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A Pertubation Matrix and Its Eigen Functions

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Abstract. On a finite graph $\mathcal{N}$ with a set of possibly non-symmetric transition indices $\{c(a,b)\}$, $c(a,b) \geq 0$, $c(a) = \sum_b c(a,b) \leq 1$, an operator $Ku(a) = (I - A)u(a) = u(a) - \sum c(a,b)u(b)$ is defined. We discuss properties of the operator K. We prove that for an eigen function $\xi(a)$ with positive entries, $K\xi(a) = \rho\xi(a)$ where $\rho > 0$ and show that the eigen value ρ is the smallest in the following sense: if for an eigen function $\eta(a)$, $K\eta(a) = \beta\eta(a)$ then $Re\beta > \rho$. This result establishes the uniqueness and minimality of the positive eigenvalue associated with the positive eigenfunction. Finally, it is proven that the set $\mathfrak{F} = \{u : Ku(a) \geq 0\}$ forms a convex cone that is a lattice under the natural order.

1. Introduction

A random walk $\{\mathcal{N}, p(a,b)\}$ on a finite graph $\mathcal{N}$, generally $p(a) = \sum_b p(a,b) = 1$ for every state a (from Pickarddello and Woess [10] and Saloff-Coste, L. [11]). But in a reflective random walk it is possible that $p(a) < 1$ for some a. A similar situation arises when we consider discrete Schrödinger equation on a finite network $\{\mathcal{N}, c(a,b)\}$ [Bendito et al. [4]] with the Laplacian $\Delta u(a) = \sum_b c(a,b)[u(b) - u(a)] = q(a)u(a)$, $q \geq 0$, $q \not\equiv 0$. Setting $p(a,b) = \frac{c(a,b)}{c(a)}$, $c(a) = \sum_b c(a,b)$, the equation reads $\sum_b p(a,b)u(b) = [1 + \frac{q(a)}{c(a)}]u(a)$ (or) $\sum_b p'(a,b)u(b) = u(a)$ where $p'(a,b) = \frac{p(a,b)}{1+\frac{q(a)}{c(a)}} \leq p(a,b)$ and $p'(a) = \sum_b p'(a,b) \leq 1$, $p'(z) < 1$ for atleast one $a = z$.

Considering these examples, we set out in this article a function theory on a finite network $\{\mathcal{N}, c(a,b)\}$, $c(a) = \sum_b c(a,b) \leq 1$ and $c(a) < 1$ for atleast one vertex a; $c(a,b) \geq 0$ and $c(a,b) > 0$ if and only if $a \sim b$ (neighbours); $c(a,b)$ and $c(b,a)$ may have different values.

Define the matrix $K = (K_{ab})$, $k_{aa} = 1$, $k_{ab} = -c(a,b)$ representing the finite network $\{\mathcal{N}, c(a,b)\}$. By using the Perron-Frobenius theorem, (see Anandam. V and M. Damlakhi [2], C. Araúz et al. [3], Gantmacher [7]) we prove that for an eigen function $\xi(a)$ with positive entries $K\xi(a) = \rho\xi(a)$,

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where $\rho > 0$; we show that the eigen value ρ is the smallest in the following sense: if for an eigen function $\eta(a)$, $K\eta(a) = \beta\eta(a)$ then $Re\beta > \rho$. Also we show that if $\sigma(a)$ is an eigen function with all its entries real, $K\sigma(a) = \alpha\sigma(a)$, then $\sigma(a)$ has both positive and negative entries. We prove also that if $\mathfrak{F}$ is the set of all functions on $\mathcal{N}$, $\mathfrak{F} = \{u : Ku(a) \geq 0\}$ then the convex cone $\mathfrak{F}$ represents a lattice of natural order.

