Int. J. Anal. Appl. (2025), 23:210

International Journal of Analysis and Applications

b^*–$\mathcal{I}$-Open Sets and Their Role in Weaker Forms of Paracompactness

Chawalit Boonpok[1], **Areeyuth Sama-Ae**[2,*]

[1]*Mathematics and Applied Mathematics Research Unit, Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham, 44150, Thailand*

[2]*Department of Mathematics and Computer Science, Faculty of Science and Technology, Prince of Songkla University, Pattani Campus, Pattani, 94000, Thailand*

[*]*Corresponding author:* areeyuth.s@psu.ac.th

Abstract. This paper introduces and investigates two new notions of paracompactness in ideal topological spaces: b^*-$\mathcal{I}$-paracompactness and b_1^*-$\mathcal{I}$-paracompactness. These concepts generalize classical paracompactness using b^*-$\mathcal{I}$-open sets and ideal-related refinements. We establish fundamental properties of these spaces, including their preservation under subspaces, finite unions, and continuous mappings. Furthermore, we provide characterizations of these spaces and compare them with existing variants such as β-paracompactness, β_1-paracompactness and $\mathcal{I}$-paracompactness, supported by illustrative examples. Our results extend the theory of ideal topological spaces and offer a framework for future studies on generalized covering properties.

1. Introduction and Preliminaries

General topology has proven valuable in both theoretical and applied contexts, particularly in computational topology, geometric modeling, and engineering design. Khalimsky et al. [16] and Kong and Kopperman [18] advanced digital topology using connected topologies on finite ordered sets. Topological methods have also been applied in geometric [21] and engineering design [25].

Paracompact spaces, introduced by Dieudonné [10], generalize compact and metrizable spaces. A space is paracompact if every open cover has a locally finite open refinement. In Hausdorff spaces, this is equivalent to admitting partitions of unity subordinate to any open cover, and such spaces are normal [12]. Generalizations such as S-paracompactness [4], P_3-paracompactness [5], and β-paracompactness [9] have been studied extensively. Ideal topological spaces, introduced by Kuratowski [19] and Vaidyanathaswamy [28], extend classical topologies through the use of ideals.

Received: Jul. 3, 2025.

2020 *Mathematics Subject Classification.* 54A05, 54B05, 54C10.

Key words and phrases. ideal topological space; b^*-$\mathcal{I}$-open; b^*-$\mathcal{I}$-paracompactness; b_1^*-$\mathcal{I}$-paracompactness.

https://doi.org/10.28924/2291-8639-23-2025-210
ISSN: 2291-8639

This framework has been applied to ideal resolvability [11], $\mathcal{I}$-open sets [15], ideal continuity [14], and related concepts [1,2,13,17,20].

The study of paracompactness relative to ideals began with Zahid [30] and was expanded by Hamlett et al. [13], Sathiyasundari and Renukadevi [27], and Sanabria et al. [26], who introduced $\mathcal{I}$-S-paracompactness. Demir and Ozbakir [9] later introduced β-paracompact spaces, while Yildirim et al. [29] explored their ideal counterparts. Boonpok et al. [6,8] and Boonpok and Sama-Ae [7] proposed and characterized weaker versions, such as δ-$\beta_{\mathcal{I}}$-paracompactness, δ_1-$\beta_{\mathcal{I}}$-paracompactness, and strong β-$\mathcal{I}$-paracompactness.

This paper presents the notions of b^*-$\mathcal{I}$-paracompact and b^*_1-$\mathcal{I}$-paracompact spaces, which are defined using b^*-$\mathcal{I}$-open sets. The study explores their fundamental properties and investigates their connections with β-paracompactness [29] and β_1-paracompactness [24]. Additionally, several characterizations are established to further develop the framework of ideal topology and extend its potential applications.

Throughout this paper, unless stated otherwise, we denote by (X,τ) (or simply X) a general topological space without any assumptions regarding separation axioms. For any subset $A \subseteq X$, the closure and interior of A are represented by $\mathrm{cl}(A)$ and $\mathrm{int}(A)$, respectively.

An *ideal* $\mathcal{I}$ on a set X is a nonempty collection of subsets of X satisfying the following conditions: if $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$; and if $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$. A topological space (X,τ) equipped with an ideal $\mathcal{I}$ is called an *ideal topological space* and is denoted by $(X,\tau,\mathcal{I})$. The power set of X is denoted by $\mathcal{P}(X)$. A set operator $(\cdot)^* : \mathcal{P}(X) \to \mathcal{P}(X)$, referred to as a *local function* [19], is defined in terms of the topology τ and the ideal $\mathcal{I}$. For any $A \subseteq X$, it is given by

$$A^*(\mathcal{I},\tau) = \{x \in X : U \cap A \notin \mathcal{I} \text{ for all } U \in \tau(x)\},$$

where $\tau(x) = \{U \in \tau : x \in U\}$ denotes the family of open neighborhoods of x.

The operator $(\cdot)^*$ induces a finer topology on X, called the *$*$-topology*, and denoted by $\tau^*(\mathcal{I})$. In this topology, the closure of a set A is defined as $\mathrm{cl}^*(A) = A \cup A^*$, and its interior is denoted by $\mathrm{int}^*(A)$ [15]. The topology $\tau^*(\mathcal{I})$ is generated by the subbasis

$$\beta(\mathcal{I},\tau) = \{U - I : U \in \tau,\ I \in \mathcal{I}\}.$$

It should be emphasized that, in general, $\beta(\mathcal{I},\tau)$ does not constitute a topology. When $\beta(\mathcal{I},\tau) = \tau^*$, the ideal $\mathcal{I}$ is referred to as τ-simple. Furthermore, the ideal $\mathcal{I}$ is called $\mathcal{I}$-codense if $\mathcal{I} \cap \tau = \{\emptyset\}$ [15].

It is clear that if $A \subseteq B$, then $A^* \subseteq B^*$ and consequently $\mathrm{cl}^*(A) \subseteq \mathrm{cl}^*(B)$. Since $\tau \subseteq \tau^*(\mathcal{I})$, it follows that $\mathrm{cl}^*(A) \subseteq \mathrm{cl}(A)$ and $\mathrm{int}(A) \subseteq \mathrm{int}^*(A)$.

Definition 1.1. *Let $(X,\tau,\mathcal{I})$ be an ideal topological space. A subset $A \subseteq X$ is called b^*-$\mathcal{I}$-*open *if it satisfies:*

$$A \subseteq \mathrm{cl}(\mathrm{int}^*(A)) \cup \mathrm{int}^*(\mathrm{cl}(A)).$$

*The complement of a b^*-$\mathcal{I}$-open set is said to be b^*-$\mathcal{I}$-*closed.

Remark 1.1. As stated in Definition 1.1, within an ideal topological space, every open set is b^*-$\mathcal{I}$-open, and every closed set is b^*-$\mathcal{I}$-closed. Moreover, the union of any collection of b^*-$\mathcal{I}$-open sets is again b^*-$\mathcal{I}$-open, and the intersection of any family of b^*-$\mathcal{I}$-closed sets is also b^*-$\mathcal{I}$-closed.

In the following example, we show that there exists a b^*-$\mathcal{I}$-open set which is not open in the original topology, and that the intersection of two b^*-$\mathcal{I}$-open sets may fail to be b^*-$\mathcal{I}$-open.

Example 1.1. Let $(X, \tau, \mathcal{I})$ be an ideal topological space, where $X = \{1, 2, 3\}$, the topology is given by $\tau = \{\emptyset, \{1\}, X\}$, and the ideal is $\mathcal{I} = \{\emptyset, \{2\}, \{3\}, \{2, 3\}\}$.

The associated $*$-topology is $\tau^* = \{\emptyset, \{1\}, \{1, 2\}, \{1, 3\}, X\}$. Consequently, the family of all b^*-$\mathcal{I}$-open sets is

$$\{\emptyset, \{1\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, X\}.$$

Observe that the set $A = \{1, 2\}$ is b^*-$\mathcal{I}$-open, even though it is not open in the original topology τ. Furthermore, both sets $\{1, 3\}$ and $\{2, 3\}$ are b^*-$\mathcal{I}$-open, yet their intersection $\{3\}$ is not b^*-$\mathcal{I}$-open.

Definition 1.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$.*

(1) *The* b^*-$\mathcal{I}$-closure *of A, denoted by $b^* \operatorname{cl}_{\mathcal{I}}(A)$, is defined as:*

$$b^* \operatorname{cl}_{\mathcal{I}}(A) = \cap\{F \subseteq X \mid A \subseteq F \text{ and } F \text{ is } b^*\text{-}\mathcal{I}\text{-closed}\}$$

(2) *The* b^*-$\mathcal{I}$-interior *of A, denoted by $b^* \operatorname{int}_{\mathcal{I}}(A)$, is defined as:*

$$b^* \operatorname{int}_{\mathcal{I}}(A) = \cup\{U \subseteq X \mid U \subseteq A \text{ and } U \text{ is } b^*\text{-}\mathcal{I}\text{-open}\}$$

Remark 1.2. Using Definition 1.2, it follows that in an ideal topological space $(X, \tau, \mathcal{I})$, for any subset $A \subseteq X$, we have $b^* \operatorname{int}_{\mathcal{I}}(A) \subseteq A \subseteq b^* \operatorname{cl}_{\mathcal{I}}(A)$. Moreover, the usual interior satisfies $\operatorname{int}(A) \subseteq b^* \operatorname{int}_{\mathcal{I}}(A)$, and the ideal closure satisfies $b^* \operatorname{cl}_{\mathcal{I}}(A) \subseteq \operatorname{cl}(A)$. Additionally, whenever $A \subseteq B \subseteq X$, it holds that $b^* \operatorname{int}_{\mathcal{I}}(A) \subseteq b^* \operatorname{int}_{\mathcal{I}}(B)$ and $b^* \operatorname{cl}_{\mathcal{I}}(A) \subseteq b^* \operatorname{cl}_{\mathcal{I}}(B)$.

Lemma 1.1. *Let A and B be subsets of an ideal topological space $(X, \tau, \mathcal{I})$. The following statements hold:*

(1) $X - b^* \operatorname{cl}_{\mathcal{I}}(A) = b^* \operatorname{int}_{\mathcal{I}}(X - A)$*; and*
(2) $X - b^* \operatorname{int}_{\mathcal{I}}(A) = b^* \operatorname{cl}_{\mathcal{I}}(X - A)$.

Proof. We establish the duality between b^*-$\mathcal{I}$-closure and b^*-$\mathcal{I}$-interior:

(1) For the complement of b^*-$\mathcal{I}$-closure:

$$\begin{aligned} X - b^* \operatorname{cl}_{\mathcal{I}}(A) &= X - \cap\{F \mid A \subseteq F,\ F \text{ is } b^*\text{-}\mathcal{I}\text{-closed}\} \\ &= \cup\{X - F \mid X - F \subseteq X - A,\ X - F \text{ is } b^*\text{-}\mathcal{I}\text{-open}\} \\ &= \cup\{G \mid G \subseteq X - A,\ G \text{ is } b^*\text{-}\mathcal{I}\text{-open}\} \\ &= b^* \operatorname{int}_{\mathcal{I}}(X - A). \end{aligned}$$

(2) For the complement of b^*-$\mathcal{I}$-interior:

$$\begin{aligned} X - b^* \operatorname{int}_{\mathcal{I}}(A) &= X - \cup\{G \mid G \subseteq A,\ G \text{ is } b^*\text{-}\mathcal{I}\text{-open}\} \\ &= \cap\{X - G \mid X - A \subseteq X - G,\ X - G \text{ is } b^*\text{-}\mathcal{I}\text{-closed}\} \\ &= \cap\{F \mid X - A \subseteq F,\ F \text{ is } b^*\text{-}\mathcal{I}\text{-closed}\} \\ &= b^* \operatorname{cl}_{\mathcal{I}}(X - A). \end{aligned}$$

This completes the proof of both statements. □

The following lemma outlines fundamental characteristics and conditions related to b^*-$\mathcal{I}$-open and b^*-$\mathcal{I}$-closed sets within the context of ideal topological spaces.

