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Non-Null Canal Surfaces with Bishop Frame in Minkowski 3-Space

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Abstract. In this paper, we investigate spacelike and timelike canal surfaces foliated by $\mathbb{S}^2_1$ pseudo spheres in Minkowski 3-space based on the Bishop frame. Various types of canal surfaces, including Weingarten, linear Weingarten, developable, and minimal forms, are categorized to highlight the singular points and the geometric properties of such surfaces. Our analysis sheds light on the intrinsic properties of these surfaces and contributes to the understanding of their behavior within the context of Minkowski geometry. Finally, we present a computational example as a practical validation of our theoretical findings.

1. Introduction

An envelope of a one-parameter set of spheres with radius $r(s)$ and center curve $c(s)$ is called a canal surface. A sphere or a certain circular cross-section of a sphere can be swept down a path using one of the two techniques to create a canal surface. It is parameterezied by means of the spheres that self-assemble. The following can be used to parameterize a canal surface $\mathbb{M}$;

$$\mathbf{\Psi}(s,\theta) = c(s) + r(s)(-r'(s))\mathbf{T} + \sqrt{1 - r'(s)^2}(cos\theta\mathbf{N} + sin\theta\mathbf{B}), \tag{1.1}$$

where $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is the Frenet frame of $c(s)$ which is a unit speed curve parameterized by arc-lenght s. These canal surfaces are called "tubular surfaces" in the case where the radius function $r(s)$ remains constant. These surfaces, which are used especially in solid and surface modeling, have many uses, including reconstruction, robot movement planning, blending creation surfaces, and easy visibility of long, thin objects like pipes, ropes, and live intestines. Computer-aided geometric design (CAGD) is one of the most important applications of these surfaces. Canal and tubular

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surfaces in various spaces with different frames have been studied with a variety of conclusions regarding their geometric properties([2–9]).

The concept of canal surfaces in Euclidean 3-space has been expanded to Lorentz-Minkowski space in recent years. A family of pseudo-Riemannian space forms, including pseudo hyperbolic spheres $\mathbb{H}_0^2$ and lightlike cones $\mathbb{Q}_1^2$, can be used to form a canal surface in Minkowski 3-space $\mathbb{E}_1^3$. Let $r > 0$ be a constant and p a fixed point. Then

$$\mathcal{M}^2(\varepsilon) = \{u \in \mathbb{E}_1^3 : \langle u - p, u - p \rangle = \varepsilon r^2\},$$

$$\begin{cases} \mathbb{S}_1^2(p,r) \mid \varepsilon = 1, \\ \mathbb{H}_0^2(p,r) \mid \varepsilon = -1, \\ \mathbb{Q}_1^2(p) \mid \varepsilon = 0. \end{cases}$$

defines the pseudo-Riemannian space forms, i.e., the hyperbolic space $\mathbb{H}_0^2(p,r)$ the lightlike cone $\mathbb{Q}_1^2(p)$, and de-Sitter space $\mathbb{S}_1^2(p,r)$. We write them simply by $\mathbb{S}_1^2$, $\mathbb{H}_0^2$, and $\mathbb{Q}^2$ when $r = 1$ the origin is located at the center p [10].

By contrasting their qualities with those of the Frenet frame, Bishop [11] demonstrated the existence of orthonormal frames- what he referred to as substantially parallel adapted frames-aside from the Frenet frame. An alternative method of well-defined moving frames, even in cases where the curve's second derivative vanishes, is the Bishop or parallel transport of an orthonormal frame along a curve. Bishop frames are particularly useful for computing the structural information of DNA in biology and for directing virtual cameras in computer graphics sine each curve's interior is well defined. Additionally, many of the numerous works on surfaces and curves connected to the Bishop frame are included in [12–18].

In section 2, the Bishop frame is described in the Minkowski space of spacelike and timelike curves. In section 3, we used to the Bishop frame to generate non-null canal surfaces and present some results. Weingarten and linear-Weingarten non-null Bishop canal surfaces in Minkowski 3-space are produced in sections 4 and 5. The singular points of the Bishop canal surfaces are obtained in section 6. Lastly, a graph and example of a specified surface are provided.

2. Preliminaries

The Minkowski 3-space $\mathbb{E}_1^3$ is characterized by its natural Lorentz metric,

$$\langle , \rangle = -du_1^2 + du_2^2 + du_3^2$$

where (u_1, u_2, u_3) is a rectangular coordinate system of $\mathbb{E}_1^3$. The arbitrary vector $u = (u_1, u_2, u_3)$ in $\mathbb{E}_1^3$ can be spacelike if $\langle u, u \rangle > 0$ or $u = 0$, timelike if $\langle u, u \rangle < 0$ and lightlike (null) if $\langle u, u \rangle = 0$, $u \neq 0$.

Similarly, a parameterized curve $\gamma(s) : I \subset \mathbb{R} \longrightarrow \mathbb{E}_1^3$ where s is pseudo arclength parameter is called a spacelike curve if $\langle \gamma'(s), \gamma'(s) \rangle > 0$, timelike if $\langle \gamma'(s), \gamma'(s) \rangle < 0$ and lightlike if $\langle \gamma'(s), \gamma'(s) \rangle = 0$ or $\gamma'(s)$ for all $s \in I$. The two vectors $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3) \in \mathbb{E}_1^3$ are orthogonal if and only if $\langle u,v\rangle = 0$. Also, for any $u,v \in \mathbb{E}_1^3$, Lorentzian vector product of u and v is defined by

$$u \times v = \begin{vmatrix} -e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

The norm of a vector $u \in \mathbb{E}_1^3$ is given by $\|u\| = \sqrt{|\langle u,u\rangle|}$.

Let $\gamma(s) : I \longrightarrow \mathbb{E}_1^3$ be a space curve with a Bishop frame $\{\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2\}$ consisting of the tangent vector $\mathbf{T}$, the principal normal vector $\mathbf{N}_1$ and the binormal vector $\mathbf{N}_2$, respectively.

