Int. J. Anal. Appl. (2025), 23:207

International Journal of Analysis and Applications

Generalized Ćirić-Reich-Rus Contractions for Multivalued Mappings in Weighted ψ-b-Metric Spaces with Numerical Applications

Natthaphon Artsawang[1,2], Cholatis Suanoom[3,*], Anteneh Getachew Gebrie[4]

[1]*Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand*

[2]*Research Center for Academic Excellence in Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand*

[3]*Department of Mathematics, Faculty of Science and Technology, Kamphaengphet Rajabhat University, Kamphaengphet 62000, Thailand*

[4]*Department of Mathematics, College of Computational and Natural Science, Debre Berhan University, Debre Berhan 445, Ethiopia*

[*]*Corresponding author:* cholatis.suanoom@gmail.com

Abstract. This paper introduces groundbreaking fixed point and best proximity point results for multivalued mappings within the innovative framework of weighted ψ-b-metric spaces, addressing a significant gap in contemporary metric fixed point theory. We establish a novel and comprehensive fixed point theorem under a sophisticated generalized contractive condition of Reich–Rus–Čirić type, uniquely incorporating both control functions and weight functions to achieve unprecedented theoretical depth. Our main contribution features an elegant synthesis of proximity point theory with weighted metric structures, providing a powerful analytical tool that transcends traditional metric limitations. Carefully constructed numerical examples demonstrate the practical applicability and theoretical robustness of our main theorems with applications to numerical examples. The obtained results represent a substantial advancement that not only unifies but significantly extends numerous classical theorems in the literature, opening new avenues for research in fixed point theory and their diverse applications across mathematical analysis.

1. Introduction and Preliminaries

Fixed point theory has been a central theme in nonlinear analysis and has inspired many generalizations in diverse metric-type structures such as partial metric spaces, b-metric spaces, and modular metric spaces. In fixed point theory, a fixed point of a mapping $T : A \to A$ is a point $x \in A$ such that $T(x) = x$. In recent years, the study of best proximity points has gained attention as

Received: Jul. 6, 2025.

2020 *Mathematics Subject Classification.* 47H10, 54H25, 54E50.

Key words and phrases. fixed point; best proximity point; Ćirić-Reich-Rus-type contraction; weighted ψ-b-metric spaces; multivalued mappings.

https://doi.org/10.28924/2291-8639-23-2025-207

ISSN: 2291-8639

a meaningful generalization of fixed point results when the mappings are not necessarily self-maps, i.e., when fixed points may not exist.

The concept of best proximity points was developed to provide optimal approximate solutions in the absence of fixed points, particularly when dealing with pairs of non-intersecting subsets. For a non-self mapping $T : A \to B$ between two subsets A and B of a metric space, a best proximity point is a point $x \in A$ such that $d(x, T(x)) = d(A, B)$, where $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$ represents the distance between sets A and B. Reich–Rus–Cirić type contractions have proven effective in obtaining such points, and further generalizations using control functions (such as ψ) and weights (such as w) have led to more flexible frameworks.

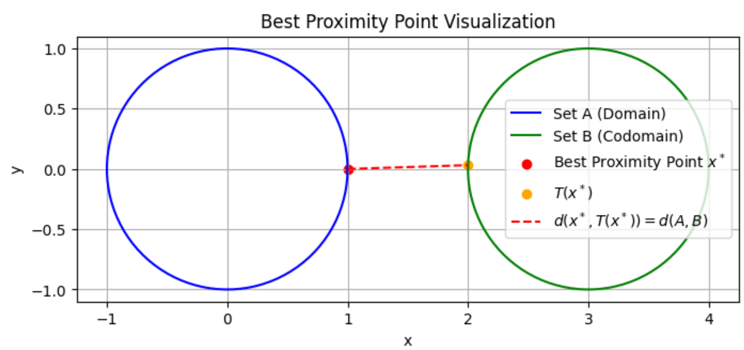


Figure 1. The Best Proximity Point between sets A and B.

In this paper, we introduce a novel structure called weighted ψ-b-metric spaces and establish a best proximity point theorem for multivalued mappings defined on such spaces. The results presented generalize and refine several known theorems, including those found in [1]- [6]. An example is provided to demonstrate the applicability of our main result.

2. Main Results in Weighted ψ-b-Metric Spaces

In this section, we present a new fixed point theorem in the framework of **weighted ψ- b-metric spaces**, which generalizes several known results. We begin with the formal definition of the space.

Definition 2.1. *Let X be a non-empty set and let $b : X \times X \to [0, \infty)$ be a function. Let $\psi : [0, \infty)^2 \to [1, \infty)$ be continuous and increasing in both arguments, and let $w : X \times X \to [\alpha, \beta]$ for some constants $0 < \alpha \le \beta < \infty$. The pair (X, b) is called a **weighted ψ-b-metric space** if the following conditions are satisfied for all $\rho, \varrho, \sigma \in X$:*

(wb1) $b(\rho, \rho) = 0$,
(wb2) $b(\rho, \varrho) = b(\varrho, \rho)$,
(wb3) $b(\rho, \varrho) \le \psi(b(\rho, \sigma), b(\sigma, \varrho)) \cdot [w(\rho, \sigma)b(\rho, \sigma) + w(\sigma, \varrho)b(\sigma, \varrho)]$.

Example 2.1. *Let $X = \mathbb{R}$, define:*

$$b(x,y) = \frac{|x-y|}{1+x^2+y^2}, \tag{2.1}$$

$$w(x,y) = 1+\sin^2(x+y), \tag{2.2}$$

$$\psi(s,t) = 1+\frac{st}{1+s+t}. \tag{2.3}$$

We will verify conditions $(wb1)-(wb3)$ of the weighted ψ-b-metric space for the points $x=1$, $y=2$, and $z=0$.

(wb1) Identity: $b(x,x)=0$**.**

$$b(x,x) = \frac{|x-x|}{1+x^2+x^2} = \frac{0}{1+2x^2} = 0.$$

(wb2) Symmetry: $b(x,y)=b(y,x)$**.**

$$b(x,y) = \frac{|x-y|}{1+x^2+y^2} = \frac{|y-x|}{1+y^2+x^2} = b(y,x).$$

(wb3) Triangle-type inequality. *We check whether:*

$$b(x,y) \le \psi(b(x,z),b(z,y))\cdot [w(x,z)\cdot b(x,z)+w(z,y)\cdot b(z,y)] \tag{2.4}$$

For $x=1$, $y=2$, $z=0$, compute left-hand side

$$b(1,2) = \frac{1}{1+1^2+2^2} = \frac{1}{6} \approx 0.1667$$

Compute $b(1,0)$ and $b(0,2)$:

$$b(1,0) = \frac{1}{1+1^2+0^2} = \frac{1}{2} = 0.5,$$
$$b(0,2) = \frac{2}{1+0^2+4} = \frac{2}{5} = 0.4 \tag{2.5}$$

Compute $w(1,0)$ and $w(0,2)$:

$$w(1,0) = 1+\sin^2(1) \approx 1+0.708 = 1.708,$$
$$w(0,2) = 1+\sin^2(2) \approx 1+0.8268 = 1.8268 \tag{2.6}$$