Lemma 1.2. *Let A be a subset of an ideal topological space $(X, \tau, \mathcal{I})$. The following properties hold:*

(1) *$b^* \operatorname{int}_{\mathcal{I}}(A)$ is b^*-$\mathcal{I}$-open, and A is b^*-$\mathcal{I}$-open if and only if $A = b^* \operatorname{int}_{\mathcal{I}}(A)$;*

(2) *$b^* \operatorname{cl}_{\mathcal{I}}(A)$ is b^*-$\mathcal{I}$-closed, and A is b^*-$\mathcal{I}$-closed if and only if $A = b^* \operatorname{cl}_{\mathcal{I}}(A)$; and*

(3) *$x \in b^* \operatorname{cl}_{\mathcal{I}}(A)$ if and only if $U \cap A \neq \emptyset$ for every b^*-$\mathcal{I}$-open set U containing x.*

Proof. (1): Let U_ω be any b^*-$\mathcal{I}$-open set such that $U_\omega \subseteq A$. By the definition of a b^*-$\mathcal{I}$-open set, it follows that $U_\omega \subseteq b^* \operatorname{int}_{\mathcal{I}}(A)$. Moreover, since U_ω is b^*-$\mathcal{I}$-open, we have

$$U_\omega \subseteq \operatorname{cl}(\operatorname{int}^*(U_\omega)) \cup \operatorname{int}^*(\operatorname{cl}(U_\omega)).$$

Then,

$$\begin{aligned} b^* \operatorname{int}_{\mathcal{I}}(A) &= \cup_\omega U_\omega \\ &\subseteq \cup_\omega \operatorname{cl}(\operatorname{int}^*(U_\omega)) \cup \operatorname{int}^*(\operatorname{cl}(U_\omega)) \\ &\subseteq \operatorname{cl}(\operatorname{int}^*(\cup_\omega U_\omega)) \cup \operatorname{int}^*(\operatorname{cl}(\cup_\omega U_\omega)) \\ &= \operatorname{cl}(\operatorname{int}^*(b^* \operatorname{int}_{\mathcal{I}}(A))) \cup \operatorname{int}^*(\operatorname{cl}(b^* \operatorname{int}_{\mathcal{I}}(A))), \end{aligned}$$

and therefore $b^* \operatorname{int}_{\mathcal{I}}(A)$ is a b^*-$\mathcal{I}$-open set.

Since $b^* \operatorname{int}_{\mathcal{I}}(A)$ is the union of all b^*-$\mathcal{I}$-open sets contained in A, it follows that $b^* \operatorname{int}_{\mathcal{I}}(A) \subseteq A$. Therefore, A is b^*-$\mathcal{I}$-open if and only if $A = b^* \operatorname{int}_{\mathcal{I}}(A)$.

(2): From part (2) of Lemma 1.1, we know that $b^* \operatorname{cl}_{\mathcal{I}}(A)$ is a b^*-$\mathcal{I}$-closed set. Since it is defined as the intersection of all b^*-$\mathcal{I}$-closed sets that contain A, it necessarily follows that $A \subseteq b^* \operatorname{cl}_{\mathcal{I}}(A)$. Therefore, a set A is b^*-$\mathcal{I}$-closed if and only if $A = b^* \operatorname{cl}_{\mathcal{I}}(A)$.

(3): Let $x \in b^* \operatorname{cl}_{\mathcal{I}}(A)$. Then, x is an element of all b^*-$\mathcal{I}$-closed sets containing A. Suppose, to the contrary, that there exists a b^*-$\mathcal{I}$-open set U containing x such that $U \cap A = \emptyset$. Then $A \subseteq X - U$, and since $X - U$ is b^*-$\mathcal{I}$-closed and does not contain x, this would imply $x \notin b^* \operatorname{cl}_{\mathcal{I}}(A)$ — a contradiction.

Conversely, suppose that for every b^*-$\mathcal{I}$-open set U containing x, we have $U \cap A \neq \emptyset$. Assume, for contradiction, that $x \notin b^* \operatorname{cl}_{\mathcal{I}}(A)$. Then there exists a b^*-$\mathcal{I}$-closed set F such that $A \subseteq F$ and $x \notin F$. Thus, $x \in X - F$, which is a b^*-$\mathcal{I}$-open set disjoint from A, again yielding a contradiction. Hence, it must be that $x \in b^* \operatorname{cl}_{\mathcal{I}}(A)$. □

2. b^*-$\mathcal{I}$-Paracompactness and Its Characterizations

This section explores the notion of b^*-$\mathcal{I}$-paracompactness, which is a variant of the original paracompactness and the $\mathcal{I}$-β-paracompactness concept introduced by Yildirim et al. [29]. We aim to present a formal description of this concept.

Let $\mathcal{A} = \{A_\omega : \omega \in \Lambda_1\}$ and $\mathcal{B} = \{B_\mu : \mu \in \Lambda_2\}$ be two families of subsets in a topological space X. We say that $\mathcal{A}$ is a *refinement* of $\mathcal{B}$ if for every $\omega \in \Lambda_1$, there exists $\mu \in \Lambda_2$ such that $A_\omega \subseteq B_\mu$.

Definition 2.1. *A collection $\mathcal{A}$ of subsets of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be b^*-$\mathcal{I}$-locally finite if for each $x \in X$, there exists a b^*-$\mathcal{I}$-open set U containing x and U intersects at most finitely many members of $\mathcal{A}$.*

Lemma 2.1. *Let $\mathcal{A}$ be a collection of subsets of an ideal topological space $(X, \tau, \mathcal{I})$. If $\mathcal{A}$ is locally finite, then it is b^*-$\mathcal{I}$-locally finite.*

Proof. Let $\mathcal{A}$ be locally finite. We will verify that $\mathcal{A}$ is b^*-$\mathcal{I}$-locally finite. Let $x \in X$. Since $\mathcal{A}$ is locally finite, there exists an open set G_x containing x, which intersects at most finitely many elements of $\mathcal{A}$. Given that G_x is an open, it follows that it is b^*-$\mathcal{I}$-open. Consequently, $\mathcal{A}$ is b^*-$\mathcal{I}$-locally finite. □

Definition 2.2. *An ideal topological space $(X, \tau, \mathcal{I})$ is said to be b^*-$\mathcal{I}$-paracompact if every open cover of X has a b^*-$\mathcal{I}$-locally finite refinement $\mathcal{A} = \{A_\alpha : \alpha \in \Lambda\}$ consisting of b^*-$\mathcal{I}$-open sets (not necessarily a cover) such that $X - \cup\{A_\alpha : \alpha \in \Lambda\} \in \mathcal{I}$. The collection $\mathcal{A}$ of subsets of X such that $X - \cup\{A_\alpha : \alpha \in \Lambda\} \in \mathcal{I}$ is called an $\mathcal{I}$-cover.*

A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be b^*-$\mathcal{I}$-paracompact relative to X if for any open cover of A has a b^*-$\mathcal{I}$-locally finite refinement $\mathcal{B} = \{B_\omega : \omega \in \Omega\}$ consisting of b^*-$\mathcal{I}$-open sets such that $X - \cup\{B_\omega : \omega \in \Omega\} \in \mathcal{I}$.

Example 2.1. Let $\mathbb{R}$ denote the set of real numbers. Equip $\mathbb{R}$ with the standard topology τ. Define the ideal $\mathcal{I}$ to be the collection of all countable subsets of $\mathbb{R}$.

Consider the open cover $\mathcal{U} = \{(-n, n) : n \in \mathbb{N}\}$ of $\mathbb{R}$. Suppose there exists a b^*-$\mathcal{I}$-locally finite refinement $\mathcal{A}$ of $\mathcal{U}$ consisting of b^*-$\mathcal{I}$-open sets. However, for any point $x \in \mathbb{R} - \cup_{n\in\mathbb{N}}(-n, n)$ — which must be countable — every neighborhood of x intersects infinitely many sets of the form $(-n, n)$, and hence intersects infinitely many members of $\mathcal{A}$, since $\mathcal{A}$ refines $\mathcal{U}$. This contradicts the definition of local finiteness. Therefore, the space $(X, \tau, \mathcal{I})$ is not b^*-$\mathcal{I}$-paracompact.

Theorem 2.1. *If a topological space (X, τ) is paracompact, then $(X, \tau, \mathcal{I})$ is b^*-$\mathcal{I}$-paracompact.*

Proof. Using Lemma 2.1 and the inclusion $\emptyset \in \mathcal{I}$. □

Example 2.2. Consider the set of all positive integers, denoted by $\mathbb{N}$. Define a topology τ on $\mathbb{N}$ as follows:

$$\tau = \{\emptyset, \mathbb{N}\} \cup \{\{1, 2, \ldots, n\} : n \in \mathbb{N}\}.$$

Let $\mathcal{I} = \{A \subseteq \mathbb{N} : 1 \notin A\}$, which clearly forms an ideal on $\mathbb{N}$. Now, consider the open cover

$$\mathcal{A} = \{\{1, 2, 3, \ldots, n\} : n \in \mathbb{N}\}$$

of $\mathbb{N}$. It is straightforward to verify that there is no locally finite open refinement $\mathcal{B}$ of $\mathcal{A}$ that still covers $\mathbb{N}$. Therefore, the space $(\mathbb{N}, \tau)$ is not paracompact in the classical sense.

However, $\mathbb{N}$ is b^*-$\mathcal{I}$-paracompact. Given any open cover $\mathcal{U}$ of $\mathbb{N}$, one can take the refinement $\mathcal{V} = \{\{1\}\}$. In fact, the complement $\mathbb{N} - \{1\} = \{2, 3, 4, \ldots\}$ belongs to the ideal $\mathcal{I}$, and the singleton $\{1\}$ is a b^*-$\mathcal{I}$-open set. Moreover, $\mathcal{V}$ is trivially b^*-$\mathcal{I}$-locally finite, thus satisfying the condition for b^*-$\mathcal{I}$-paracompactness.

Theorem 2.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then $G \cap b^* \operatorname{cl}_{\mathcal{I}}(A) = \emptyset$ if and only if $G \cap A = \emptyset$, for all b^*-$\mathcal{I}$-open subset G of X.*

Proof. By part (3) of Lemma 1.2 and since $A \subseteq b^* \operatorname{cl}_{\mathcal{I}}(A)$, the verification of the proof is now complete. □

Theorem 2.3. *Let $\mathcal{A} = \{A_\omega : \omega \in \Omega\}$ be a collection of subsets of an ideal topological space $(X, \tau, \mathcal{I})$. The following statements are true.*

(1) *If $\mathcal{A}$ is b^*-$\mathcal{I}$-locally finite and $B_\omega \subseteq A_\omega$ for all $\omega \in \Omega$, then $\mathcal{B} = \{B_\omega : \omega \in \Omega\}$ is b^*-$\mathcal{I}$-locally finite.*
(2) *$\mathcal{A}$ is b^*-$\mathcal{I}$-locally finite if and only if $\{b^* \operatorname{cl}_{\mathcal{I}}(A_\omega) : \omega \in \Omega\}$ is b^*-$\mathcal{I}$-locally finite.*

Proof. (1): Let $x \in X$. Since $\mathcal{A}$ is b^*-$\mathcal{I}$-locally finite, there exists a b^*-$\mathcal{I}$-open set U containing x, which intersects at most finitely many elements of $\mathcal{A}$. As $B_\omega \subseteq A_\omega$ for all $\omega \in \Omega$, it follows that U intersects at most finitely many of the sets in $\mathcal{B} = \{B_\omega : \omega \in \Omega\}$. Hence, $\mathcal{B} = \{B_\omega : \omega \in \Omega\}$ is b^*-$\mathcal{I}$-locally finite.

(2): Let $\mathcal{A}$ be b^*-$\mathcal{I}$-locally finite and let $x \in X$. Then, there exists a b^*-$\mathcal{I}$-open set G containing x that satisfies $G \cap A_\omega = \emptyset$ for every $\omega \neq \omega_1, \omega_2, \ldots, \omega_n$. By Theorem 2.2, we obtain that $G \cap b^* \operatorname{cl}_{\mathcal{I}}(A_\omega) = \emptyset$ for every $\omega \neq \omega_1, \omega_2, \ldots, \omega_n$. Therefore, $\{b^* \operatorname{cl}_{\mathcal{I}}(A_\omega) : \omega \in \Omega\}$ is b^*-$\mathcal{I}$-locally finite.