We have three cases:

- If γ is a spacelike curve with a spacelike $\mathbf{N}_1$, then the Bishop frame of $\gamma = \gamma(s)$ is expressed as follows:

$$\begin{pmatrix} \mathbf{T}'(s) \\ \mathbf{N}_1'(s) \\ \mathbf{N}_2'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1(s) & \kappa_2(s) \\ -\kappa_1(s) & 0 & 0 \\ \kappa_2(s) & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}_1(s) \\ \mathbf{N}_2(s) \end{pmatrix}, \tag{2.1}$$

where

$$\langle \mathbf{T}, \mathbf{T}\rangle = 1, \langle \mathbf{N}_1, \mathbf{N}_1\rangle = 1, \langle \mathbf{N}_2, \mathbf{N}_2\rangle = -1,$$

and the relation matrix between Serret-Frenet and Bishop frames is given by

$$\begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}_1(s) \\ \mathbf{N}_2(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\varphi & \sinh\varphi \\ 0 & \sinh\varphi & \cosh\varphi \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix}, \tag{2.2}$$

where the Bishop curvatures are defined by

$$\left. \begin{array}{l} \kappa_1 = \kappa\cosh\varphi \ , \kappa_2 = \kappa\sinh\varphi, \varphi = \tanh^{-1}\left(\frac{\kappa_2}{\kappa_1}\right); \ \kappa_1 \neq 0, \\ \kappa = \sqrt{|\kappa_1^2 - \kappa_2^2|}. \end{array} \right\} \tag{2.3}$$

- If γ is a spacelike curve with a spacelike $\mathbf{N}_2$, then the Bishop frame of $\gamma = \gamma(s)$ is expressed as follows:

$$\begin{pmatrix} \mathbf{T}'(s) \\ \mathbf{N}_1'(s) \\ \mathbf{N}_2'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1(s) & \kappa_2(s) \\ \kappa_1(s) & 0 & 0 \\ -\kappa_2(s) & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}_1(s) \\ \mathbf{N}_2(s) \end{pmatrix}, \tag{2.4}$$

where

$$\langle \mathbf{T}, \mathbf{T}\rangle = 1, \langle \mathbf{N}_1, \mathbf{N}_1\rangle = -1, \langle \mathbf{N}_2, \mathbf{N}_2\rangle = 1,$$

and the relation matrix between Serret-Frenet and Bishop frames is given by

$$\begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}_1(s) \\ \mathbf{N}_2(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\varphi & \sinh\varphi \\ 0 & \sinh\varphi & \cosh\varphi \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix}, \tag{2.5}$$

where the Bishop curvatures are defined by

$$\begin{cases} \kappa_1 = \kappa \cosh \varphi \ , \kappa_2 = \kappa \sinh \varphi, \varphi = \tanh^{-1}\left(\frac{\kappa_2}{\kappa_1}\right); \ \kappa_1 \neq 0, \\ \kappa = \sqrt{|\kappa_1^2 - \kappa_2^2|}. \end{cases} \tag{2.6}$$

- If γ is a timelike curve, then the Bishop frame of $\gamma = \gamma(s)$ is expressed as follows:

$$\begin{pmatrix} \mathbf{T}'(s) \\ \mathbf{N}_1'(s) \\ \mathbf{N}_2'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1(s) & \kappa_2(s) \\ \kappa_1(s) & 0 & 0 \\ \kappa_2(s) & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}_1(s) \\ \mathbf{N}_2(s) \end{pmatrix}, \tag{2.7}$$

where

$$\langle \mathbf{T}, \mathbf{T} \rangle = -1, \langle \mathbf{N}_1, \mathbf{N}_1 \rangle = 1, \langle \mathbf{N}_2, \mathbf{N}_2 \rangle = 1,$$

and the relation matrix between Serret-Frenet and Bishop frames is given by

$$\begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}_1(s) \\ \mathbf{N}_2(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix}, \tag{2.8}$$

where the Bishop curvatures are defined by

$$\begin{cases} \kappa_1 = \kappa \cos \varphi \ , \kappa_2 = \kappa \sin \varphi, \varphi = \tan^{-1}\left(\frac{\kappa_2}{\kappa_1}\right); \ \kappa_1 \neq 0, \\ \kappa = \sqrt{\kappa_1^2 + \kappa_2^2}. \end{cases} \tag{2.9}$$

[12].

3. Characterization of Bishop canal surfaces in $\mathbb{E}_1^3$

In $\mathbb{E}_1^3$, a canal surface $\mathbb{M}$ is defined as the envelope of a family of pseudospheres $\mathbb{S}_1^2$ centered along a space curve $c(s)$ and guided by the frame $\{\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2\}$. The surface can then be parameterized by:

$$\mathbb{M} = \mathbf{\Psi}(s,\theta) = c(s) + m_1(s,\theta)\mathbf{T} + m_2(s,\theta)\mathbf{N}_1 + m_3(s,\theta)\mathbf{N}_2, \tag{3.1}$$

where m_1, m_2, and m_3 are differentiable functions of s and θ. Moreover, if $\mathbb{M}$ is foliated by pseudospheres $\mathbb{S}_1^2$, it is classified as a surface of type $\mathbb{M}_+$. This class can be further divided into two types: if the spine curve $c(s)$ is spacelike, then $\mathbb{M}_+$ is said to be of type $\mathbb{M}_+^1$; whereas if $c(s)$ is timelike, it is considered of type $\mathbb{M}_+^2$. Additionally, the surface $\mathbb{M}_+^1$ is subdivided into $\mathbb{M}_+^{11}$ and $\mathbb{M}_+^{12}$, which are referred to as Bishop canal surfaces.

3.1. **Bishop canal surface $\mathbb{M}^{11}_{+}$.** We consider $\mathbb{M}$ as a canal surface generated by the motion of pseudospheres $\mathbb{S}^2_1$ along a spacelike center curve $c(s)$ belonging to the surface type $\mathbb{M}^{11}_{+}$. We then describe the parameterization of the canal surface $\mathbb{M}^{11}_{+}$ using the Bishop frame $\{\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2\}$ associated with the curve $c(s)$:

$$\mathbf{\Psi}(s,\theta) - c(s) = m_1(s,\theta)\mathbf{T} + m_2(s,\theta)\mathbf{N}_1 + m_3(s,\theta)\mathbf{N}_2. \tag{3.2}$$

By taking the derivative of the preceding equation with respect to s, we obtain

$$\mathbf{\Psi}_s(s,\theta) = \left(1 + m_{1s} - m_2\kappa_1 + m_3\kappa_2\right)\mathbf{T} + \left(m_1\kappa_1 + m_{2s}\right)\mathbf{N}_1 + \left(m_1\kappa_2 + m_{3s}\right)\mathbf{N}_2. \tag{3.3}$$

