Compatto, Rend. Sem. Mat. Univ. Padova 24 (1955), 84–92. http://www.numdam.org/item?id=RSMUP_1955__24__84_0.

[9] A.A. Gillespie, B.B. Williams, V. Lakshmikantham, Fixed Point Theorems for Expanding Maps, Appl. Anal. 14 (1983), 161–165. https://doi.org/10.1080/00036818308839419.

[10] J. Górnicki, Fixed Point Theorems for $\mathcal{F}$-Expanding Mappings, Fixed Point Theory Appl. 2017 (2016), 9. https://doi.org/10.1186/s13663-017-0602-3.

[11] M. Jleli, B. Samet, A New Generalization of the Banach Contraction Principle, J. Inequal. Appl. 2014 (2014), 38. https://doi.org/10.1186/1029-242x-2014-38.

[12] M. Jleli, E. Karapinar, D. O'Regan, B. Samet, Some Generalizations of Darbo's Theorem and Applications to Fractional Integral Equations, Fixed Point Theory Appl. 2016 (2016), 11. https://doi.org/10.1186/s13663-016-0497-4.

[13] S.M. Kang, Fixed Point for Expansion Mappings, Math. Japonica 38 (1993), 713–717. https://cir.nii.ac.jp/crid/1573668924494060288.

[14] C. Kuratowski, Sur les Espaces Complets, Fundam. Math. 15 (1930), 301–309. https://doi.org/10.4064/fm-15-1-301-309.

[15] S. Park, B.E. Rhoades, Some Fixed Point Theorems for Expansion Mappings, Math. Japon. 33 (1988), 129–132.

[16] V. Parvaneh, N. Hussain, M. Khorshidi, N. Mlaiki, H. Aydi, Fixed Point Results for Generalized $\mathcal{F}$-Contractions in Modular b-Metric Spaces with Applications, Mathematics 7 (2019), 887. https://doi.org/10.3390/math7100887.

[17] J. Schauder, Der Fixpunktsatz in Funktionalraümen, Stud. Math. 2 (1930), 171–180. http://eudml.org/doc/217247.

[18] D. Wardowski, Fixed Points of a New Type of Contractive Mappings in Complete Metric Spaces, Fixed Point Theory Appl. 2012 (2012), 94. https://doi.org/10.1186/1687-1812-2012-94.

[19] D. Wardowski, Solving Existence Problems via $\mathcal{F}$-Contractions, Proc. Am. Math. Soc. 146 (2017), 1585–1598. https://doi.org/10.1090/proc/13808.

[20] S.Z. Wang, B.Y. Li, Z.M. Gao, K.K. Iséki, Some Fixed Point Theorems on Expansion Mappings, Math. Japon. 29 (1984), 631–636.

[21] M. Al-Refai, K. Pal, New Aspects of Caputo-Fabrizio Fractional Derivative, Prog. Fract. Differ. Appl. 5 (2019), 157–166. https://doi.org/10.18576/pfda/050206.

[22] M. Younis, D. Singh, D. Gopal, A. Goyal, M.S. Rathore, On Applications of Generalized $\mathcal{F}$-Contraction to Differential Equations, Nonlinear Funct. Anal. Appl. 24 (2019), 155–174.

[23] M.B. Zada, M. Sarwar, T. Abdeljawad, F. Jarad, Generalized Darbo-Type $\mathcal{F}$-Contraction and $\mathcal{F}$-Expansion and Its Applications to a Nonlinear Fractional-Order Differential Equation, J. Funct. Spaces 2020 (2020), 4581035. https://doi.org/10.1155/2020/4581035.