2. Preliminaries

Let $\{\mathcal{N}, C\}$ represent a finite connected network where $\{c(a,b)\}$ denote a collection of transition functions over $\mathcal{N}$ such that $c(a,b)$ is non-negative, $c(a,b)$ is positive if only if a and b are adjacent, and $c(a,a) = 0$ for all $a \in \mathcal{N}$. Additionally, $c(a) = \sum_b c(a,b)$ must be less than or equal to one, and there exists at least one vertex $a = z$ such that $c(z) < 1$. A vertex a in $\{\mathcal{N}, c(a,b)\}$ is considered interior to a subset $G \subset \mathcal{N}$ if a and all its neighbouring vertices i.e, $b \sim a$ belong to the subset G; the set of all interior vertices of G is represented as $\mathring{G}$, while the boundary is denoted as $\partial G = G \backslash \mathring{G}$. A set G is defined as connected if, for any two distinct vertices a and b within G, there exists a path $\{a = e_0, e_1, ..., e_n = b\}, e_i \sim e_{i+1}$ where each adjacent pair $e_i \sim e_{i+1}$ is contained in G, thus linking a to b.

Definition 2.1. *Laplacian*(Δ)*: Let* $s(x)$ *be a real valued function defined on* $\{\mathcal{N}, C\}$*. For* $a \in \mathring{G}$, $G \subset \mathcal{N}$*, the Laplacian* (Δ) *of* s *at* a *is defined as*

$$\Delta s(a) = \sum_{b \sim a} c(a,b)[s(b) - s(a)]$$

Example 2.1. *Finite network with its Laplacian:*

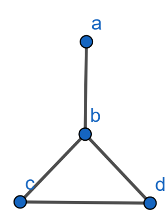

Figure 1. Finite network

The vertex set of the given finite network is $\{a,b,c,d\}$ *and the edge set is* $\{(a,b),(b,c),(b,d),(c,d)\}$ *with the transition probabilities* $c(b,a) = 0.6$, $c(a,b) = 0.6$, $c(c,b) = 0.5$, $c(d,b) = 0.5$, $c(b,c) = 0.2$, $c(d,c) = 0.5$, $c(c,d) = 0.5$, $c(b,d) = 0.2$, *we see that* $c(b) = c(c) = c(d) = 1$ *and* $c(a) < 1$.
Then Laplacian matrix(L) is,
$L = D(Degree\ matrix) - A(Probability\ transition\ matrix)$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 0.6 & 0 & 0 \\ 0.6 & 0 & 0.5 & 0.5 \\ 0 & 0.2 & 0 & 0.5 \\ 0 & 0.2 & 0.5 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & -0.6 & 0 & 0 \\ -0.6 & 3 & -0.5 & -0.5 \\ 0 & -0.2 & 2 & -0.5 \\ 0 & -0.2 & -0.5 & 2 \end{bmatrix}$$

Definition 2.2. *Eigen function: An eigen function of a linear operator L is a non-zero function f that, when acted upon by L, results in a scalar multiple of itself. This scalar multiple is called the eigenvalue λ associated with that eigenfunction.*
This is expressed by the eigenvalue equation: $L[f(a)] = \lambda f(a)$.
where:

- *L is a linear operator.*
- *$f(a)$ is the eigenfunction.*
- *λ is the eigenvalue, a scalar (which can be real or complex).*

Example 2.2. *The Eigen functions for the above finite network with the Laplacian matrix*
$L = \begin{bmatrix} 1 & -0.6 & 0 & 0 \\ -0.6 & 3 & -0.5 & -0.5 \\ 0 & -0.2 & 2 & -0.5 \\ 0 & -0.2 & -0.5 & 2 \end{bmatrix}$ *is,*

$v_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ *for the eigen value* $\lambda_1 = \frac{5}{2}$

$v_2 = \begin{bmatrix} 10.914 \\ 3.448 \\ 1 \\ 1 \end{bmatrix}$ *for the eigen value* $\lambda_2 = 0.810$

$v_3 = \begin{bmatrix} -0.587 \\ 0.409 \\ 1 \\ 1 \end{bmatrix}$ *for the eigen value* $\lambda_3 = 1.418$

$$v_4 = \begin{bmatrix} 2.340 \\ -8.857 \\ 1 \\ 1 \end{bmatrix} \textit{for the eigen value } \lambda_4 = 3.271$$