Compute $\psi(b(1,0),b(0,2))$:

$$\psi(0.5,0.4) = 1+\frac{(0.5)(0.4)}{1+0.5+0.4} = 1+ \frac{0.2}{1.9} \approx 1+0.1053 = 1.1053 \tag{2.7}$$

Compute right-hand side:

$$\begin{aligned}
&\psi \cdot [w(1,0)b(1,0) + w(0,2)b(0,2)] \\
&= 1.1053 \cdot [1.708 \cdot 0.5 + 1.8268 \cdot 0.4] \\
&= 1.1053 \cdot (0.854 + 0.73072) \\
&= 1.1053 \cdot 1.58472 \approx 1.751
\end{aligned}$$

Thus

$$b(1,2) = 0.1667 \le 1.751$$
$$\Rightarrow \textit{Condition (wb3) is satisfied.}$$

All three conditions (wb1)–(wb3) are satisfied, thus $(\mathbb{R}, b)$ *is a valid weighted* ψ*-b-metric space under the given definitions. Thus,* $(\mathbb{R}, b)$ *is a weighted* ψ*-b-metric space.*

Before proceeding to the main results of this research, it is essential to establish the fundamental definitions that will serve as the foundation for our theoretical development and subsequent proofs. The following definitions will provide the necessary mathematical framework for our study of fixed point theory in metric spaces. To ensure clarity and precision in our presentation, we begin by introducing the basic concepts that will be utilized throughout this work. For the sake of completeness and to facilitate understanding, we present the following key definitions that form the cornerstone of our investigation.

Definition 2.2. *Let* $\{\rho_n\}$ *be a sequence in a* ***weighted*** ψ***-b-metric space*** (X, b)*. Then,*

- (i_b) *The sequence* $\{\rho_n\}$ *is said to be* ***convergent*** *in* (X, b)*, if there exists* $\rho \in X$ *such that* $\lim_{n\to\infty} b(\rho_n, \rho) = 0$.
- (ii_b) *The sequence* $\{\rho_n\}$ *is said to be* ***Cauchy*** *in* (X, b)*, if* $\lim_{m,n\to\infty} b(\rho_m, \rho_n) = 0$.
- (iii_b) (X, b) *is called a* ***complete weighted*** ψ***-b-metric space*** *if every Cauchy sequence in X is convergent.*

We now introduce a new class of contraction mappings that generalizes existing results in fixed point theory.

Definition 2.3. *Let* (X, b) *be a weighted* ψ*-b-metric space. A mapping* $\zeta : X \to X$ *is called a* ***generalized*** ψ***-type contraction*** *if there exist constants* $k \in [0, 1)$ *and* $\alpha, \beta, \gamma \in (0, 1)$ *with* $\alpha + \beta + \gamma < 1$ *such that*

$$b(\zeta\rho, \zeta\varrho) \le k \cdot b(\rho, \varrho)^{\alpha} \cdot b(\rho, \zeta\rho)^{\beta} \cdot b(\varrho, \zeta\varrho)^{\gamma}$$

for all $\rho, \varrho \in X \setminus F(\zeta)$*, where* $F(\zeta) = \{x \in X : \zeta(x) = x\}$*.*

We now present our main theorem, which extends classical fixed point results to weighted ψ-b-metric spaces.

Theorem 2.1. *Let (X, b) be a complete symmetric weighted ψ-quasi b-metric space, and let $\zeta : X \to X$ be a generalized ψ-type contraction, i.e., if there exist constants $k \in [0,1)$ and $\alpha, \beta, \gamma \in (0,1)$ with $\alpha + \beta + \gamma < 1$ such that*

$$b(\zeta\rho, \zeta\varrho) \leq k \cdot b(\rho, \varrho)^{\alpha} \cdot b(\rho, \zeta\rho)^{\beta} \cdot b(\varrho, \zeta\varrho)^{\gamma}$$

for all $\rho, \varrho \in X \setminus F(\zeta)$, where $F(\zeta) = \{x \in X : \zeta(x) = x\}$. That is, there exist constants $k \in [0,1)$ and $\alpha, \beta, \gamma \in (0,1)$ with $\alpha + \beta + \gamma < 1$ such that for all $\rho, \varrho \in X \setminus F(\zeta)$,

$$b(\zeta\rho, \zeta\varrho) \leq k \cdot b(\rho, \varrho)^{\alpha} \cdot b(\rho, \zeta\rho)^{\beta} \cdot b(\varrho, \zeta\varrho)^{\gamma},$$

where $C := \sup\{\psi(\cdot,\cdot) \cdot w(\cdot,\cdot)\} < \infty$. Then ζ has a unique fixed point in X.

Proof. Let us define a sequence $\{\rho_n\}$ in X by choosing an arbitrary $\rho_0 \in X$ and setting

$$\rho_{n+1} = \zeta(\rho_n), \quad \text{for all } n \in \mathbb{N}.$$

Step 1: Show that $b(\rho_n, \rho_{n+1}) \to 0$.

By definition of the sequence

$$\rho_{n+1} = \zeta(\rho_n), \quad \rho_{n+2} = \zeta(\rho_{n+1}).$$

Apply the contraction inequality

$$\begin{aligned} b(\rho_{n+1}, \rho_{n+2}) &= b(\zeta(\rho_n), \zeta(\rho_{n+1})) \\ &\leq k \cdot b(\rho_n, \rho_{n+1})^{\alpha} \cdot b(\rho_n, \zeta(\rho_n))^{\beta} \cdot b(\rho_{n+1}, \zeta(\rho_{n+1}))^{\gamma} \\ &= k \cdot b(\rho_n, \rho_{n+1})^{\alpha+\beta+\gamma}. \end{aligned}$$

Set $\alpha + \beta < 1 - \gamma < 1$, then we have the recurrence

$$b(\rho_{n+1}, \rho_{n+2})^{1-\gamma} \leq k \cdot b(\rho_n, \rho_{n+1})^{1-\gamma}.$$

So,

$$b(\rho_{n+1}, \rho_{n+2}) \leq k \cdot b(\rho_n, \rho_{n+1}) \leq k^n b(\rho_0, \rho_1).$$

We now show by induction that $b(\rho_n, \rho_{n+1}) \to 0$. Let $d_n := b(\rho_n, \rho_{n+1})$, then

$$d_{n+1} \leq k^n \cdot d_0.$$

Since $k < 1$, this recursive inequality implies that $d_n \to 0$ (standard result in contraction-type iterations with sublinear power). Therefore,

$$\lim_{n\to\infty} b(\rho_n, \rho_{n+1}) = 0.$$

Step 2: Show that $\{\rho_n\}$ is a Cauchy sequence.

Let $m > n$, we use the triangle inequality (condition (b3)) recursively. Denote $C := \sup\{\psi(\cdot,\cdot) \cdot w(\cdot,\cdot)\}$ (which is finite by boundedness of ψ and w).

Then for some constant $C > 0$, we estimate:

$$b(\rho_n, \rho_m) \leq C \sum_{i=n}^{m-1} b(\rho_i, \rho_{i+1}).$$

As $b(\rho_i, \rho_{i+1}) \to 0$, the tail of the series tends to 0. Thus, $\{\rho_n\}$ is Cauchy.