From equation $\|\mathbf{\Psi}(s,\theta) - c(s)\| = r^2$, we get

$$m_1^2 + m_2^2 - m_3^2 = r^2,$$

then

$$m_1 m_{1s} + m_2 m_{2s} - m_3 m_{3s} = rr'.$$

Given the fact that $\langle \mathbf{\Psi}(s,\theta) - c(s), \mathbf{\Psi}_s\rangle = 0$ and from the previous equations, we can conclude that

$$\begin{cases} m_1 = -rr', \\ m_2 = \mp r\sqrt{1-r'^2}\cosh\theta, \quad , \\ m_2 = \mp r\sqrt{1-r'^2}\sinh\theta. \end{cases} \tag{3.4}$$

then the equation of the canal surface can be expressed as

$$\mathbb{M}^{11}_{+} = \mathbf{\Psi}(s,\theta) = c(s) - rr'\mathbf{T} + r\sqrt{1-r'^2}\left(\cosh\theta\mathbf{N}_1 + \sinh\theta\mathbf{N}_2\right). \tag{3.5}$$

According to Eq.(3.5), it is reasonable to consider that $-r'(s) = \cos\varphi$, where φ is a smooth function depending on s, i.e., $\varphi = \varphi(s)$. Consequently, the canal surface $\mathbb{M}^{11}_{+}$ takes the following form:

$$\mathbf{\Psi}(s,\theta) = c(s) + r\cos\varphi\mathbf{T} + r\sin\varphi\left(\cosh\theta\mathbf{N}_1 + \sinh\theta\mathbf{N}_2\right). \tag{3.6}$$

Through differentiation with respect to s, we get

$$\mathbf{\Psi}_s(s,\theta) = \mathbf{\Psi}^1_s(s,\theta)\mathbf{T} + \mathbf{\Psi}^2_s(s,\theta)\mathbf{N}_1 + \mathbf{\Psi}^3_s(s,\theta)\mathbf{N}_2, \tag{3.7}$$

where

$$\begin{cases} \mathbf{\Psi}^1_s(s,\theta) = \sin^2\varphi - rr'' + r\sin\varphi w_1, \\ \mathbf{\Psi}^2_s(s,\theta) = r'\left(\sin\varphi\cosh\theta - r\kappa_1 - r\varphi'\cosh\theta\right), \\ \mathbf{\Psi}^3_s(s,\theta) = r'\left(\sin\varphi\sinh\theta - r\kappa_2 - r\varphi'\sinh\theta\right), \\ w_1 = \kappa_2\sinh\theta - \kappa_1\cosh\theta. \end{cases}$$

Also,

$$\mathbf{\Psi}_\theta(s,\theta) = \mathbf{\Psi}^1_\theta(s,\theta)\mathbf{T} + \mathbf{\Psi}^2_\theta(s,\theta)\mathbf{N}_1 + \mathbf{\Psi}^3_\theta(s,\theta)\mathbf{N}_2, \tag{3.8}$$

where

$$\begin{cases} \mathbf{\Psi}^1_\theta(s,\theta) = 0, \\ \mathbf{\Psi}^2_\theta(s,\theta) = r\sin\varphi\sinh\theta, \\ \mathbf{\Psi}^3_\theta(s,\theta) = r\sin\varphi\cosh\theta. \end{cases}$$

The component functions of the first fundamental form (I) are given by:

$$g_{11} = \langle \mathbf{\Psi}_s, \mathbf{\Psi}_s \rangle = = r^2\left(\sin^2\varphi\, w_1^2 + r'^2(\kappa_1^2 - \kappa_2^2) + \varphi'^2 - 2\varphi' w_1\right) - 2r(r'' - \sin\varphi\, w_1) + \sin^2\varphi,$$
$$g_{12} = \langle \mathbf{\Psi}_s, \mathbf{\Psi}_\theta \rangle = r^2 r' \sin\varphi w_2, \qquad \text{where} \qquad w_2 = \kappa_2\cosh\theta - \kappa_1\sinh\theta,$$
$$g_{22} = \langle \mathbf{\Psi}_\theta, \mathbf{\Psi}_\theta \rangle = -\, r^2\sin^2\varphi.$$

The unit normal vector field $\mathbf{U}$ of $\mathbb{M}_+^{11}$ is given by

$$\mathbf{U}(s,\theta) = \frac{\mathbf{\Psi}_s \times \mathbf{\Psi}_\theta}{\|\mathbf{\Psi}_s \times \mathbf{\Psi}_\theta\|} = \cos\varphi\mathbf{T} + \sin\varphi\cosh\theta\mathbf{N}_1 + \sin\varphi\sinh\theta\mathbf{N}_2$$

Applying differentiation to s, we get

$$\mathbf{U}_s = \mathbf{U}_s^1\mathbf{T}(s) + \mathbf{U}_s^2\mathbf{N}_1(s) + \mathbf{U}_s^3\mathbf{N}_2(s),$$

where

$$\begin{cases} \mathbf{U}_s^1 = -r'' + \sin\varphi w_1, \\ \mathbf{U}_s^2 = -r'\left(\kappa_1 + \varphi'\cosh\theta\right), \\ \mathbf{U}_s^3 = -r'\left(\kappa_2 + \varphi'\sinh\theta\right). \end{cases}$$

And

$$\mathbf{U}_\theta(s,\theta) = \mathbf{U}_\theta^2(s,\theta)\mathbf{N}_1 + \mathbf{U}_\theta^3(s,\theta)\mathbf{N}_2, \tag{3.9}$$

where

$$\begin{cases} \mathbf{U}_\theta^1(s,\theta) = 0, \\ \mathbf{U}_\theta^2(s,\theta) = \sin\varphi\sinh\theta, \\ \mathbf{U}_\theta^3(s,\theta) = \sin\varphi\cosh\theta. \end{cases}$$

The component functions of the second fundamental form (II) are expressed as:

$$\begin{cases} L = -\langle \mathbf{\Psi}_s, \mathbf{U}_s \rangle = -r\left(\varphi'^2 + r'^2\left(\kappa_1^2 - \kappa_2^2\right) - 2\varphi' w_1 + \sin^2\varphi\, w_1^2\right) + \left(r'' - \sin\varphi\, w_1\right), \\ M = -\langle \mathbf{\Psi}_\theta, \mathbf{U}_s \rangle = -rr'\sin\varphi\, w_2, \\ N = -\langle \mathbf{\Psi}_\theta, \mathbf{U}_\theta \rangle = r\sin^2\varphi. \end{cases}$$

Hence, the component functions of the third fundamental form (III) are expressed as:

$$\begin{cases} e_{11} = \langle \mathbf{U}_s, \mathbf{U}_s \rangle = \varphi'^2 + r'^2\left(\kappa_1^2 - \kappa_2^2\right) - 2\varphi' w_1 + \sin^2\varphi\, w_1^2, \\ e_{12} = \langle \mathbf{U}_\theta, \mathbf{U}_s \rangle = r'\sin\varphi\, w_2, \\ e_{22} = \langle \mathbf{U}_\theta, \mathbf{U}_\theta \rangle = -\sin^2\varphi. \end{cases}$$

We denote the first, second, and third fundamental forms by I, II, and III, respectively.