Definition 2.3. *Random walk(from [12]):*

Let $\{\mathcal{N}, P\}$ be a random walk with a finite number of states $\mathcal{N}$ and the probability transition matrix $P = \{p(a,b)\}$, the transition probability from state a to state b is denoted as $p(a,b)$. We assume $\{\mathcal{N}, P\}$ to be

- *Connected (i.e, for any two distinct states in the random walk there exists a path connecting them).*
- *There exist no path from a state to itself (without self loops).*

We say two states a and b are neighbours if there exists an edge between them and it is denoted by $a \sim b$ and $p(a) = \sum_{b \sim a} p(a,b) = 1$ for every $a \in \mathcal{N}$. We will define $[a,b]$ as an edge if and only if the transition probability of the $[a,b]$ is positive.

Example 2.3. *Consider a particle on a clock face with 12 positions numbered 1 to 12. The particle starts at position 1 at each time step, it moves either forward (clockwise) one position or backward (counterclockwise) one position. The state space is $\mathcal{N} = \{0, 1, 2, ..12\}$ with the set $P = \{p(a,b)\}$ of transition probabilities given by $p(n, n+1) = p(n, n-1) = \frac{1}{2}$, for $n \geq 0$. The particle's movement forms a path, which is the random walk, where $\sum_{b \sim a} p(a,b) = 1$ for every state $a \in \mathcal{N}$.*

Definition 2.4. *Lower directed family: Let $(S, \leq)$ is a partially ordered set. A non-empty set $F \subseteq S$ is called a lower directed family (or directed downwards) if, for every pair of elements $a, b \in F$, there exists an element x in F such that $x \leq a$ and $x \leq b$.*

Definition 2.5. *Convex cone: A set C is a convex cone if, for any vectors x and y in C, and any non-negative scalars α and β, the linear combination $\alpha x + \beta y$ is also in C.*

For a real-valued function $u(a)$ on $\mathcal{N}$, write $Au(a) = \sum_b c(a,b)u(b)$ and the operator $Ku(a) = (I - A)u(a) = u(a) - \sum_b c(a,b)u(b)$. With $\{\mathcal{N}, c(a,b)\}$ we associate a random walk $\{\mathcal{N}, p(a,b)\}$, taking $p(a,b) = \frac{c(a,b)}{c(a)}$. The Laplacian Δ of this random walk is $\Delta u(a) = \sum_b p(a,b)[u(b) - u(a)]$.

Then,

$$\begin{aligned} -\Delta u(a) =& u(a) - \sum p(a,b)u(b) \\ =& u(a) - \sum \frac{c(a,b)}{c(a)} u(b) \\ =& u(a) - \frac{1}{c(a)} Au(a) \\ =& u(a) + \frac{1}{c(a)} [K - I]u(a) \\ \textit{so that } Ku(a) =& [1 - c(a)]u(a) - c(a)\Delta u(a). \end{aligned}$$

Hence in particular $-c(a)\Delta u(a) \leq Ku(a)$ if $u(a) \geq 0$.

Suppose $K\xi(a) = \rho\xi(a)$ where ρ is a constant and $\xi(a) > 0$, then $-c(a)\Delta\xi(a) \leq K\xi(a) = \rho\xi(a)$. Suppose now $\rho \leq 0$, then $-c(a)\Delta\xi(a) \leq 0$. Since $\Delta\xi(a) \geq 0$, on a finite network $\xi(a)$ is a constant $\alpha > 0$. From $K\xi(a) = \rho\xi(a)$, we get $(I - A)\alpha = \rho\alpha$. Hence $[1 - c(a)]\alpha = \rho\alpha$. Since $0 < c(z) < 1$, we conclude $\rho = 1 - c(z) > 0$, contradicting the assumption $\rho \leq 0$. consequently we conclude that $\rho > 0$. (For Laplace eigen values of finite graphs, see for example Mohar. B [9] and Biyikoglu et al. [5]).

