Int. J. Anal. Appl. (2025), 23:231

International Journal of Analysis and Applications

A Second Order NSFD Method for a Malaria Propagation Model

Calisto B. Marime, Justin B. Munyakazi*

Department of Mathematics and Applied Mathematics, University of the Western Cape, Private Bag X17, Bellville 7535, South Africa

**Corresponding author:* jmunyakazi@uwc.ac.za

Abstract. Standard numerical methods such as the implicit and explicit Euler and the Runge-Kutta methods have been used to approximate solutions of continuous-time transmission dynamics of many diseases. However, their convergence is conditional. Also, they do not always preserve the key features of the continuous-time model. Most times, they require a small time step which may increase the computational complexities especially for a long time horizon. In this paper we construct a nonstandard finite difference (NSFD) method to approximate the solution of a malaria propagation model. NSFD methods do not suffer from the drawback of time step restriction and preserve the physics of the problem under consideration. However their accuracy and rate of convergence remain a point of concern. In the construction of the NSFD scheme that we propose, we consider weights and denominator functions that depend not only on the time step but also iteratively on the state variables of the discrete model. This guarantees a second order convergence as opposed to earlier NSFD schemes which were independent of weights and their denominator functions were solely dependent of the time step. Numerical experiments confirm that the proposed scheme outperforms the first order NSFD in terms of accuracy and rate of convergence.

1. Introduction

Standard numerical schemes such as Euler and Runge-Kutta methods fail to provide meaningful approximations of the solutions to continuous-time models for certain step sizes, leading to numerical instabilities [4, 18, 19, 23]. Numerical instabilities occur when a numerical method generates chaos, oscillations, or inaccurate results. In 1980 Mickens proposed Nonstandard Finite Difference (NSFD) schemes to overcome those weaknesses often encountered when standard numerical methods are used [14, 20]. Nowadays, these methods have been applied to solve mathematical models in real-world situations to describe different phenomena and processes [2, 3, 5, 8, 13, 21].

Received: Jul. 3, 2025.

2020 *Mathematics Subject Classification.* 34A45, 65L05, 65L12, 65L20, 65L70.

Key words and phrases. malaria model; denominator function; NSFD method; second order; positivity; stability; error analysis.

https://doi.org/10.28924/2291-8639-23-2025-231

ISSN: 2291-8639

Unfortunately, most of these dynamically consistent NSFD techniques are only first-order accurate [11, 17, 25]. Owing to this, researchers have focused on improving the accuracy of NSFD schemes, leading to the proposal of higher-order NSFD techniques (see, for example [9,10,15,16]).

The noteworthy findings that center on the high-order NSFD scheme are mentioned below. In recent works, second-order NSFD methods that preserve positivity and local asymptotic stability have been developed in [10]. Moreover, the second-order NSFD technique developed in [12] preserves the positivity, local asymptotic stability, and global asymptotic stability of a general single-species model. Such methods are said to be dynamically consistent with the corresponding differential equations; that is, the discrete scheme enjoys the same qualitative features as the continuous model. Newly proposed NSFD schemes are not only convergent of order two but are also dynamically consistent with the model under study [9,10,13]. This leads to advancements in more accurate numerical solutions of the continuous models.

In this paper, we consider the nonlinear system of differential equations for malaria transmission dynamics which was studied in [24]. In this compartmental model, the human population is divided into susceptible humans (S_h), exposed humans (E_h), infected humans (I_h), and recovered humans (R_h) and the vector population is divided into susceptible mosquitoes (S_m), exposed mosquitoes (E_m), and infected mosquitoes (I_m).
The following is the flow diagram for the model showing the malaria transmission dynamics.

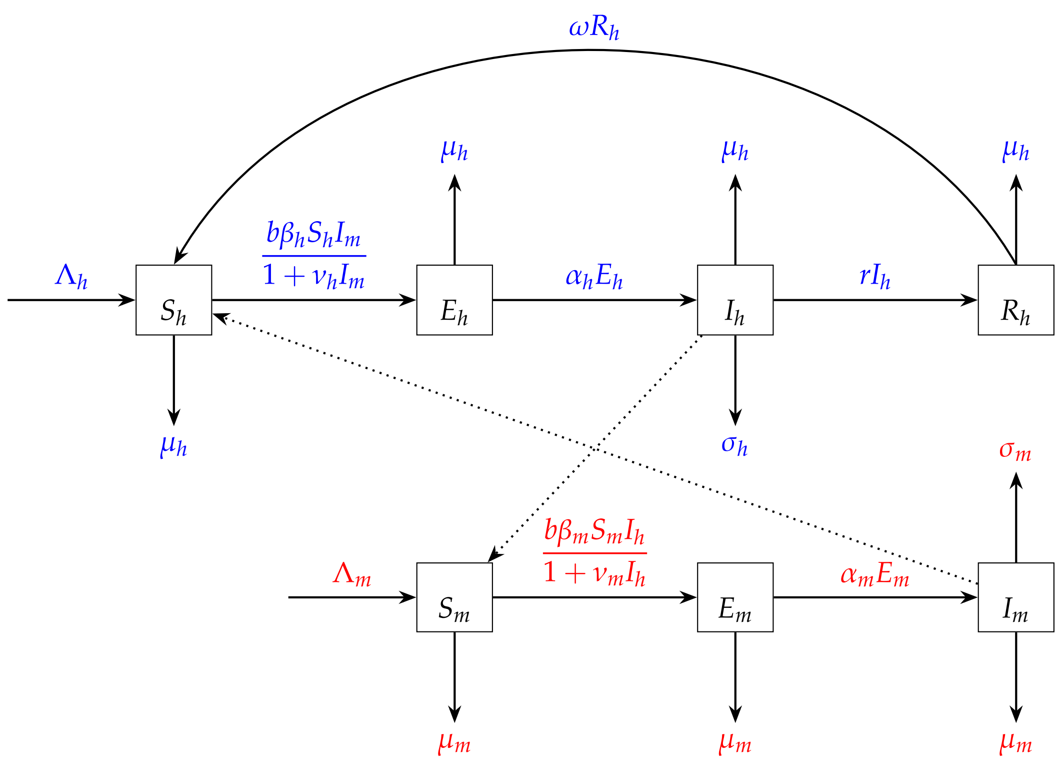

FIGURE 1. Compartmental model of malaria transmission between humans and mosquitoes

State variables and parameters for malaria transmission are given in the following tables.

Table 1. Description of state variables

Variable	Description
$S_h(t)$	Number of susceptible humans at time t
$E_h(t)$	Number of exposed humans at time t
$I_h(t)$	Number of infectious humans at time t
$R_h(t)$	Number of recovered humans at time t
$S_m(t)$	Number of susceptible mosquitoes at time t
$E_m(t)$	Number of exposed mosquitoes at time t
$I_m(t)$	Number of infectious mosquitoes at time t

Table 2. Description of model parameters

Parameter	Description
Λ_h	Recruitment rate of susceptible humans
Λ_m	Recruitment rate of susceptible mosquitoes
b	Mosquito biting rate
β_h	Transmission probability from infectious mosquito to human
β_m	Transmission probability from infectious human to mosquito
μ_h	Natural death rate of humans
μ_m	Natural death rate of mosquitoes
δ_h	Disease-induced death rate of humans
δ_m	Disease-induced death rate of mosquitoes
α_h	Progression rate from exposed to infectious in humans
α_m	Progression rate from exposed to infectious in mosquitoes
r	Recovery rate of humans
ω	Rate of loss of immunity in humans
ν_h	Saturation constant (antibody response) in humans
ν_m	Saturation constant (antibody response) in mosquitoes

The propagation of malaria is described by the following non-linear differential equations based on the flow diagram in Figure 1:

$$\begin{aligned}
\frac{dS_h(t)}{dt} &= \Lambda_h - \frac{b\beta_h S_h(t) I_m(t)}{1+v_h I_m(t)} - \mu_h S_h(t) + \omega R_h(t), \\
\frac{dE_h(t)}{dt} &= \frac{b\beta_h S_h(t) I_m(t)}{1+v_h I_m(t)} - (\alpha_h + \mu_h) E_h(t), \\
\frac{dI_h(t)}{dt} &= \alpha_h E_h(t) - (r + \mu_h + \sigma_h) I_h(t),
\end{aligned}$$

$$\begin{aligned}
\frac{dR_h(t)}{dt} &= rI_h(t) - (\mu_h + \omega)R_h(t), \\
\frac{dS_m(t)}{dt} &= \Lambda_m - \frac{b\beta_m S_m(t) I_h(t)}{1 + v_m I_h(t)} - \mu_m S_m(t), \\
\frac{dE_m(t)}{dt} &= \frac{b\beta_m S_m(t) I_h(t)}{1 + v_m I_h(t)} - (\alpha_m + \mu_m)E_m(t), \\
\frac{dI_m(t)}{dt} &= \alpha_m E_m(t) - (\mu_m + \sigma_m)I_m(t),
\end{aligned} \tag{1.1}$$

with initial conditions:

$$\begin{aligned}
S_h(0) = S_{0_h}, \quad I_h(0) = I_{0_h}, \quad E_h(0) = E_{0_h}, \quad R_h(0) = R_{0_h} \\
S_m(0) = S_{0_m}, \quad E_m(0) = E_{0_m}, \quad I_m(0) = I_{0_m}.
\end{aligned} \tag{1.2}$$

The quantities Λ_h and Λ_m are birth rates of humans and mosquitoes respectively. The model takes into account the latency period, after mosquito bites at a rate b, humans and mosquitoes move from the class of susceptible to exposed with probability β_h and β_m respectively. At the end of the latency period, the exposed humans and mosquitoes move to the infected class at the rate α_h, and α_m respectively. The population is decreased by natural death in each compartment at a rate μ_h for humans, and μ_m for mosquitoes. In addition to natural death, the infected mosquitoes and humans are reduced by induced death at a rate σ_m and σ_h respectively. The infected humans transfer to the recovery class at the rate r, and after some time individuals lose immunity and become susceptible again at a rate ω. Infected mosquitoes transmit the virus to susceptible humans at a ratio of $\dfrac{I_m(t)}{1 + v_h I_m(t)}$. The pace at which humans generate antibodies in response to the clash of antigens released by infected mosquitoes is represented by $v_h \in [0, 1]$. Likewise, $v_m \in [0, 1]$ represents the mosquitoes' rate of antibody production against the antigens they come into touch with from infectious humans.

To the best of our knowledge, the implementation of the high-order NSFD scheme to solve malaria propagation models has never been attempted before. In this paper, our objective is to construct a second-order NSFD method that preserves the qualitative properties of the continuous model (1.1). To achieve this goal, we employ non-local approximations with weights and denominator functions that are not only dependent on the step size. Indeed, we prove that the proposed second-order NSFD scheme is stable and preserves the positivity and boundedness of the solutions. Moreover, we establish that the equilibrium points and the basic reproduction number of the discrete model coincide with those of the continuous-time model. Moreover, numerical experiments exhibit the impact of weights and how they affect the accuracy of the method. Furthermore, we establish numerically the condition for weights that minimizes errors.

The structure of the paper is as follows: in Section 2, we develop a second-order NSFD (2NSFD) method, prove the positivity and boundedness of solutions, calculate the equilibrium points, examine the stability, and prove the order of convergence. In Section 3, we perform numerical simulations of the model. In Section 4, we provide some concluding remarks of our work.