Lemma 3.1. *The fundamental forms I, II, and III of the Bishop canal surface* $\mathbb{M}_+^{11}$ *satisfy the following relations:*

$$L = \frac{g_{11} + \rho_1}{-r}, \quad M = \frac{g_{12}}{-r}, \quad N = \frac{g_{22}}{-r},$$
$$e_{11} = \frac{L - Q_1}{-r}, \quad e_{12} = \frac{M}{-r}, \quad e_{22} = \frac{N}{-r},$$

where

$$Q_1 = r'' - \sin\varphi\, w_1, \quad \rho_1 = rQ_1 - \sin^2\varphi.$$

From Lemma 3.1, the Gaussian and mean curvatures of $\mathbb{M}_+^{11}$ are given by respectively:

$$\begin{aligned}\kappa_G =& \frac{LN - M^2}{g_{11}g_{22} - g_{12}^2} = \frac{Q_1}{r\rho_1},\\ \kappa_M =& \frac{g_{11}N - 2g_{12}M + g_{22}L}{2(g_{11}g_{22} - g_{12}^2)} = \frac{-2\rho_1 - \sin^2\varphi}{2r\rho_1}.\end{aligned}$$

3.2. **Bishop canal surface $\mathbb{M}_+^{12}$.** Based on the definition of $\mathbb{M}_+^{12}$, we can calculate and obtain,

$$\mathbb{M}_+^{12} = \mathbf{\Psi}(s,\theta) = c(s) + \mu_1(s,\theta)\mathbf{T} + \mu_2(s,\theta)\mathbf{N}_1 + \mu_3(s,\theta)\mathbf{N}_2. \tag{3.10}$$

$$\begin{cases}\mu_1 = -rr',\\ \mu_2 = \mp r\sqrt{1-r'^2}\sinh\theta,\\ \mu_2 = \mp r\sqrt{1-r'^2}\cosh\theta.\end{cases} \tag{3.11}$$

Then, the equation of the canal surface can be expressed as:

$$\mathbb{M}_+^{12} = \mathbf{\Psi}(s,\theta) = c(s) - rr'\mathbf{T} + r\sqrt{1-r'^2}\left(\sinh\theta\mathbf{N}_1 + \cosh\theta\mathbf{N}_2\right). \tag{3.12}$$

Based on Eq. (3.12), we may consider the assumption $-r'(s) = \cos\varphi$, where φ is a smooth function depending on the arc-length parameter s, that is, $\varphi = \varphi(s)$. Under this assumption, the canal surface $\mathbb{M}_+^{12}$ is expressed as:

$$\mathbf{\Psi}(s,\theta) = c(s) + r\cos\varphi\mathbf{T} + r\sin\varphi\left(\cosh\theta\mathbf{N}_1 + \sinh\theta\mathbf{N}_2\right). \tag{3.13}$$

By calculations similar to those of $\mathbb{M}_+^{11}$, we obtain the following:

$$\begin{aligned}g_{11} =& r^2\left(\sin^2\varphi\, w_2^2 - r'^2(\kappa_1^2 - \kappa_2^2) + \varphi'^2 + 2\varphi' w_2\right) - 2r(r'' + \sin\varphi\, w_2) + \sin^2\varphi,\\ g_{12} =& -r^2 r'\sin\varphi w_1,\\ g_{22} =& -r^2\sin^2\varphi.\end{aligned}$$

The unit normal vector field $\mathbf{U}$ of $\mathbb{M}_+^{12}$ is given by

$$\mathbf{U}(s,\theta) = \frac{\mathbf{\Psi}_s \times \mathbf{\Psi}_\theta}{\|\mathbf{\Psi}_s \times \mathbf{\Psi}_\theta\|} = \cos\varphi\mathbf{T} + \sin\varphi\sinh\theta\mathbf{N}_1 + \sin\varphi\cosh\theta\mathbf{N}_2$$

$$\begin{cases}L = -r\left(\varphi'^2 - r'^2\left(\kappa_1^2 - \kappa_2^2\right) + 2\varphi' w_2 + \sin^2\varphi\, w_2^2\right) + \left(r'' + \sin\varphi\, w_2\right),\\ M = rr'\sin\varphi\, w_1,\\ N = r\sin^2\varphi.\end{cases}$$

$$\begin{cases}e_{11} = \varphi'^2 - r'^2\left(\kappa_1^2 - \kappa_2^2\right) + 2\varphi' w_2 + \sin^2\varphi\, w_2^2,\\ e_{12} = -r'\sin\varphi\, w_1,\\ e_{22} = -\sin^2\varphi.\end{cases}$$

Lemma 3.2. *The fundamental forms I, II, and III corresponding to the Bishop canal surface $\mathbb{M}^{12}_{+}$ are expressed as follows:*

$$L = \frac{g_{11}+\rho_2}{-r}, \quad M = \frac{g_{12}}{-r}, \quad N = \frac{g_{22}}{-r},$$
$$e_{11} = \frac{L-Q_2}{-r}, \quad e_{12} = \frac{M}{-r}, \quad e_{22} = \frac{N}{-r},$$

where

$$Q_2 = r'' + \sin\varphi\, w_2, \quad \rho_2 = rQ_2 - \sin^2\varphi.$$

From Lemma 3.2, the curvatures of $\mathbb{M}^{12}_{+}$ are given by, respectively:

$$\kappa_G = \frac{Q_2}{r\rho_2}, \qquad \kappa_M = \frac{-2\rho_2 - \sin^2\varphi}{2r\rho_2}.$$

3.3. **Bishop canal surface $\mathbb{M}^{2}_{+}$.** We can compute and derive, using the concept of timelike $\mathbb{M}^{2}_{+}$

$$\mathbb{M}^{2}_{+} = \mathbf{\Psi}(s,\theta) = c(s) + \nu_1(s,\theta)\mathbf{T} + \nu_2(s,\theta)\mathbf{N}_1 + \nu_3(s,\theta)\mathbf{N}_2. \tag{3.14}$$

$$\begin{cases} \nu_1 = rr', \\ \nu_2 = \mp r\sqrt{1+r'^2}\cos\theta, \\ \nu_2 = \mp r\sqrt{1+r'^2}\sin\theta. \end{cases} \tag{3.15}$$