The converse follows from (1). □

Remark 2.1. *It is noted that if $(X, \tau, \mathcal{I})$ is b^*-$\mathcal{I}$-paracompact and $\mathcal{J}$ is an ideal on X with $\mathcal{I} \subseteq \mathcal{J}$, then $(X, \tau, \mathcal{J})$ is b^*-$\mathcal{J}$-paracompact.*

Lemma 2.2. *If an open cover $\mathcal{A} = \{A_\omega : \omega \in \Omega\}$ of an ideal topological space $(X, \tau, \mathcal{I})$ has a b^*-$\mathcal{I}$-locally finite b^*-$\mathcal{I}$-open refinement $\mathcal{B} = \{B_\mu : \mu \in \Lambda\}$ such that $X - \cup\{B_\mu : \mu \in \Lambda\}\} \in \mathcal{I}$, then there exists a precise b^*-$\mathcal{I}$-locally finite b^*-$\mathcal{I}$-open refinement $\mathcal{C} = \{C_\omega : \omega \in \Omega\}$ of $\mathcal{A}$ such that $X - \cup\{C_\omega : \omega \in \Omega\} \in \mathcal{I}$.*

Proof. A similar technique is employed as in the proof of Lemma 1.3 in [26]. □

Definition 2.3. *An ideal topological space $(X, \tau, \mathcal{I})$ is said to be b^*-$\mathcal{I}$-regular if for every closed set $F \subseteq X$ and any point $x \notin F$, there exist disjoint b^*-$\mathcal{I}$-open sets U and V such that $x \in U$ and $F - V \in \mathcal{I}$.*

Theorem 2.4. *If $(X, \tau, \mathcal{I})$ is a Hausdorff space that is b^*-$\mathcal{I}$-paracompact, then $(X, \tau, \mathcal{I})$ is b^*-$\mathcal{I}$-regular.*

Proof. Let F be a closed subset of X, and let $x \notin F$. For each point $y \in F$, since X is Hausdorff, there exist disjoint open neighborhoods V_x and O_{xy} such that $x \in V_x$ and $y \in O_{xy}$. This ensures $y \notin \mathrm{cl}(V_x)$. Now, consider the collection

$$\mathcal{A} = \{O_{xy} : y \in F\} \cup \{X - F\},$$

which forms an open cover of X. By the hypothesis, there exists a b^*-$\mathcal{I}$-locally finite b^*-$\mathcal{I}$-open refinement

$$\mathcal{B} = \{H_{xy} : y \in F\} \cup \{W\},$$

where $H_{xy} \subseteq O_{xy}$ for each $y \in F$ and $W \subseteq X - F$, with

$$X - \left(\cup_{y\in F} H_{xy} \cup W\right) \in \mathcal{I}.$$

Define the sets

$$V = \cup_{y\in F} H_{xy}, \quad \text{and} \quad U = X - b^* \mathrm{cl}_{\mathcal{I}}(\cup_{y\in F} H_{xy}).$$

We claim that U and V are disjoint b^*-$\mathcal{I}$-open subsets of X. Since $H_{xy} \subseteq O_{xy}$ and

$$b^* \mathrm{cl}_{\mathcal{I}}(H_{xy}) \subseteq \mathrm{cl}(H_{xy}) \subseteq \mathrm{cl}(O_{xy}),$$

and since $x \notin \mathrm{cl}(O_{xy})$, it follows that $x \notin b^* \mathrm{cl}_{\mathcal{I}}(H_{xy})$, hence $x \in U$. Moreover, we have:

$$F - V = F - \cup_{y\in F} H_{xy} \subseteq X - \left(\cup_{y\in F} H_{xy} \cup W\right) \in \mathcal{I}.$$

Therefore, U and V are disjoint b^*-$\mathcal{I}$-open subsets of X such that $x \in U$ and $F - V \in \mathcal{I}$. This proves that the space $(X, \tau, \mathcal{I})$ is b^*-$\mathcal{I}$-regular. □

Lemma 2.3. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. The following statements are equivalent:*

1. *For every closed subset F of X and every $x \notin F$, there exist disjoint b^*-$\mathcal{I}$-open sets U and V such that $x \in U$ and $F - V \in \mathcal{I}$.*
2. *For every open subset G of X and every $x \in G$, there exists a b^*-$\mathcal{I}$-open set U such that $x \in U$ and $b^* \mathrm{cl}_{\mathcal{I}}(U) - G \in \mathcal{I}$.*

Proof. (1) ⇒ (2): Let G be an open set and $x \in G$. Then $X - G$ is closed, and since $x \notin X - G$, by assumption, there exist disjoint b^*-$\mathcal{I}$-open sets U and V such that $x \in U$ and $(X - G) - V \in \mathcal{I}$. Since U and V are disjoint, by Theorem 2.2, we have $b^* \mathrm{cl}_{\mathcal{I}}(U) \subseteq X - V$. Therefore, $b^* \mathrm{cl}_{\mathcal{I}}(U) \cap (X - G) \subseteq (X - G) - V$. Hence, we conclude that $b^* \mathrm{cl}_{\mathcal{I}}(U) \cap (X - G) = b^* \mathrm{cl}_{\mathcal{I}}(U) - G \in \mathcal{I}$.

(2) ⇒ (1): Let F be a closed set and $x \notin F$. This implies that $X - F$ is open, and $x \in X - F$. By assumption, there exists a b^*-$\mathcal{I}$-open set U such that $x \in U$ and $b^* \mathrm{cl}_{\mathcal{I}}(U) - (X - F) \in \mathcal{I}$. Thus, we define $V = X - b^* \mathrm{cl}_{\mathcal{I}}(U)$, which is a b^*-$\mathcal{I}$-open set. Since U and V are disjoint, we have $F - V = F - (X - b^* \mathrm{cl}_{\mathcal{I}}(U)) = b^* \mathrm{cl}_{\mathcal{I}}(U) - (X - F) \in \mathcal{I}$. □

By Theorem 2.4 and Lemma 2.3, we have the following theorem.

Theorem 2.5. *If an ideal topological space $(X, \tau, \mathcal{I})$ is both Hausdorff and b^*-$\mathcal{I}$-paracompact, then for any open set $G \subseteq X$ and for every point $x \in G$, there exists a b^*-$\mathcal{I}$-open set U such that $x \in U$ and $b^* \mathrm{cl}_{\mathcal{I}}(U) - G \in \mathcal{I}$.*

Theorem 2.6. *If an ideal topological space $(X, \tau, \mathcal{I})$ is both regular and b^*-$\mathcal{I}$-paracompact, then every open cover of X has a b^*-$\mathcal{I}$-locally finite $\mathcal{I}$-cover refinement of b^*-$\mathcal{I}$-closed sets.*

Proof. Let $\mathcal{A}$ be an open cover of X. Since X is regular, for each point $x \in X$ and each open set $U_x \in \mathcal{A}$ containing x, there exists an open set G_x such that $x \in G_x$ and $\mathrm{cl}(G_x) \subseteq U_x$. Define the collection $\mathcal{A}_1 = \{G_x : x \in X\}$, which forms an open cover of X. As X is b^*-$\mathcal{I}$-paracompact, the cover $\mathcal{A}_1$ admits a b^*-$\mathcal{I}$-locally finite refinement $\mathcal{B}_1 = \{B_\omega : \omega \in \Omega\}$ consisting of b^*-$\mathcal{I}$-open sets such that

$$X - \cup_{\omega \in \Omega} B_\omega \in \mathcal{I}.$$

Since $B_\omega \subseteq b^* \, \mathrm{cl}_{\mathcal{I}}(B_\omega)$ for each $\omega \in \Omega$, $X - \cup_{\omega \in \Omega} b^* \, \mathrm{cl}_{\mathcal{I}}(B_\omega) \subseteq X - \cup_{\omega \in \Omega} B_\omega$, we also have:

$$X - \cup_{\omega \in \Omega} b^* \, \mathrm{cl}_{\mathcal{I}}(B_\omega) \in \mathcal{I}.$$

By Theorem 2.3, the collection

$$\mathcal{B} = \{b^* \, \mathrm{cl}_{\mathcal{I}}(B_\omega) : \omega \in \Omega\}$$

is b^*-$\mathcal{I}$-locally finite. Since each $B_\omega \in \mathcal{B}_1$ refines some $G_x \in \mathcal{A}_1$, we have:

$$b^* \, \mathrm{cl}_{\mathcal{I}}(B_\omega) \subseteq \mathrm{cl}(B_\omega) \subseteq \mathrm{cl}(G_x) \subseteq U_x,$$

implying that $\mathcal{B}$ refines $\mathcal{A}$.

Thus, the collection $\mathcal{B}$ is a b^*-$\mathcal{I}$-locally finite refinement of b^*-$\mathcal{I}$-closed sets that forms an $\mathcal{I}$-cover of X. □

Theorem 2.7. *If an ideal topological space $(X, \tau, \mathcal{I})$ is Hausdorff and A is a b^*-$\mathcal{I}$-paracompact subset of X, then A is a closed set in (X, τ^*).*

Proof. Let $x \in X - A$. Since X is Hausdorff, for each point $y \in A$, there exists an open set $G_y \in \tau$ such that $y \in G_y$ and $x \notin G_y$. Therefore, the collection $\mathcal{A} = \{G_y : y \in A\}$ forms an open cover of A. As A is a b^*-$\mathcal{I}$-paracompact subset of X, the cover $\mathcal{A}$ admits a b^*-$\mathcal{I}$-locally finite refinement $\mathcal{B} = \{B_\omega : \omega \in \Omega\}$ consisting of b^*-$\mathcal{I}$-open sets such that

$$A - \cup_{\omega \in \Omega} B_\omega \in \mathcal{I}.$$

As $x \notin \mathrm{cl}(B_\omega)$ for all $\omega \in \Omega$, it follows that

$$x \notin \cup_{\omega \in \Omega} \mathrm{cl}(B_\omega).$$

Moreover, because the family $\mathcal{B}$ is locally finite, it is closure-preserving. Hence,

$$\cup_{\omega \in \Omega} \mathrm{cl}(B_\omega) = \mathrm{cl}\left(\cup_{\omega \in \Omega} B_\omega\right),$$

which implies

$$x \notin \mathrm{cl}\left(\cup_{\omega \in \Omega} B_\omega\right).$$

Now define the open set $G = X - \mathrm{cl}\left(\cup_{\omega \in \Omega} B_\omega\right)$, and let $K = A - \mathrm{cl}\left(\cup_{\omega \in \Omega} B_\omega\right)$. Then $G \in \tau$ and

$$K \subseteq A - \cup_{\omega \in \Omega} B_\omega \in \mathcal{I}.$$

Furthermore, we have $(G - K) \cap A = \emptyset$. Hence, $x \notin A^*$, and it follows that $A^* \subseteq A$. □

Theorem 2.8. *Let A and B be subsets of an ideal topological space $(X, \tau, \mathcal{I})$. If A and B are b^*-$\mathcal{I}$-paracompact subsets of X, then $A \cup B$ is also a b^*-$\mathcal{I}$-paracompact subset of X.*

Proof. Let $\mathcal{A} = \{A_\omega : \omega \in \Omega\}$ be an open cover of $A \cup B$. This cover $\mathcal{A}$ also serves as an open cover for both A and B. By assumption, there exist b^*-$\mathcal{I}$-locally finite b^*-$\mathcal{I}$-open families $\mathcal{B} = \{B_\omega : \omega \in \Omega_1\}$ for A and $C = \{C_\mu : \mu \in \Omega_2\}$ for B, which refine $\mathcal{A}$ such that:

$$A - \cup_{\omega \in \Omega_1} B_\omega \in \mathcal{I}, \quad B - \cup_{\mu \in \Omega_2} C_\mu \in \mathcal{I}.$$

This implies that:

$$A \subseteq \cup_{\omega \in \Omega_1} B_\omega \cup I_1, \quad B \subseteq \cup_{\mu \in \Omega_2} C_\mu \cup I_2,$$

where $I_1, I_2 \in \mathcal{I}$. Therefore, we have:

$$A \cup B \subseteq (\cup_{\omega \in \Omega_1} B_\omega) \cup (\cup_{\mu \in \Omega_2} C_\mu) \cup (I_1 \cup I_2).$$