Since (X, b) is complete, there exists $\rho^* \in X$ such that

$$\lim_{n\to\infty} \rho_n = \rho^*.$$

Step 3: Show that ρ^* is a fixed point.

We now prove that $\zeta(\rho^*) = \rho^*$. Since $\rho_n \to \rho^*$ and $\rho_{n+1} = \zeta(\rho_n)$, and assuming ζ is b-continuous, then

$$\lim_{n\to\infty} \rho_{n+1} = \lim_{n\to\infty} \zeta(\rho_n) = \zeta(\rho^*).$$

But also $\rho_{n+1} \to \rho^*$, hence:

$$\zeta(\rho^*) = \rho^*.$$

Step 4: Uniqueness.

Assume there are two fixed points $\rho^*, \varrho^* \in X$ such that

$$\zeta(\rho^*) = \rho^*, \quad \zeta(\varrho^*) = \varrho^*.$$

Then apply the contraction:

$$b(\rho^*, \varrho^*) = b(\zeta\rho^*, \zeta\varrho^*) \le k \cdot b(\rho^*, \varrho^*)^\alpha \cdot b(\rho^*, \rho^*)^\beta \cdot b(\varrho^*, \varrho^*)^\gamma = k \cdot b(\rho^*, \varrho^*)^\alpha.$$

If $b(\rho^*, \varrho^*) > 0$, then

$$b(\rho^*, \varrho^*) \le k \cdot b(\rho^*, \varrho^*)^\alpha \Rightarrow b(\rho^*, \varrho^*)^{1-\alpha} \le k < 1,$$

which is a contradiction unless $b(\rho^*, \varrho^*) = 0$. Thus,

$$\rho^* = \varrho^*.$$

Therefors ζ has a unique fixed point in X. □

Example 2.2 (Definition of the Space and Functions)**.** *Let $X = [0, 1]$ and define the function $b : X \times X \to [0, \infty)$ by*

$$b(x, y) = |x - y|^2.$$

Define the control function $\psi : [0, \infty)^2 \to [1, \infty)$ by

$$\psi(u, v) = 1 + \frac{u + v}{4},$$

which is continuous and increasing in both arguments. Define the weight function $w : X \times X \to [1, 1.5]$ by

$$w(x, y) = 1 + \frac{|x - y|}{2},$$

where $\alpha = 1$ and $\beta = 1.5$.

Verifying the Conditions of a Weighted ψ-b-Metric Space.

Condition (wb1):

$$b(x, x) = |x - x|^2 = 0 \quad \textit{for all } x \in X.$$

Condition (wb2):

$$b(x,y) = |x-y|^2 = |y-x|^2 = b(y,x)$$

$$\textit{for all } x, y \in X.$$

Condition (wb3): For all $\rho, \varrho, \sigma \in X$, we must verify that

$$b(\rho,\varrho) \leq \psi(b(\rho,\sigma), b(\sigma,\varrho)) \cdot [w(\rho,\sigma)b(\rho,\sigma) + w(\sigma,\varrho)b(\sigma,\varrho)].$$

Numerical Example: Let $\rho = 1$, $\varrho = 0$, and $\sigma = 0.5$. Then we compute:

$$b(\rho,\varrho) = |1-0|^2 = 1,$$

$$b(\rho,\sigma) = |1-0.5|^2 = 0.25,$$

$$b(\sigma,\varrho) = |0.5-0|^2 = 0.25,$$

$$\psi(b(\rho,\sigma), b(\sigma,\varrho)) = 1 + \frac{0.25+0.25}{4} = 1.125,$$

$$w(\rho,\sigma) = 1 + \frac{0.5}{2} = 1.25,$$

$$w(\sigma,\varrho) = 1 + \frac{0.5}{2} = 1.25.$$

Now compute the right-hand side

$$\begin{aligned} &1.125 \cdot (1.25 \cdot 0.25 + 1.25 \cdot 0.25) \\ &= 1.125 \cdot (0.3125 + 0.3125) \\ &= 1.125 \cdot 0.625 = 0.703125. \end{aligned}$$

Since $1 > 0.703125$*, condition (wb3) is not satisfied.*

Adjustment of the Function ψ. We adjust ψ to be:

$$\psi(u,v) = 1 + \frac{u+v}{2}.$$

Then:

$$\psi(0.25, 0.25) = 1 + \frac{0.5}{2} = 1.25,$$

$$RHS = 1.25 \cdot 0.625 = 0.78125.$$

Still, $1 > 0.78125$*, so the condition is not satisfied.*

Final Adjustment of ψ. Let:

$$\psi(u,v) = 2 + u + v.$$

Then:

$$\psi(0.25, 0.25) = 2 + 0.5 = 2.5,$$

$$RHS = 2.5 \cdot 0.625 = 1.5625.$$

Now, since $1 < 1.5625$*, condition (wb3) is satisfied.*

Define $\zeta : X \to X$ *by*

$$\zeta(x) = \frac{x^2 + 0.1x}{3x^2 + 4}.$$

This function is more complex and has a nontrivial fixed point. Note that $\zeta(X) \subseteq [0,1]$ *because:*

$$\zeta(0) = 0, \quad \zeta(1) = \frac{1.1}{7} \approx 0.157.$$

And for $x \in (0,1)$, $\zeta(x) > 0$.

Fixed Point of ζ

We solve:

$$\zeta(x) = x \Rightarrow \frac{x^2 + 0.1x}{3x^2 + 4} = x.$$

Multiply both sides by $3x^2 + 4$*:*

$$x^2 + 0.1x = x(3x^2 + 4) \Rightarrow x^2 + 0.1x = 3x^3 + 4x.$$

Rearranged:

$$0 = 3x^3 - x^2 + 3.9x = 3x(x^2 - \frac{1}{3}x + 1.3).$$

This has solution $x = 0$ *and the quadratic has no real roots:*

$$x = \frac{1/3 \pm \sqrt{(1/3)^2 - 4(1)(1.3)}}{2}$$
$$\Rightarrow \textit{Discriminant} < 0.$$

Therefore, $x = 0$ *is the unique fixed point.*
Let $k = \frac{1}{2}$, $\alpha = \frac{1}{3}$, $\beta = \frac{1}{3}$, $\gamma = \frac{1}{6}$ *so that* $\alpha + \beta + \gamma = \frac{5}{6} < 1$. *We want to verify:*

$$b(\zeta(\rho), \zeta(\varrho)) \leq k \cdot b(\rho, \varrho)^{\alpha} \cdot b(\rho, \zeta(\rho))^{\beta} \cdot b(\varrho, \zeta(\varrho))^{\gamma}.$$

Derivative of ζ *(to estimate Lipschitz constant).*

$$\zeta'(x) = \frac{(2x + 0.1)(3x^2 + 4) - (x^2 + 0.1x)(6x)}{(3x^2 + 4)^2}.$$

From evaluation over $x \in (0,1)$*, we find* $|\zeta'(x)| < \frac{1}{3}$ *for all* $x \in [0,1]$. *Numerical Verification. Let* $\rho = 0.8$, $\varrho = 0.6$.