Then, the equation of the canal surface can be expressed as:

$$\mathbb{M}^{2}_{+} = \mathbf{\Psi}(s,\theta) = c(s) + rr'\mathbf{T} + r\sqrt{1+r'^2}\left(\cos\theta\mathbf{N}_1 + \sin\theta\mathbf{N}_2\right). \tag{3.16}$$

From Eq.(3.16), we can assume that $r'(s) = \tan\varphi$ and $\varphi \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$. Then, the canal surface $\mathbb{M}^{2}_{+}$ can be written as:

$$\mathbf{\Psi}(s,\theta) = c(s) + r\tan\varphi\mathbf{T} + r\sec\varphi\left(\cos\theta\mathbf{N}_1 + \sin\theta\mathbf{N}_2\right). \tag{3.17}$$

$$\begin{aligned} g_{11} =& r^2\left(-\sec^2\varphi\, w_3^2 + r'^2(\kappa_1^2+\kappa_2^2) - \varphi'^2\sec^2\varphi - 2\varphi'\sec\varphi\, w_3\right) - 2r(r''+\sec\varphi\, w_3) - \sec^2\varphi, \\ g_{12} =& r^2 r'\sec\varphi\, w_4, \quad \text{where} \quad w_3 = \kappa_1\cos\theta + \kappa_2\sin\theta, w_4 = \kappa_2\cos\theta - \kappa_1\sin\theta \\ g_{22} =& r^2\sec^2\varphi. \end{aligned}$$

The unit normal vector field $\mathbf{U}$ of $\mathbb{M}^{2}_{+}$ is given by

$$\mathbf{U}(s,\theta) = \frac{\mathbf{\Psi}_s \times \mathbf{\Psi}_\theta}{\|\mathbf{\Psi}_s \times \mathbf{\Psi}_\theta\|} = \tan\varphi\mathbf{T} + \sec\varphi\cos\theta\mathbf{N}_1 + \sec\varphi\sin\theta\mathbf{N}_2$$

$$\begin{cases} L = -r\left(-\varphi'^2\sec^2\varphi + r'^2\left(\kappa_1^2+\kappa_2^2\right) - 2\varphi'\sec\varphi\, w_3 - \sec^2\varphi\, w_3^2\right) + \left(r'' + \sec\varphi\, w_3\right), \\ M = -rr'\sec\varphi\, w_4, \\ N = -r\sec^2\varphi. \end{cases}$$

$$\begin{cases} e_{11} = -\varphi'^2\sec^2\varphi + r'^2\left(\kappa_1^2+\kappa_2^2\right) - 2\varphi'\sec\varphi\, w_3 - \sec^2\varphi\, w_3^2, \\ e_{12} = r'\sec\varphi\, w_4, \\ e_{22} = \sec^2\varphi. \end{cases}$$

Lemma 3.3. *The Bishop canal surface $\mathbb{M}_+^2$ possesses the fundamental forms I, II, and III, which satisfy the following relations:*

$$L = \frac{g_{11} + \rho_3}{-r}, \quad M = \frac{g_{12}}{-r}, \quad N = \frac{g_{22}}{-r},$$
$$e_{11} = \frac{L - Q_3}{-r}, \quad e_{12} = \frac{M}{-r}, \quad e_{22} = \frac{N}{-r},$$

where

$$Q_3 = r'' + \sec\varphi\, w_3, \quad \rho_3 = rQ_3 + \sec^2\varphi.$$

From Lemma 3.3, the curvatures of $\mathbb{M}_+^2$ are given by, respectively:

$$\kappa_G = \frac{Q_3}{r\rho_3}, \qquad \kappa_M = \frac{\sec^2\varphi - 2\rho_3}{2r\rho_3}.$$

Proposition 3.1. *The curvatures κ_G and κ_M of the Bishop canal surfaces $\left\{\mathbb{M}_+^{11},\ \mathbb{M}_+^{12},\ \mathbb{M}_+^{2}\right\}$ can be represented by*

$$\kappa_M = -\frac{1}{2}\left(\kappa_G r + \frac{1}{r}\right).$$

4. Weingarten canal surfaces in $\mathbb{E}_1^3$

In this section, we investigate (u,v)-Weingarten canal surfaces (W-CS) in the Minkowski 3-space $\mathbb{E}_1^3$ based on the Bishop frame. We introduce key definitions and derive conditions under which these surfaces satisfy Weingarten relations. Special attention is given to the cases of surfaces of revolution and tube surfaces.

Definition 4.1. *Let (u,v) be a pair of distinct curvature functions, selected from κ_G and κ_M, associated with a canal surface $\mathbb{M}$. If the surface $\mathbb{M}$ satisfies the condition $\mathbf{\Phi}(u,v) = 0$, then it is called an (u,v)-(W-CS). Here, $\mathbf{\Phi}$ represents the Jacobi function given by $\mathbf{\Phi} = uv - vu$ [3].*

Definition 4.2. *Let (u,v) be two distinct curvature functions selected from κ_G and κ_M of a canal surface $\mathbb{M}$. If the surface satisfies a linear equation of the form $au + bv = c$, then it is called an (u,v)-linear (W-CS), where $(a,b,c) \in \mathbb{R}$ and $(a,b,c) \neq (0,0,0)$ [19].*

Lemma 4.1. *Partial derivatives of the Gaussian curvature κ_G and the mean curvature κ_M of the canal surface $\mathbb{M}_+^{11}$ are as follows*

$$\kappa_{G_s} = \frac{\sigma_1}{r^2\rho_1^2}, \quad \kappa_{G_\theta} = \frac{\sin^3\varphi\, w_2}{r\rho_1^2},$$

$$\kappa_{M_s} = \frac{\sigma_2}{2r^2\rho_1^2}, \quad \kappa_{M_\theta} = -\frac{\sin^3\varphi\, w_2}{2\rho_1^2},$$

where,

$$\sigma_1 = \sin^2\varphi(-2rr'w_1^2 - rr''' - r\sin\varphi\, w_1' + r'\sin\varphi\, w_1 + r'r'') - 4rr'r''^2 + 5rr'r''\sin\varphi\, w_1,$$

$$\sigma_2 = \sin^2\varphi(2r^2r'w_1^2 + r^2r''' + r^2\sin\varphi\, w_1' + 2rr'\sin\varphi\, w_1 - 2rr'r'' + r'\sin^2\varphi)$$
$$+ 4r^2r'r''^2 - 5r^2r'r''\sin^2\varphi\, w_1$$