Now $(I - K)$ is a matrix with all its entries non-negative, see example 2.1, the probability transition matrix (A) is, $\begin{bmatrix} 0 & 0.6 & 0 & 0 \\ 0.6 & 0 & 0.5 & 0.5 \\ 0 & 0.2 & 0 & 0.5 \\ 0 & 0.2 & 0.5 & 0 \end{bmatrix}$. Hence by the Perron-Frobenius theorem there is the largest eigen value λ with associated eigen vectors $\xi(a)$; the entries of $\xi(a)$ are all of the same sign so that we can take $\xi(a) > 0$ and $\sum_a \xi(a) = 1$ (refer Theorem.2.2, [2]). Note also the eigen space associated with $\mathcal{N}$ is one, hence the eigen vector $\xi(a) > 0$ and $\sum_a \xi(a) = 1$ is uniquely determined. Now $(I - K)\xi(a) = \lambda\xi(a) \Rightarrow K\xi(a) = (1 - \lambda)\xi(a) = \rho\xi(a)$. Then as we just saw above, $\rho > 0$. Remark also that ρ is the smallest eigen value of the matrix K. Note also that $p < 1$. The reason for calling ρ the "smallest" eigen value is: $\rho = 1 - \lambda$ and λ is the largest eigen value of $(1 - k)$.

Proposition 2.1. *Suppose $K\sigma(a) = \beta\sigma(a)$ for some β real or complex. Then $Re\beta > \rho$.*

Proof. If $K\sigma(a) = \beta\sigma(a)$
Then

$$\begin{aligned}(I - K)\sigma(a) &= I\sigma(a) - K\sigma(a) \\ &= \sigma(a) - \beta\sigma(a) \\ &= (1 - \beta)\sigma(a)\end{aligned}$$

By Perron-Frobenius theorem,

$$|1 - \beta| < \lambda = 1 - \rho$$

By the property of complex numbers,

$$\begin{aligned}Re(1 - \beta) &\leq |1 - \beta| \\ 1 - Re\beta \leq |1 - \beta| &< 1 - \rho \\ \rho &< Re\beta\end{aligned}$$

□

Proposition 2.2. *Any real eigenvector of K other than $\xi(a)$ has both positive and negative entries.*

Proof. Suppose $K\eta(a) = \beta\eta(a)$ where $\eta(a)$ has only real entries. Then β is real so that $\beta > \rho$. where ρ is the smallest eigen value $K\xi(a) = \rho\xi(a)$. Suppose now every entry of $\eta(a)$ is non-negative, then

$$\begin{aligned}\xi(a)K\eta(a) &= \xi(a)[\beta\eta(a)] \\ &\geq \xi(a)[\rho\eta(a)] \\ &= [\rho\xi(a)]\eta(a) \\ &= [K\xi(a)]\eta(a)\end{aligned}$$

Hence

$$\begin{aligned}\xi(a)[\eta(a) - \sum_b c(a,b)\eta(b)] &\geq [\xi(a) - \sum c(a,b)\xi(b)]\eta(a) \\ \sum_b c(a,b)[\xi(b)\eta(a) - \xi(a)\eta(b)] &\geq 0 \\ \sum_b c(a,b)\xi(a)\xi(b)[\frac{\eta(a)}{\xi(a)} - \frac{\eta(b)}{\xi(b)}] &\geq 0 \\ \sum_b c(a,b)\xi(b)[\frac{\eta(a)}{\xi(a)} - \frac{\eta(b)}{\xi(b)}] &\geq 0\end{aligned}$$

That is, $-\Delta^*[\frac{\eta(a)}{\xi(a)}] \geq 0$ where Δ^* is the Laplacian associated with the finite network $\{\mathcal{N}, c^*(a,b) = c(a,b)\xi(b)\}$. Hence, for all a in $\mathcal{N}$, $\frac{\eta(a)}{\xi(a)} = \alpha$, a constant, $\alpha \geq 0$.
Clearly α cannot be 0 so that $\eta(a) = \alpha\xi(a), \alpha > 0$. Then $\beta\eta(a) = K\eta(a) = \alpha K\xi(a) = \alpha[\rho\xi(a)] = \rho\eta(a)$ for all a in $\mathcal{N}$. Since $\eta(a) > 0$ for atleast one $a = z$, we conclude that $\beta = \rho$, not valid.