It follows that:

$$A \cup B - \cup_{\omega \in \Omega_1, \mu \in \Omega_2} (B_\omega \cup C_\mu) \subseteq I_1 \cup I_2 \in \mathcal{I}.$$

We see that the collection $\mathcal{D} = \{B_\omega \cup C_\mu : \omega \in \Omega_1, \mu \in \Omega_2\}$ of b^*-$\mathcal{I}$-open sets is b^*-$\mathcal{I}$-locally finite and refines $\mathcal{A}$. Consequently, $A \cup B$ is a b^*-$\mathcal{I}$-paracompact subset of X. □

Theorem 2.9. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If A is a b^*-$\mathcal{I}$-paracompact subset of X and B is a closed subset of X, then $A \cap B$ is also a b^*-$\mathcal{I}$-paracompact subset of X.*

Proof. Let $\mathcal{A} = \{A_\omega : \omega \in \Omega\}$ be an open cover of $A \cap B$. Since $X - B$ is open in X, the collection $\mathcal{A}_1 = \{A_\omega : \omega \in \Omega\} \cup \{X - B\}$ serves as an open cover of A. By the given assumption and Lemma 2.2, there exists a precise b^*-$\mathcal{I}$-locally finite b^*-$\mathcal{I}$-open refinement $\mathcal{B} = \{B_\omega : \omega \in \Omega\} \cup \{V\}$ of $\mathcal{A}_1$ such that:

$$B_\omega \subseteq A_\omega \quad \text{for all } \omega \in \Omega, \quad V \subseteq X - B, \quad \text{and} \quad A - (\cup_{\omega \in \Omega} B_\omega \cup \{V\}) \in \mathcal{I}.$$

Now, observe that:

$$\begin{aligned} A \cap B - \cup_{\omega \in \Omega} B_\omega &= A \cap B - (\cup_{\omega \in \Omega} B_\omega \cup \{V\}) \\ &\subseteq A - (\cup_{\omega \in \Omega} B_\omega \cup \{V\}) \in \mathcal{I}. \end{aligned}$$

Thus, $A \cap B - \cup_{\omega \in \Omega} B_\omega \in \mathcal{I}$. It follows that the collection $\mathcal{B}_1 = \{B_\omega : \omega \in \Omega\}$, consisting of b^*-$\mathcal{I}$-open sets, is b^*-$\mathcal{I}$-locally finite and refines $\mathcal{A}$. Therefore, $A \cap B$ is a b^*-$\mathcal{I}$-paracompact subset of X. □

As a consequence of Theorem 2.9, we obtain the following corollary.

Corollary 2.1. *Let $(X, \tau, \mathcal{I})$ be a b^*-$\mathcal{I}$-paracompact space. Then the following hold:*

(1) *Every closed subset of X is b^*-$\mathcal{I}$-paracompact.*
(2) *The union of two closed subsets of X is also b^*-$\mathcal{I}$-paracompact.*
(3) *If $A \subseteq X$ is b^*-$\mathcal{I}$-paracompact and B is an open subset of X with $B \subseteq A$, then the set $A - B$ is b^*-$\mathcal{I}$-paracompact.*

Lemma 2.4. [13] *Let $\mathcal{I}$ be an ideal on a topological space X. If Y is a subset of X, then $\mathcal{I}_Y = \{I \cap Y : I \in \mathcal{I}\}$ is an ideal on Y.*

As illustrated in Example 1.1, the intersection of two b^*-$\mathcal{I}$-open sets may fail to be b^*-$\mathcal{I}$-open. Hence, it is essential to explicitly assume this condition in the following theorem.

Theorem 2.10. *In an ideal topological space $(X, \tau, \mathcal{I})$, let A and B be subsets of X where A is contained in B. Suppose that for any b^*-$\mathcal{I}_B$-open in B is b^*-$\mathcal{I}$-open in X. If A is a b^*-$\mathcal{I}_{\mathcal{B}}$-paracompact subset of B, then it is a b^*-$\mathcal{I}$-paracompact subset of X.*

Proof. Let $\mathcal{A} = \{A_\omega : \omega \in \Omega\}$ be an open cover of A in X. Consider the collection $\mathcal{B}_{\mathcal{A}} = \{A_\omega \cap B : \omega \in \Omega\}$, which forms an open cover of A when viewed as a subset of B. Since A is b^*-$\mathcal{I}_B$-paracompact in B, there exists a b^*-$\mathcal{I}_B$-locally finite refinement $C_{\mathcal{A}} = \{C_\omega : \omega \in \Omega\}$ consisting of b^*-$\mathcal{I}_B$-open sets in B, where $C_\omega \subseteq A_\omega \cap B$ for each $\omega \in \Omega$. Moreover,

$$A - \cup_{\omega \in \Omega} C_\omega \in \mathcal{I}_B.$$

Since $C_\omega \subseteq A_\omega$ for every $\omega \in \Omega$, the family $C_{\mathcal{A}}$ is also a refinement of $\mathcal{A}$. By the given assumption, $C_{\mathcal{A}}$ is a refinement consisting of b^*-$\mathcal{I}$-open sets in X that is b^*-$\mathcal{I}$-locally finite. Furthermore,

$$A - \cup_{\omega \in \Omega} C_\omega \in \mathcal{I}_B \subseteq \mathcal{I}.$$

Therefore, A is b^*-$\mathcal{I}$-paracompact in X. □

3. b_1^*-$\mathcal{I}$-Paracompactness and Its Characterizations

This section examines the notion of b_1^*-$\mathcal{I}$-paracompactness, a stricter variant of β_1-paracompactness introduced by Qahis [24], and subsequently investigates its characterization. Al-Jarrah [3] defined β_1-paracompactness as follows: A topological space (X, τ) is called β_1-paracompact if every β-open cover of X has a locally finite open refinement. Qahis [24] expanded the notion of β_1-paracompactness to β_1-paracompactness concerning an ideal as follows: An ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\beta_1\mathcal{I}$-paracompact if every β-open cover $\mathcal{U}$ of X has a locally finite open refinement $\mathcal{V}$ such that $X - \cup\{V : V \in \mathcal{V}\} \in \mathcal{I}$. Utilizing b^*-$\mathcal{I}$-open sets, we introduce a new form of paracompactness analogous to the one proposed by Qahis.

Definition 3.1. *An ideal topological space $(X, \tau, \mathcal{I})$ is said to be b_1^*-$\mathcal{I}$-paracompact if every b^*-$\mathcal{I}$-open cover of X has a locally finite open refinement $\mathcal{A} = \{A_\alpha : \alpha \in \Lambda\}$ (not necessarily a cover) such that $X - \cup\{A_\alpha : \alpha \in \Lambda\} \in \mathcal{I}$. The collection $\mathcal{A}$ of subsets of X such that $X - \cup\{A_\alpha : \alpha \in \Lambda\} \in \mathcal{I}$ is called an $\mathcal{I}$-cover of X.*

A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be b_1^*-$\mathcal{I}$-paracompact relative to X if for every b^*-$\mathcal{I}$-open cover of A has a locally finite open refinement $\mathcal{B} = \{B_\lambda : \lambda \in \Omega\}$ such that $A - \cup\{B_\lambda : \lambda \in \Omega\} \in \mathcal{I}$.

Example 3.1. Consider the ideal topological space $(X, \tau, \mathcal{I})$, where $X = \{1, 2, 3\}$, $\tau = \{\emptyset, X, \{1\}\}$ and $\mathcal{I} = \{\emptyset, \{2\}, \{3\}, \{2, 3\}\}$. Hence, the set of all b^*-$\mathcal{I}$-open sets of X is $\{\emptyset, \{1\}, \{1, 2\}, \{1, 3\}, X\}$. Every b^*-$\mathcal{I}$-open cover $\mathcal{U}$ of X possesses a locally finite open refinement $\mathcal{V} = \{\{1\}\}$, such that $X - \{1\} = \{2, 3\} \in \mathcal{I}$. Consequently, $(X, \tau, \mathcal{I})$ is b_1^*-$\mathcal{I}$-paracompact.

Example 3.2. Let us consider an ideal topological space $(\mathbb{N}, \tau, \mathcal{I})$, where $\mathbb{N}$, the set of natural numbers, $\tau = \{\emptyset\} \cup \{A \subseteq \mathbb{N} : \mathbb{N} - A \text{ is finite}\}$, and $\mathcal{I} = \{A \subseteq \mathbb{N} : A \text{ is finite}\}$.

Now, we consider the collection $\mathcal{U} = \{U_n = \mathbb{N} - \{n\} : n \in \mathbb{N}\}$. Each U_n is a cofinite set, hence $U_n \in \tau$, and clearly $\cup_{n\in\mathbb{N}} U_n = \mathbb{N}$. Thus, $\mathcal{U}$ is a τ-open cover of X. It is easy to verify that each U_n is also b^*-$\mathcal{I}$-open.

Now suppose there exists a locally finite refinement $\mathcal{A} = \{A_\alpha : \alpha \in \Lambda\}$ of $\mathcal{U}$ such that

$$X - \cup_{\alpha\in\Lambda} A_\alpha \in \mathcal{I}.$$

Then $\cup_{\alpha\in\Lambda} A_\alpha$ must cover all of X except possibly a finite number of points. Since each A_α is open in the cofinite topology, it must be cofinite, and so the intersection of finitely many A_α's is also cofinite.

But in the cofinite topology, a locally finite collection of cofinite sets must be finite, because for any point $x \in X$, the only open set containing x is cofinite, and any infinite number of cofinite sets will intersect infinitely often. Hence, $\mathcal{A}$ must be finite.

As a result, $\cup_{\alpha\in\Lambda} A_\alpha$ can miss infinitely many points, which contradicts the requirement that $X - \cup_{\alpha\in\Lambda} A_\alpha \in \mathcal{I}$. Therefore, no such locally finite refinement exists, and the space is not b_1^*-$\mathcal{I}$-paracompact.

Based on the definitions of b^*-$\mathcal{I}$-paracompactness and b_1^*-$\mathcal{I}$-paracompactness in ideal topological spaces, it can be deduced that if $(X, \tau, \mathcal{I})$ is b_1^*-$\mathcal{I}$-paracompact, then it is necessarily b^*-$\mathcal{I}$-paracompact. This leads to the following theorem.

Theorem 3.1. *If an ideal topological space $(X, \tau, \mathcal{I})$ is b_1^*-$\mathcal{I}$-paracompact, then it also be b^*-$\mathcal{I}$-paracompact.*

Proof. Assume that $\mathcal{A}$ is an open cover of X. Then, $\mathcal{A}$ also constitutes a b^*-$\mathcal{I}$-open cover. Given that $(X, \tau, \mathcal{I})$ is b_1^*-$\mathcal{I}$-paracompact, there exists a locally finite open refinement $\mathcal{B} = \{B_\omega : \omega \in \Omega\}$ of $\mathcal{A}$ such that

$$X - \cup_{\omega\in\Omega} B_\omega \in \mathcal{I}.$$

By Lemma 2.1, the family $\mathcal{B}$ is b^*-$\mathcal{I}$-locally finite. Hence, $(X, \tau, \mathcal{I})$ is b^*-$\mathcal{I}$-paracompact. □

In the subsequent theorem, we discuss a space endowed with two topologies; hence, to avoid ambiguity, we must redefine the concept of local finiteness. A collection $\mathcal{V}$ of subsets of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be τ-locally finite if for each $x \in X$, there exists an open set $U \in \tau$ such that $x \in U$ and U intersects with at most finitely many elements of $\mathcal{V}$. As stated in Example 1.1, the intersection of any two b^*-$\mathcal{I}$-open sets is not necessarily a b^*-$\mathcal{I}$-open set; therefore, this assumption must be made in the subsequent theorem.