2. The Second Order NSFD Method

In this section, after constructing the NSFD method, we discuss the qualitative properties of the discrete solution as well as the stability and convergence of the method.

2.1. **Construction of the method.** We follow the ideas of [10,14] to construct a second-order NSFD scheme for the continuous model (1.1). First, we consider the continuous model (1.1) on a finite interval $[0, T]$ and divide the interval by a uniform mesh such that

$$0 = t^0 < t^1 < ..... < t^{N-1} < t^N = T, \tag{2.1}$$

where $\Delta t = t^k - t^{k-1}$ for $k = 1, 2, 3, ..., N$ is the step size. Let us denote $(S_h^k, E_h^k, I_h^k, R_h^k, S_m^k, E_m^k, I_m^k)^T$ the intended approximation of $(S_h(t^k), E_h(t^k), I_h(t^k), R_h(t^k), S_m(t^k), E_m(t^k), I_m(t^k))^T$. Using the theory of NSFD methods(see for example [18,22]), we discretize the model (1.1) as follows:

$$\begin{aligned}
\frac{S_h^{k+1} - S_h^k}{\phi_1(\Delta_t, S_h^k, R_h^k, I_m^k)} &= \Lambda_h - \frac{b\beta_h S_h^{k+1} I_m^k}{1 + v_h I_m^k} - \mu_h S_h^{k+1} + \omega R_h^k + \Phi_1 S_h^k - \Phi_1 S_h^{k+1}, \\
\frac{E_h^{k+1} - E_h^k}{\phi_2(\Delta t, S_h^k, E_h^k, I_m^k)} &= \frac{b\beta_h S_h^k I_m^k}{1 + v_h I_m^k} - (\alpha_h + \mu_h) E_h^{k+1} + \Phi_2 E_h^k - \Phi_2 E_h^{k+1}, \\
\frac{I_h^{k+1} - I_h^k}{\phi_3(\Delta t, E_h^k, I_h^k)} &= \alpha_h E_h^k - (r + \mu_h + \sigma_h) I_h^{k+1} + \Phi_3 I_h^k - \Phi_3 I_h^{k+1}, \\
\frac{R_h^{k+1} - R_h^k}{\phi_4(\Delta t, I_h^k, R_h^k)} &= r I_h^k - (\mu_h + \omega) R_h^{k+1} + \Phi_4 R_h^k - \Phi_4 R_h^{k+1}, \\
\frac{S_m^{k+1} - S_m^k}{\phi_5(\Delta t, S_m^k, I_h^k)} &= \Lambda_m - \frac{b\beta_m S_m^{k+1} I_h^k}{1 + v_m I_h^k} - \mu_m S_m^{k+1} + \Phi_5 S_m^k - \Phi_5 S_m^{k+1}, \\
\frac{E_m^{k+1} - E_m^k}{\phi_6(\Delta t, S_m^k, E_m^k, I_h^k)} &= \frac{b\beta_m S_m^k I_h^k}{1 + v_m I_h^k} - (\alpha_m + \mu_m) E_m^{k+1} + \Phi_6 E_m^k - \Phi_6 E_m^{k+1}, \\
\frac{I_m^{k+1} - I_m^k}{\phi_7(\Delta t, E_m^k, I_m^k)} &= \alpha_m E_m^k - (\mu_m + \sigma_m) I_m^{k+1} + \Phi_7 I_m^k - \Phi_7 I_m^{k+1}.
\end{aligned} \tag{2.2}$$

where $\phi_i(\Delta t, S_h^k, E_h^k, I_h^k, R_h^k, S_m^k, E_m^k, I_m^k) = \Delta t + O(\Delta t^2)$ as $\Delta t \to 0$ $(i = 1, 2, 3, 4, 5, 6, 7)$ is the nonstandard denominator function and Φ_i $(i = 1, 2, 3, 4, 5, 6, 7)$ is a real number that plays the role of weight. Nonstandard discretization of the right-hand side of (2.2) uses $\Phi_i \in \mathbb{R}$ as weights in the discretization of the zero function. For example, 0 can be discretized as $S_h - S_h = \Phi_1 S_h^k - \Phi_1 S_h^{k+1}$. In scheme (2.2), one notices that if the denominator function ϕ_i depends only on the step size Δt, and $\Phi_i = 0$, then NSFD method (2.2) reduces to the first-order NSFD scheme:

$$\begin{aligned}
\frac{S_h^{k+1} - S_h^k}{\phi(\Delta t)} &= \Lambda_h - \frac{b\beta_h S_h^{k+1} I_m^k}{1 + v_h I_m^k} - \mu_h S_h^{k+1} + \omega R_h^k, \\
\frac{E_h^{k+1} - E_h^k}{\phi(\Delta t)} &= \frac{b\beta_h S_h^k I_m^k}{1 + v_h I_m^k} - (\alpha_h + \mu_h) E_h^{k+1},
\end{aligned}$$

$$\begin{aligned}
\frac{I_h^{k+1}-I_h^k}{\phi(\Delta t)} &= \alpha_h E_h^k-(r+\mu_h+\sigma_h)I_h^{k+1},\\
\frac{R_h^{k+1}-R_h^k}{\phi(\Delta t)} &= rI_h^k-(\mu_h+\omega)R_h^{k+1},\\
\frac{S_m^{k+1}-S_m^k}{\phi(\Delta t)} &= \Lambda_m-\frac{b\beta_m S_m^{k+1}I_h^k}{1+v_m I_h^k}-\mu_m S_m^{k+1},\\
\frac{E_m^{k+1}-E_m^k}{\phi(\Delta t)} &= \frac{b\beta_m S_m^k I_h^k}{1+v_m I_h^k}-(\alpha_m+\mu_m)E_m^{k+1},\\
\frac{I_m^{k+1}-I_m^k}{\phi(\Delta t)} &= \alpha_m E_m^k-(\mu_m+\sigma_m)I_m^{k+1}.
\end{aligned} \tag{2.3}$$

In the NSFD scheme (2.2), the denominator function ϕ_i depends iteratively on the solution.

2.2. **Qualitative properties of the discrete solution.** We prove the positivity and boundedness of the solutions and calculate the equilibrium points.

Positivity of solutions

Theorem 2.1. *Let $\Phi_i \in \mathbb{R}$, satisfy $\Phi_i \geq 0$. Then for all values of the step size Δt, $S_h^k, E_h^k, I_h^k, R_h^k, S_m^k, E_m^k, I_m^k \geq 0$ for all values of $k \geq 0$ when*

$$S_h^0, E_h^0, I_h^0, R_h^0, S_m^0, E_m^0, I_m^0 \geq 0.$$

Hence, positivity of the model (1.1) is upheld for all finite step sizes by the scheme (2.2).

Proof. For simplicity, we omit arguments of the denominator functions $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7$. We rewrite the scheme (2.2) in the explicit form

$$\begin{aligned}
S_h^{k+1} &= \frac{(S_h^k+\phi_1)(\Lambda_h+\omega R_h^k+\Phi_1 S_h^k))(1+v_h I_m^k)}{1+v_h I_m^k+\phi_1(b\beta_h I_m^k)+\phi_1(\mu_h+\Phi_1)(1+v_h I_m^k)},\\
E_h^{k+1} &= \frac{E_h^k(1+v_h I_m^k)+\phi_2(b\beta_h S_h^k I_m^k+\Phi_2 E_h^k(1+v_h I_m^k))}{1+v_h I_m^k+\phi_2(1+v_h I_m^k)[(\alpha_h+\mu_h+\Phi_2]},\\
I_h^{k+1} &= \frac{I_h^k+\phi_3(\alpha_h E_h^k+\Phi_3 I_h^k)}{1+\phi_3((r+\mu_h+\sigma_h)+\Phi_3)},\\
R_h^{k+1} &= \frac{R_h^k+\phi_4(\Delta t, I_h^k, R_h^k)(rI_h^k+\Phi_4 R_h^k)}{1+\phi_4(\Delta t, I_h^k, R_h^k)(\mu_h+\omega+\Phi_4)},\\
S_m^{k+1} &= \frac{(S_m^k+\phi_5(\Lambda_m+\Phi_5 S_m^k))(1+v_m I_h^k)}{1+v_m I_h^k+\phi_5(b\beta_m I_h^k+(1+v_m I_h^k)(\mu_m+\Phi_5))},\\
E_h^{k+1} &= \frac{(1+v_m I_h^k)(E_m^k)+\phi_6(b\beta_m S_m^k I_h^k+(1+v_m I_h^k)\Phi_6 E_m^k)}{1+v_m I_h^k+\phi_6(1+v_m I_h^k)(\alpha_m+\mu_m+\Phi_6)},
\end{aligned} \tag{2.4}$$

$$I_h^{k+1} = \frac{I_m^k + \phi_7(\alpha_m E_m^k + \Phi_7 I_m^k)}{1 + \phi_7(\mu_m + \sigma_m + \Phi_7)}.$$

Since all parameters in the system (2.4) are positive, it is clear that if

$$S_h^k, E_h^k, I_h^k, R_h^k, S_m^k, E_m^k, I_m^k \geq 0$$

then

$$S_h^{k+1}, E_h^{k+1}, I_h^{k+1}, R_h^{k+1}, S_m^{k+1}, E_m^{k+1}, I_m^{k+1} \geq 0$$

holds unconditionally for all state variables. This concludes the proof.

Boundedness of solutions

Theorem 2.2. *The solution* $(S_h^k, E_h^k, I_h^k, R_h^k, S_m^k, E_m^k, I_m^k) \in \mathbb{R}_+^7$ *of the scheme (2.2) is bounded.*

Proof. Let us suppose that $V_h^k = S_h^k + E_h^k + I_h^k + R_h^k$ and $V_m^k = S_m^k + E_m^k + I_m^k$, then from (2.2) we have

$$\begin{aligned}
\frac{V_h^{k+1} - V_h^k}{\phi_1(\Delta t, I_h^k)} &= \Lambda_h - \mu_h V_h^{k+1} - \sigma I_h^k + \Phi_1 V_h^k - \Phi_1 V_h^{k+1}, \\
\frac{V_m^{k+1} - V_m^k}{\phi_2(\Delta t, I_m)} &= \Lambda_m - \mu_m V_m^{k+1} - \sigma I_m^k + \Phi_2 V_m^k - \Phi V_m^{k+1}.
\end{aligned} \tag{2.5}$$

From (2.5), we have

$$\begin{aligned}
V_h^{k+1} &= \frac{V_h^k(1 + \phi_1\Phi_1)}{1 + \phi_1(\mu_h + \Phi_1)} + \frac{\phi_1\Lambda_h}{1 + \phi_1(\mu_h + \Phi_1)} - \frac{\phi_1\sigma_h I_h^k}{1 + \phi_1(\mu_h + \Phi_1)}, \\
V_v^{k+1} &= \frac{V_m^k(1 + \phi_2\Phi_2)}{1 + \phi_2(\mu_m + \Phi_2)} + \frac{\phi_2\Lambda_m}{1 + \phi_2(\mu_m + \Phi_2)} - \frac{\phi_1\sigma I_m^k}{1 + \phi_1(\mu_m + \Phi_2)}.
\end{aligned} \tag{2.6}$$

