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FIXED POINT THEOREMS FOR CIRIC'S AND GENERALIZED CONTRACTIONS IN b-METRIC SPACES

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Abstract. In this article we obtained b-metric variant of common fixed point results for Ciric's and generalized contractions. We have also proved some fixed point results for rational contractive type conditions in the context of b-metric space. A particular example is also given in the support of our established result regarding Ciric's type contraction.

1. Introduction

Fixed point theory is one of the most important topic in the development of non linear analysis. In this area, the first important and significant result was proved by Banach in 1922 for contraction mapping in complete metric space. A comprehensive literature and generalization of the Banach's contraction theorem can be found in [7] and [11]. Some problems particularly the problem of the convergence of measurable functions with respect to measure leads Czerwik (see [5],[6]) to the generalization of metric space and introduce the concept of b-metric space After Czerwik([5], [6]) many papers have been published containing fixed point results on b-metric spaces for single value and multivalued functions(see[1], [2], [10]).

In this paper we proved b-metric variant of fixed point results for Ciric's and generalized contraction. We also studied some results involving rational contractive type condition.

Through out the paper $\mathbb{R}^+$ will represent the set of non negative real numbers.

2. Preliminaries

Definition 1. [8]. Let X be a non-empty set and let $d : X \times X \to \mathbb{R}^+$ be a function satisfying the conditions,

d_1) $d(x, y) = 0 \quad \Leftrightarrow \quad x = y$;

d_2) $d(x, y) = d(y, x)$;

d_3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$. Then d it is called metric on X, and the pair (X, d) is called metric space.

Definition 2. [6]. Let X be a non empty set, let $k \geq 1$ be a real number then a mapping $b : X \times X \to \mathbb{R}^+$ is called b-metric if $\forall\, x, y, z \in X$, the following conditions are satisfied:

b_1) $b(x, y) = 0 \quad \Leftrightarrow \quad x = y$;

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b_2) $b(x,y) = b(y,x)$;
b_3) $b(x,y) \leq k[b(x,z) + b(z,y)]$. And the pair (X,b) is called b-metric space.

It is clear from the definition of b-metric that every metric space is b-metric for $k = 1$, but the converse is not true as clear from the following example.

Example 2.1. [2]. Let $X = \{0,1,2\}$. Defined $b : X \times X \to \mathbb{R}^+$ as follows
$b(0,0) = b(1,1) = b(2,2) = 0$, $b(1,2) = b(2,1) = b(0,1) = b(1,0) = 1, b(2,0) = b(0,2) = m \geq 2$ for $k = \frac{m}{2}$ where $m \geq 2$
the function defined above is a b-metric space but not a metric for $m > 2$.

For more examples of b-metric space (see [2], [10]).

In our main work we will use the following definitions which can be found in [2] and [10].

Definition 3. A sequence $\{x_n\}$ in b-metric space (X,b) is called Cauchy sequence if for $\epsilon > 0$ there exist a positive integer N such that for $m,n \geq N$ we have $b(x_m, x_n) < \epsilon$.

Definition 4. A sequence $\{x_n\}$ is called convergent in b-metric space (X,b) if for $\epsilon > 0$ and $n \geq N$ we have $b(x_n, x) < \epsilon$ where x is called the limit point of the sequence $\{x_n\}$.

Definition 5. A b-metric space (X,b) is said to be complete if every Cauchy sequence in X converge to a point of X.