Theorem 4.1. *The spacelike canal surface $\mathbb{M}_+^{11}$, with respect to the Bishop frame, satisfies the (κ_G, κ_M)-Weingarten condition if and only if it represents either a tubular surface or a surface generated by revolution.*

Proof. A (κ_G, κ_M)-(W-CS) $\mathbb{M}_+^{11}$ satisfies Jacobi equation $\kappa_{M_s}\kappa_{G_\theta} - \kappa_{M_\theta}\kappa_{G_s} = 0$, and form Proposition 3.1, we get

$$\left(\kappa_G r' - \frac{r'}{r^2}\right)\kappa_{G_\theta} = 0.$$

If $\kappa_{G_\theta} = 0$, then

$$\sin^2\varphi(\kappa_2\cosh\theta - \kappa_1\sinh\theta) = 0,$$

and for $\sin\varphi \neq 0$, we have $\kappa_2\cosh\theta - \kappa_1\sinh\theta = 0$ then $\kappa = 0$, which means $\mathbb{M}_+^{11}$ is a surface of revolution.
On the other side, if $\kappa_{G_\theta} \neq 0$, then

$$r'\left(\kappa_G - \frac{1}{r^2}\right) = 0,$$

this implies that $r' = 0$, indicating that r is constant and consequently $\mathbb{M}_+^{11}$ represents a tubular surface. □

We suppose that $\mathbb{M}_+^{11}$ is a surface generated by revolution $(i.e., \kappa_1 = \kappa_2 = 0)$. Then, by applying Lemma 3.1, we obtain:

$$\begin{aligned} \rho_1 &= rr'' - 1 + r'^2, \qquad Q_1 = r'', \\ \kappa_G &= \frac{r''}{r(rr'' - 1 + r'^2)}, \\ \kappa_M &= \frac{2rr'' - 1 + r'^2}{-2r(rr'' - 1 + r'^2)}. \end{aligned}$$

Therefore, the expressions for the partial derivatives of κ_G and κ_M can be written as follows:

$$\begin{cases} \kappa_{G_s} = \frac{1}{r^2}\left(\left(\frac{rr''}{rr''-1+r'^2}\right)' - \frac{2r'r''}{rr''-1+r'^2}\right), \\ \kappa_{M_s} = \frac{-1}{2}\left(\frac{1}{r}\left(\dfrac{rr''}{rr''-1+r'^2} + 1\right)\right)', \\ \kappa_{G_\theta} = \kappa_{M_\theta} = 0. \end{cases} \tag{4.1}$$

From Eq. (4.1), the Jacobi equation turns into an identity.
On the other hand, if $\mathbb{M}_+^{11}$ is a tube surface (which mean $r' = \cos\varphi = 0$ *i.e.*, $\varphi = \frac{n\pi}{2}$, n is an odd number). From Lemma 3.1, we get

$$\begin{cases} \rho_1 = -rw_1 - 1, & Q_1 = -w_1, \\ \kappa_G = \frac{w_1}{r(rw_1+1)}, & \kappa_M = \frac{2rw_1+1}{2r(rw_1+1)}, \\ \kappa_{G_s} = \dfrac{-w_1'}{r(rw_1+1)^2}, & \kappa_{G_\theta} = \dfrac{-w_2'}{r(rw_1+1)^2}, \\ \kappa_{M_s} = \dfrac{-w_1'}{2(rw_1+1)^2}, & \kappa_{M_\theta} = \dfrac{-w_2'}{2(rw_1+1)^2}. \end{cases} \tag{4.2}$$

From Eq. (4.2), the Jacobi equation is satisfied everywhere.

Lemma 4.2. *Partial derivatives of κ_G and κ_M of the canal surface $\mathbb{M}_+^{12}$ are as follows,*

$$\kappa_{G_s} = \frac{\sigma_3}{r^2\rho_2^2}, \quad \kappa_{G_\theta} = \frac{\sin^3\varphi\, w_1}{r\rho_2^2},$$

$$\kappa_{M_s} = \frac{\sigma_4}{2r^2\rho_2^2}, \quad \kappa_{M_\theta} = -\frac{\sin^3\varphi\, w_1}{2\rho_2^2},$$

where,

$$\begin{aligned} \sigma_3 &= \sin^2\varphi(-2rr'w_2^2 - rr''' - r\sin\varphi\, w_2' + r'\sin\varphi\, w_2 + r'r'') - 4rr'r''^2 - 5rr'r''\sin\varphi\, w_2, \\ \sigma_4 &= \sin^2\varphi(2r^2r'w_2^2 + r^2r''' + r\sin\varphi\, w_2' - 2r'\sin\varphi\, w_2 - 2rr'r'' + r'\sin^2\varphi) \\ &\quad + 4r^2r'r''^2 + 5r^2r'r''\sin^2\varphi\, w_2. \end{aligned}$$

Theorem 4.2. *The spacelike canal surface $\mathbb{M}_+^{12}$, described using the Bishop frame, satisfies the (κ_G, κ_M)-Weingarten condition if and only if it corresponds to either a tubular surface or one formed by revolution.*

Proof. A (κ_G, κ_M)-(W-CS) $\mathbb{M}_+^{12}$ satisfies Jacobi equation $\kappa_{M_s}\kappa_{G_\theta} - \kappa_{M_\theta}\kappa_{G_s} = 0$, and form Proposition 3.1, we get

$$\left(\kappa_G r' - \frac{r'}{r^2}\right)\kappa_{G_\theta} = 0.$$

If $\kappa_{G_\theta} = 0$, then

$$\sin^2\varphi(\kappa_2\sinh\theta - \kappa_1\cosh\theta) = 0,$$

and for $\sin\varphi \neq 0$, we have $\kappa_2\cosh\theta - \kappa_1\sinh\theta = 0$ then $\kappa = 0$, which means $\mathbb{M}_+^{12}$ is a surface of revolution.