From the first equation of (2.6), we have

$$\begin{aligned}
V_h^{k+1} &\leq \frac{V_h^k(1 + \phi_1\Phi_1)}{1 + \phi_1(\mu_h + \Phi_1)} + \frac{\phi_1\Lambda_h}{1 + \phi_1(\mu_h + \Phi_1)} \\
&\leq \frac{1 + \phi_1\Phi_1}{1 + \phi_1(\mu_h + \Phi_1)}\left[\frac{(1 + \phi_1\Phi_1)V_h^{k-1}}{1 + \phi_1(\mu_h + \Phi_1)} + \frac{\phi_1\Lambda_h}{1 + \phi(\mu_h + \Phi_1)}\right] + \frac{\phi_1\Lambda_h}{1 + \phi_1(\mu_h + \Phi_1)} \\
&= \left[\frac{1 + \phi_1\Phi_1}{1 + \phi_1(\mu_h + \Phi_1)}\right]^2 V_h^{k-1} + \frac{\phi_1\Lambda_h}{1 + \phi_1(\mu_h + \Phi_1)}\left[\frac{1 + \phi_1\Phi_1}{1 + \phi_1(\mu_h + \Phi_1)} + 1\right] \\
&\leq \cdots \leq \left[\frac{1 + \phi_1\Phi_1}{1 + \phi_1(\mu_h + \Phi_1)}\right]^{k+1} V_h^0 + \frac{\phi_1\Lambda_h}{1 + \phi_1(\mu_h + \Phi_1)}\sum_{j=0}^{k}\left[\frac{1 + \phi_1\Phi_1}{1 + \phi_1(\mu_h + \Phi_1)}\right]^j \\
&= \left[\frac{1 + \phi_1\Phi_1}{1 + \phi_1(\mu_h + \Phi_1)}\right]^{k+1} V_h^0 + \frac{\phi_1\Lambda_h}{1 + \phi_1(\mu_h + \Phi_1)}\left[\frac{1 - \left(\frac{1+\phi_1\Phi_1}{1+\phi_1(\mu_h+\Phi_1)}\right)^{k+1}}{1 - \frac{1+\phi_1\Phi_1}{1+\phi_1(\mu_h+\Phi_1)}}\right].
\end{aligned} \tag{2.7}$$

As $k \to \infty$,

$$\lim_{k\to\infty} V_h^k \leq \frac{\Lambda_h}{\mu_h}. \tag{2.8}$$

In the same way, for the second equation of (2.6) we have $V_m^k \to \frac{\Lambda_m}{\mu_m}$ as $k \to \infty$. This completes the proof.

Equilibrium points

The disease-free equilibrium [6] of the continuous model (1.1) is given by

$$E_0 = \left(\frac{\Lambda_h}{\mu_h}, 0, 0, 0, \frac{\Lambda_m}{\mu_m}, 0, 0\right), \tag{2.9}$$

and the unique Endemic Equilibrium E^* exists if and only if $R_0 > 1$ where

$$R_0 = \sqrt{\frac{b^2\alpha_h\beta_h\Lambda_h\alpha_m\beta_m\Lambda_m}{\mu_h(\alpha_h+\mu_h)(r+\sigma_h+\mu_h)\mu_m(\sigma_m+\mu_m)(\alpha_m+\mu_m)}}, \tag{2.10}$$

with

$$E^* = \left(S_h^*, E_h^*, I_h^*, R_h^*, S_m^*, E_m^*, I_m^*\right), \tag{2.11}$$

and

$$S_h^* = \frac{\Lambda_h + \omega R_h^*}{\frac{b\beta_h I_m^*}{1+v_h I_m^*} + \mu_h}, \quad E_h^* = \frac{(r+\delta_h+\mu_h)I_h^*}{\alpha_h}, \quad R_h^* = \frac{rI_h^*}{\mu_h+\omega},$$
$$S_m^* = \frac{\Lambda_m}{\frac{b\beta_m I_h^*}{1+\nu_m I_h^{**}} + \mu_m}, \quad E_m^* = \frac{b\beta_m S_m^* I_h^*}{(1+\nu_m I_h^*)(\alpha_m+\mu_m)}, \quad I_m^* = \frac{\alpha_m E_m^*}{\mu_m+\sigma_m}.$$

Now we need to show that the equilibrium points of the discrete model (2.2) are the same as those of the continuous model. To find the steady-state solutions of the NSFD scheme (2.2), we set $S_h^{k+1} = S_h^k$, $E_h^{k+1} = E_h^k$, $I_h^{k+1} = I_h^k$, $R_h^{k+1} = R_h^k$, $S_m^{k+1} = S_m^k$, $E_m^{k+1} = E_m^k$ and $I_m^{k+1} = I_m^k$. The Scheme (2.2) becomes

$$\begin{aligned}
&\Lambda_h - \frac{b\beta_h S_h^k I_m^k}{1+v_h I_m^k} - \mu_h S_h^k + \omega R_h^k = 0, \quad \frac{b\beta_h S_h^k I_m^k}{1+v_h I_m^k} - (\alpha_h+\mu_h)E_h^k = 0,\\
&\alpha_h E_h^k - (r+\mu_h+\sigma_h)I_h^k = 0, \quad rI_h^k - (\mu_h+\omega)R_h^k = 0,\\
&\Lambda_m - \frac{b\beta_m S_m^k I_h^k}{1+v_m I_h^k} - \mu_m S_m^k = 0, \quad \frac{b\beta_m S_m^k I_h^k}{1+v_m I_h^k} - (\alpha_m+\mu_m)E_m^k = 0,\\
&\alpha_m E_m^k - (\mu_m+\sigma_m)I_m^k = 0.
\end{aligned} \tag{2.12}$$

From (2.12), it is easy to notice that the set of equilibria of (1.1) and (2.2) are identical.

2.3. **Stability of the scheme.** To determine the stability of the scheme (2.2), we can rewrite the scheme (2.4) in another form. The first equation in the scheme (2.4) can take the form

$$S_h^{k+1} = \frac{S_h^k + \phi_1\left(\Lambda_h + \omega R_h^k - \frac{b\beta_h S_h^k I_m^k}{1+v_h I_m^k} - \mu_h S_h^k + \Phi_1 S_h^k + \frac{b\beta_h S_h^k I_m^k}{1+v_h I_m^k} + \mu_h S_h^k\right)}{1 + \phi_1\left(\frac{b\beta_h I_m^k}{1+v_h I_m^k} + \mu_h + \Phi_1\right)}.$$

By rearranging and factoring out S_h^k, we have

$$S_h^{k+1} = \frac{S_h^k\left[1 + \phi_1\left(\frac{b\beta_h I_m^k}{1+v_h I_m^k} + \mu_h + \Phi_1\right)\right] + \phi_1\left(\Lambda_h + \omega R_h^k - \frac{b\beta_h S_h^k I_m^k}{1+v_h I_m^k} - \mu_h S_h^k\right)}{1 + \phi_1\left(\frac{b\beta_h I_m^k}{1+v_h I_m^k} + \mu_h + \Phi_1\right)},$$

which reduces to

$$S_h^{k+1} = S_h^k + \phi_1 \frac{f_1(S_h^k, R_h^k, I_m^k)}{1 + \phi_1(\Delta t, S_h^k, R_h^k, I_m^k)\left(\frac{b\beta_h I_m^k}{1+v_h I_m^k} + \mu_h + \Phi_1\right)}, \tag{2.13}$$

where

$$f_1(S_h^k, R_h^k, I_m^k) = \Lambda_h - \frac{b\beta_h S_h^k I_m^k}{1 + v_h I_m^k} - \mu_h S_h^k + \omega R_h^k. \tag{2.14}$$

By applying the same reasoning to the remaining equations of the scheme (2.4), we obtain

$$\begin{aligned}
E_h^{k+1} &= E_h^k + \phi_2 \frac{f_2(S_h^k, E_h^k, I_m^k)}{1 + \phi_2(\Delta t, S_h^k, E_h^k, I_m^k)(\alpha_h + \mu_h + \Phi_2)}, \\
I_h^{k+1} &= I_h^k + \phi_3 \frac{f_3(E_h^k, I_h^k)}{1 + \phi_3(\Delta t, E_h^k, I_h^k)(r + \mu_h + \sigma_h + \Phi_3)}, \\
R_h^{k+1} &= R_h^k + \phi_4 \frac{f_4(I_h^k, R_h^k)}{1 + \phi_4(\Delta t, I_h^k, R_h^k)(\mu_h + \omega + \Phi_4)}, \\
S_m^{k+1} &= S_m^k + \phi_5 \frac{f_5(S_m^k, I_h^k)}{1 + \phi_5(\Delta t, S_m^k, I_h^k)(\frac{b\beta_m I_h^k}{1+v_m I_h^k} + \mu_m + \Phi_5)}, \\
E_h^{k+1} &= E_m^k + \phi_6 \frac{f_6(S_m^k, E_m^k, I_h^k)}{1 + \phi_6(\Delta t, S_m^k, E_m^k, I_h^k)(\alpha_m + \mu_m + \Phi_6)}, \\
I_h^{k+1} &= I_m^k + \phi_7 \frac{f_7(E_m^k, I_m^k)}{1 + \phi_7(\Delta t, E_m^k, I_m^k)(\mu_m + \sigma_m + \Phi_7)},
\end{aligned} \tag{2.15}$$

where

$$\begin{aligned}
f_2(S_h^k, E_h^k, I_m^k) &= \frac{b\beta_h S_h^k I_m^k}{1 + v_h I_m^k} - (\alpha_h + \mu_h)E_h^k, \\
f_3(E_h^k, I_h^k) &= \alpha_h E_h^k - (r + \mu_h + \sigma_h)I_h^k, \\
f_4(I_h^k, R_h^k) &= r I_h^k - (\mu_h + \omega)R_h^k,
\end{aligned}$$

$$\begin{aligned} f_5(S_m^k, I_h^k) &= \Lambda_m - \frac{b\beta_m S_m^k I_h^k}{1+v_m I_h^k} - \mu_m S_m^k, \\ f_6(S_m^k, E_m^k, I_h^k) &= \frac{b\beta_m S_m^k I_h^k}{1+v_m I_h^k} - (\alpha_m+\mu_m)E_m^k, \\ f_7(E_m^k, I_m^k) &= \alpha_m E_m^k - (\mu_m+\sigma_m) I_h^k. \end{aligned} \tag{2.16}$$