Definition 6. [4]. Let (X,d) be a metric space, a self mapping $T : X \to X$ is called generalized contraction if and only for all $x,y \in X$, there exist c_1, c_2, c_3, c_4 such that $\sup\{c_1 + c_2 + c_3 + 2c_4\} < 1$ and

$$d(Tx,Ty) \leq c_1.d(x,y) + c_2.d(x,Tx) + c_3.d(y,Ty) + c_4.[d(x,Ty) + d(y,Tx)]$$

Definition 7. [11]. Let (X,d) be a metric space, a self mapping $T : X \to X$ is called Ciric's type contraction if and only if for all $x,y \in X$, there exist $h < 1$ and,

$$d(Tx,Ty) \leq h.\max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2}\right\}$$

Definition 8. [4]. Let (X,d) be a metric space, a self mapping $T : X \to X$ is called quasi contraction if and only if for all $x,y \in X$, there exist $h < 1$ and,

$$d(Tx,Ty) \leq h.\max\left\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty) + d(y,Tx)\right\}$$

Remark.

- In [4] the author shown by an example that every quasi contraction need not to be generalized contraction in metric spaces.

Theorem 2.1. [10]. *Let* (X,b) *be a complete b- metric space with* $k \geq 1$ *and let* $T : X \to X$ *be a contraction with* $\alpha \in [0,1)$ *and* $k\alpha < 1$ *then* T *has a unique fixed point in* X.

Definition 9. [3]. Let (X,d) be a metric space a mapping $T : X \to X$ is called weak contraction if there exist constants $\alpha \in (0,1)$ and some $\beta \geq 0$ such that

$$d(Tx,Ty) \leq \alpha.d(x,y) + \beta.d(y,Tx) \tag{1}$$

for all $x,y \in X$.

Due to the symmetry of distance the weak contraction clearly include the following

$$d(Tx, Ty) \leq \alpha.d(x, y) + \beta.d(x, Ty) \tag{2}$$

for all $x, y \in X$.

Therefore to check the weak contraction of the mapping, we have to check both (1) and (2).

Remarks.

- It is clear that every contraction in metric space is necessarily weak contraction but the converse is not always true(see[3]).
- Unlike metric space every mapping satisfying the contractive condition in b-metric space need not to be weak contraction necessarily(see[10]).

On the other hand Khan[9] proved the following fixed point result for complete metric spaces.

Theorem 2.2. *Let (X, d) be a complete metric space and T be a self mapping on X satisfying the following condition*

$$d(Tx, Ty) \leq \mu.\frac{d(x, Tx).d(x, Ty) + d(y, Ty).d(y, Tx)}{d(x, Ty) + d(y, Tx)}$$

$\forall\ x, y \in X$ *and* $\mu \in [0, 1)$, *then T has a unique fixed point.*

3. Main Results

Lemma 1. *Let (X, b) be a b-metric space and $\{x_n\}$ be a sequence in b-metric space such that*

$$b(x_n, x_{n+1}) \leq \alpha.b(x_{n-1}, x_n) \tag{3}$$

for $n = 1, 2, 3, ...$ and $0 \leq \alpha k < 1$, $\alpha \in [0, 1)$, and k is defined in b-metric space then $\{x_n\}$ is a Cauchy sequence in X.

Proof. Let $n, m \in N$ and $m > n$ we have

$$\begin{aligned} b(x_n, x_m) &\leq k[b(x_n, x_{n+1}) + b(x_{n+1}, x_{n+2}) \ldots\ldots\ldots\ldots\ldots\ldots] \\ &\leq kb(x_n, x_{n+1}) + k^2[b(x_{n+1}, x_{n+2}) + b(x_{n+2}, x_{n+3}) \ldots\ldots\ldots\ldots] \\ &\leq kb(x_n, x_{n+1}) + k^2 b(x_{n+1}, x_{n+2}) + k^3 b(x_{n+2}, x_{n+3}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

Now using (3) we have

$$\begin{aligned} b(x_n, x_m) &\leq k\alpha^n b(x_0, x_1) + k^2\alpha^{n+1} b(x_0, x_1) + k^3\alpha^{n+2} b(x_0, x_1) + \ldots\ldots\ldots\ldots \\ &\leq (1 + k\alpha + (k\alpha)^2 + \ldots\ldots\ldots)k\alpha^n b(x_0, x_1) \\ &\leq k\alpha^n \left(\frac{1}{1 - k\alpha}\right) b(x_0, x_1). \end{aligned}$$

Since $k\alpha < 1$ therefore taking limit $m, n \to \infty$ we have

$$\lim_{m,n\to\infty} b(x_n, x_m) = 0.$$

Hence $\{x_n\}$ is a Cauchy sequence in b-metric space X. □

The next theorem is about to hold a unique fixed point for generalized contraction in complete b-metric space.