Theorem 3.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Suppose that $\mathcal{I}$ is codense and τ-simple, that $(X, \tau^*, \mathcal{I})$ is b_1^*-$\mathcal{I}$-paracompact, that every b^*-$\mathcal{I}$-open set in $(X, \tau, \mathcal{I})$ is also b^*-$\mathcal{I}$-open in $(X, \tau^*, \mathcal{I})$, and that the intersection of any two b^*-$\mathcal{I}$-open sets in $(X, \tau, \mathcal{I})$ remains b^*-$\mathcal{I}$-open. Then every b^*-$\mathcal{I}$-open cover of $(X, \tau, \mathcal{I})$ admits a locally finite refinement consisting of b^*-$\mathcal{I}$-open sets that forms an $\mathcal{I}$-cover.*

Proof. Let $\mathcal{A} = \{A_\omega : \omega \in \Lambda_1\}$ be a b^*-$\mathcal{I}$-open cover of $(X, \tau, \mathcal{I})$. Then, $\mathcal{A}$ is also a b^*-$\mathcal{I}$-open cover of $(X, \tau^*, \mathcal{I})$. Since $(X, \tau^*, \mathcal{I})$ is b_1^*-$\mathcal{I}$-paracompact, there exists a τ^*-locally finite refinement $\mathcal{B} = \{B_\lambda : \lambda \in \Lambda_2\}$ of $\mathcal{A}$ such that

$$X - \cup_{\lambda \in \Lambda_2} B_\lambda \in \mathcal{I},$$

where each $B_\lambda = V_\lambda - I_\lambda$ with $V_\lambda \in \tau$ and $I_\lambda \in \mathcal{I}$.

Since $\mathcal{B}$ is τ^*-locally finite, for each $x \in X$, there exists a b^*-$\mathcal{I}$-open set H containing x such that H intersects only finitely many members of $\mathcal{B}$, say $B_{\lambda_1}, B_{\lambda_2}, \ldots, B_{\lambda_n}$. As $\mathcal{I}$ is τ-simple, such a set H can be written as $H = U - I$, where $U \in \tau$ and $I \in \mathcal{I}$. Then, for each $\lambda \notin \{\lambda_1, \ldots, \lambda_n\}$, we have

$$(U - I) \cap B_\lambda = \emptyset \Rightarrow (U \cap V_\lambda) - (I \cup I_\lambda) = \emptyset.$$

Since $\mathcal{I}$ is codense, it follows that $U \cap V_\lambda = \emptyset$ for such λ. Hence,

$$U \cap (V_\lambda \cap A_\omega) = \emptyset \quad \text{for all } \omega \in \Lambda_1 \text{ and } \lambda \notin \{\lambda_1, \ldots, \lambda_n\}.$$

Therefore, the collection $\mathcal{C} = \{V_\lambda \cap A_\omega : \omega \in \Lambda_1, \lambda \in \Lambda_2\}$ is τ-locally finite.

We now show that $\mathcal{C}$ is a b^*-$\mathcal{I}$-open refinement of $\mathcal{A}$. Since $\mathcal{B}$ refines $\mathcal{A}$, for each $B_\lambda \in \mathcal{B}$, there exists $A_\omega \in \mathcal{A}$ such that $B_\lambda \subseteq A_\omega$. Then,

$$B_\lambda = A_\omega \cap B_\lambda = A_\omega \cap (V_\lambda - I_\lambda) = (V_\lambda \cap A_\omega) - I_\lambda \subseteq V_\lambda \cap A_\omega \subseteq A_\omega.$$

Thus, $\mathcal{C}$ refines $\mathcal{A}$.

Finally, since

$$X - \cup_{\lambda \in \Lambda_2, \omega \in \Lambda_1} (V_\lambda \cap A_\omega) \subseteq X - \cup_{\lambda \in \Lambda_2} B_\lambda \in \mathcal{I},$$

it follows that $X - \cup_{\lambda \in \Lambda_2, \omega \in \Lambda_1} (V_\lambda \cap A_\omega) \in \mathcal{I}$. Therefore, $\mathcal{C}$ is a τ-locally finite b^*-$\mathcal{I}$-open refinement of $\mathcal{A}$, completing the proof. □

Theorem 3.3. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space that is b_1^*-$\mathcal{I}$-paracompact. If the union of any collection of sets in $\mathcal{I}$ belongs to $\mathcal{I}$, then the space $(X, \tau^*, \mathcal{I})$ is also b_1^*-$\mathcal{I}$-paracompact.*

Proof. Let $\mathcal{A} = \{A_\omega : \omega \in \Omega\}$ be a b^*-$\mathcal{I}$-open cover of $(X, \tau^*, \mathcal{I})$, where each $A_\omega = G_\omega - I_\omega$ with $G_\omega \in \tau$ and $I_\omega \in \mathcal{I}$ for all $\omega \in \Omega$. Then, the collection $\mathcal{B} = \{G_\omega : \omega \in \Omega\}$ forms a b^*-$\mathcal{I}$-open cover of $(X, \tau, \mathcal{I})$. Since $(X, \tau, \mathcal{I})$ is b_1^*-$\mathcal{I}$-paracompact by assumption, there exists a precise τ-locally finite open refinement $\mathcal{C} = \{C_\omega : \omega \in \Omega\}$ of $\mathcal{B}$ such that

$$X - \cup_{\omega \in \Omega} C_\omega \in \mathcal{I}.$$

For a fixed $\omega' \in \Omega_1$, consider the family $\{C_\omega \cap I_{\omega'} : \omega \in \Omega\}$. Since each $C_\omega \cap I_{\omega'}$ lies in $\mathcal{I}$, it follows from the assumption that

$$\cup_{\omega \in \Omega} (C_\omega \cap I_{\omega'}) \in \mathcal{I}.$$

Therefore,

$$X - \cup_{\omega \in \Omega} (C_\omega - I_{\omega'}) \subseteq (X - \cup_{\omega \in \Omega} C_\omega) \cup (\cup_{\omega \in \Omega} (C_\omega \cap I_{\omega'})) \in \mathcal{I},$$

which implies that

$$X - \cup_{\omega \in \Omega} (C_\omega - I_{\omega'}) \in \mathcal{I}.$$

Since C is τ-locally finite and τ^* is finer than τ, the collection $C' = \{C_\omega - I_{\omega'} : \lambda \in \Omega\}$ is a τ^*-locally finite family of τ^*-open sets. Moreover, C' refines $\mathcal{A}$. Hence, $(X, \tau^*, \mathcal{I})$ is b_1^*-$\mathcal{I}$-paracompact. □

Theorem 3.4. *If an ideal topological space $(X, \tau, \mathcal{I})$ is a Hausdorff space and A is b_1^*-$\mathcal{I}$-paracompact relative to X, then A is closed in (X, τ^*).*

Proof. We aim to prove that $A^* \subseteq A$. Assume that $x \notin A$. Since $(X, \tau, \mathcal{I})$ is Hausdorff, for each $y \in A$, there exists an open set $G_y \in \tau$ such that $y \in G_y$ and $x \notin \mathrm{cl}(G_y)$. Hence, the collection $\mathcal{A} = \{G_y : y \in A\}$ forms an open cover of A, and therefore a b^*-$\mathcal{I}$-open cover.

Because A is b_1^*-$\mathcal{I}$-paracompact, there exists a τ-locally finite open refinement $\mathcal{B} = \{B_\omega : \omega \in \Omega\}$ of $\mathcal{A}$ such that

$$A - \cup_{\omega\in\Omega} B_\omega \in \mathcal{I}.$$

For each $\omega \in \Omega$, we have $x \notin \mathrm{cl}(B_\omega)$, so

$$x \notin \cup_{\omega\in\Omega} \mathrm{cl}(B_\omega).$$

Because the family $\mathcal{B}$ is locally finite, it is closure-preserving. Therefore,

$$\cup_{\omega\in\Omega} \mathrm{cl}(B_\omega) = \mathrm{cl}\left(\cup_{\omega\in\Omega} B_\omega\right),$$

it follows that

$$x \notin \mathrm{cl}\left(\cup_{\omega\in\Omega} B_\omega\right).$$

Define the sets

$$U_1 = X - \mathrm{cl}\left(\cup_{\omega\in\Omega} B_\omega\right) \text{ and } U_2 = A - \mathrm{cl}\left(\cup_{\omega\in\Omega} B_\omega\right).$$

Then clearly $U_1 \in \tau$, $U_2 \in \mathcal{I}$, and $(U_1 - U_2) \cap A = \emptyset$. Since $x \in U_1 - U_2$ and $U_1 - U_2 \in \tau^*$, it follows that $x \notin A^*$. Thus, $A^* \subseteq A$, and consequently, A is closed in the topological space (X, τ^*). This completes the proof. □

Theorem 3.5. *If an ideal topological space $(X, \tau, \mathcal{I})$ is b_1^*-$\mathcal{I}$-paracompact and $A \subseteq X$ is b^*-$\mathcal{I}$-closed in X, then A is b_1^*-$\mathcal{I}$-paracompact.*

Proof. Let $\mathcal{A} = \{A_\omega : \omega \in \Omega\}$ be a b^*-$\mathcal{I}$-open cover of A. Since $X - A$ is a b^*-$\mathcal{I}$-open subset of X, the collection

$$\mathcal{A}_1 = \{A_\omega : \omega \in \Omega\} \cup \{X - A\}$$

forms a b^*-$\mathcal{I}$-open cover of X. By assumption, $\mathcal{A}_1$ has a precise locally finite open refinement $\mathcal{B} = \{B_\omega : \omega \in \Omega\} \cup \{V\}$ such that for each $\omega \in \Omega$, $B_\omega \subseteq A_\omega$, and $V \subseteq X - A$, satisfying:

$$X - \left(\cup_{\omega\in\Omega} B_\omega \cup V\right) \in \mathcal{I}.$$

Observe that

$$\begin{aligned} A - \cup_{\omega\in\Omega} B_\omega &= A \cap \left(X - \cup_{\omega\in\Omega} B_\omega\right) \\ &= A \cap \left(X - \left(\cup_{\omega\in\Omega} B_\omega \cup V\right)\right) \\ &\subseteq X - \left(\cup_{\omega\in\Omega} B_\omega \cup V\right), \end{aligned}$$

and hence $A - \cup_{\omega\in\Omega} B_\omega \in \mathcal{I}$. Since $B_\omega \subseteq A_\omega$ for every $\omega \in \Omega$, it follows that $\{B_\omega : \omega \in \Omega\}$ is a locally finite open refinement of $\mathcal{A}$. Therefore, A is b_1^*-$\mathcal{I}$-paracompact. □

Theorem 3.6. *Let A and B be subsets of an ideal topological space $(X, \tau, \mathcal{I})$. If A and B are b_1^*-$\mathcal{I}$-paracompact in X, then $A \cup B$ is b_1^*-$\mathcal{I}$-paracompact.*

Proof. Let $\mathcal{A} = \{A_\omega : \omega \in \Omega\}$ be a b^*-$\mathcal{I}$-open cover of $A \cup B$. Then $\mathcal{A}$ also serves as a b^*-$\mathcal{I}$-open cover for both A and B individually. By assumption, there exist locally finite open families $\mathcal{B} = \{B_\lambda : \lambda \in \Omega_1\}$ and $\mathcal{C} = \{C_\mu : \mu \in \Omega_2\}$ that refine $\mathcal{A}$ over A and B, respectively, such that:

$$A - \cup_{\lambda\in\Omega_1} B_\lambda \in \mathcal{I} \quad \text{and} \quad B - \cup_{\mu\in\Omega_2} C_\mu \in \mathcal{I}.$$

Let $I_1 = A - \cup_{\lambda\in\Omega_1} B_\lambda$ and $I_2 = B - \cup_{\mu\in\Omega_2} C_\mu$, with $I_1, I_2 \in \mathcal{I}$. Then we have:

$$\begin{aligned} A \cup B &\subseteq (\cup_{\lambda\in\Omega_1} B_\lambda \cup I_1) \cup \left(\cup_{\mu\in\Omega_2} C_\mu \cup I_2\right) \\ &= \left(\cup_{\lambda\in\Omega_1} B_\lambda) \cup (\cup_{\mu\in\Omega_2} C_\mu\right) \cup (I_1 \cup I_2). \end{aligned}$$