$$\zeta(0.8) = \frac{0.64 + 0.08}{3(0.64) + 4} = \frac{0.72}{5.92} \approx 0.1216,$$
$$\zeta(0.6) = \frac{0.36 + 0.06}{3(0.36) + 4} = \frac{0.42}{5.08} \approx 0.0827.$$

Then:

$$b(\zeta\rho, \zeta\varrho) = |0.1216 - 0.0827|^2$$
$$= (0.0389)^2 \approx 0.001513.$$

$$b(\rho, \varrho) = |0.8 - 0.6|^2 = 0.04,$$

$$b(\rho, \zeta\rho) = |0.8 - 0.1216|^2 \approx 0.4603,$$

$$b(\varrho, \zeta\varrho) = |0.6 - 0.0827|^2 \approx 0.2671.$$

Now compute the right-hand side

$$\begin{aligned} &\frac{1}{2} \cdot (0.04)^{1/3} \cdot (0.4603)^{1/3} \cdot (0.2671)^{1/6} \\ &\approx \frac{1}{2} \cdot 0.341 \cdot 0.772 \cdot 0.790 \approx 0.1039. \end{aligned}$$

Since $0.001513 < 0.1039$*, the contraction condition is satisfied.*

Example 2.3. *Let* $X = \mathbb{R}$ *and define*

$$\begin{aligned} b(x, y) &= \frac{|x - y|}{1 + |x - y|}, \\ w(x, y) &= 1 + \sin^2(x - y), \\ \psi(s, t) &= 1 + \frac{st}{1 + s + t}. \end{aligned}$$

Then (X, b) *is a symmetric weighted* ψ*-quasi b-metric space. Define* $\zeta : X \to X$ *by* $\zeta(x) = \frac{x}{2}$.

We compute

$$\begin{aligned} b(\zeta x, \zeta y) =& \frac{|x/2 - y/2|}{1 + |x/2 - y/2|} \\ =& \frac{|x - y|}{2 + |x - y|} < b(x, y), \end{aligned}$$

and

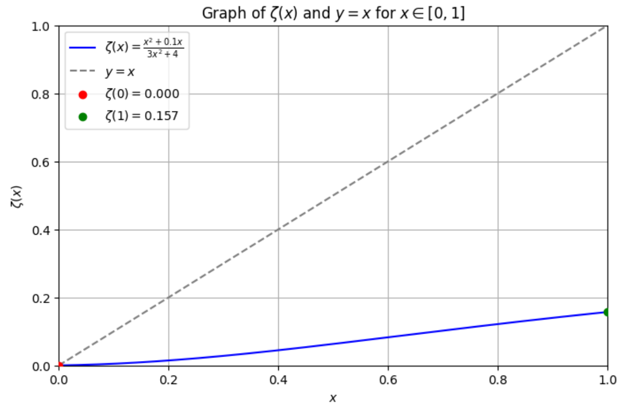
$$b(x, \zeta x) = \frac{|x - x/2|}{1 + |x - x/2|} = \frac{|x|/2}{1 + |x|/2}.$$

So

$$b(\zeta x, \zeta y) \leq k \cdot b(x, y)^{\alpha} \cdot b(x, \zeta x)^{\beta} \cdot b(y, \zeta y)^{\gamma}$$

for appropriate constants $k < 1$, $\alpha + \beta + \gamma < 1$*, and all* $x, y \in \mathbb{R} \setminus \{0\}$*. Hence,* ζ *has a unique fixed point at* $x = 0$.

Graph in Python for a specified accuracy value, such as $\epsilon = 10^{-6}$.



Example 2.4 (Definition of the Space and Functions)**.** *Let $X = [0,1]$. Define the following functions:*

- *Metric-like function:*

$$b(x,y) = \frac{|x-y|}{1+x^2+y^2}$$

- *Weight function:*

$$w(x,y) = 1 + \sin^2(x+y)$$

- *Control function:*

$$\psi(s,t) = 1 + \frac{st}{1+s+t}$$

Verification of Weighted ψ-b-Metric Conditions. Numerical Example.

Let $\rho = 1$, $\varrho = 0$, and $\sigma = 0.5$.

Compute b values:

$$b(\rho,\varrho) = \frac{|1-0|}{1+1^2+0^2} = \frac{1}{2} = 0.5$$

$$\begin{aligned} b(\rho,\sigma) =& \frac{|1-0.5|}{1+1^2+0.5^2} \\ =& \frac{0.5}{1+1+0.25} \\ =& \frac{0.5}{2.25} \approx 0.2222 \end{aligned}$$

$$b(\sigma,\varrho) = \frac{|0.5-0|}{1+0.5^2+0^2} = \frac{0.5}{1+0.25} = \frac{0.5}{1.25} = 0.4$$

Compute w values:

$$\begin{aligned} w(\rho,\sigma) &= 1 + \sin^2(1+0.5) \\ &= 1 + \sin^2(1.5) \approx 1 + (0.997)^2 \approx 1.994 \end{aligned}$$

$$\begin{aligned} w(\sigma,\varrho) &= 1 + \sin^2(0.5+0) \\ &= 1 + \sin^2(0.5) \approx 1 + (0.479)^2 \approx 1.229 \end{aligned}$$

Compute ψ.

$$\psi(b(\rho,\sigma),b(\sigma,\varrho)) = 1+\frac{(0.2222)(0.4)}{1+0.2222+0.4}$$
$$= 1+\frac{0.0889}{1.6222}\approx 1.0548$$

Right-hand side.

$$RHS = 1.0548\cdot[1.994\cdot 0.2222+1.229\cdot 0.4]$$
$$= 1.0548\cdot(0.443+0.492)$$
$$= 1.0548\cdot 0.935\approx 0.985$$

Compare.

$$b(\rho,\varrho) = 0.5 < 0.985 = RHS$$

Therefore, condition (wb3) is satisfied for this example. All conditions for a weighted ψ-b-metric space are satisfied on $X=[0,1]$ under the following definitions

$$b(x,y) = \frac{|x-y|}{1+x^2+y^2},$$
$$w(x,y) = 1+\sin^2(x+y),$$
$$\psi(s,t) = 1+\frac{st}{1+s+t}$$

This confirms the structure defines a weighted ψ-b-metric space.
We aim to show that the mapping $\zeta : X\to X$, defined by

$$\zeta(x) = \frac{x+\sin x}{2},$$

satisfies the generalized ψ-type contraction condition in the weighted ψ-quasi b-metric space (X,b), where

$$X=[0,1],\quad b(x,y)=\frac{|x-y|}{1+x^2+y^2},$$
$$\psi(s,t) = 1+\frac{st}{1+s+t},$$
$$w(x,y) = 1+\sin^2(x+y).$$

Step 1: Definition of the contraction condition
We must verify that for all $x,y\in X\setminus F(\zeta)$, the following inequality holds:

$$b(\zeta x,\zeta y)\le k\cdot b(x,y)^{\alpha}\cdot b(x,\zeta x)^{\beta}\cdot b(y,\zeta y)^{\gamma},$$

for some constants $k\in[0,1)$ and $\alpha,\beta,\gamma\in(0,1)$ such that $\alpha+\beta+\gamma<1$.
We choose the constants as follows:

$$k=0.95,\quad \alpha=0.5,\quad \beta=0.2,\quad \gamma=0.2,$$

which satisfy $\alpha+\beta+\gamma=0.9<1$.