Theorem 3.1. *Let (X, b) be a complete b-metric space with coefficient $k \geq 1$ and T be a self mapping $T : X \to X$ satisfying the condition*

$$b(Tx, Ty) \leq \alpha.b(x, y) + \beta.b(x, Tx) + \gamma.b(y, Ty) + \mu.[b(x, Ty) + b(y, Tx)] \tag{4}$$

$\forall\ x, y \in X$, where $\alpha, \beta, \gamma, \mu \geq 0$, with

$$k\alpha + k\beta + \gamma + (k^2 + k)\mu < 1 \tag{5}$$

then T has a unique fixed point.

Proof. Let x_0 be arbitrary in X we define a sequence $\{x_n\}$ in X by the rule

$$x_0, x_1 = Tx_0, x_2 = Tx_1, ..., x_{n+1} = Tx_n,$$

consider

$$b(x_n, x_{n+1}) = b(Tx_{n-1}, Tx_n)$$

by using condition (4) we have

$$b(x_n, x_{n+1}) \leq \alpha.b(x_{n-1}, x_n) + \beta.b(x_{n-1}, x_n) + \gamma.b(x_n, x_{n+1}) + \mu.[b(x_{n-1}, x_{n+1}) + b(x_n, x_n)]$$

$$\leq \alpha.b(x_{n-1}, x_n) + \beta.b(x_{n-1}, x_n) + \gamma.b(x_n, x_{n+1}) + \mu.k[b(x_{n-1}, x_n) + b(x_n, x_{n+1}]$$

So

$$b(x_n, x_{n+1}) \leq \frac{(\alpha + \beta + k\mu)}{(1 - (\gamma + k\mu))} b(x_{n-1}, x_n)$$

$$\leq \lambda b(x_{n-1}, x_n)$$

where

$$\lambda = \frac{\alpha + \beta + k\mu}{1 - (\gamma + k\mu)}$$

From (5) it is clear that $\lambda < 1/k$.

Now from Lemma 1 we can say that $\{x_n\}$ is a Cauchy sequence in complete b-metric space, so there exist $u \in X$ such that $\lim_{n\to\infty} x_n = u$. Now we have to show that u is the fixed point of T for this consider,

$$b(Tu, Tx_n) \leq \alpha.b(u, x_n) + \beta.b(u, Tu) + \gamma.b(x_n, Tx_n) + \mu.[b(u, Tx_n) + b(x_n, Tu)]$$

$$\leq \alpha.b(u, x_n) + \beta.b(u, Tu) + \gamma.b(x_n, x_{n+1}) + \mu.[b(u, x_{n+1}) + b(x_n, Tu)].$$

Taking limit $n \to \infty$ we have

$$b(Tu, u) \leq \alpha.b(u, u) + \beta.b(u, Tu) + \gamma.b(u, u) + \mu.[b(u, u) + b(u, Tu)]$$

$$\leq (\beta + \mu).b(Tu, u)$$

the above inequality is possible only if $b(Tu, u) = 0 \Rightarrow Tu = u$.
Hence u is the fixed point of T.

Uniqueness. Let $u \neq v$ be two fixed points of T then

$$b(u, v) = b(Tu, Tv) \leq \alpha.b(u, v) + \beta.b(u, Tu) + \gamma.b(v, Tv) + \mu.[b(u, Tv) + b(v, Tu)]$$

$$= (\alpha + 2\mu).b(u, v)$$

since u, v are fixed points of T so finally we get using (5) the above inequality is possible only if $b(u, v) = 0 \Rightarrow u = v$.
Hence fixed point of T is unique in X. □

Theorem 3.1 yields the following corollaries.