Theorem 2.3. *The NSFD scheme (2.2) is elementary stable.*

Proof. Assuming that E_0^* is the equilibrium point of (2.2), then from (2.15), the Jacobian matrix of (2.2) evaluated at E_0^* is

$$J^D(E_0^*) = \begin{bmatrix} A_{11} & 0 & 0 & A_{14} & 0 & 0 & A_{17} \\ A_{21} & A_{22} & 0 & 0 & 0 & 0 & A_{27} \\ 0 & A_{32} & A_{33} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{43} & A_{44} & 0 & 0 & 0 \\ 0 & 0 & A_{53} & 0 & A_{55} & 0 & 0 \\ 0 & 0 & A_{63} & 0 & A_{65} & A_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{76} & A_{77} \end{bmatrix}, \tag{2.17}$$

where

$$\begin{aligned}
A_{11} &= 1 + \frac{\phi_1 \frac{\partial f_1(S_h^k, R_h^k, I_m^k)}{\partial S_h^k}(E_0^*)}{1+\phi_1\left(\frac{b\beta_h I_m^k}{1+v_h I_m^k} + \mu_h + \Phi_1\right)}, \quad & A_{14} &= \frac{\phi_1 \frac{\partial f_1(S_h^k, R_h^k, I_m^k)}{\partial R_h^k}(E_0^*)}{1+\phi_1\left(\frac{b\beta_h I_m^k}{1+v_h I_m^k} + \mu_h + \Phi_1\right)}, \\
A_{17} &= \frac{\phi_1 \frac{\partial f_1(S_h^k, R_h^k, I_m^k)}{\partial I_m^k}(E_0^*)}{1+\phi_1\left(\frac{b\beta_h I_m^k}{1+v_h I_m^k} + \mu_h + \Phi_1\right)}, \quad & A_{21} &= \frac{\phi_2 \frac{\partial f_2(S_h^k, E_h^k, I_m^k)}{\partial S_h^k}(E_0^*)}{1+\phi_2(\alpha_h+\mu_h+\Phi_2)}, \\
A_{22} &= 1 + \frac{\phi_2 \frac{\partial f_2(S_h^k, E_h^k, I_m^k)}{\partial E_h^k}(E_0^*)}{1+\phi_2(\alpha_h+\mu_h+\Phi_2)}, \quad & A_{27} &= \frac{\phi_2 \frac{\partial f_2(S_h^k, E_h^k, I_m^k)}{\partial I_m^k}(E_0^*)}{1+\phi_2(\alpha_h+\mu_h+\Phi_2)}, \\
A_{32} &= \frac{\phi_3 \frac{\partial f_3(E_h^k, I_h^k)}{\partial E_h^k}(E_0^*)}{1+\phi_3(r+\mu_h+\sigma_h+\Phi_3)}, \quad & A_{33} &= 1 + \frac{\phi_3 \frac{\partial f_3(E_h^k, I_h^k)}{\partial I_h^k}(E_0^*)}{1+\phi_3(r+\mu_h+\sigma_h+\Phi_3)}, \\
A_{43} &= \frac{\phi_4 \frac{\partial f_4(I_h^k, R_h^k)}{\partial I_h^k}(E_0^*)}{1+\phi_4(\mu_h+\omega+\Phi_4)}, \quad & A_{44} &= 1 + \frac{\phi_4 \frac{\partial f_4(I_h^k, R_h^k)}{\partial R_h^k}(E_0^*)}{1+\phi_4(\mu_h+\omega+\Phi_4)}, \\
A_{53} &= \frac{\phi_5 \frac{\partial f_5(S_m^k, I_h^k)}{\partial I_h^k}(E_0^*)}{1+\phi_5\left(\frac{b\beta_h I_h^k}{1+v_m I_h^k} + \mu_m + \Phi_5\right)}, \quad & A_{55} &= 1 + \frac{\phi_5 \frac{f_5(S_m^k, I_h^k)}{\partial S_m^k}(E_0^*)}{1+\phi_5\left(\frac{b\beta_h I_h^k}{1+v_m I_h^k} + \mu_m + \Phi_5\right)},
\end{aligned} \tag{2.18}$$

$$A_{63} = \frac{\phi_6 \frac{f_6(S_m^k,E_m^k,I_h^k)}{\partial I_h^k}(E_0^*)}{1+\phi_6(\alpha_m+\mu_m+\Phi_6)}, \qquad A_{65} = \frac{\phi_6 \frac{f_6(S_m^k,E_m^k,I_h^k)}{\partial S_m^k}(E_0^*)}{1+\phi_6(\alpha_m+\mu_m+\Phi_6)},$$

$$A_{66} = 1+\frac{\phi_6 \frac{f_6(S_m^k,E_m^k,I_h^k)}{\partial E_m^k}(E_0^*)}{1+\phi_6(\alpha_m+\mu_m+\Phi_6)}, \qquad A_{76} = \frac{\phi_7 \frac{f_7(E_m^k,I_m^k)}{\partial E_m^k}(E_0^*)}{1+\phi_7(\mu_m+\sigma_m+\Phi_7)},$$

$$A_{77} = 1+\frac{\phi_7 \frac{f_7(E_m^k,I_m^k)}{\partial I_m^k}(E_0^*)}{1+\phi_7(\mu_m+\sigma_m+\Phi_7)}.$$

Now we have to find partial derivatives of the system (2.16) with respect to each state variable as follows

$$\begin{aligned}
&\frac{\partial f_1(S_h^k,R_h^k,I_m^k)}{\partial S_h^k} = -\frac{b\beta_h I_m^k}{1+v_h I_m^k}-\mu_h, \qquad \frac{\partial f_1(S_h^k,R_h^k,I_m^k)}{\partial R_h^k} = \omega,\\
&\frac{\partial f_1(S_h^k,R_h^k,I_m^k)}{\partial I_m^k} = \frac{(b\beta_h S_h^k I_m^k)v_h-(1+v_h I_m^k)(b\beta_h S_h^k)}{(1+v_h I_m^k)^2},\\
&\frac{\partial f_2(S_h^k,E_h^k,I_m^k)}{\partial S_h^k} = \frac{b\beta_h I_m^k}{1+v_h I_m^k}, \qquad \frac{\partial f_2(S_h^k,E_h^k,I_m^k)}{\partial E_h^k} = -(\alpha_h+\mu_h),\\
&\frac{\partial f_2(S_h^k,E_h^k,I_m^k)}{\partial I_m^k} = \frac{(1+v_h I_m^k)(b\beta_h S_h^k)-(b\beta_h S_h^k I_m^k)v_h)}{(1+v_h I_m^k)^2},\\
&\frac{\partial f_3(E_h^k,I_h^k)}{\partial E_h^k} = \alpha_h, \quad \frac{\partial f_3(E_h^k,I_h^k)}{\partial I_h^k} = -(r+\mu_h+\sigma_h), \qquad \frac{\partial f_4(I_h^k,R_h^k)}{\partial I_h^k} = r,\\
&\frac{\partial f_4(I_h^k,R_h^k)}{\partial R_h^k} = -(\mu_h+\omega), \qquad \frac{\partial f_5(S_m^k,I_h^k)}{\partial I_h^k} = \frac{(b\beta_m I_h^k S_m^k)v_m-(1+v_m I_h^k)(b\beta_m S_m^k)}{(1+v_m I_h^k)^2},\\
&\frac{\partial f_5(S_m^k,I_h^k)}{\partial S_m^k} = -\frac{b\beta_m I_h^k}{1+v_m I_h^k}-\mu_m, \qquad \frac{\partial f_6(S_m^k,E_m^k,I_h^k)}{\partial E_m^k} = -(\alpha_m+\mu_m),\\
&\frac{\partial f_6(S_m^k,E_m^k,I_h^k)}{\partial I_h^k} = \frac{(b\beta_m S_m^k)(1+v_m I_h^k)-(b\beta_m I_h^k S_m^k)v_m}{(1+v_m I_h^k)^2},\\
&\frac{\partial f_6(S_m^k,E_m^k,I_h^k)}{\partial S_m^k} = \frac{b\beta_m I_h^k}{1+v_m I_h^k}, \qquad \frac{\partial f_7(E_m^k,I_m^k)}{\partial I_m^k} = -(\mu_m+\sigma_m), \quad \frac{\partial f_7(E_m^k,I_m^k)}{\partial S_m^k} = \alpha_m.
\end{aligned} \tag{2.19}$$

Moving forward, we analyze the stability of the scheme (2.2) at the disease-free equilibrium (DFE), $E_0 = \left(\frac{\Lambda_h}{\mu_h},0,0,0,\frac{\Lambda_m}{\mu_m},0,0\right)$. The entries of the Jacobian matrix (2.17) are:

$$\begin{aligned}
&A_{11} = 1-\frac{\phi_1(\Delta t,\frac{\Lambda_h}{\mu_h},0,0)\mu_h}{1+\phi_1(\Delta t,\frac{\Lambda_h}{\mu_h},0,0)(\mu_h+\Phi_1)}, \qquad A_{14} = \frac{\phi_1(\Delta t,\frac{\Lambda_h}{\mu_h},0,0)\omega}{1+\phi_1(\Delta t,\frac{\Lambda_h}{\mu_h},0,0)(\mu_h+\Phi_1)},\\
&A_{17} = -\frac{\phi_1(\Delta t,\frac{\Lambda_h}{\mu_h},0,0)\frac{b\beta_h\Lambda_h}{\mu_h}}{1+\phi_1(\Delta t,\frac{\Lambda_h}{\mu_h},0,0)(\mu_h+\Phi_1)}, \qquad A_{22} = 1-\frac{\phi_2(\Delta t,\frac{\Lambda_h}{\mu_h},0,0)(\alpha_h+\mu_h)}{1+\phi_2(\Delta t,\frac{\Lambda_h}{\mu_h},0,0)(\alpha_h+\mu_h+\Phi_2)},
\end{aligned} \tag{2.20}$$

$$A_{27} = \frac{\phi_2(\Delta t, \frac{\Lambda_h}{\mu_h}, 0, 0)\frac{b\beta_h\Lambda_h}{\mu_h}}{1 + \phi_2(\Delta t, \frac{\Lambda_h}{\mu_h}, 0, 0)(\alpha_h + \mu_h + \Phi_2)}, \qquad A_{32} = \frac{\phi_3(\Delta t, 0, 0)\alpha_h}{1 + \phi_3(\Delta t, 0, 0)(r + \mu_h + \sigma_h + \Phi_3)},$$

$$A_{33} = 1 - \frac{\phi_3(\Delta t, 0, 0)(r + \mu_h + \sigma_h)}{1 + \phi_3(\Delta t, 0, 0)(r + \mu_h + \sigma_h + \Phi_3)}, \qquad A_{43} = \frac{\phi_4(\Delta t, 0, 0)r}{1 + \phi_4(\Delta t, 0, 0)(\mu_h + \omega + \Phi_4)},$$

$$A_{44} = 1 - \frac{\phi_4(\Delta t, 0, 0)(\mu_h + \omega)}{1 + \phi_4(\Delta t, 0, 0)(\mu_h + \omega + \Phi_4)}, \qquad A_{53} = -\frac{\phi_5(\Delta t, 0, \frac{\Lambda_m}{\mu_m})\frac{b\beta_m\Lambda_m}{\mu_m}}{1 + \phi_5(\Delta t, 0, \frac{\Lambda_m}{\mu_m})(\mu_m + \Phi_5)},$$

$$A_{55} = 1 - \frac{\phi_5(\Delta t, 0, \frac{\Lambda_m}{\mu_m})\mu_m}{1 + \phi_5(\Delta t, 0, \frac{\Lambda_m}{\mu_m})(\mu_m + \Phi_5)}, \qquad A_{63} = \frac{\phi_6(\Delta t, 0, \frac{\Lambda_m}{\mu_m}, 0)\frac{b\beta_m\Lambda_m}{\mu_m}}{1 + \phi_6(\Delta t, 0, \frac{\Lambda_m}{\mu_m}, 0)(\alpha_m + \mu_m + \Phi_6)},$$

$$A_{66} = 1 - \frac{\phi_6(\Delta t, 0, \frac{\Lambda_m}{\mu_m}, 0)(\mu_m + \alpha_m)}{1 + \phi_6(\Delta t, 0, \frac{\Lambda_m}{\mu_m}, 0)(\alpha_m + \mu_m + \Phi_6)}, \qquad A_{76} = \frac{\phi_7(\Delta t, 0, 0)\alpha_m}{1 + \phi_7(\Delta t, 0, 0)(\mu_m + \sigma_m + \Phi_7)},$$

$$A_{77} = 1 - \frac{\phi_7(\Delta t, 0, 0)(\mu_m + \sigma_m)}{1 + \phi_7(\Delta t, 0, 0)(\mu_m + \sigma_m + \Phi_7)}.$$

It is clear that all the eigenvalues of the Jacobian matrix (2.17) satisfy $|J^D(E_0^*) - \Lambda I| < 1$. This shows that the DFE is locally asymptotically stable. This completes the proof.