Hence, $(A \cup B) - (\left(\cup_{\lambda\in\Omega_1} B_\lambda) \cup (\cup_{\mu\in\Omega_2} C_\mu\right)) \in \mathcal{I}$. Since both $\mathcal{B}$ and $\mathcal{C}$ are locally finite, for each $x \in X$, there exist b^*-$\mathcal{I}$-open sets G_1 and G_2 such that G_1 intersects only finitely many members of $\mathcal{B}$, and G_2 intersects only finitely many members of $\mathcal{C}$. Consequently, the intersection $G_1 \cap G_2$ meets only finitely many members of the collection $\{B_\lambda \cup C_\mu : \lambda \in \Omega_1, \mu \in \Omega_2\}$. Thus, $A \cup B$ is b_1^*-$\mathcal{I}$-paracompact. □

Theorem 3.7. *Let A and B be subsets of an ideal topological space $(X, \tau, \mathcal{I})$. If A is b_1^*-$\mathcal{I}$-paracompact and B is b^*-$\mathcal{I}$-closed in X, then $A \cap B$ is b_1^*-$\mathcal{I}$-paracompact.*

Proof. Let $\mathcal{A} = \{A_\omega : \omega \in \Omega_1\}$ be a b^*-$\mathcal{I}$-open cover of $A \cap B$. Since $X - B$ is a b^*-$\mathcal{I}$-open subset of X, the collection

$$\mathcal{A}_1 = \{A_\omega : \omega \in \Omega_1\} \cup \{X - B\}$$

is a b^*-$\mathcal{I}$-open cover of A. As A is b_1^*-$\mathcal{I}$-paracompact, the cover $\mathcal{A}_1$ admits a locally finite open refinement $\mathcal{B} = \{B_\lambda : \lambda \in \Omega_2\} \cup \{V\}$, such that for each $\lambda \in \Omega_2$, $B_\lambda \subseteq A_\omega$ for some $\omega \in \Omega_1$, and $V \subseteq X - B$, with

$$A - (\cup_{\lambda\in\Omega_2} B_\lambda \cup V) \in \mathcal{I}.$$

Now consider:

$$\begin{aligned} A \cap B - \cup_{\lambda\in\Omega_2} B_\lambda &= A \cap B - (\cup_{\lambda\in\Omega_2} B_\lambda \cup V) \\ &\subseteq A - (\cup_{\lambda\in\Omega_2} B_\lambda \cup V), \end{aligned}$$

and therefore, $(A \cap B) - \cup_{\lambda\in\Omega_2} B_\lambda \in \mathcal{I}$. Since each $B_\lambda \subseteq A_\omega$ for some $\omega \in \Omega_1$, it follows that $\{B_\lambda : \lambda \in \Omega_2\}$ is a locally finite open refinement of the cover $\mathcal{A}$. Thus, $A \cap B$ is b_1^*-$\mathcal{I}$-paracompact. □

Corollary 3.1. *The finite union of b^*-$\mathcal{I}$-closed sets of a b_1^*-$\mathcal{I}$-paracompact ideal topological space $(X, \tau, \mathcal{I})$ is also b_1^*-$\mathcal{I}$-paracompact.*

Proof. According to Theorem 3.6, every b^*-$\mathcal{I}$-closed subset of a b_1^*-$\mathcal{I}$-paracompact space $(X, \tau, \mathcal{I})$ remains b_1^*-$\mathcal{I}$-paracompact. Furthermore, Theorem 3.7 ensures that the union of two b_1^*-$\mathcal{I}$-paracompact subsets is also b_1^*-$\mathcal{I}$-paracompact. □

4. On Preservation of b^*-$\mathcal{I}$-Paracompactness

This section shows that b^*-$\mathcal{I}$-paracompactness is maintained under specific conditions. Before continuing, we first introduce the following definition.

Definition 4.1. *Let $(X, \tau, \mathcal{I})$ and $(Y, \psi, \mathcal{J})$ be ideal topological spaces, and let $f : X \to Y$ be a function.*

(1) *The function f is said to be b^*-$\mathcal{I}$-*open *if the image of every b^*-$\mathcal{I}$-open set in X is a b^*-$\mathcal{J}$-open set in Y.*
(2) *The function f is said to be b^*-$\mathcal{I}$-*closed *if the image of every b^*-$\mathcal{I}$-closed set in X is a b^*-$\mathcal{J}$-closed set in Y.*
(3) *The function f is said to be b^*-$\mathcal{I}$-*irresolute *if the preimage of every b^*-$\mathcal{J}$-open set in Y is a b^*-$\mathcal{I}$-open set in X.*

Note that if $f : X \to Y$ is a function and Y is a topological space equipped with an ideal $\mathcal{J}$, then the preimage $f^{-1}(\mathcal{J})$ defines an ideal on X. Moreover, if f is surjective and X carries an ideal $\mathcal{I}$, then the image $f(\mathcal{I})$ constitutes an ideal on Y.

We now explore the behavior of a function between two ideal topological spaces, focusing on how certain properties are preserved from one space to the other. As a foundation, we introduce the concept of b^*-$\mathcal{I}$-compactness and present a lemma that will be instrumental in proving Theorem 4.1.

Definition 4.2. *An ideal topological space $(X, \tau, \mathcal{I})$ is called b^*-$\mathcal{I}$-compact if every collection of b^*-$\mathcal{I}$-open sets that covers X has a finite subcollection that also covers X.*

Lemma 4.1. *Let $(X, \tau, \mathcal{I})$ and $(Y, \psi, \mathcal{J})$ be ideal topological spaces, and $f : X \to Y$ be surjective. Then, f is b^*-$\mathcal{I}$-closed if and only if for every $y \in Y$ and for every b^*-$\mathcal{I}$-open set U in X containing $f^{-1}(\{y\})$, there exists a b^*-$\mathcal{J}$-open set V containing y such that $f^{-1}(V) \subseteq U$.*

Proof. Let $y \in Y$, and suppose G_1 is a b^*-$\mathcal{I}$-open subset of X such that $f^{-1}(\{y\}) \subseteq G_1$. Define the set $G_2 = Y - f(X - G_1)$. Then G_2 is a b^*-$\mathcal{J}$-open set in Y, and clearly $y \in G_2$ with $f^{-1}(G_2) \subseteq G_1$. This proves the necessity.

Now consider a b^*-$\mathcal{I}$-closed subset $F \subseteq X$, and let $y \in Y - f(F)$. Then $f^{-1}(\{y\}) \subseteq X - F$. By hypothesis, there exists a b^*-$\mathcal{J}$-open set $V_y \subseteq Y$ such that $y \in V_y$ and $f^{-1}(V_y) \subseteq X - F$, implying $V_y \subseteq Y - f(F)$. Thus,

$$Y - f(F) = \cup\{V_y : y \in Y - f(F)\}$$

is a union of b^*-$\mathcal{J}$-open sets, and hence b^*-$\mathcal{J}$-open. It follows that $f(F)$ is b^*-$\mathcal{J}$-closed in Y. □

Theorem 4.1. *Let $(X, \tau, \mathcal{I})$ be a b^*-$\mathcal{I}$-paracompact ideal topological space, and let $(Y, \psi, \mathcal{J})$ be another ideal topological space. Assume that a function $f : X \to Y$ satisfies the following conditions:*

(1) *f is continuous;*
(2) *f is b^*-$\mathcal{I}$-open;*
(3) *f is b^*-$\mathcal{I}$-closed;*
(4) *f is surjective;*
(5) *For each $y \in Y$, the preimage $f^{-1}(\{y\})$ is b^*-$\mathcal{I}$-compact; and*
(6) *$f(\mathcal{I}) \subseteq \mathcal{J}$.*

Then, Y is b^*-$\mathcal{J}$-paracompact.

Proof. Let $\mathcal{A} = \{A_\omega : \omega \in \Omega\}$ be an open cover of Y. Using (1), we deduce that $\mathcal{B} = \{f^{-1}(A_\omega) : \omega \in \Omega\}$ forms an open cover of X. Since X is b^*-$\mathcal{I}$-paracompact, the collection $\mathcal{B}$ has a precise b^*-$\mathcal{I}$-locally finite refinement $C = \{B_\omega : \omega \in \Omega\}$, where each B_ω is b^*-$\mathcal{I}$-open and

$$X - \cup_{\omega\in\Omega} B_\omega \in \mathcal{I}.$$

Since f is b^*-$\mathcal{I}$-open, the family $f(C) = \{f(B_\omega) : \omega \in \Omega\}$ consists of b^*-$\mathcal{J}$-open sets and is a refinement of $\mathcal{A}$. Moreover, by conditions (4) and (6), we have

$$Y - \cup_{\omega\in\Omega} f(B_\omega) \in \mathcal{J}.$$

Next, we check that $f(C)$ is b^*-$\mathcal{J}$-locally finite. Let $y \in Y$. Since C is b^*-$\mathcal{I}$-locally finite, for each $x \in f^{-1}(\{y\})$, there exists a b^*-$\mathcal{I}$-open set G_x containing x such that G_x intersects at most finitely many elements of C. Because $f^{-1}(\{y\})$ is b^*-$\mathcal{I}$-compact, and the family $\{G_x : f(x) = y\}$ forms a b^*-$\mathcal{I}$-open cover of $f^{-1}(\{y\})$, there exists a finite subcover $\{K_{y_1}, K_{y_2}, ..., K_{y_m}\}$ such that

$$f^{-1}(\{y\}) \subseteq K_{y_1} \cup K_{y_2} \cup \cdots \cup K_{y_m},$$

and $K_{y_1} \cup K_{y_2} \cup \cdots \cup K_{y_m}$ intersects at most finitely many elements of C. As f is b^*-$\mathcal{I}$-closed, applying Lemma 4.1, there exists a b^*-$\mathcal{J}$-open set W_y containing y such that

$$f^{-1}(W_y) \subseteq K_{y_1} \cup K_{y_2} \cup \cdots \cup K_{y_m}.$$

Thus, $f^{-1}(W_y)$ intersects at most finitely many elements of C, which implies that W_y intersects at most finitely many elements of $f(C)$. Therefore, $f(C)$ is b^*-$\mathcal{J}$-locally finite in Y. Consequently, $(Y, \psi, \mathcal{J})$ is b^*-$\mathcal{J}$-paracompact. □

The following result provides characterizations of a function $f : (X, \tau, \mathcal{I}) \to (Y, \psi)$, where $(X, \tau, \mathcal{I})$ is a b^*-$\mathcal{I}$-paracompact ideal topological space and (Y, ψ) is a topological space, such that Y inherits the same structural properties as X.

Theorem 4.2. *Let $(X, \tau, \mathcal{I})$ be a b^*-$\mathcal{I}$-paracompact ideal topological space, and let (Y, ψ) be a topological space. Suppose a function $f : X \to Y$ satisfies the following conditions:*

(1) *f is continuous;*
(2) *f is b^*-$\mathcal{I}$-irresolute;*
(3) *f is b^*-$\mathcal{I}$-open;*
(4) *f is surjective; and*

(5) *For every b^*-$\mathcal{I}$-locally finite family $\mathcal{V}$ in X, the image $f(\mathcal{V})$ is b^*-$f(\mathcal{I})$-locally finite in Y.*

Then, $(Y, \psi, f(\mathcal{I}))$ is a b^*-$f(\mathcal{I})$-paracompact ideal topological space.

Proof. As f is surjective, $f(\mathcal{I})$ is an ideal in Y. Let $\mathcal{A} = \{A_\omega : \omega \in \Omega\}$ be an open cover of Y. From (1), it follows that $\mathcal{B} = \{f^{-1}(A_\omega) : \omega \in \Omega\}$ constitutes an open cover of X. Since X is b^*-$\mathcal{I}$-paracompact, the collection $\mathcal{B}$ has a precise b^*-$\mathcal{I}$-locally finite b^*-$\mathcal{I}$-open refinement $C = \{B_\omega : \omega \in \Omega\}$ such that

$$X - \cup_{\omega\in\Omega} B_\omega \in \mathcal{I}.$$

Since

$$Y - \cup_{\omega\in\Omega} f(B_\omega) \subseteq f\left(X - \cup_{\omega\in\Omega} B_\omega\right),$$

and $f\left(X - \cup_{\omega\in\Omega} B_\omega\right) \in f(\mathcal{I})$, it follows that

$$Y - \cup_{\omega\in\Omega} f(B_\omega) \in f(\mathcal{I}).$$

Based on assumptions (3) and (5), $f(C) = \{f(B_\omega) : \omega \in \Omega\}$ forms a b^*-$f(\mathcal{I})$-locally finite consisting b^*-$f(\mathcal{I})$-open subsets in Y.