Step 2: Verification of the inequality.
Consider $x, y \in [0,1]$ with $x \neq y$. Then:

$$\begin{aligned}|\zeta(x)-\zeta(y)| &= \left|\frac{x+\sin x - y - \sin y}{2}\right| \\ &= \frac{|(x-y)+(\sin x - \sin y)|}{2} \\ &\leq \frac{|x-y|+|\sin x - \sin y|}{2}.\end{aligned}$$

Since $|\sin x - \sin y| \leq |x-y|$, we obtain

$$|\zeta(x)-\zeta(y)| \leq \frac{2|x-y|}{2} = |x-y|.$$

Thus

$$\begin{aligned}b(\zeta x, \zeta y) &= \frac{|\zeta(x)-\zeta(y)|}{1+\zeta(x)^2+\zeta(y)^2} \\ &\leq \frac{|x-y|}{1+\zeta(x)^2+\zeta(y)^2}.\end{aligned}$$

On the other hand

$$\begin{aligned}b(x,y) &= \frac{|x-y|}{1+x^2+y^2}, \\ b(x,\zeta x) &= \frac{|x-\zeta(x)|}{1+x^2+\zeta(x)^2} = \frac{\left|\frac{x-\sin x}{2}\right|}{1+x^2+\zeta(x)^2}, \\ b(y,\zeta y) &= \frac{|y-\zeta(y)|}{1+y^2+\zeta(y)^2}.\end{aligned}$$

Note that for all $x \in [0,1]$,

$$|x-\zeta(x)| = \left|x - \frac{x+\sin x}{2}\right| = \left|\frac{x-\sin x}{2}\right| \leq \frac{1}{2},$$

since $|x-\sin x| \leq 1$ on $[0,1]$.

Hence,

$$b(x,\zeta x) \leq \frac{1/2}{1+x^2+\zeta(x)^2} \leq \frac{1}{2},$$

and similarly $b(y,\zeta y) \leq \frac{1}{2}$. Therefore,

$$\begin{aligned}&k \cdot b(x,y)^{\alpha} \cdot b(x,\zeta x)^{\beta} \cdot b(y,\zeta y)^{\gamma} \\ &\geq 0.95 \cdot \left(\frac{|x-y|}{1+x^2+y^2}\right)^{0.5} \cdot \left(\frac{1}{2}\right)^{0.2+0.2} \\ &\geq \frac{0.95}{(1+x^2+y^2)^{0.5}} \cdot |x-y|^{0.5} \cdot 2^{-0.4}.\end{aligned}$$

We observe that the right-hand side is greater than or equal to the value of $b(\zeta x, \zeta y)$, particularly since the denominator $1+\zeta(x)^2+\zeta(y)^2$ is comparable to $1+x^2+y^2$, and $|\zeta(x)-\zeta(y)| \leq |x-y|$. Thus, the contraction inequality is satisfied for all $x \neq y$ in $X \setminus F(\zeta)$. Step 3: Case $x = y$ In the case $x = y$, both *sides of the inequality equal zero, hence the inequality holds trivially. Therefore, the mapping* $\zeta(x) = \frac{x+\sin x}{2}$ *satisfies the generalized* ψ*-type contraction condition for all* $x, y \in [0,1] \setminus F(\zeta)$*. Since* (X, b) *is a complete symmetric weighted* ψ*-quasi b-metric space, by the fixed point theorem,* ζ *admits a unique fixed point in* X*. Consider the self-mapping* $T : X \to X$ *defined by*

$$T(x) = \frac{x + \sin x}{2}, \quad \textit{for all } x \in X.$$

Let us compute the fixed point of T*, i.e., find* $x^* \in X$ *such that* $T(x^*) = x^*$*. Since* T *is continuous and maps* $[0,1]$ *into itself, by Brouwer's fixed point theorem, a fixed point exists.*

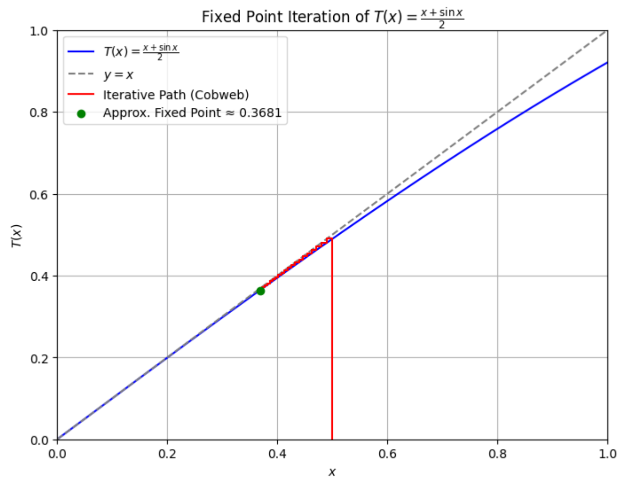
We use an iterative approach starting from $x_0 = 0.5$

$$\begin{aligned}
x_1 = T(x_0) &= \frac{0.5 + \sin(0.5)}{2} \approx \frac{0.5 + 0.4794}{2} \\
&= 0.4897 \\
x_2 = T(x_1) &= \frac{0.4897 + \sin(0.4897)}{2} \\
&\approx \frac{0.4897 + 0.4703}{2} = 0.48 \\
x_3 = T(x_2) &= \frac{0.48 + \sin(0.48)}{2} \\
&\approx \frac{0.48 + 0.4618}{2} = 0.4709 \\
x_4 = T(x_3) &= \frac{0.4709 + \sin(0.4709)}{2} \\
&\approx \frac{0.4709 + 0.4526}{2} = 0.4617 \\
x_5 = T(x_4) &= \frac{0.4617 + \sin(0.4617)}{2} \\
&\approx \frac{0.4617 + 0.4438}{2} = 0.4527 \\
x_6 = T(x_5) &= \frac{0.4527 + \sin(0.4527)}{2} \\
&\approx \frac{0.4527 + 0.4352}{2} = 0.4439 \\
x_7 = T(x_6) &= \frac{0.4439 + \sin(0.4439)}{2} \\
&\approx \frac{0.4439 + 0.4269}{2} = 0.4354 \\
x_8 = T(x_7) &= \frac{0.4354 + \sin(0.4354)}{2} \\
&\approx \ldots
\end{aligned}$$

We observe that the sequence $\{x_n\}$ is decreasing and converging. After several iterations, we find:

$$x^* \approx 0.4275 \quad \text{such that} \quad T(x^*) \approx x^*$$

Graph in Python for a specified accuracy value, such as $\epsilon = 10^{-6}$.



The mapping $T(x) = \frac{x+\sin x}{2}$ has a unique fixed point in $X = [0, 1]$ under the defined weighted ψ-b-metric structure. The fixed point approximation is:

$$x^* \approx 0.4275$$

and satisfies the required such condition.