Corollary 2.1. If $\Phi_i, i = 1, \ldots, 7$, are real numbers satisfying

$$\begin{aligned} &\Phi_1 \geq \frac{-\mu_h}{2}, \quad \Phi_2 \geq \frac{-(\alpha_h + \mu_h)}{2}, \quad \Phi_3 \geq \frac{-(r + \mu_h + \sigma_h)}{2}, \quad \Phi_4 \geq \frac{-(\mu_h + \omega)}{2}, \\ &\Phi_5 \geq \frac{-\mu_m}{2}, \quad \Phi_6 \geq \frac{-(\mu_m + \alpha_m)}{2}, \quad \Phi_7 \geq \frac{-(\mu_m + \sigma_m)}{2}, \end{aligned} \tag{2.21}$$

then the equilibrium point E_0^* of (2.2) is locally asymptotically stable.

Proof. See [12], Theorem 2.1.

Computation of the basic reproduction number (R_0)

We compute the basic reproduction number for the NSFD scheme (2.2) using the next-generation matrix technique [1]. After reordering the equations of the scheme (2.2), the Jacobian matrix evaluated at the DFE is

$$M = \begin{bmatrix} A_{22} & 0 & 0 & A_{27} & 0 & 0 & 0 \\ 0 & A_{66} & A_{63} & 0 & 0 & 0 & 0 \\ A_{32} & 0 & A_{33} & 0 & 0 & 0 & 0 \\ 0 & A_{76} & 0 & A_{77} & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{17} & A_{11} & 0 & 0 \\ 0 & 0 & A_{53} & 0 & 0 & A_{55} & 0 \\ 0 & 0 & A_{43} & 0 & 0 & 0 & A_{44} \end{bmatrix} \tag{2.22}$$

Where entries of the matrix (2.22) are defined in (2.20). This matrix has the form

$$M = \begin{bmatrix} F+T & 0 \\ A & C \end{bmatrix} \tag{2.23}$$

where

$$F = \begin{bmatrix} 0 & 0 & 0 & A_{27} \\ 0 & 0 & A_{63} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} A_{22} & 0 & 0 & 0 \\ 0 & A_{66} & 0 & 0 \\ A_{32} & 0 & A_{33} & 0 \\ 0 & A_{76} & 0 & A_{77} \end{bmatrix} \tag{2.24}$$

The matrices F and T are non-negative and irreducible. We now compute the basic reproduction number R_{0_D}. The inverse of $(\mathbf{I}-T)$, where $\mathbf{I}$ is the identity matrix, is

$$(\mathbf{I}-T)^{-1} = \begin{bmatrix} \frac{1}{1-A_{22}} & 0 & 0 & 0 \\ 0 & \frac{1}{1-A_{66}} & 0 & 0 \\ \frac{A_{32}}{(1-A_{22})(1-A_{33})} & 0 & \frac{1}{1-A_{33}} & 0 \\ 0 & \frac{A_{76}}{(1-A_{66})(1-A_{77})} & 0 & \frac{1}{1-A_{77}} \end{bmatrix}. \tag{2.25}$$

So,

$$F(\mathbf{I}-T)^{-1} = \begin{bmatrix} 0 & \frac{A_{27}A_{76}}{(A_{66})(1-A_{77})} & 0 & \frac{A_{27}}{1-A_{77}} \\ \frac{A_{63}A_{32}}{(1-A_{22})(1-A_{33})} & 0 & \frac{A_{63}}{1-A_{33}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.26}$$

The nonzero eigenvalues of $F(\mathbf{I}-T)^{-1}$ are

$$\lambda_a = \sqrt{\frac{A_{63}A_{32}A_{27}A_{76}}{(1-A_{22})(1-A_{33})(1-A_{66})(1-A_{77})}}, \quad \lambda_b = \sqrt{\frac{A_{66}A_{27}}{(1-A_{33})(1-A_{77})}}.$$

By simplifying arguments inside λ_b, we find it depends on the step size, and this can be proven numerically that for any given step size, λ_b is always smaller than λ_a. From (2.20), the numerator of λ_a is

$$\frac{\phi_2\phi_3\phi_6\phi_7 \frac{b\beta_m\Lambda_m}{\mu_m}\alpha_h \frac{b\beta_h\Lambda_h}{\mu_h}\alpha_m}{[1+\phi_6(\alpha_m+\mu_m+\Phi_6)][1+\phi_3(r+\mu_h+\sigma_h+\Phi_3)][1+\phi_2(\alpha_h+\mu_h+\Phi_3)][1+\phi_7(\mu_m+\sigma_m+\Phi_7)]}, \tag{2.27}$$

and the denominator is

$$\frac{\phi_2\phi_3\phi_6\phi_7(\alpha_h+\mu_h)(r+\mu_h+\sigma_h)(\mu_m+\alpha_m)(\mu_m+\sigma_m)}{[1+\phi_6(\alpha_m+\mu_m+\Phi_6)][1+\phi_3(r+\mu_h+\sigma_h+\Phi_3)][1+\phi_2(\alpha_h+\mu_h+\Phi_3)][1+\phi_7(\mu_m+\sigma_m+\Phi_7)]}. \tag{2.28}$$

By dividing the numerator and the denominator, we have

$$\lambda_a = \sqrt{\frac{b^2\alpha_h\beta_h\Lambda_h\alpha_m\beta_m\Lambda_m}{\mu_h(\alpha_h+\mu_h)(r+\sigma_h+\mu_h)\mu_m(\sigma_m+\mu_m)(\alpha_m+\mu_m)}} = R_0^D. \tag{2.29}$$

We note that $R_0 = R_0^D$, which means that the NSFD (2.2) preserves the basic reproduction number of the continuous model.

2.4. **Convergence of the NSFD scheme.** By employing the technique in [14], we now show that the NSFD scheme (2.2) is convergent of order 2.

Theorem 2.4. *The NSFD scheme (2.2) is accurate of order* 2 *with global error* $O(\Delta t^2)$ *for all* $(S_h^k, E_h^k, I_h^k, R_h^k, S_m^k, E_m^k, I_m^k) \in R_+^7$ *and* $f_i \neq 0$, *for* $i = 1,2,3,4,5,6,7$ *given in (2.14 and 2.16).*

Proof. For the solution components $(S_h(t), E_h(t), I_h(t), R_h(t), S_m(t), E_m(t), I_m(t))$, using Taylor series expansion in the neighborhood of $t = t^k$, we have the following:

$$\begin{aligned}
S_h(t^{k+1}) &= S_h(t^k+\Delta t) = S_h(t^k) + \Delta t S_h'(t^k) + \frac{\Delta t^2}{2}S_h''(t^k) + O(\Delta t^3),\\
E_h(t^{k+1}) &= E_h(t^k+\Delta t) = E_h(t^k) + \Delta t E_h'(t^k) + \frac{\Delta t^2}{2}E_h''(t^k) + O(\Delta t^3),\\
I_h(t^{k+1}) &= I_h(t^k+\Delta t) = I_h(t^k) + \Delta t I_h'(t^k) + \frac{\Delta t^2}{2}I_h''(t^k) + O(\Delta t^3),\\
R_h(t^{k+1}) &= R_h(t^k+\Delta t) = R_h(t^k) + \Delta t R_h'(t^k) + \frac{\Delta t^2}{2}R_h''(t^k) + O(\Delta t^3),\\
S_m(t^{k+1}) &= S_m(t^k+\Delta t) = S_m(t^k) + \Delta t S_m'(t^k) + \frac{\Delta t^2}{2}S_m''(t^k) + O(\Delta t^3),\\
E_m(t^{k+1}) &= E_m(t^k+\Delta t) = E_m(t^k) + \Delta t E_m'(t^k) + \frac{\Delta t^2}{2}E_m''(t^k) + O(\Delta t^3),\\
I_m(t^{k+1}) &= I_m(t^k+\Delta t) = I_m(t^k) + \Delta t I_m'(t^k) + \frac{\Delta t^2}{2}I_m''(t^k) + O(\Delta t^3).
\end{aligned} \tag{2.30}$$

By making use of (2.14) and (2.16) and letting $g_1(\Delta t, S_h^k, R_h^k, I_m^k)$, $g_2(\Delta t, S_h^k, E_h^k, I_m^k)$, $g_3(\Delta t, E_h^k, I_h^k)$, $g_4(\Delta t, I_h^k, R_h^k)$, $g_5(\Delta t, S_m^k, I_h^k)$, $g_6(\Delta t, S_m^k, E_m^k, I_h^k)$, $g_7(\Delta t, E_m^k, I_m^k)$ to be the right-hand side of the system (2.13) and (2.15). From (2.13), it is easy to notice that

$$g_1(0, S_h^k, R_h^k, I_m^k) = S_h^k \tag{2.31}$$

Furthermore, by considering the first derivative of (2.13), we have

$$\frac{\partial g_1(\Delta t, S_h^k, R_h^k, I_m^k)}{\partial \Delta t} = \frac{\left[1+\phi_1\left(\frac{b\beta_h I_m^k}{1+v_h I_m^k}+\mu_h+\Phi_1\right)\right]\frac{\partial\phi_1}{\partial\Delta t}f_1 - \phi_1 f_1\frac{\partial\phi_1}{\partial\Delta t}\left(\frac{b\beta_h I_m^k}{1+v_h I_m^k}+\mu_h+\Phi_1\right)}{\left[1+\phi_1\left(\frac{b\beta_h I_m^k}{1+v_h I_m^k}+\mu_h+\Phi_1\right)\right]^2} \tag{2.32}$$

Remark 2.1. *Since* $\phi(\Delta t) = \Delta t + O(\Delta t^2)$*, then* $\dfrac{\partial \phi}{\partial \Delta t} = 1 + O(\Delta t)$*. By neglecting the higher orders of* $\dfrac{\partial \phi}{\partial \Delta t}$*, then we have* $\dfrac{\partial \phi}{\partial \Delta t} \approx 1$ *.*

By the Remark 2.1, the equation (2.32) reduces to

$$\frac{\partial g_1(0, S_h^k, R_h^k, I_m^k)}{\partial \Delta t} = f_1(S_h^k, R_h^k, I_m^k) \tag{2.33}$$