Consequently the assumption that all entries of $\eta(a)$ are non-negative is not valid. That is, $\eta(a)$ contains atleast one negative entry. Similarly $\eta(a)$ should have atleast one positive entry. So, $\eta(a)$ has both positive and negative entries. □

Theorem 2.1. *(Poisson) If $f(a)$ is given on $\{\mathcal{N}, c(a,b)\}$, then there exists a unique function $u(a)$ such that $Ku(a) = f(a)$ on $\mathcal{N}$.*

Proof. Since the smallest eigenvalue ρ of the matrix K is $\rho > 0$, 0 is not eigenvalue of K, hence K is invertible, so the theorem. □

Proposition 2.3. *If $u(a) \geq 0$ and $Ku(a) \leq 0$, then $u = 0$.*

Proof. Since $-c(a)\Delta u(a) = [c(a) - 1]u(a) + Ku(a) \leq 0$, then $\Delta u(a) \geq 0$, hence $u(a)$ is a constant $\alpha \geq 0$. But then $0 \geq Ku(a) = \alpha K1 = \alpha[1 - c(a)]$. In particular, $\alpha[1 - c(a)] \leq 0$ implying that $\alpha \leq 0$ so that $\alpha = 0$. □

Remark 2.1. *In the context of potential theory on finite graphs, (see Anandam [1], chapter 2) $K\xi(a) = \rho\xi(a) \geq 0$ means that $\xi(a)$ is a K-subharmonic function. From the above proposition if $u(a)$ is a K-superharmonic function such that $0 \leq u(a) \leq \xi(a)$, then $u(a) = 0$. Thus, actually the function $\xi(a)$ is a K-potential.*

Green's Function: Given any vertex e, there exists a unique function $G_e(a)$ on $\mathcal{N}$ such that $KG_e(a) = \delta_e(a)$. The uniqueness of $G_e(a)$ follows from the fact that for the invertible K, if $Kf = 0$ then $f = 0$.

Remark 2.2. *For any real-valued function $f(a)$ on $\mathcal{N}$, the unique Poisson solution of $Ku(a) = f(a)$ is given by $u(a) = \sum_b f(b)G_b(a)$.*

Theorem 2.2. *(Minimum Principle): Let E be a subset of $\mathcal{N}$. If $u(a)$ is a function on $\mathcal{N}$ such that $Ku(a) \geq 0$ for each a in E and $u(a) \geq 0$ on $\mathcal{N}\setminus E$, then $u \geq 0$ on $\mathcal{N}$.*

Proof. Suppose $u(a)$ takes negative values. If $min\ u(a) = -m < 0$, then there exist $z \in E$, where $u(z) = -m$. Since $-m = u(z) \geq \sum_{b\sim z} c(z,b)u(b)$ then $\sum_{b\sim z} c(z,b)[u(b) + m] + m[1 - c(z)] \leq 0$. Since $u(b) + m \geq 0$ and $m[1 - c(z)] \geq 0$, then $u(b) = -m$ for all $b \sim z$.
Let e be a vertex in $\mathcal{N}\setminus E$. Then there exists a path $\{z = z_0, z_1, \ldots\ldots, z_n = e\}$ connecting z to e. Let i be the smallest index such that $z_i \in E$ and $z_{i+1} \in \mathcal{N}\setminus E$. Note that $u(z_i) = -m$, hence $u(z_{i+1}) = -m$, contradicting $u(z_{i+1}) \geq 0$ since $z_{i+1} \in \mathcal{N}\setminus E$. This shows that $u(a)$ cannot take negative values. □

A variation: Let $u(a)$ be defined on a subset E. If $Ku \geq 0$ on E and $u(a) \geq -\alpha$, $\alpha \geq 0$, on ∂E then $u(a) \geq -\alpha$ on E.