Corollary 3.1. *Let (X, b) be a complete b-metric space with coefficient $k \geq 1$ and T be a self mapping $T : X \to X$ satisfying the condition*

$$b(Tx, Ty) \leq \alpha.b(x, y) + \beta.b(x, Tx) + \gamma.b(y, Ty)$$

$\forall\ x, y \in X$, where $\alpha, \beta, \gamma \geq 0$ with $k\alpha + k\beta + \gamma < 1$, then T has a unique fixed point.

Proof. Putting $\mu = 0$ in Theorem(3.1) we get the required result easily. □

Corollary 3.2. *Let (X, b) be a complete b-metric space with coefficient $k \geq 1$ and T be a self mapping $T : X \to X$ satisfying the condition*

$$b(Tx, Ty) \leq \alpha.b(x, y) + \beta.b(x, Tx)$$

$\forall\ x, y \in X$, where $\alpha, \beta \geq 0$ with $k\alpha + k\beta < 1$, then T has a unique fixed point.

Proof. By putting $\mu = \gamma = 0$ in Theorem(3.1) we get the required result. □

Corollary 3.3. *Let (X, b) be a complete b-metric space with coefficient $k \geq 1$ and T be a self mapping $T : X \to X$ satisfying the condition*

$$b(Tx, Ty) \leq \alpha.b(x, y)$$

$\forall\ x, y \in X$, where $\alpha \geq 0$ with $k\alpha < 1$ then T has a unique fixed point.

Proof. By putting $\beta = \gamma = \mu = 0$ in Theorem(3.1) we get the require result. □

Remark.

- Corollary (3.3) is the result of [10].

Now we present the modified form of Khan's Theorem[9] in the context of b-metric spaces.

Theorem 3.2. *Let (X, b) be a complete b-metric space with coefficient $k \geq 1$ and T be a self mapping $T : X \to X$ satisfying the condition*

$$b(Tx, Ty) \leq \beta.b(x, y) + \mu.\frac{b(x, Tx).b(x, Ty) + b(y, Ty).b(y, Tx)}{b(x, Ty) + b(y, Tx)} \tag{6}$$

$\forall\ \ x, y \in X$ and $\beta, \mu \geq 0$, $b(x, Ty) + b(y, Tx) \neq 0$ with $k(\beta + \mu) < 1$, then T has a unique fixed point.

Proof. Let x_0 be arbitrary in X, we define a sequence $\{x_n\}$ by the rule,

$$x_0,\ x_1 = Tx_0,\ x_2 = Tx_1, ..., \ x_{n+1} = Tx_n \quad \text{for all} \quad n \in N$$

Now to show that $\{x_n\}$ is a Cauchy sequence in X then consider,

$$b(x_n, x_{n+1}) = b(Tx_{n-1}, Tx_n)$$

From equation (6) we have

$$\begin{aligned}
& b(x_n, x_{n+1}) \\
\leq\ & \beta.b(x_{n-1}, x_n) + \mu.\frac{b(x_{n-1}, Tx_{n-1}).b(x_{n-1}, Tx_n) + b(x_n, Tx_n).b(x_n, Tx_{n-1})}{b(x_{n-1}, Tx_n) + b(x_n, Tx_{n-1})} \\
\leq\ & \beta.b(x_{n-1}, x_n) + \mu.\frac{b(x_{n-1}, x_n).b(x_{n-1}, x_{n+1}) + b(x_n, x_{n+1}).b(x_n, x_n)}{b(x_{n-1}, x_{n+1}) + b(x_n, x_n)} \\
\leq\ & (\beta + \mu).\ b(x_{n-1}, x_n).
\end{aligned}$$