Moreover, by taking the second derivative of equation (2.32), we have

$$\frac{\partial^2 g_1}{\partial \Delta t^2} = \frac{D^2 \frac{\partial^2 \phi_1}{\partial \Delta t^2} f_1 - 2 f_1 \left(\frac{\partial \phi_1}{\partial \Delta t}\right)^2 \left(\frac{b\beta_h I_m^k}{1+v_h I_m^k} + \mu_h + \Phi_1\right)(D)}{D^4}, \tag{2.34}$$

where $D = 1 + \phi_1 \left(\dfrac{b\beta_h I_m^k}{1 + v_h I_m^k} + \mu_h + \Phi_1\right)$. Using Remark 2.1, the equation (2.34) reduces to

$$\frac{\partial^2 g_1(0, S_h^k, R_h^k, I_m^k)}{\partial \Delta t^2} = \left[\frac{\partial^2 \phi_1}{\partial \Delta t^2} - 2\left(\frac{b\beta_h I_m^k}{1 + v_h I_m^k} + \mu_h + \Phi_1\right)\right] f_1. \tag{2.35}$$

From the first equation of (2.15), we obtain the following by considering the first and second derivatives:

$$\begin{aligned} &g_2(0, S_h^k, E_h^k, I_m^k) = E_h^k, \\ &\frac{\partial g_2(0, S_h^k, E_h^k, I_m^k)}{\partial \Delta t} = f_2(S_h^k, E_h^k, I_m^k), \\ &\frac{\partial^2 g_2(0, S_h^k, E_h^k, I_m^k)}{\partial \Delta t^2} = \left[\frac{\partial^2 \phi_2}{\partial \Delta t^2} - 2\left(\alpha_h + \mu_h + \Phi_2\right)\right] f_2. \end{aligned} \tag{2.36}$$

From the second equation of (2.15), we obtain the following by considering the first and second derivatives:

$$\begin{aligned} &g_3(0, E_h^k, I_h^k) = I_h^k, \\ &\frac{\partial g_3(0, E_h^k, I_h^k)}{\partial \Delta t} = f_3(E_h^k, I_h^k), \\ &\frac{\partial^2 g_3(0, E_h^k, I_h^k)}{\partial \Delta t^2} = \left[\frac{\partial^2 \phi_3}{\partial \Delta t^2} - 2\left(r + \mu_h + \sigma_h + \Phi_3\right)\right] f_3. \end{aligned} \tag{2.37}$$

From the third equation of (2.15), we obtain the following by considering the first and second derivatives:

$$\begin{aligned} &g_4(0, I_h^k, R_h^k) = R_h^k, \\ &\frac{\partial g_4(0, I_h^k, R_h^k)}{\partial \Delta t} = f_4(I_h^k, R_h^k), \\ &\frac{\partial^2 g_4(0, I_h^k, R_h^k)}{\partial \Delta t^2} = \left[\frac{\partial^2 \phi_4}{\partial \Delta t^2} - 2\left(\mu_h + \omega + \Phi_4\right)\right] f_4. \end{aligned} \tag{2.38}$$

From the fourth equation of (2.15), we obtain the following by considering the first and second derivatives:

$$\begin{aligned}
&g_5(0, S_m^k, I_h^k) = S_m^k,\\
&\frac{\partial g_5(0, S_m^k, I_h^k)}{\partial \Delta t} = f_5(S_m^k, I_h^k),\\
&\frac{\partial^2 g_5(0, S_m^k, I_h^k)}{\partial \Delta t^2} = \left[\frac{\partial^2 \phi_5}{\partial \Delta t^2} - 2\left(\frac{b\beta_m I_h^k}{1+v_m I_h^k} + \mu_m + \Phi_5\right)\right] f_5.
\end{aligned} \tag{2.39}$$

From the fifth equation of (2.15), we obtain the following by considering the first and second derivatives:

$$\begin{aligned}
&g_6(0, S_m^k, E_m^k, I_h^k) = E_m^k,\\
&\frac{\partial g_6(0, S_m^k, E_m^k, I_h^k)}{\partial \Delta t} = f_6(S_m^k, E_h^k, I_h^k),\\
&\frac{\partial^2 g_6(0, S_m^k, E_m^k, I_h^k)}{\partial \Delta t^2} = \left[\frac{\partial^2 \phi_6}{\partial \Delta t^2} - 2\left(\alpha_m + \mu_m + \Phi_6\right)\right] f_6.
\end{aligned} \tag{2.40}$$

From the sixth equation, we obtain the following by considering the first and second derivatives:

$$\begin{aligned}
&g_7(0, E_m^k, I_m^k) = I_m^k,\\
&\frac{\partial g_7(0, E_m^k, I_m^k)}{\partial \Delta t} = f_6(E_m^k, I_m^k),\\
&\frac{\partial^2 g_7(0, E_m^k, I_m^k)}{\partial \Delta t^2} = \left[\frac{\partial^2 \phi_7}{\partial \Delta t^2} - 2\left(\mu_m + \sigma_m + \Phi_7\right)\right] f_7.
\end{aligned} \tag{2.41}$$

Combining (2.31) through to (2.41) with Taylor theorem, we have the following:

$$\begin{aligned}
S_h^{k+1} &= S_h^k + \Delta t f_1 + \frac{\Delta t^2}{2}\left[\frac{\partial^2 \phi_1}{\partial \Delta t^2} - 2\left(\frac{b\beta_h I_m^k}{1+v_h I_m^k} + \mu_h + \Phi_1\right)\right] f_1 + O(\Delta t^3),\\
E_h^{k+1} &= E_h^k + \Delta t f_2 + \frac{\Delta t^2}{2}\left[\frac{\partial^2 \phi_2}{\partial \Delta t^2} - 2\left(\alpha_h + \mu_h + \Phi_2\right)\right] f_2 + O(\Delta t^3),\\
I_h^{k+1} &= I_h^k + \Delta t f_3 + \frac{\Delta t^2}{2}\left[\frac{\partial^2 \phi_3}{\partial \Delta t^2} - 2\left(r + \mu_h + \sigma_h + \Phi_3\right)\right] f_3 + O(\Delta t^3),\\
R_h^{k+1} &= R_h^k + \Delta t f_4 + \frac{\Delta t^2}{2}\left[\frac{\partial^2 \phi_4}{\partial \Delta t^2} - 2\left(\mu_h + \omega + \Phi_4\right)\right] f_4 + O(\Delta t^3),\\
S_m^{k+1} &= S_m^k + \Delta t f_5 + \frac{\Delta t^2}{2}\left[\frac{\partial^2 \phi_5}{\partial \Delta t^2} - 2\left(\frac{b\beta_m I_h^k}{1+v_m I_h^k} + \mu_m + \Phi_5\right)\right] f_5 + O(\Delta t^3),\\
E_m^{k+1} &= E_m^k + \Delta t f_6 + \frac{\Delta t^2}{2}\left[\frac{\partial^2 \phi_6}{\partial \Delta t^2} - 2\left(\alpha_m + \mu_m + \Phi_6\right)\right] f_6 + O(\Delta t^3),\\
I_m^{k+1} &= I_m^k + \Delta t f_7 + \frac{\Delta t^2}{2}\left[\frac{\partial^2 \phi_7}{\partial \Delta t^2} - 2\left(\mu_m + \sigma_m + \Phi_7\right)\right] f_7 + O(\Delta t^3).
\end{aligned} \tag{2.42}$$

Hence, from (2.30) and (2.42), we have the local truncation error given by

$$
\begin{aligned}
&S_h^{k+1} - S_h(t^{k+1}) = O(\Delta t^3), \quad E_h^{k+1} - E_h(t^{k+1}) = O(\Delta t^3),\\
&I_h^{k+1} - I_h(t^{k+1}) = O(\Delta t^3), \quad R_h^{k+1} - R_h(t^{k+1}) = O(\Delta t^3),\\
&S_m^{k+1} - S_m(t^{k+1}) = O(\Delta t^3), \quad E_m^{k+1} - E_m(t^{k+1}) = O(\Delta t^3),\\
&I_m^{k+1} - I_m(t^{k+1}) = O(\Delta t^3).
\end{aligned} \tag{2.43}
$$

By examining the largest possible error for all time steps, we have the global error given by

$$
\begin{aligned}
&\max_{0\le k\le N}\left|S_h^{k+1} - S_h(t^{k+1})\right| = O(\Delta t^2), \quad \max_{0\le k\le N}\left|E_h^{k+1} - E_h(t^{k+1})\right| = O(\Delta t^2),\\
&\max_{0\le k\le N}\left|I_h^{k+1} - I_h(t^{k+1})\right| = O(\Delta t^2), \quad \max_{0\le k\le N}\left|R_h^{k+1} - R_h(t^{k+1})\right| = O(\Delta t^2),\\
&\max_{0\le k\le N}\left|S_m^{k+1} - S_m(t^{k+1})\right| = O(\Delta t^2), \quad \max_{0\le k\le N}\left|E_m^{k+1} - E_m(t^{k+1})\right| = O(\Delta t^2),\\
&\max_{0\le k\le N}\left|I_m^{k+1} - I_m(t^{k+1})\right| = O(\Delta t^2).
\end{aligned} \tag{2.44}
$$

This completes the proof.

Remark 2.2. It is important to note that ϕ_i, for $i = 1, \cdots, 7$ satisfy (2.46) for $(S_h^k, E_h^k, I_h^k, R_h^k, S_m^k, E_m^k, I_m^k) \in R_+^7$ and $f_i \neq 0$ given in (2.14) and (2.16).

$$
\begin{aligned}
\frac{\partial^2\phi_1(0, S_h^k, R_h^k, I_m^k)}{\partial\Delta t^2} &= \rho_1(S_h^k, R_h^k, I_m^k) = 2\left(\frac{b\beta_h I_m^k}{1+v_h I_m^k} + \mu_h + \Phi_1\right) + \frac{\partial f_1}{\partial S_h^k} + \frac{\partial f_1}{\partial R_h^k}\frac{f_4}{f_1} + \frac{\partial f_1}{\partial I_h^k}\frac{f_7}{f_1},\\
\frac{\partial^2\phi_2(0, S_h^k, E_h^k, I_m^k)}{\partial\Delta t^2} &= \rho_2(S_h^k, E_h^k, I_m^k) = 2\left(\alpha_h + \mu_h + \Phi_2\right) + \frac{\partial f_2}{\partial E_h^k} + \frac{\partial f_2}{\partial S_h^k}\frac{f_1}{f_2} + \frac{\partial f_2}{\partial I_m^k}\frac{f_7}{f_2},\\
\frac{\partial^2\phi_3(0, E_h^k, I_h^k)}{\partial\Delta t^2} &= \rho_3(E_h^k, I_h^k) = 2\left(r + \mu_h + \sigma_h + \Phi_3\right) + \frac{\partial f_3}{\partial I_h^k} + \frac{\partial f_3}{\partial E_h^k}\frac{f_2}{f_3},\\
\frac{\partial^2\phi_4(0, I_h^k, R_h^k)}{\partial\Delta t^2} &= \rho_4(I_h^k, R_h^k) = 2\left(\mu_h + \omega + \Phi_4\right) + \frac{\partial f_4}{\partial R_h^k} + \frac{\partial f_4}{\partial I_h^k}\frac{f_3}{f_4},\\
\frac{\partial^2\phi_5(0, S_m^k, I_h^k)}{\partial\Delta t^2} &= \rho_5(S_m^k, I_h^k) = 2\left(\frac{b\beta_m I_h^k}{1+v_m I_h^k} + \mu_m + \Phi_5\right) + \frac{\partial f_5}{\partial S_m^k} + \frac{\partial f_5}{\partial I_h^k}\frac{f_3}{f_5},\\
\frac{\partial^2\phi_6(0, S_m^k, E_m^k, I_h^k)}{\partial\Delta t^2} &= \rho_6(S_m^k, E_m^k, I_h^k) = 2\left(\alpha_m + \mu_m + \Phi_6\right) + \frac{\partial f_6}{\partial E_m^k} + \frac{\partial f_6}{\partial S_m^k}\frac{f_5}{f_6} + \frac{\partial f_6}{\partial I_h^k}\frac{f_3}{f_6},\\
\frac{\partial^2\phi_7(0, E_m^k, I_m^k)}{\partial\Delta t^2} &= \rho_7(E_m^k, I_m^k) = 2\left(\mu_m + \sigma_m + \Phi_7\right) + \frac{\partial f_7}{\partial I_m^k} + \frac{\partial f_7}{\partial E_m^k}\frac{f_6}{f_7}.
\end{aligned} \tag{2.45}
$$