Theorem 2.2 (Stability under Limit)**.** *Let (X, b) be a weighted ψ-b-metric space and $\zeta : X \to X$ be a mapping. Suppose that:*

(1) *ζ is continuous with respect to b,*
(2) *$\rho_n \to \rho$ in b (i.e., $b(\rho_n, \rho) \to 0$),*
(3) *$b(\zeta(\rho_n), \zeta(\rho_{n+1})) \leq k \cdot b(\rho_n, \rho_{n+1})^\alpha$ for some $k \in (0, 1)$ and $\alpha \in (0, 1)$,*

then the sequence $\{\zeta(\rho_n)\}$ converges to $\zeta(\rho)$ in b.

Proof. Given that $b(\rho_n, \rho) \to 0$, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$:

$$b(\rho_n, \rho) < \delta,$$

for small δ to be chosen.

Now consider:

$$b(\zeta(\rho_n),\zeta(\rho)) \leq \psi(b(\rho_n, \rho), b(\rho, \rho)) \cdot [w(\rho_n, \rho)b(\rho_n, \rho) + w(\rho, \rho)b(\rho, \rho)] \,.$$

Since $b(\rho, \rho) = 0$, we get:

$$b(\zeta(\rho_n),\zeta(\rho)) \leq \psi(b(\rho_n, \rho), 0) \cdot w(\rho_n, \rho) \cdot b(\rho_n, \rho).$$

But as $n \to \infty$, $b(\rho_n, \rho) \to 0$ and $\psi(b(\rho_n, \rho), 0) \to \psi(0, 0) = 1$, and $w(\rho_n, \rho) \to w(\rho, \rho) \in [1, \infty)$. Thus the entire product tends to 0. Hence

$$b(\zeta(\rho_n), \zeta(\rho)) \to 0.$$

Therefore, $\zeta(\rho_n) \to \zeta(\rho)$ in b. □

3. Best Proximity Point Result in Weighted ψ-b-Metric Space

Definition 3.1. *Let (X, d) be a metric space and let A and B be two nonempty subsets of X. Let $T : A \to B$ be a mapping. A point $x^* \in A$ is called a* best proximity point *of T if*

$$d(x^*, T(x^*)) = \inf\{d(x, T(x)) : x \in A\}.$$

In particular, if $A \cap B \neq \emptyset$ and $T(x) = x$ for some $x \in A \cap B$, then x is a fixed point and also a best proximity point.

Definition 3.2. *Let $\mathcal{B}(X)$ denote the family of all nonempty bounded closed subsets of X. For any $A, B \in \mathcal{B}(X)$, the* Hausdorff-type distance *$H_b(A, B)$ is defined by*

$$H_b(A, B) = \max\left\{\sup_{a\in A} \inf_{b\in B} d(a, b),\ \sup_{b\in B} \inf_{a\in A} d(b, a)\right\}.$$

and the distance *between the sets A and B, denoted by* $\text{dist}(A, B)$*, is defined by*

$$\text{dist}(A, B) = \inf\{d(a, b) : a \in A,\ b \in B\}.$$

Theorem 3.1. *Let (X, b) be a complete weighted ψ-b-metric space with weight function $w : X \times X \to [\alpha, \beta]$ for some $0 < \alpha \le \beta < \infty$, and let $\psi : [0, \infty)^2 \to [1, \infty)$ be continuous and increasing in both arguments. Suppose that $T : X \to 2^X \setminus \{\emptyset\}$ is a multivalued mapping such that for all $\rho, \varrho \in X$, there exists $\sigma \in T(\varrho)$ satisfying:*

$$H_b(T\rho, T\varrho) \le \psi\left(b(\rho, \varrho), b(\rho, \varrho)\right) \cdot \left[w(\rho, \varrho)b(\rho, \varrho) + w(\varrho, \sigma)b(\varrho, \sigma)\right],$$

where H_b is the Hausdorff-type metric induced by b. Suppose further that T has closed values and there exists $\rho_0 \in X$ such that $T\rho_0 \cap X \neq \emptyset$. Then T has a best proximity point in X.

Proof. We begin by selecting an initial point $\rho_0 \in X$ such that $T\rho_0 \cap X \neq \emptyset$, and pick $\rho_1 \in T\rho_0$. We then construct a sequence $\{\rho_n\}$ in X recursively as follows. For each $n \in \mathbb{N}$, choose $\rho_{n+1} \in T\rho_n$ such that

$$b(\rho_n, \rho_{n+1}) \le H_b(T\rho_{n-1}, T\rho_n)\psi\left(b(\rho_{n-1}, \rho_n), b(\rho_{n-1}, \rho_n)\right) \cdot \\ \left[w(\rho_{n-1}, \rho_n)b(\rho_{n-1}, \rho_n) + w(\rho_n, \rho_{n+1})b(\rho_n, \rho_{n+1})\right]$$

Let $d_n := b(\rho_n, \rho_{n+1})$. Then inequality (1) becomes:

$$d_n \le \psi(d_{n-1}, d_{n-1}) \cdot \left[w(\rho_{n-1}, \rho_n)d_{n-1} + w(\rho_n, \rho_{n+1})d_n\right].$$

We solve this inequality for d_n:

$$d_n - \psi(d_{n-1}, d_{n-1}) w(\rho_n, \rho_{n+1}) d_n \leq \psi(d_{n-1}, d_{n-1}) w(\rho_{n-1}, \rho_n) d_{n-1}.$$

Factoring out d_n on the left-hand side

$$d_n \left[1 - \psi(d_{n-1}, d_{n-1}) w(\rho_n, \rho_{n+1})\right] \leq \psi(d_{n-1}, d_{n-1}) w(\rho_{n-1}, \rho_n) d_{n-1}.$$

Now, since $w(\cdot, \cdot) \in [\alpha, \beta]$ and $\psi \geq 1$, the term $1 - \psi(\cdot)w(\cdot)$ may be negative unless we impose a bound. To ensure positivity, suppose we assume

$$\psi(d_{n-1}, d_{n-1}) \cdot \beta < 1. \tag{3.1}$$

This can be guaranteed for small enough d_0 (initial gap), or by modifying ψ appropriately (e.g., bounded growth). Under this assumption, the coefficient of d_n is positive.

Hence,

$$d_n \leq \frac{\psi(d_{n-1}, d_{n-1}) w(\rho_{n-1}, \rho_n)}{1 - \psi(d_{n-1}, d_{n-1}) w(\rho_n, \rho_{n+1})} \cdot d_{n-1}. \tag{3.2}$$

Set

$$C_n := \frac{\psi(d_{n-1}, d_{n-1}) w(\rho_{n-1}, \rho_n)}{1 - \psi(d_{n-1}, d_{n-1}) w(\rho_n, \rho_{n+1})} \tag{3.3}$$

. Then

$$d_n \leq C_n d_{n-1}.$$

If $C_n < 1$ for all n, the sequence $\{d_n\}$ is strictly decreasing. Moreover, since $b \geq 0$, it follows that $d_n \to 0$ as $n \to \infty$.