Proof. The function $v(a) = u(a) + \alpha$ on E extended by 0 on $\mathcal{N}\setminus E$ satisfies $Kv(a) \geq 0$ for $a \in E$ and $v(a) \geq 0$ if $a \in \partial E$. Hence $v(a) \geq 0$ so that $u(a) \geq -\alpha$ on E. □

Corollary 2.1. *If $u(a)$ is a function on $\mathcal{N}$ such that $Ku(a) = 0$ at each vertex of a subset E and $u(a) = 0$ on $\mathcal{N}\setminus E$, then $u = 0$ on $\mathcal{N}$.*

Corollary 2.2. *For a function $u(a)$ on $\mathcal{N}$ with $Ku(a) \geq 0$, write $A = \{a : Ku(a) > 0\}$. Let $s(a)$ be a function having $Ks(a) \geq 0$. If $s(a) \geq u(a)$ on A, then $s(a) \geq u(a)$ on $\mathcal{N}$.*

Proof. Let $v(a) = inf[s(a), u(a)]$. Then $Kv(a) \geq 0$ and $v(a) = u(a)$ on A. Let $f(a) = u(a) - v(a)$ on $\mathcal{N}$. Then $f(a) = 0$ on A and $Kf(a) \leq 0$ on $\mathcal{N}/A$. Hence by the minimum principle, $f(a) \leq 0$ on $\mathcal{N}$ which implies that $v = u$ on $\mathcal{N}$, so that $u \leq s$ on $\mathcal{N}$. □

Remark 2.3. (1) *For a vertex e in $\mathcal{N}$, the Green's function $G_e(a) \leq G_e(e)$ for all a.*
(2) *If a non-zero function $s(a)$ is defined on $\mathcal{N}$, $Ks \geq 0$, then $s > 0$ on $\mathcal{N}$ and $\frac{s(a)}{s(e)} \geq \frac{G_e(a)}{G_e(e)}$.*

Proof. Let $v(a) = \frac{s(e)}{G_e(e)}G_e(a)$. Then $A = \{a : kv(a) > 0\} = \{e\}$. Now at e, $s(e) = v(e)$. Hence by corollary 2.2, $s(a) \geq v(a)$ on $\mathcal{N}$, thus proving the Remark. □

Theorem 2.3. *(Dirichlet Solution) For a subset F of $\{\mathcal{N}, c(a,b)\}$ and $E \subset \mathring{F}$. Suppose $f(a)$ is a function on $F\setminus E$. Then, a unique function $s(a)$ exists on F such that if $a \in E$ then $Ks(a) = 0$ and $s = f$ on $F\setminus E$.*

Proof. For some positive M, let $|f(a)| \leq M$ on $F\setminus E$. Then the function $v(a)$ on F satisfies $Kv \geq 0$ on E such that $v = f$ on $F\setminus E$ and $v = M$ on E. Assume that the family of all functions $u(a)$ on F is denoted by $\mathfrak{F}$ such that $u = f$ on $F\setminus E$ and $\Delta u \geq 0$ on E. Take $s(a) = \inf_{u\in\mathfrak{F}} u(a)$. Note that if $u_1, u_2 \in \mathfrak{F}$

then $\inf(u_1, u_2) \in \mathfrak{F}$ so that we can extract a subsequence $\{u_n\}$ from $\mathfrak{F}$ such that $s(a) = \lim u_n(a)$ on F. Consequently $s(a) = f(a)$ on $F \backslash E$ and $Ks \geq 0$ on E.

Actually $Ks(a) = 0$ for every $a \in E$. For take $z \in E$ and consider the function s_z on F such that $s_z(a) = s(a)$ and $s_z(z) = \sum_{b \sim z} s(b)c(z,b)$ if $a \in F$ and $a \neq z$. Then $s_z \in \mathfrak{F}$ and $s_z \leq s$ on F. This means, since s is the infimum in $\mathfrak{F}$, $s = s_z$ on F so that $Ks(a) = 0$ if $a = z$. The minimum principle implies that the solution $s(a)$ is unique (corollary 2.1). □

3. The family of all functions $\mathfrak{F}$

Note: The family of all functions $u(a)$ on $\mathcal{N}$ for which $Ku(a) \geq 0$ on $\mathcal{N}$ is denoted as $\mathfrak{F}$.