Since $\beta + \mu < \frac{1}{k}$, therefore by Lemma 1 $\{x_n\}$ is a Cauchy sequence in complete b-metric space X, so there must exist $u \in X$ such that $\lim_{n\to\infty} x_n = u$.
Now to show that u is the fixed point of T for this consider,

$$\begin{aligned} b(Tx_n, Tu) &\leq \beta.b(x_n, u) + \mu.\frac{b(x_n, Tx_n).b(x_n, Tu) + b(u, Tu).b(u, Tx_n)}{b(x_n, Tu) + b(u, Tx_n)} \\ &\leq \beta.b(x_n, u) + \mu.\frac{b(x_n, x_{n+1}).b(x_n, Tu) + b(u, Tu).b(u, x_{n+1})}{b(x_n, Tu) + b(u, x_{n+1})} \end{aligned}$$

taking limit $n \to \infty$ we have,

$$\beta.b(x_n, u) + \mu.\frac{b(x_n, x_{n+1}).b(x_n, Tu) + b(u, Tu).b(u, x_{n+1})}{b(x_n, Tu) + b(u, x_{n+1})} \to 0.$$

Hence

$$b(u, Tu) = 0 \quad \Rightarrow \quad Tu = u$$

Thus u is the fixed point of T.
Uniqueness. Let u, v are two distinct fixed points of T, for $u \neq v$ consider,

$$b(u, v) = b(Tu, Tv) \leq \beta.b(u, v) + \mu.\frac{b(u, Tu).b(u, Tv) + b(v, Tv).b(v, Tu)}{b(u, Tv) + b(v, Tu)}$$

since u, v are fixed points of T we get

$$b(u, v) = b(Tu, Tv) \leq \beta.b(u, v) + \mu.\frac{b(u, u).b(u, v) + b(v, v).b(v, u)}{b(u, v) + b(v, u)} = \beta.b(u, v)$$

using the restriction in the theorem the above inequality is possible only if,

$$b(u, v) = 0 \quad \Rightarrow \quad u = v.$$

Therefore fixed point of T is unique. This complete the proof of the theorem.
Remark.

- In order to hold Khan's theorem in b-metric space we made no restriction.

□

Theorem 3.3. *Let (X, b) be a complete b-metric space with coefficient $k \geq 1$ and T be a self mapping $T : X \to X$ satisfying the condition*

$$b(Tx, Ty) \leq \alpha.\ b(x, y) + \beta.\ \frac{b(y, Ty)[1 + b(x, Tx)]}{1 + b(x, y)} + \gamma.\ \frac{b(y, Ty) + b(y, Tx)}{1 + b(y, Ty).b(y, Tx)} \tag{7}$$

$\forall\ x, y \in X$, where $\alpha, \beta, \gamma \geq 0$ and $k\alpha + \beta + \gamma < 1$
Then T has a unique fixed point.

Proof. Let x_0 be arbitrary in X, we define a sequence $\{x_n\}$ by the rule,

$$x_0, x_1 = Tx_0, x_2 = Tx_1, ..., x_{n+1} = Tx_n \quad \text{for all} \quad n \in N$$

Now to show that $\{x_n\}$ is a Cauchy sequence in X then consider,

$$b(x_n, x_{n+1}) = b(Tx_{n-1}, Tx_n)$$

From equation (7) we have

$$b(x_n, x_{n+1})$$

$$\leq \quad \alpha.\ b(x_{n-1}, x_n) + \beta.\ \frac{b(x_n, x_{n+1})[1 + b(x_{n-1}, x_n)]}{1 + b(x_{n-1}, x_n)} + \gamma.\ \frac{b(x_n, x_{n+1}) + b(x_n, x_n)}{1 + b(x_n, x_{n+1}).b(x_n, x_n)}$$

$$\leq \quad \alpha.\ b(x_{n-1}, x_n) + \beta.\ \frac{b(x_n, x_{n+1})[1 + b(x_{n-1}, x_n)]}{1 + b(x_{n-1}, x_n)} + \gamma.\ b(x_n, x_{n+1})$$