The function ϕ_i can be selected using the techniques in [14, 18] such that:

$$
\phi_i(\Delta t, \rho_i) = \begin{cases} \frac{1-e^{-\rho_i\Delta t}}{\rho_i}, & \text{if } \rho_i \neq 0\\ \Delta t, & \text{if } \rho_i = 0, \end{cases} \tag{2.46}
$$

The denominator function (2.46) also satisfies the following properties: $\phi_i(\Delta t, \rho_i) = \Delta t^2 + O(\Delta t^3)$ as $\Delta t \to 0$ and $\phi_i(\Delta t, \rho_i) > 0$ for all $\Delta t > 0, \rho_i > 0$ [10].

Corollary 2.2. Since weight ensures stability and convergence of the scheme, therefore, choice of weight is determined by the stability of the method. By letting $(S_h^0, E_h^0, I_h^0, R_h^0, S_m^0, E_m^0, I_m^0)^T$ to be the initial solutions of the scheme (2.2) with right-hand side $\mathbf{f}_i, i = 1, \ldots, 7$, and assuming that the second argument of the step by step update function is subject to the Lipschitz condition, then we have:

$$\left\|\mathbf{f}_i(\mathbf{X}) - \mathbf{f}_i(\mathbf{Y})\right\| \leq L^D \|\mathbf{X} - \mathbf{Y}\|, \textit{ for all } \{\mathbf{X}, \mathbf{Y}\} \in \mathbb{R}_+^7 \tag{2.47}$$

where $\mathbf{X} = (S_{h1}^k, E_{h1}^k, I_{h1}^k, R_{h1}^k, S_{m1}^k, E_{m1}^k, I_{m1}^k)^T$ and $\mathbf{Y} = (S_{h2}^k, E_{h2}^k, I_{h2}^k, R_{h2}^k, S_{m2}^k, E_{m2}^k, I_{m2}^k)^T$. The NSFD (2.15) step by step update in the scheme provides $\mathbf{f}_i$ such that:

$$\mathbf{f_i} = \begin{pmatrix} \frac{f_1(S_h^k, R_h^k, I_m^k)}{1+\phi_1(\Delta t, S_h^k, R_h^k, I_m^k)\left(\frac{b\beta_h I_m^k}{1+v_h I_m^k} + \mu_h + \Phi_1\right)} \\ \frac{f_2(S_h^k, E_h^k, I_m^k)}{1+\phi_2(\Delta t, S_h^k, E_h^k, I_m^k)(\alpha_h + \mu_h + \Phi_2)} \\ \frac{f_3(E_h^k, I_h^k)}{1+\phi_3(\Delta t, E_h^k, I_h^k)(r + \mu_h + \sigma_h + \Phi_3)} \\ \frac{f_4(I_h^k, R_h^k)}{1+\phi_4(\Delta t, I_h^k, R_h^k)(\mu_h + \omega + \Phi_4)} \\ \frac{f_5(S_m^k, I_h^k)}{1+\phi_5(\Delta t, S_m^k, I_h^k)\left(\frac{b\beta_m I_h^k}{1+v_m I_h^k} + \mu_m + \Phi_5\right)} \\ \frac{f_6(S_m^k, E_m^k, I_h^k)}{1+\phi_6(\Delta t, S_m^k, E_m^k, I_h^k)(\alpha_m + \mu_m + \Phi_6)} \\ \frac{f_7(E_m^k, I_m^k)}{1+\phi_7(\Delta t, E_m^k, I_m^k)(\mu_m + \sigma_m + \Phi_7)}. \end{pmatrix} \tag{2.48}$$

By obtaining the partial derivatives of (2.48) and substituting DFE, we have the following Jacobian matrix:

$$J_{\mathbf{f_i}} = \begin{bmatrix} \frac{\partial \mathbf{f_1}}{\partial S_h} & 0 & 0 & \frac{\partial \mathbf{f_1}}{\partial R_h} & 0 & 0 & \frac{\partial \mathbf{f_1}}{\partial I_m} \\ 0 & \frac{\partial \mathbf{f_2}}{\partial E_h} & 0 & 0 & 0 & 0 & \frac{\partial \mathbf{f_2}}{\partial I_m} \\ 0 & \frac{\partial \mathbf{f_3}}{\partial E_h} & \frac{\partial \mathbf{f_3}}{\partial I_h} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial \mathbf{f_4}}{\partial I_h} & \frac{\partial \mathbf{f_4}}{\partial R_h} & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial \mathbf{f_5}}{\partial I_h} & 0 & \frac{\partial \mathbf{f_5}}{\partial S_m} & 0 & 0 \\ 0 & 0 & \frac{\partial \mathbf{f_6}}{\partial I_h} & 0 & 0 & \frac{\partial \mathbf{f_6}}{\partial E_m} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\partial \mathbf{f_7}}{\partial E_m} & \frac{\partial \mathbf{f_7}}{\partial I_m} \end{bmatrix} \tag{2.49}$$

entries in the Jacobian matrix (2.49) are in the system (2.19). One can notice that from the Jacobian matrix (2.49) all eigenvalues are negative. This shows that all perturbations around the DFE decay exponentially over time. Moving forward, to obtain the Lipschitz constant L^D, we calculate the maximum row sum of the Jacobian matrix (2.49) which is given by

$$L^D = \|J_{\mathbf{f_i}}\|_{\text{max row sum}} = \max\left(\sum_{j=1}^{7} |J_{ij}|\right) \tag{2.50}$$

Remark 2.3. The Lipschitz constant determines the convergence and stability properties. A smaller Lipschitz constant shows that the scheme converges close to the true solution.

3. Numerical Simulations

This section showcases the advantages of the second-order NSFD (2NSFD) method and provides numerical evidence to back-up the theoretical assertions from the preceding section. We use the data and parameter values provided in Table 3 below (see [23]). By using parameter values in the Table 3 and substituting them into the Jacobian matrix (2.49), the maximum row sum (2.50) is $L^D = 0.39$. In our simulations, we use the denominator function given by (2.46) and $t \in [0, 300]$. As for the weight, we choose $\Phi_i = 0$ for $i = 1, \ldots, 7$ from Theorem 2.3 since it satisfies both positivity and stability of the NSFD scheme (2.2).

Table 3. Value of parameters for numerical results

$S_h(0) = 700$	$E_h(0) = 400$	$I_h(0) = 250$	$R_h(0) = 100$
$S_m(0) = 90$	$E_m(0) = 40$	$I_m(0) = 20$	
$\Lambda_h = 0.17$	$\Lambda_m = 1.7$	$\beta_h = 0.1$	$\beta_m = 0.3$
$\mu_h = 0.0001$	$\mu_m = 0.010$	$\sigma_h = 0.2 \times 10^{-4}$	$\sigma_m = 0.05$
$\alpha_h = 0.2$	$\alpha_m = 0.33$	$r = 0.012$	$\omega = 0.0011$
$v_h = 0.5$	$v_m = 0.01$	$b = 0.01$	

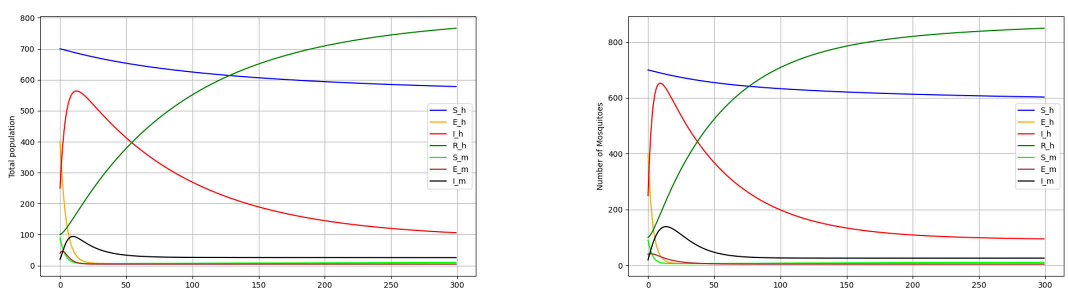

Figure 2. Solution profiles generated by first-order NSFD (2.3) (left) and second-order NSFD (2.2)(right) schemes for $\Delta t = 11$.

Figure 2 shows that the second-order NSFD scheme for humans preserves the positivity, boundedness, and stability of the solutions for large step sizes. This validates Theorem 2.1 and Theorem 2.3. The same is observed for the first-order NSFD method.

Due to the unavailability of the exact solution, we employ the double-mesh principle [7] to estimate the error E_{DM} as follows.

$$E_{DM} = |S_{h(\Delta t)}(t_{\text{end}}) - S_{h(\frac{\Delta t}{2})}(t_{\text{end}})| + |E_{h(\Delta t)}(t_{\text{end}}) - E_{h(\frac{\Delta t}{2})}(t_{\text{end}})| + \cdots + |I_{m(\Delta t)}(t_{\text{end}}) - I_{m(\frac{\Delta t}{2})}(t_{\text{end}})|, \tag{3.1}$$

where $t_{\text{end}} = 300$. The convergence rate is approximated by

$$\text{Rate} = \log_2\left(\frac{E_{DM}(\Delta t)}{E_{DM}(\Delta t/2)}\right). \tag{3.2}$$

TABLE 4. Errors and convergence rates

Δt	2NSFD Errors	Rate	1NSFD errors	Rate
2^{2}	5.60×10^{-1}	–	1.03×10^{0}	–
2^{1}	1.69×10^{-1}	1.7311	5.31×10^{-1}	0.9615
2^{0}	4.57×10^{-2}	1.8829	2.69×10^{-1}	0.9808
2^{-1}	1.17×10^{-2}	1.9712	1.35×10^{-1}	0.9901
2^{-2}	2.90×10^{-3}	2.0085	6.80×10^{-2}	0.9950
2^{-3}	7.17×10^{-4}	2.0152	3.40×10^{-2}	0.9975
2^{-4}	1.78×10^{-4}	2.0114	1.70×10^{-2}	0.9988
2^{-5}	4.42×10^{-5}	2.0068	8.52×10^{-3}	0.9994
2^{-6}	1.10×10^{-5}	2.0037	4.26×10^{-3}	0.9997
2^{-7}	2.75×10^{-6}	2.0019	2.13×10^{-3}	0.9999
2^{-8}	6.88×10^{-7}	2.0010	1.07×10^{-3}	0.9999
2^{-9}	1.72×10^{-7}	2.0006	5.33×10^{-4}	1.0004
2^{-10}	4.30×10^{-8}	2.0000	2.66×10^{-4}	0.9992

Despite both the 1NSFD and the 2NSFD schemes being dynamically consistent with the continuous-time model, Table 4 confirms that the 2NSFD method is more accurate than the 1NSFD method. The errors are computed using the equation (3.1) and the convergence rate using (3.2). This fact is further highlighted in Figure 3 where the error vectors estimated via the double-mesh principle are plotted using 200 and 400 subintervals.