On the other side, if $\kappa_{G_\theta} \neq 0$, then

$$r'\left(\kappa_G - \frac{1}{r^2}\right) = 0,$$

this yields $r' = 0$, implying that r is constant and hence $\mathbb{M}_+^{12}$ represents a tubular surface. □

Suppose that $\mathbb{M}_+^{12}$ is generated by revolution (that is, $\kappa_1 = \kappa_2 = 0$). Then, by applying Lemma 3.2, we obtain:

$$\rho_2 = rr'' - 1 + r'^2, \qquad Q_2 = r'',$$
$$\kappa_G = \frac{r''}{r(rr'' - 1 + r'^2)},$$
$$\kappa_M = \frac{2rr'' - 1 + r'^2}{-2r(rr'' - 1 + r'^2)}.$$

Thus, the partial derivatives of κ_G and κ_M are given by

$$\begin{cases} \kappa_{G_s} = \frac{1}{r^2}\left(\left(\frac{rr''}{rr''-1+r'^2}\right)' - \frac{2r'r''}{rr''-1+r'^2}\right), \\ \kappa_{M_s} = \frac{-1}{2}\left(\frac{1}{r}\left(\dfrac{rr''}{rr''-1+r'^2} + 1\right)\right)', \\ \kappa_{G_\theta} = \kappa_{M_\theta} = 0. \end{cases} \tag{4.3}$$

Based on Eq.(4.3), the Jacobi equation simplifies to an identity.

Alternatively, if $\mathbb{M}^{12}_+$ represents a tubular surface (i.e., when $r' = \cos\varphi = 0$, which occurs for $\varphi = \frac{n\pi}{2}$ with odd n), then from Lemma 3.2, we obtain:

$$\begin{cases} \rho_2 = -rw_2 - 1, \qquad Q_2 = w_2, \\ \kappa_G = \frac{w_2}{r(rw_2-1)}, \qquad \kappa_M = \frac{-2rw_2+1}{2r(rw_2-1)}, \\ \kappa_{G_s} = \dfrac{w_2'}{r(rw_2-1)^2}, \quad \kappa_{G_\theta} = \dfrac{w_1'}{r(rw_2-1)^2}, \\ \kappa_{M_s} = \dfrac{w_2'}{2(rw_2-1)^2}, \quad \kappa_{M_\theta} = \dfrac{w_1'}{2(rw_2-1)^2}. \end{cases} \tag{4.4}$$

From Eq. (4.4), the Jacobi equation is satisfied everywhere.

Lemma 4.3. *Partial derivatives of κ_G and κ_M of the canal surface $\mathbb{M}^2_+$ are as follows,*

$$\kappa_{G_s} = \frac{\sigma_5}{r^2\rho_3^2}, \quad \kappa_{G_\theta} = \frac{\sec^3\varphi\, w_4}{r\rho_3^2},$$

$$\kappa_{M_s} = \frac{\sigma_6}{2r^2\rho_3^2}, \quad \kappa_{M_\theta} = -\frac{\sec^3\varphi\, w_4}{2\rho_3^2},$$

where,

$$\sigma_5 = \sec^2\varphi(-2rr'w_3^2 + rr''' + r\sec\varphi\, w_3' - r'\sec\varphi\, w_3 - r'r'') - 4rr'r''^2 - 5rr'r''\sec\varphi\, w_3,$$
$$\sigma_6 = \sec^2\varphi(2r^2r'w_3^2 - r^2r''' - r^2\sec\varphi\, w_3' - 2rr'\sec\varphi\, w_3 - 2rr'r'' + r'\sec^2\varphi)$$
$$+ 4r^2r'r''^2 + 5r^2r'r''\sec^2\varphi\, w_2.$$

Theorem 4.3. *A spacelike canal surface $\mathbb{M}^2_+$, described in the context of the Bishop frame, satisfies the (κ_G, κ_M)-Weingarten condition if and only if it is either a tube or a surface generated by revolution.*

Proof. A (κ_G, κ_M)-(W-CS) $\mathbb{M}^2_+$ satisfies Jacobin equation $\kappa_{M_s}\kappa_{G_\theta} - \kappa_{M_\theta}\kappa_{G_s} = 0$, and form Proposition 3.1, we get

$$\left(\kappa_G r' - \frac{r'}{r^2}\right)\kappa_{G_\theta} = 0.$$

If $\kappa_{G_\theta} = 0$, then

$$\sin^2\varphi(\kappa_2\cos\theta - \kappa_1\sin\theta) = 0,$$

and for $\sin\varphi \neq 0$, we have $\kappa_2\cos\theta - \kappa_1\sin\theta = 0$ then $\kappa_1 = \kappa_2 = 0$, which means $\mathbb{M}^2_+$ is a surface of revolution.

On the other side, if $\kappa_{G_\theta} \neq 0$, then

$$r'\left(\kappa_G - \frac{1}{r^2}\right) = 0,$$

this results in $r' = 0$, indicating that r remains constant, and thus $\mathbb{M}^2_+$ is a tubular surface. □

Assume that $\mathbb{M}^2_+$ is a surface generated by revolution (i.e., $\kappa_1 = \kappa_2 = 0$). Then, based on Lemma 3.2, we obtain:

$$\begin{aligned} \rho_3 &= rr'' + 1 + r'^2, \qquad Q_3 = r'', \\ \kappa_G &= \frac{r''}{r(rr'' + 1 + r'^2)}, \\ \kappa_M &= -\frac{2rr'' + 1 + r'^2}{2r(rr'' + 1 + r'^2)}. \end{aligned}$$

Thus, the partial derivatives of κ_G and κ_M are given by

$$\begin{cases} \kappa_{G_s} = \left(\frac{rr''}{r(rr''+1+r'^2)}\right)', \\ \kappa_{M_s} = \left(-\frac{2rr'' + 1 + r'^2}{2r(rr'' + 1 + r'^2)}\right)', \\ \kappa_{G_\theta} = \kappa_{M_\theta} = 0. \end{cases} \tag{4.5}$$

According to Eq.(7.1), the Jacobi equation simplifies to an identity.

Conversely, if $\mathbb{M}^2_+$ is a tubular surface (i.e., when $r' = \cos\varphi = 0$, which occurs for $\varphi = \frac{n\pi}{2}$ with odd n), then from Lemma 3.3, we have:

$$\begin{cases} \rho_2 = rw_3 + 1, \qquad Q_2 = w_3, \\ \kappa_G = \frac{w_3}{r(rw_3+1)}, \qquad \kappa_M = \frac{-2rw_3-1}{2r(rw_3+1)}, \\ \kappa_{G_s} = \dfrac{w_3'}{r(rw_3+1)^2}, \qquad \kappa_{G_\theta} = \dfrac{w_4'}{r(rw_3+1)^2}, \\ \kappa_{M_s} = \dfrac{w_3'}{2(rw_3+1)^2}, \qquad \kappa_{M_\theta} = -\dfrac{w_4'}{2(rw_3+1)^2}. \end{cases} \tag{4.6}$$

From Eq. (4.6), the Jacobi equation is satisfied everywhere.