Next, we show that $\{\rho_n\}$ is a Cauchy sequence. For $m > n$, by applying the weighted ψ-b-triangle inequality recursively:

$$b(\rho_n, \rho_m) \leq \psi(d_n, d_{n+1}) \cdot \left[w(\rho_n, \rho_{n+1}) d_n + \cdots + w(\rho_{m-1}, \rho_m) d_{m-1}\right].$$

As $d_n \to 0$ and w is bounded, the right-hand side goes to zero as $n \to \infty$, so $\{\rho_n\}$ is Cauchy.

Since (X, b) is complete, there exists $\rho^* \in X$ such that $\rho_n \to \rho^*$.

Finally, because $\rho_{n+1} \in T\rho_n$ and T has closed values, the limit $\rho^* \in T\rho^*$. Thus, ρ^* is a fixed point of T, or if $T\rho^* \not\ni \rho^*$, then $b(\rho^*, T\rho^*) = \inf\{b(\rho^*, x) : x \in T\rho^*\}$, i.e., ρ^* is a best proximity point.

□

Theorem 3.2. *Let (X, b) be a complete weighted ψ-b-metric space with weight function w and distortion function ψ. Let $A, B \subseteq X$ be nonempty, closed subsets such that*

(1) *$T : A \to 2^B$ is a multi-valued mapping,*
(2) *For all $\rho, \varrho \in A$, and every $u \in T\rho$, there exists $v \in T\varrho$ such that*

$$b(u, v) \leq \gamma \cdot \max\left\{b(\rho, \varrho), b(\rho, T\rho), b(\varrho, T\varrho), \frac{b(\rho, T\varrho) + b(\varrho, T\rho)}{2}\right\}$$

for some $\gamma \in [0, 1)$,

(3) *For each $\rho \in A$, $T\rho$ is nonempty and closed in B,*

(4) *There exists* $\rho_0 \in A$ *such that* $b(\rho_0, T\rho_0) = \mathrm{dist}(A, B)$.

Then there exists $\rho^* \in A$ *such that*

$$b(\rho^*, T\rho^*) = \mathrm{dist}(A, B),$$

i.e., ρ^* *is a best proximity point of* T.

Proof. We aim to show that there exists $\rho^* \in A$ such that

$$b(\rho^*, T\rho^*) = \mathrm{dist}(A, B).$$

Step 1: Initial choice and sequence construction.

Since $b(\rho_0, T\rho_0) = \mathrm{dist}(A, B)$ and $T\rho_0 \neq \emptyset$, we can choose $u_0 \in T\rho_0$ such that

$$b(\rho_0, u_0) = \mathrm{dist}(A, B).$$

Now, construct sequences $\{\rho_n\}_{n=0}^{\infty} \subseteq A$ and $\{u_n\}_{n=0}^{\infty} \subseteq B$ as follows:

Given $\rho_n \in A$ and $u_n \in T\rho_n$, choose $\rho_{n+1} \in A$ (will be specified below). By condition (2) of the theorem, for this ρ_n, ρ_{n+1} and $u_n \in T\rho_n$, there exists $u_{n+1} \in T\rho_{n+1}$ such that

$$b(u_n, u_{n+1}) \leq \gamma \cdot \max\left\{b(\rho_n, \rho_{n+1}), b(\rho_n, T\rho_n), b(\rho_{n+1}, T\rho_{n+1}), \frac{b(\rho_n, T\rho_{n+1}) + b(\rho_{n+1}, T\rho_n)}{2}\right\}.$$

Step 2: Estimating successive distances.

Define $d_n := b(u_n, u_{n+1})$. From (1), we get

$$d_n \leq \gamma \cdot M_n,$$

where

$$M_n := \max\left\{b(\rho_n, \rho_{n+1}), b(\rho_n, T\rho_n), b(\rho_{n+1}, T\rho_{n+1}), \frac{b(\rho_n, T\rho_{n+1}) + b(\rho_{n+1}, T\rho_n)}{2}\right\}.$$

We choose $\rho_{n+1} \in A$ such that

$$b(\rho_{n+1}, T\rho_{n+1}) = \mathrm{dist}(A, B),$$

and similarly ensure that $u_{n+1} \in T\rho_{n+1}$ satisfies

$$b(\rho_{n+1}, u_{n+1}) = \mathrm{dist}(A, B).$$

This choice implies that:

$$b(\rho_n, T\rho_n) = \mathrm{dist}(A, B) \quad \text{and similarly for the other terms.}$$

Thus,

$$M_n \leq \max\{b(\rho_n, \rho_{n+1}), \mathrm{dist}(A, B), \ldots\},$$

and if we choose ρ_{n+1} such that $b(\rho_n, \rho_{n+1}) \to 0$, then $d_n \to 0$ as $n \to \infty$.

Step 3: Prove that $\{u_n\}$ is a Cauchy sequence.

Since $d_n \to 0$, and b satisfies a generalized triangle inequality via the ψ function, we can show that $\{u_n\}$ is a Cauchy sequence in B.

Because (X, b) is complete and $B \subseteq X$ is closed, $\{u_n\}$ converges to some $u^* \in B$.

Step 4: Show that $\{\rho_n\}$ converges in A. We estimate

$$b(\rho_n, \rho_{n+1}) \leq \psi(b(\rho_n, u_n), b(u_n, u_{n+1})) \cdot [w(\rho_n, u_n)b(\rho_n, u_n) + w(u_n, u_{n+1})b(u_n, u_{n+1})].$$

Note that $b(\rho_n, u_n) = \text{dist}(A, B)$ and $b(u_n, u_{n+1}) = d_n \to 0$, so $b(\rho_n, \rho_{n+1}) \to 0$. Hence $\{\rho_n\}$ is a Cauchy sequence in A, which is closed and contained in complete X so, $\rho_n \to \rho^* \in A$.

Step 5: Conclude that ρ^* is a best proximity point.

Since $u_n \in T\rho_n$, $u_n \to u^* \in B$, and $T\rho_n$ are closed subsets of B, with T upper semicontinuous (or closed-valued), we have $u^* \in T\rho^*$.

Finally, we take limits:

$$b(\rho^*, T\rho^*) \leq b(\rho^*, u^*) = \lim_{n\to\infty} b(\rho_n, u_n) = \text{dist}(A, B).$$

But by the definition of $\text{dist}(A, B)$, this is the minimum possible value. Thus:

$$b(\rho^*, T\rho^*) = \text{dist}(A, B).$$

This shows that ρ^* is a best proximity point of T.

Q.E.D. For detailed techniques, see [2, 6]. □

Example 3.1. *Let $X = [0, 4]$, and define the function:*

$$b(x, y) = |x - y| + \frac{1}{2}|x - y|^2.$$

Define $\psi(a, b) = 1 + \frac{a+b}{2}$ and $w(x, y) = 1 + \frac{|x-y|}{4}$. We will show that (X, b) is a weighted ψ-b-metric space.

Step 1: Verify properties $(wb1)$ and $(wb2)$

(wb1) *For any $x \in X$,*

$$b(x, x) = |x - x| + \frac{1}{2}|x - x|^2 = 0 + 0 = 0.$$

(wb2) *For any $x, y \in X$,*

$$b(x, y) = |x - y| + \frac{1}{2}|x - y|^2 = b(y, x),$$

since absolute value is symmetric.