Therefore

$$\begin{aligned} b(x_n, x_{n+1}) &\leq \frac{\alpha}{1 - (\beta + \gamma)}.\ b(x_{n-1}, x_n) \\ &= h.\ b(x_{n-1}, x_n) \end{aligned}$$

where $h = \frac{\alpha}{1-(\beta+\gamma)}$ with $h < \frac{1}{k}$, because $k\alpha + \beta + \gamma < 1$
similarly we have

$$b(x_n, x_{n+1}) \leq h^2.\ b(x_{n-2}, x_{n-1})$$

continuing the same process we get

$$b(x_n, x_{n+1}) \leq h^n.\ b(x_0, x_1)$$

since $0 \leq h < 1$ so $h^n \to 0$ as $n \to \infty$, which shows that $\{x_n\}$ is a Cauchy sequence in complete b-metric space so there exist $z \in X$ such that $x_n \to z$ as $n \to \infty$
Now to show that z is the fixed point of T for this consider,
(8)

$$b(Tx_n, Tz) \leq \alpha.\ b(x_n, z) + \beta.\ \frac{b(z, Tz)[1 + b(x_n, Tx_n)]}{1 + b(x_n, z))} + \gamma.\ \frac{b(z, Tz) + b(z, Tx_n)}{1 + b(z, Tz).b(z, Tx_n)}$$

From the construction it is clear that $Tx_n = x_{n+1}$ and also $\{x_n\}$ is a Cauchy sequence converges to z. Therefore taking limit $n \to \infty$ equation (8) become

$$b(z, Tz) \leq (\beta + \gamma)b(z, Tz)$$

which is possible only if $b(z, Tz) = 0$, thus $Tz = z$.
Hence z is the fixed point of T.

Uniqueness. Suppose that T has two fixed points z and w for $z \neq w$ Consider,

$$b(z, w) = b(Tz, Tw) \leq \alpha.\ b(z, w) + \beta.\ \frac{b(w, Tw)[1 + b(z, Tz)]}{1 + b(z, w)} + \gamma.\ \frac{b(w, Tw) + b(w, Tz)}{1 + b(w.Tw).b(w, Tz)}$$

So the above inequality become

$$b(z, w) \leq (\alpha + \gamma).\ b(z, w)$$

the above inequality is possible only if $b(z, w) = 0 \quad \Rightarrow \quad z = w$. Hence fixed point of T is unique. □

Our next theorem is about b-metric variant of Ciric's type contraction.

Theorem 3.4. *Let (X, b) be a complete b-metric space with coefficient $k \geq 1$ and T be a self mapping $T : X \to X$ satisfying the condition*

$$b(Tx, Ty) \leq h. \max\left\{b(x, y), b(x, Tx), b(y, Ty), \frac{1}{2k}[b(x, Ty) + b(y, Tx)]\right\} \tag{9}$$

$\forall\ x, y \in X$, where $h \in [0, 1)$ and $kh < 1$
Then T has a unique fixed point.

Proof. Let x_0 be arbitrary in X we define a sequence $\{x_n\}$ in X by the rule

$$x_0,\ x_1 = Tx_0,\ x_2 = Tx_1, \ldots\ldots.., \ x_{n+1} = Tx_n.$$

Consider

$$b(x_n, x_{n+1}) = b(Tx_{n-1}, Tx_n)$$

using (11) we have

$$\begin{aligned}
&b(x_n, x_{n+1})\\
&\le h.\max\left\{b(x_{n-1}, x_n), b(x_{n-1}, Tx_{n-1}), b(x_n, Tx_n), \frac{1}{2k}[b(x_{n-1}, Tx_n)+b(x_n, Tx_{n-1})]\right\}\\
&= h.\max\left\{b(x_{n-1}, x_n), b(x_{n-1}, x_n), b(x_n, x_{n+1}), \frac{1}{2k}[b(x_{n-1}, x_{n+1}) + b(x_n, x_n)]\right\}\\
&\le h.\max\left\{b(x_{n-1}, x_n), b(x_{n-1}, x_n), b(x_n, x_{n+1}), \frac{1}{2}[b(x_{n-1}, x_n) + b(x_n, x_{n+1})]\right\}
\end{aligned}$$

$$= h.\max\left\{b(x_{n-1}, x_n), b(x_n, x_{n+1}), \frac{1}{2}[b(x_{n-1}, x_n) + b(x_n, x_{n+1})]\right\} \tag{10}$$