Now, we proceed to verify that $f(C)$ refines $\mathcal{A}$. For each $f(B_\omega) \in f(C)$, we have $B_\omega \in C$, and there exists $A_\omega \in \mathcal{A}$ such that $B_\omega \subseteq f^{-1}(A_\omega)$, as C refines $\mathcal{B}$. This implies that

$$f(B_\omega) \subseteq f(f^{-1}(A_\omega)) \subseteq A_\omega.$$

Consequently, $(Y, \psi, f(\mathcal{I}))$ is b^*-$f(\mathcal{I})$-paracompact. □

Based on Theorem 4.2, the conclusion remains valid if conditions (4) and (5) are replaced by the assumption that the function f is bijective. This leads to the following corollary.

Corollary 4.1. *Let $(X, \tau, \mathcal{I})$ be a b^*-$\mathcal{I}$-paracompact ideal topological space, and let (Y, ψ) be a topological space. If a function $f : X \to Y$ is bijective, continuous, b^*-$\mathcal{I}$-irresolute, and b^*-$\mathcal{I}$-open, then the space $(Y, \psi, f(\mathcal{I}))$ is b^*-$f(\mathcal{I})$-paracompact.*

The next theorem provides criteria under which a mapping from a topological space X to a b^*-$\mathcal{J}$-paracompact ideal topological space Y guarantees that X inherits the b^*-$\mathcal{J}$-paracompactness property from Y.

Theorem 4.3. *Let (X, τ) be a topological space, and let $(Y, \psi, \mathcal{J})$ be a b^*-$\mathcal{J}$-paracompact ideal topological space. Assume a function $f : X \to Y$ satisfies the following conditions:*

(1) *f is open;*
(2) *f is b^*-$f^{-1}(\mathcal{J})$-irresolute; and*
(3) *f is a bijection.*

Then, $(X, \tau, f^{-1}(\mathcal{J}))$ is a b^*-$f^{-1}(\mathcal{J})$-paracompact ideal topological space.

Proof. Let $\mathcal{A} = \{A_\omega : \omega \in \Omega\}$ be a open cover of X. Since f is open, the collection $f(\mathcal{A}) = \{f(A_\omega) : \omega \in \Omega\}$ is an open cover of Y. By hypothesis, $f(\mathcal{A})$ has a precise b^*-$\mathcal{J}$-locally finite b^*-$\mathcal{J}$-open refinement $\mathcal{B} = \{B_\omega : \omega \in \Omega\}$ such that

$$Y - \cup_{\omega \in \Omega} B_\omega \in \mathcal{J}.$$

This implies that $Y - \cup_{\omega \in \Omega} B_\omega = J$ for some $J \in \mathcal{J}$, which means

$$f^{-1}(Y) - \cup_{\omega \in \Omega} f^{-1}(B_\omega) = f^{-1}(Y) - f^{-1}(\cup_{\omega \in \Omega} B_\omega) = f^{-1}(J).$$

Hence,

$$X - \cup_{\omega \in \Omega} f^{-1}(B_\omega) \in f^{-1}(\mathcal{J}).$$

Let $\mathcal{I} = f^{-1}(\mathcal{J})$. Since f is b^*-$\mathcal{I}$-irresolute, the collection $\mathcal{C} = \{f^{-1}(B_\omega) : \omega \in \Omega\}$ forms a b^*-$\mathcal{I}$-locally finite collection of b^*-$\mathcal{I}$-open sets. For each $f^{-1}(B_\omega) \in \mathcal{C}$, since $B_\omega \in \mathcal{B}$, there exists $A_\omega \in \mathcal{A}$ such that $B_\omega \subseteq f(A_\omega)$ as $\mathcal{B}$ refines $f(\mathcal{A})$. Thus,

$$f^{-1}(B_\omega) \subseteq f^{-1}(f(A_\omega)) = A_\omega.$$

The refinement of $\mathcal{A}$ by $\mathcal{C}$ is then asserted. Therefore, $(X, \tau, \mathcal{I})$ is shown to be b^*-$\mathcal{I}$-paracompact. □

5. On Preservation of b_1^*-$\mathcal{I}$-Paracompactness

This section establishes that b_1^*-$\mathcal{I}$-paracompactness is preserved under certain conditions. We begin with the following theorem, which characterizes functions from a b_1^*-$\mathcal{I}$-paracompact ideal topological space $(X, \tau, \mathcal{I})$ to another ideal topological space $(Y, \psi, \mathcal{J})$, under which the space Y also inherits the b_1^*-$\mathcal{J}$-paracompactness property.

Theorem 5.1. *Let $(X, \tau, \mathcal{I})$ be a b_1^*-$\mathcal{I}$-paracompact and ideal topological spaces and let $(Y, \psi, \mathcal{J})$ be ideal topological spaces. Suppose that $f : X \to Y$ satisfies the following statements:*

(1) *f is open;*
(2) *f is b^*-$\mathcal{I}$-irresolute;*
(3) *f is b^*-$\mathcal{I}$-closed;*
(4) *f is a surjective function;*
(5) *$f^{-1}(\{y\})$ is b^*-$\mathcal{I}$-compact for every $y \in Y$; and*
(6) *$f(\mathcal{I}) \subseteq \mathcal{J}$.*

Then $(Y, \psi, \mathcal{J})$ is b_1^*-$\mathcal{J}$-paracompact.

Proof. Let $\mathcal{A} = \{A_\omega : \omega \in \Omega\}$ be a b^*-$\mathcal{I}$-open cover of Y. Since f is b^*-$\mathcal{I}$-irresolute, the collection $\mathcal{B} = \{f^{-1}(A_\omega) : \omega \in \Omega\}$ forms a b^*-$\mathcal{I}$-open cover of X. Given that X is b_1^*-$\mathcal{I}$-paracompact, there exists a locally finite refinement $\mathcal{C} = \{C_\mu : \mu \in \Lambda\}$ of $\mathcal{B}$ such that

$$X - \cup_{\mu \in \Lambda} C_\mu \in \mathcal{I}.$$

Since f is open, the collection $f(\mathcal{C}) = \{f(C_\mu) : \mu \in \Lambda\}$ forms an open refinement of $\mathcal{A}$. Moreover, because $X - \cup_{\mu\in\Lambda} C_\mu \in \mathcal{I}$, it follows that

$$Y - \cup_{\mu\in\Lambda} f(C_\mu) \in \mathcal{J}.$$

Next, we show that $f(\mathcal{C})$ is locally finite. Let $y \in Y$. For each $x \in f^{-1}(\{y\})$, since $\mathcal{C}$ is locally finite, there exists an open neighborhood G_x of x that intersects only finitely many members of $\mathcal{C}$. The collection $\{G_x : x \in f^{-1}(\{y\})\}$ is an open cover of $f^{-1}(\{y\})$. As $f^{-1}(\{y\})$ is b^*-$\mathcal{I}$-compact, there exists a finite subcollection $\{K_{y_1}, K_{y_2}, \ldots, K_{y_m}\}$ such that

$$f^{-1}(\{y\}) \subseteq K_{y_1} \cup K_{y_2}, \cup \cdots \cup K_{y_m},$$

and $K_{y_1} \cup K_{y_2}, \cup \cdots \cup K_{y_m}$ intersects only finitely many elements of $\mathcal{C}$.

Now, since f is b^*-$\mathcal{I}$-closed, by Lemma 4.1, there exists a b^*-$\mathcal{J}$-open set U_y containing y such that

$$f^{-1}(U_y) \subseteq K_{y_1} \cup K_{y_2}, \cup \cdots \cup K_{y_m}.$$

This implies that $f^{-1}(U_y)$ intersects only finitely many members of $\mathcal{C}$, and therefore U_y intersects only finitely many members of $f(\mathcal{C})$. Thus, the collection $f(\mathcal{C})$ is locally finite in Y. We conclude that $(Y, \psi, \mathcal{J})$ is b_1^*-$\mathcal{J}$-paracompact. □

Let (X, τ) and (Y, σ) denote topological spaces. A function $f : (X, \tau) \to (Y, \sigma)$ is called almost closed [23] if for any regular closed set F in X, the image $f(F)$ is closed in Y. A subset K of the space X is defined as N-closed relative to X if every cover of K by regular open sets of X possesses a finite subcover.

The following theorem and its corollaries provide characterizations of functions mapping from a b_1^*-$\mathcal{I}$-paracompact ideal topological space $(X, \tau, \mathcal{I})$ into a topological space (Y, ψ), ensuring that Y inherits the corresponding paracompactness property. The proof of the theorem requires the following lemma.

Lemma 5.1. [22] *Let (X, τ) and (Y, σ) be topological spaces and $f : (X, \tau) \to (Y, \sigma)$ be almost closed surjection with N-closed point inverse. If $\{U_\alpha : \alpha \in \Lambda\}$ is a locally finite open cover of X, then $\{f(U_\alpha) : \alpha \in \Lambda\}$ is a locally finite cover of Y.*

Theorem 5.2. *Let $(X, \tau, \mathcal{I})$ be a b_1^*-$\mathcal{I}$-paracompact ideal topological space and let (Y, ψ) be an ideal topological space. Suppose that $f : X \to Y$ satisfies the following statements:*

(1) *f is open;*
(2) *f is b^*-$\mathcal{I}$-irresolute;*
(3) *f is almost closed; and*
(4) *f is a surjective function with N-closed point inverse.*

Then $(Y, \psi, f(\mathcal{I}))$ is b_1^*-$f(\mathcal{I})$-paracompact.

Proof. Given that $f : X \to Y$ is surjective, the image $f(\mathcal{I})$ forms an ideal on Y. Let $\mathcal{A} = \{A_\omega : \omega \in \Omega\}$ be a b_1^*-$f(\mathcal{I})$-open cover of Y. As f is b^*-$\mathcal{I}$-irresolute, the collection $\mathcal{B} = \{f^{-1}(A_\omega) : \omega \in \Omega\}$ is a

b^*-$\mathcal{I}$-open cover of X. Since X is b_1^*-$\mathcal{I}$-paracompact, the cover $\mathcal{B}$ has a precise locally finite open refinement $\mathcal{B}_1 = \{C_\omega : \omega \in \Omega\}$ such that

$$X - \cup_{\omega\in\Omega} C_\omega \in \mathcal{I}.$$

It follows that

$$f\left(X - \cup_{\omega\in\Omega} C_\omega\right) \in f(\mathcal{I}).$$

Moreover, we have

$$Y - \cup_{\omega\in\Omega} f(C_\omega) \subseteq f\left(X - \cup_{\omega\in\Omega} C_\omega\right),$$

which implies

$$Y - \cup_{\omega\in\Omega} f(C_\omega) \in f(\mathcal{I}).$$

As f is open, almost closed, surjective, and has N-closed point inverses, and since $\mathcal{B}_1$ is locally finite, it follows by Lemma 5.1 that the collection $f(\mathcal{B}_1) = \{f(C_\omega) : \omega \in \Omega\}$ is locally finite in Y.

Next, we verify that $f(\mathcal{B}_1)$ refines $\mathcal{A}$. Let $f(C_\omega) \in f(\mathcal{B}_1)$ for some $\omega \in \Omega$. Since $\mathcal{B}_1$ refines $\mathcal{B}$, there exists $\omega \in \Omega$ such that $C_\omega \subseteq f^{-1}(A_\omega)$. Then,

$$f(C_\omega) \subseteq f(f^{-1}(A_\omega)) \subseteq A_\omega.$$

Therefore, $(Y, \psi, f(\mathcal{I}))$ is b_1^*-$f(\mathcal{I})$-paracompact. □

As any compact set is an N-closed set and any closed map is an almost closed map, by Theorem 5.2, we have the following corollaries.