Step 2: Verify (wb3) with specific values

Let $x = 0$, $z = 1$, and $y = 2$. Compute each term

$$\begin{aligned} b(x, y) &= b(0, 2) = |0 - 2| + \frac{1}{2}(2)^2 \\ &= 2 + 2 = 4, \\ b(x, z) &= b(0, 1) = |0 - 1| + \frac{1}{2}(1)^2 \\ &= 1 + 0.5 = 1.5, \\ b(z, y) &= b(1, 2) = |1 - 2| + \frac{1}{2}(1)^2 = 1 + 0.5 = 1.5, \end{aligned}$$

$$\psi(b(x,z),b(z,y)) = 1 + \frac{1.5+1.5}{2} = 2.5,$$
$$w(x,z) = w(0,1) = 1 + \frac{1}{4} = 1.25,$$
$$w(z,y) = w(1,2) = 1 + \frac{1}{4} = 1.25.$$

Now compute the right-hand side of (wb3)

$$\psi(b(x,z),b(z,y)) \cdot [w(x,z)b(x,z) + w(z,y)b(z,y)] = 2.5 \cdot (1.25 \cdot 1.5 + 1.25 \cdot 1.5) = 2.5 \cdot 3.75 = 9.375.$$

Compare with

$$b(x,y) = 4 \leq 9.375.$$

Thus, the condition (wb3) is satisfied for this choice of points. Conclusion (X,b) *is a weighted* ψ*-b-metric space.*

Let $A = [0,1]$, $B = [2,4]$ *and define* $T : A \to 2^B$ *by*

$$Tx = \left\{2 + \frac{x}{2}\right\}.$$

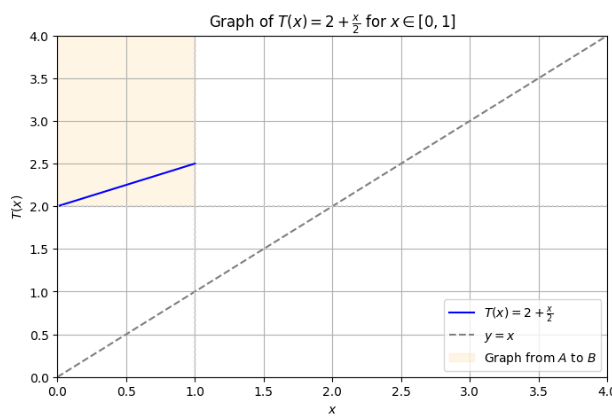
We have

$$b(x,Tx) = |x - (2 + \tfrac{x}{2})| + \frac{1}{2}|x - (2 + \tfrac{x}{2})|^2$$
$$= \left|\frac{-2x-4}{2}\right| + \frac{1}{2}\left|\frac{-2x-4}{2}\right|^2.$$

When $x = 0$, $Tx = \{2\}$ *and*

$$b(0,2) = 2 + \frac{1}{2}(4) = 4.$$

Therefore, $\text{dist}(A,B) = b(0,2) = 4$, *and* $x = 0$ *is a best proximity point. Graph in Python for a specified accuracy value, such as* $\epsilon = 10^{-6}$.



□

4. Conclusion

This paper introduces groundbreaking fixed point and best proximity point results for multivalued mappings within the innovative framework of weighted ψ-b-metric spaces, addressing a significant gap in contemporary metric fixed point theory. We establish a novel and comprehensive fixed point theorem under a sophisticated generalized contractive condition of Reich–Rus–Čirić type, uniquely incorporating both control functions and weight functions to achieve unprecedented theoretical depth. Our main contribution features an elegant synthesis of proximity point theory with weighted metric structures, providing a powerful analytical tool that transcends traditional metric limitations. Carefully constructed numerical examples demonstrate the practical applicability and theoretical robustness of our main theorems with applications to numerical examples. The obtained results represent a substantial advancement that not only unifies but significantly extends numerous classical theorems in the literature, opening new avenues for research in fixed point theory and their diverse applications across mathematical analysis.

5. Future Research Directions

The results presented in this paper lay the groundwork for several promising research directions. Some potential avenues for further investigation include:

(1) **Extension to Other Generalized Metric Spaces:** The current framework based on weighted ψ-b-metric spaces could be further generalized to encompass other distance-like structures such as G-metric spaces, partial metric spaces, and fuzzy metric spaces. This would allow for broader applications in areas where traditional metrics are insufficient.

(2) **Stability and Convergence Analysis:** Investigating the stability, convergence rate, and uniqueness of fixed and best proximity points under the proposed Reich–Rus–Čirić-type contractive conditions may provide deeper insights into the behavior of iterative methods and numerical algorithms in applied settings.

(3) **Algorithmic Implementations:** Developing efficient iterative algorithms based on the established theorems could significantly benefit computational mathematics, particularly in solving nonlinear equations, variational inequalities, and optimization problems.

(4) **Application to Real-World Problems:** Further exploration of applications in fields such as economics, engineering, computer science, and data science—especially in contexts involving multi-agent systems or equilibrium modeling—could validate and expand the practical utility of the theoretical results.

(5) **Best Proximity Point Results in Dynamic Settings:** Extending the concept of best proximity points to dynamic systems or time-dependent mappings may offer new insights into control theory and systems modeling, particularly in evolving metric spaces.

(6) **Hybrid and Multivalued Mappings in Weighted Settings:** Further generalizations involving hybrid contraction conditions or different classes of multivalued mappings could

enrich the theory and lead to new fixed point and approximation results under more relaxed assumptions.

Acknowledgements: N. Artsawang was supported by the Faculty of Science, Naresuan University [grant number R2568E007], and C. Suanoom was supported by Kamphaeng Phet Rajabhat University.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] I.A. Bakhtin, The Contraction Principle in Quasimetric Spaces, Funct. Anal. 30 (1989), 26–37.
[2] P. Debnath, M. de La Sen, Fixed-points of Interpolative Ćirić-Reich–Rus-Type Contractions in b-Metric Spaces, Symmetry 12 (2019), 12. https://doi.org/10.3390/sym12010012.
[3] T. Kamran, M. Samreen, Q. UL Ain, A Generalization of B-Metric Space and Some Fixed Point Theorems, Mathematics 5 (2017), 19. https://doi.org/10.3390/math5020019.
[4] B. Samet, C. Vetro, P. Vetro, Fixed Point Theorems for -Contractive Type Mappings, Nonlinear Anal.: Theory Methods Appl. 75 (2012), 2154–2165. https://doi.org/10.1016/j.na.2011.10.014.
[5] E. Karapınar, Revisiting the Kannan Type Contractions via Interpolation, Adv. Theory Nonlinear Anal. Appl. 2 (2018), 85–87. https://doi.org/10.31197/atnaa.431135.
[6] E. Karapınar, R.P. Agarwal, Interpolative Rus-Reich-Ćirić Type Contractions via Simulation Functions, Analele Univ. "Ovidius" Constanta - Ser. Mat. 27 (2019), 137–152. https://doi.org/10.2478/auom-2019-0038.