If

$$b(x_{n-1}, x_n) < b(x_n, x_{n+1}).$$

Then

$$b(x_{n-1}, x_n) < \frac{1}{2}[b(x_{n-1}, x_n) + b(x_n, x_{n+1})] < b(x_n, x_{n+1}).$$

So (10) implies that

$$b(x_n, x_{n+1} \le h.b(x_n, x_{n+1})$$

which is impossible since $h < 1$ for the same reason we also ignored the term $b(x_n, x_{n+1})$ thus (10) become

$$b(x_n, x_{n+1}) \le h.b(x_{n-1}, x_n).$$

By use of Lemma(1) we get that $\{x_n\}$ is a Cauchy sequence in complete b-metric space X so there exist $u \in X$ such that $\lim_{n\to\infty} x_n = u$, now we have to show that u is the fixed point of T for this consider

$$\begin{aligned}
b(Tu, Tx_n) &\le h.\max\left\{b(u, x_n), b(u, Tu), b(x_n, Tx_n), \frac{1}{2k}[b(u, Tx_n) + b(x_n, Tu)]\right\}\\
&\le h.\max\left\{b(u, x_n), b(u, Tu), b(x_n, x_{n+1}), \frac{1}{2k}[b(u, x_{n+1}) + b(x_n, Tu)]\right\}.
\end{aligned}$$

Taking limit $n \to \infty$ we have

$$\begin{aligned}
b(Tu, u) &\le h.\max\left\{b(u, Tu), \frac{1}{2k}b(u, Tu)\right\}\\
&\le h.b(Tu, u)
\end{aligned}$$

the above inequality is possible only if $b(Tu, u) = 0 \quad \Rightarrow \quad Tu = u$.
Hence u is the fixed point of T.

Uniqueness. Let $u \ne v$ be two fixed points of T then

$$b(u, v) = b(Tu, Tv) \le h.\max\left\{b(u, v), b(u, Tu), b(v, Tv), \frac{1}{2k}[b(u, Tv) + b(v, Tu)]\right\}$$

since u, v are the fixed points of T so finally we have

$$b(u, v) \le h.b(u, v)$$

the above inequality is possible only if $b(u,v)=0 \quad \Rightarrow \quad u=v$.
Therefore fixed point of T in X is unique. $\square$

The above theorem produce the following corollary for $k=1$.

Corollary 3.4. *Let (X,b) be a complete metric space with coefficient and T be a self mapping $T: X \to X$ satisfying the condition*

$$b(Tx,Ty) \leq h.\max\left\{b(x,y), b(x,Tx), b(y,Ty), \frac{1}{2}[b(x,Ty)+b(y,Tx)]\right\} \tag{11}$$

for all $x,y \in X$, where $h \in [0,1)$, then T has a unique fixed point.

Remark. Corollary 3.4 is the result of B. E. Rohades [11].

Example 3.1. Let $X=[0,1]$ and $b(x,y)=|x-y|^2$ be a b-metric with coefficient $k=2 \ \forall \ x,y \in X$, we define the mapping T by

$$Tx = \frac{2}{3} \text{ if } x \in [0,1) \text{ and } T1 = 0$$

then T satisfy all the conditions of the Theorem(3.4) for $h \in [\frac{4}{9}, \frac{1}{2})$ having $x=\frac{2}{3}$ is its unique fixed point in X.

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