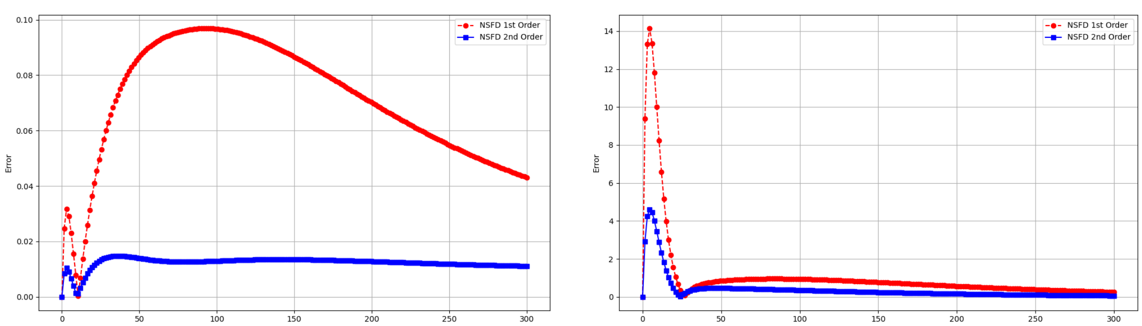

FIGURE 3. The errors versus time for 2NSFD and 1NSFD. (Left: Susceptible humans, Right: Infected humans).

Since weights (Φ_i) are presumed to be non-negative, one wonders whether any positive number can be used to compute the errors. We use different positive values of Φ which satisfy condition (2.21) in Table 5 to address this.

It turns out that Φ affects errors in the sense that the error increases with the value of Φ. We notice that the choice of weight ($\Phi = 0$) satisfies the stability condition, and it is the optimal value to minimize the errors of the NSFD scheme(2.2).

TABLE 5. Double-mesh errors, and convergence rate with different Φ values for 2NSFD

Δt	Error($\Phi = 0$)	Rate	Error($\Phi = 0.2$)	Rate	Error($\Phi = 0.5$)	Rate
2^2	5.60E01	–	8.16E-01	–	9.41E-01	–
2^1	1.69E-01	1.7311	3.02E-01	1.4355	3.97E-01	1.2452
2^0	4.57E-02	1.8829	9.65E-02	1.6445	1.45E-01	1.4515
2^{-1}	1.17E-02	1.9712	2.77E-02	1.8025	4.63E-02	1.6478
2^{-2}	2.90E-03	2.0085	7.40E-03	1.9010	1.33E-02	1.7971
2^{-3}	7.17E-04	2.0152	1.91E-03	1.9531	3.59E-03	1.8915
2^{-4}	1.78Ee-04	2.0114	4.85E-04	1.9781	9.33E-04	1.9443
2^{-5}	4.42E-05	2.0068	1.22E-04	1.9896	2.38E-04	1.9719
2^{-6}	1.10E-05	2.0037	3.07E-05	1.9950	6.01E-05	1.9859
2^{-7}	2.75E-06	2.0019	7.68E-06	1.9975	1.51E-05	1.9929
2^{-8}	6.88E-07	2.0010	1.92E-06	1.9988	3.78E-06	1.9965
2^{-9}	1.72E-07	2.0006	4.80E-07	1.9994	9.47E-07	1.9982
2^{-10}	4.30E-08	2.0000	1.20E-07	1.9995	2.37E-07	1.9992

Remark 3.1. It is worth noting that the value of the weight guarantees the stability of the underlying NSFD scheme. The weight must be a non-negative real number. In Table 5, the optimum weight is $\Phi = 0$. This is simply a coincidence. In other situation the optimum weight can be strictly positive (see for example [14]).

4. Conclusion

We developed a NSFD scheme for a malaria propagation model. The scheme comprises weights and denominator functions that depend, not only on the step-size, but also iteratively on the state variables. We showed that the constructed scheme is dynmanically consistent with the continuous-time model in that it preserves the positivity, boundedness and stability of the solution. Additionally, the equilibrium points and the basic reproduction number of the proposed discrete model coincide with those of the continuous one. Moreover we proved that the scheme is second-order convergent. Furthermore, we determined numerically the optimal weight for the scheme (for the minimal error).

Acknowledgement: The authors would like to appreciate the meaningful criticism from the reviewers that has helped improve the quality of this paper.

Author contributions: Conceptualization: JBM; Formal analysis: JBM, CBM; Investigation: CBM; Methodology: JBM, CBM; Writing original draft: CBM.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] L.J. Allen, P. van den Driessche, The Basic Reproduction Number in Some Discrete-Time Epidemic Models, J. Differ. Equ. Appl. 14 (2008), 1127–1147. https://doi.org/10.1080/10236190802332308.

[2] A.J. Arenas, G. González-Parra, B.M. Chen-Charpentier, Construction of Nonstandard Finite Difference Schemes for the SI and SIR Epidemic Models of Fractional Order, Math. Comput. Simul. 121 (2016), 48–63. https://doi.org/10.1016/j.matcom.2015.09.001.

[3] J. Calatayud, M. Jornet, An Improvement of Two Nonstandard Finite Difference Schemes for Two Population Mathematical Models, J. Differ. Equ. Appl. 27 (2021), 422–430. https://doi.org/10.1080/10236198.2021.1903888.

[4] B. Chen, F. Solis, Discretizations of Nonlinear Differential Equations Using Explicit Finite Order Methods, J. Comput. Appl. Math. 90 (1998), 171–183. https://doi.org/10.1016/s0377-0427(98)00017-x.

[5] J. Cresson, F. Pierret, Non Standard Finite Difference Scheme Preserving Dynamical Properties, J. Comput. Appl. Math. 303 (2016), 15–30. https://doi.org/10.1016/j.cam.2016.02.007.

[6] F. Dorner, R. Mosleh, Analysis of Ode Models for Malaria Propagation, Acta Marisiensis. Ser. Technol. 17 (2020), 31–39. https://doi.org/10.2478/amset-2020-0007.

[7] E.P. Doolan, J.J.H. Miller, W. Schilders, Uniform Numerical Methods for Problems with Initial and Boundary Layers, Boole Press, Dublin, 1980.

[8] S. Garba, A. Gumel, A. Hassan, J. Lubuma, Switching From Exact Scheme to Nonstandard Finite Difference Scheme for Linear Delay Differential Equation, Appl. Math. Comput. 258 (2015), 388–403. https://doi.org/10.1016/j.amc.2015.01.088.

[9] M. Gupta, J.M. Slezak, F. Alalhareth, S. Roy, H.V. Kojouharov, Second-order Nonstandard Explicit Euler Method, AIP Conf. Proc. 2302 (2020), 110003. https://doi.org/10.1063/5.0033534.

[10] M.T. Hoang, A Novel Second-Order Nonstandard Finite Difference Method for Solving One-Dimensional Autonomous Dynamical Systems, Commun. Nonlinear Sci. Numer. Simul. 114 (2022), 106654. https://doi.org/10.1016/j.cnsns.2022.106654.

[11] M.T. Hoang, Dynamical Analysis of a Generalized Hepatitis B Epidemic Model and Its Dynamically Consistent Discrete Model, Math. Comput. Simul. 205 (2023), 291–314. https://doi.org/10.1016/j.matcom.2022.10.006.

[12] M.T. Hoang, A Novel Second-Order Nonstandard Finite Difference Method Preserving Dynamical Properties of a General Single-Species Model, Int. J. Comput. Math. 100 (2023), 2047–2062. https://doi.org/10.1080/00207160.2023.2248304.

[13] M.T. Hoang, A Class of Second-Order and Dynamically Consistent Nonstandard Finite Difference Schemes for Nonlinear Volterra's Population Growth Model, Comput. Appl. Math. 42 (2023), 85. https://doi.org/10.1007/s40314-023-02230-z.

[14] M.T. Hoang, M. Ehrhardt, A Second-Order Nonstandard Finite Difference Method for a General Rosenzweig-Macarthur Predator-Prey Model, J. Comput. Appl. Math. 444 (2024), 115752. https://doi.org/10.1016/j.cam.2024.115752.

[15] J. Martín-Vaquero, A. Martín del Rey, A. Encinas, J. Hernández Guillén, A. Queiruga-Dios, G. Rodríguez Sánchez, Higher-order Nonstandard Finite Difference Schemes for a Mseir Model for a Malware Propagation, J. Comput. Appl. Math. 317 (2017), 146–156. https://doi.org/10.1016/j.cam.2016.11.044.

[16] J. Martín-Vaquero, A. Queiruga-Dios, A. Martín del Rey, A. Encinas, J. Hernández Guillén, G. Rodríguez Sánchez, Variable Step Length Algorithms with High-Order Extrapolated Non-Standard Finite Difference Schemes for a Seir Model, J. Comput. Appl. Math. 330 (2018), 848–854. https://doi.org/10.1016/j.cam.2017.03.031.

[17] M.M. Khalsaraei, A. Shokri, S. Noeiaghdam, M. Molayi, Nonstandard Finite Difference Schemes for an SIR Epidemic Model, Mathematics 9 (2021), 3082. https://doi.org/10.3390/math9233082.

[18] R.E. Mickens, Nonstandard Finite Difference Models of Differential Equations, World Scientific, 1994.

[19] R.E. Mickens, Discretizations of Nonlinear Differential Equations Using Explicit Nonstandard Methods, J. Comput. Appl. Math. 110 (1999), 181–185. https://doi.org/10.1016/s0377-0427(99)00233-2.

[20] R.E. Mickens, Applications of Nonstandard Finite Difference Schemes, World Scientific, Singapore, 2000.

[21] R.E. Mickens, T.M. Washington, NSFD Discretizations of Interacting Population Models Satisfying Conservation Laws, Comput. Math. Appl. 66 (2013), 2307–2316. https://doi.org/10.1016/j.camwa.2013.06.011.

[22] R.E. Mickens, Nonstandard Finite Difference Schemes: Methodology and Applications, World Scientific, 2020.

[23] R. Mosleh, Positivity Preserving Explicit Discrete Model for Malaria Propagation, Ann. Univ. Sci. Budapest. Sect. Math. 63 (2020), 123–133.

[24] S. Olaniyi, O. Obabiyi, Mathematical Model for Malaria Transmission Dynamics in Human and Mosquito Populations with Nonlinear Forces of Infection, Int. J. Pure Appl. Math. 88 (2013), 125–156. https://doi.org/10.12732/ijpam.v88i1.10.

[25] D.T. Wood, H.V. Kojouharov, A Class of Nonstandard Numerical Methods for Autonomous Dynamical Systems, Appl. Math. Lett. 50 (2015), 78–82. https://doi.org/10.1016/j.aml.2015.06.008.