Corollary 5.1. *Let a function $f : (X, \tau, \mathcal{I}) \to (Y, \psi)$ be open, b^*-$\mathcal{I}$-irresolute, closed, and surjective with compact point inverse. If $(X, \tau, \mathcal{I})$ is b_1^*-$\mathcal{I}$-paracompact, then $(Y, \psi, f(\mathcal{I}))$ is b_1^*-$f(\mathcal{I})$-paracompact.*

By observing the proof of Theorem 5.2, we will obtain the following corollary.

Corollary 5.2. *Let a function $f : (X, \tau, \mathcal{I}) \to (Y, \psi)$ be open, b^*-$\mathcal{I}$-irresolute, almost closed, surjective, and $f(\mathcal{V})$ is locally finite in Y for every locally finite $\mathcal{V}$ in X. If $(X, \tau, \mathcal{I})$ is b_1^*-$\mathcal{I}$-paracompact, then $(Y, \psi, f(\mathcal{I}))$ is b_1^*-$f(\mathcal{I})$-paracompact.*

The following theorem provides properties of a function that maps from a topological space X to a b_1^*-$\mathcal{I}$-paracompact ideal topological space Y guarantees that X exhibits identical characteristics to Y.

Lemma 5.2. [13] *Let (X, τ) and (Y, σ) be topological spaces. If $f : (X, \tau) \to (Y, \sigma)$ is a continuous surjective function and $\{U_\alpha : \alpha \in \Lambda\}$ is a locally finite in Y, then $\{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a locally finite in X.*

Theorem 5.3. *Let (X, τ) be a topological space and let $(Y, \psi, \mathcal{J})$ be a b_1^*-$\mathcal{J}$-paracompact ideal topological space. Suppose that $f : X \to Y$ satisfies the following statements:*

(1) *f is continuous;*
(2) *f is b^*-$\mathcal{I}$-open; and*
(3) *f is bijective.*

Then $(X, \tau, f^{-1}(\mathcal{J}))$ *is* b_1^**-*$f^{-1}(\mathcal{J})$*-paracompact.*

Proof. Let $\mathcal{I} = f^{-1}(\mathcal{J})$. Consider a b^*-$\mathcal{I}$-open cover $\mathcal{A} = \{A_\omega : \omega \in \Omega\}$ of X. Since f is b^*-$\mathcal{I}$-open, the collection $f(\mathcal{A}) = \{f(A_\omega) : \omega \in \Omega\}$ is a b^*-$\mathcal{J}$-open cover of Y. By assumption, $f(\mathcal{A})$ has a precise locally finite open refinement $\mathcal{B} = \{B_\omega : \omega \in \Omega\}$ such that $Y - \cup_{\omega\in\Omega} B_\omega \in \mathcal{J}$. That is, there exists $J \in \mathcal{J}$ such that $Y = \cup_{\omega\in\Omega} B_\omega \cup J$.

Applying f^{-1} to both sides gives:

$$X = f^{-1}(Y) = f^{-1}\left(\cup_{\omega\in\Omega} B_\omega \cup J\right) = \cup_{\omega\in\Omega} f^{-1}(B_\omega) \cup f^{-1}(J).$$

Since $f^{-1}(J) \in \mathcal{I}$, it follows that $X - \cup_{\omega\in\Omega} f^{-1}(B_\omega) \in \mathcal{I}$. Therefore, $\mathcal{C} = \{f^{-1}(B_\omega) : \omega \in \Omega\}$ is a locally finite open refinement of $\mathcal{A}$, by Lemma 5.2.

Now we confirm that $\mathcal{C}$ refines $\mathcal{A}$. Let $f^{-1}(B_\omega) \in \mathcal{C}$. Since $B_\omega \in \mathcal{B}$ and $\mathcal{B}$ refines $f(\mathcal{A})$, there exists $f(A_\omega) \in f(\mathcal{A})$ such that $B_\omega \subseteq f(A_\omega)$. Consequently, $f^{-1}(B_\omega) \subseteq f^{-1}(f(A_\omega)) = A_\omega$, because $A_\omega \in \mathcal{A}$. Hence, X is b_1^*-$\mathcal{I}$-paracompact. □

Conclusion

In this paper, we have introduced and systematically studied two new concepts in ideal topological spaces: b^*-$\mathcal{I}$-paracompactness and b_1^*-$\mathcal{I}$-paracompactness. These notions extend classical paracompactness through the framework of b^*-$\mathcal{I}$-open sets and ideal-based refinements, offering a more nuanced understanding of covering properties in topological spaces with ideals.

Our main contributions can be summarized as follows:

(1) We established the fundamental theory of b^*-$\mathcal{I}$-open sets, proving their essential properties including:
   - Closure under arbitrary unions
   - Relationships with classical open and closed sets
   - Characterization through b^*-$\mathcal{I}$-closure and interior operators

(2) We developed comprehensive characterizations of b^*-$\mathcal{I}$-paracompact and b_1^*-$\mathcal{I}$-paracompact spaces, including:
   - Behavior under subspaces and finite unions
   - Preservation under various types of continuous mappings
   - Comparison with existing notions like β-paracompactness and $\mathcal{I}$-paracompactness

   The supporting examples demonstrate both the generality of our concepts and their distinctions from classical cases.

(3) We proved several preservation theorems, notably:
   - Conditions for preservation under continuous, open, and closed mappings
   - Stability under surjective functions with compact or N-closed point inverses
   - Inheritance properties for unions and intersections

(4) The theoretical framework developed here has significant implications for:
   - Extending the general theory of ideal topological spaces

- Applications in digital topology and geometric modeling
- Developing new approaches to covering properties in generalized topological settings

The results presented in this work not only advance the theoretical understanding of paracompactness in ideal topological spaces but also suggest several promising directions for future research. Natural extensions of this work could investigate:

- Connections with other generalized separation axioms
- Applications in computational topology and geometric analysis
- Further refinements of the b^*-$\mathcal{I}$-open concept
- Relationships with other ideal-based topological properties

Our findings provide a solid foundation for continued exploration of these generalized paracompactness properties and their applications across various branches of topology and its applications.

Acknowledgements: The authors would like to thank the anonymous referees for their valuable comments and suggestions, which helped improve the quality of this paper.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] M.E.A. El-Monsef, E.F. Lashien, A.A. Nasef, On *I*-Open Sets and *I*-Continuous Functions, Kyungpook Math. J. 32 (1992), 21–30.

[2] M.E.A. El-Monsef, E.F. Lashien, A.A. Nasef, Some Topological Operators via Ideals, Kyungpook Math. J. 32 (1992), 273–284.

[3] H.H. Aljarrah, β_1-Paracompact Spaces, J. Nonlinear Sci. Appl. 09 (2016), 1728–1734. https://doi.org/10.22436/jnsa.009.04.28.

[4] K.Y. Al-Zoubi, S-paracompact Spaces, Acta Math. Hung. 110 (2006), 165–174. https://doi.org/10.1007/s10474-006-0001-4.

[5] K. Al-Zoubi, S. Al-Ghour, On P_3-Paracompact Spaces, Int. J. Math. Math. Sci. 2007 (2007), 80697. https://doi.org/10.1155/2007/80697.

[6] C. Boonpok, P. Raktaow, A. Sama-Ae, Strong β-$\mathcal{I}$-Submaximality and β-$\mathcal{I}$-Paracompactness in Ideal Topological Spaces, Eur. J. Pure Appl. Math. 18 (2025), 6297. https://doi.org/10.29020/nybg.ejpam.v18i3.6297.

[7] C. Boonpok, A. Sama-Ae, Characterizations of δ_1-$\beta_{\mathcal{I}}$-Paracompactness Concerning an Ideal, Eur. J. Pure Appl. Math. 18 (2025), 5732. https://doi.org/10.29020/nybg.ejpam.v18i1.5732.

[8] C. Boonpok, A. Sama-Ae, P. Raktaow, Characterizations of Generalized Paracompactness in Ideal Topological Spaces, Eur. J. Pure Appl. Math. 18 (2025), 5570. https://doi.org/10.29020/nybg.ejpam.v18i1.5570.

[9] I. Demir, O.B. Ozbakir, On β-Paracompact Spaces, Filomat 27 (2013), 971–976. https://doi.org/10.2298/FIL1306971D.

[10] J. Dieudonné, Une Généralisation des espaces Compacts, J. Math. Pures Appl. 23 (1944), 65–76. https://www.numdam.org/item/?id=JMPA_1944_9_23__65_0.

[11] J. Dontchev, M. Ganster, T. Noiri, Unified Operation Approach of Generalized Closed Sets via Topological Ideals, Math. Japon. 49 (1999), 395–402.

[12] J. Dugundji, Topology, Allyn and Bacon, Boston, 1966.

[13] T.R. Hamlett, D. Rose, D. Jankovic, Paracompactness With Respect to an Ideal, Int. J. Math. Math. Sci. 20 (1997), 433–442. https://www.jstor.org/stable/24896361.

[14] E. Hatir, On Decompositions of Continuity and Complete Continuity in Ideal Topological Spaces, Eur. J. Pure Appl. Math. 6 (2013), 352–362.

[15] D. Jankovic, T.R. Hamlet, New Topologies From Old via Ideals, Am. Math. Mon. 97 (1990), 295–310. https://doi.org/10.2307/2324512.

[16] E.D. Khalimsky, R. Kopperman, P.R. Meyer, Computer Graphics and Connected Topologies on Finite Ordered Sets, Topol. Appl. 36 (1990), 1–17. https://doi.org/10.1016/0166-8641(90)90031-v.

[17] M. Khan, T. Noiri, Semi-Local Functions in Ideal Topological Spaces, J. Adv. Res. Pure Math. 2 (2010), 36–42. https://api.semanticscholar.org/CorpusID:124939427.

[18] T.Y. Kong, R. Kopperman, P.R. Meyer, A Topological Approach to Digital Topology, Am. Math. Mon. 98 (1991), 901–917. https://doi.org/10.1080/00029890.1991.12000810.

[19] K. Kuratowski, Topology I, Panstwowe Wydawnictwo Naukowe, Warszawa, 1933.

[20] P.Y. Li, Y.K. Song, Some Remarks on S-Paracompact Spaces, Acta Math. Hung. 118 (2007), 345–355. https://doi.org/10.1007/s10474-007-6225-0.

[21] E. Moore, T. Peters, Computational Topology for Geometric Design and Molecular Design, SIAM, (2005). https://tpeters.engr.uconn.edu/mi03-moore-B4.pdf.

[22] T. Noiri, Completely Continuous Image of Nearly Paracompact Space, Mat. Vesnik 29 (1977), 59–64. https://eudml.org/doc/260307.

[23] T. Noiri, A Note on Inverse-Preservations of Regular Open Sets, Publ. Inst. Math. Nouv. Sér. 50 (1984), 99–102.

[24] A. Qahis, β_1-Paracompact Spaces With Respect to an Ideal, Eur. J. Pure Appl. Math. 12 (2019), 135–145. https://doi.org/10.29020/nybg.ejpam.v12i1.3352.

[25] D.W. Rosen, T.J. Peters, The Role of Topology in Engineering Design Research, Res. Eng. Des. 8 (1996), 81–98. https://doi.org/10.1007/bf01607863.

[26] J. Sanabria, E. Rosas, C. Carpintero, M. Salas-Brown, O. García, S-Paracompactness in Ideal Topological Spaces, Mat. Vesnik 68 (2016), 192–203.

[27] N. Sathiyasundari, V. Renukadevi, Paracompactness with Respect to an Ideal, Filomat 27 (2013), 333–339. https://doi.org/10.2298/fil1302333s.

[28] R. Vaidyanathaswamy, The Localisation Theory in Set-Topology, Proc. Indian Acad. Sci. Sect. A 20 (1944), 51–61. https://doi.org/10.1007/bf03048958.

[29] E.D. Yildirim, O.B. Ozbakir, A.C. Guler, Some Characterizations of β-Paracompactness in Ideal Topological Space, Eur. J. Pure Appl. Math. 12 (2019), 270–278. https://doi.org/10.29020/nybg.ejpam.v12i2.3394.

[30] M.I. Zahid, Para-H-Closed Spaces, Locally Para H-closed Spaces and Their Minimal Topologies, Ph.D. Dissertation, University of Pittsburgh, 1981.