Lemma 3.1. *If $v_1, v_2 \in \mathfrak{F}$ then $v = \inf(v_1, v_2) \in \mathfrak{F}$.*

Proof. Suppose $v(e) = v_1(e)$, at a vertex e. Then $Kv(e) = v(e) - \sum_b c(e,b)v(b) \geq v_1(e) - \sum c(e,b)v_1(b) = Kv_1(e) \geq 0$. Hence $v \in \mathfrak{F}$. □

Lemma 3.2. *If $u \in \mathfrak{F}$ then u is non-negative on $\mathcal{N}$.*

Proof. Assume that u takes negative values on $\mathcal{N}$. Then at a vertex z, $u(z) = -m = \inf_{a \in \mathcal{N}} u(a)$ for some $m > 0$. Then we see that, $u = -m$ on $\mathcal{N}$ by the minimum principle. But then $Ku(a) = -m - \sum_b c(a,b)(-m) = -[1 - c(a)]m$. Since $c(b) < 1$ atleast at one vertex b, $Ku(b) < 0$, a contradiction. □

Theorem 3.1. *The convex cone $\mathfrak{F}$ is a lattice representing a natural order.*

Proof. If $u_1, u_2 \in \mathfrak{F}$, then by the above Lemma 3.1 $\inf(u_1, u_2) \in \mathfrak{F}$.
Let $f = \sup(u_1, u_2)$. Then $f \leq u_1 + u_2 \in \mathfrak{F}$. Let $\mathbb{F}$ be the subfamily of $\mathfrak{F}$ such that $\mathbb{F} = \{v \in \mathfrak{F}, v \geq f\}$. Note that $\mathbb{F}$ is a lower directed family in the sense that if $v_1, v_2 \in \mathbb{F}$, then $\inf(v_1, v_2) \in \mathbb{F}$, so that we can extract a decreasing sequence $v_n \in \mathbb{F}$ such that if $v_0 = lim_n v_n$ then $v_0 \in \mathbb{F}$. Clearly $v_0 = u_1 \vee u_2$. Hence $\mathfrak{F}$ is lattice representing the natural order. □

Theorem 3.2. *(Maximum Principle): Let G be a subset of $\mathcal{N}$. If $u(a)$ is a function on G, $Ku(a) \leq 0$ if $a \in \mathring{G}$ and $u(a) \leq M \geq 0$ on ∂G, then $u \leq M$ on G.*

Proof. Let $v(a) = u(a) - M$ on G and on $\mathcal{N}$, $s(a) = \sup(v(a), 0)$ extended by 0 outside G. Then $Ks(a) \leq 0$ if $a \in \mathring{G}$ and $s(a) = 0$ if $a \in \mathcal{N} \backslash \mathring{G}$. Hence $s = 0 \Rightarrow v \leq 0$ on G. □

Proposition 3.1. *Let $g(a)$ be a function defined on $\mathcal{N}$, $Kg(a) = f(a)$. Then $g = s_1 - s_2$ where $s_1, s_2 \in \mathfrak{F}$; moreover $g(a)$ has a representation $g(a) = \sum_b f(b)G_b(a)$ on $\mathcal{N}$.*

Proof. Given $Kg = f$, write $Ks_1 = f^+$ and $Ks_2 = f^-$. Then $s_1, s_2 \in \mathfrak{F}$ and $K(s_1 - s_2) = f = Kg$ so that $g = s_1 - s_2 + h$ where $Kh = 0$, hence $h = 0$. Moreover, since $s_i(a) = \sum_b Ks_i(b)G_b(a)$ then $g(a) = \sum_b f(b)G_b(a)$ on $\mathcal{N}$. □

Remark 3.1. *In particular K^{-1} is the matrix $(G_b(a))$, $a, b \in \mathcal{N}$.*

Applications

(1) Random walk has wide applications in the field of Image segmentation. The random walk algorithm, used in image processing can be seen as a form of random walk on a graph (refer [6] and [8]).

(2) Eigen Functions has a wide aaplication in Image Denoising and Reconstruction. Newer methods treat images as potential functions within a discretized Schrödinger equation, and the eigen functions of the associated Hamiltonian are used for image representation and denoising (refer [13]).

(3) Modal Analysis: Engineers use modal analysis, which is based on eigenfunctions and eigenvalues, to predict how structures will respond to dynamic loads (e.g., earthquakes, wind).

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