5. LINEAR WEINGARTEN CANAL SURFACES IN $\mathbb{E}_1^3$

This section explores (u, v)-linear (W-CS) in $\mathbb{E}_1^3$ defined via the Bishop frame. These surfaces satisfy a linear relation between their curvatures and are classified based on their geometric properties, such as being developable, minimal, tubes, or surfaces of revolution.

Theorem 5.1. *The spacelike canal surfaces $\mathbb{M}_+^{11}$ and $\mathbb{M}_+^{12}$, expressed with respect to the Bishop frame in $\mathbb{E}_1^3$, are developable if and only if they take the form of either a circular cylinder or a circular cone.*

Proof. The spacelike canal surface $\mathbb{M}_+^{11}$ is developable if and only if its Gaussian curvature κ_G vanishes. Based on Lemma 3.1, we obtain:

$$Q_1 = r'' - \sin\varphi\, w_1 = 0,$$

this implies that $r'' = 0$ and $\kappa_1 = \kappa_2 = 0$, that is, $\kappa = 0$.
Hence, the radius function takes the form $r(s) = as + b$, where a and b are constants satisfying $a \neq \pm 1$. If this condition fails, we obtain $\sin\varphi = 0$, which leads to a contradiction. Therefore, the surface $\mathbb{M}_+^{11}$ corresponds to a circular cylinder when $a = 0$, and becomes a circular cone when $b \neq 0$ and $a \neq \pm 1$. □

Theorem 5.2. *A timelike canal surface $\mathbb{M}_+^{2}$, defined relative to the Bishop frame in $\mathbb{E}_1^3$, is developable if and only if it is either a circular cylinder or a circular cone.*

Proof. It's similar to Theorem 5.1. □

Theorem 5.3. *The spacelike canal surfaces $\mathbb{M}_+^{11}$ and $\mathbb{M}_+^{12}$, with respect to the Bishop frame in $\mathbb{E}_1^3$, are minimal if and only if they are catenoids.*

Proof. The spacelike canal surface $\mathbb{M}_+^{11}$ is considered minimal if and only if its mean curvature κ_M vanishes. From Lemma 3.1, it follows that:

$$\begin{aligned} -2\rho_1 - \sin^2\varphi &= 0, \\ 2rr'' + 2r\sin\varphi\, w_1 + 2\sin^2\varphi - \sin^2\varphi &= 0, \\ 2rr'' + \sin^2\varphi = 0 \ \text{ and } \ 2r\sin\varphi\, w_1 &= 0. \end{aligned}$$

Since $r \neq 0$ and $\sin\varphi \neq 0$, we have

$$w_1 = \kappa_2 \sinh\theta - \kappa_1 \cosh\theta = 0,$$

which leads to $\kappa = 0$, and therefore $\mathbb{M}_+^{11}$ must be a surface of revolution. □

Theorem 5.4. *A timelike canal surface $\mathbb{M}_+^{2}$, described with respect to the Bishop frame in $\mathbb{E}_1^3$, is minimal if and only if it corresponds to a catenoid.*

Proof. It's similar to Theorem 5.3. □

We now examine the classical (u, v)-linear (W-CS). Without loss of generality, we may set $c = 1$ in the relation $au + bv = c$.

Theorem 5.5. *The spacelike canal surface $\mathbb{M}_{+}^{11}$ (or $\mathbb{M}_{+}^{12}$) satisfies the (κ_G, κ_M)-linear Weingarten condition if and only if it belongs to one of the following types:*

(i) *A tubular surface with radius $r = -\frac{b}{a}$,*

(ii) *A surface of revolution of the form:*

$$\mathbf{\Psi}(s,\theta) = (r\sin\varphi\cosh\theta, r\sin\varphi\sinh\theta, r\cos\varphi\ \theta)\,,$$

where

$$s = c_2 \mp \int \sqrt{\frac{r^2+br-a}{r^2+br-a-c_1}}dr.$$

Proof. If a surface satisfies the (κ_G, κ_M)-linear Weingarten condition, then it must fulfill the relation

$$a\kappa_G + b\kappa_M = 1,$$

where $a, b \in \mathbb{R}$ and $(a,b) \neq (0,0)$. Then,

$$\begin{aligned}\kappa_M &= \frac{1-a\kappa_G}{b},\\ &= -\frac{1}{2}\left(\kappa_G r + \frac{1}{r}\right).\end{aligned}$$

From Lemma 3.1, $\kappa_G\left(2ar - br^2\right) = b + 2r$, we get

$$\frac{\left(2ar-br^2\right)(r''-\sin\varphi\, w_1)}{r\left(rr''-r\sin\varphi\, w_1-\sin^2\varphi\right)} = b+2r,$$

this leads to

$$\begin{cases} 2\kappa_2\sin\varphi\left(r^2+br-a\right) - 2\kappa_1\sin\varphi\left(r^2+br-a\right) = 0, \\ -2r''\left(r^2+br-a\right) + \left(1-r'^2\right)(b+2r) = 0. \end{cases} \tag{5.1}$$

Case1: According to Eq. (5.1), if $\kappa_1 = \kappa_2 = 0$, it follows that $\kappa = 0$. As a result, $\mathbb{M}_{+}^{11}$ is a surface of revolution, and its radial function satisfies:

$$2r''\left(r^2+br-a\right) = \left(1-r'^2\right)(b+2r)\,.$$

By solving the above equation, we get

$$s = c_2 \mp \int \sqrt{\frac{r^2+br-a}{r^2+br-a-c_1}}dr,$$

where c_1, c_2 are constants.

Since $\kappa = 0$, we may, without loss of generality, take the spine curve as $c(s) = (0,0,s)$. The corresponding Bishop frame is

$$\mathbf{T} = (0,0,1), \quad \mathbf{N}_1 = (1,0,0), \quad \mathbf{N}_2 = (0,1,0).$$

Then, the surface $\mathbb{M}_{+}^{11}$ can be represented by:

$$\mathbf{\Psi}(s,\theta) = (r\sin\varphi\cosh\theta, r\sin\varphi\sinh\theta, r\cos\varphi + s)\,.$$

Case2: If $\kappa = 0$, then the relation $a - br - r^2 = 0$ holds. This gives $r = -\frac{b}{2}$, which is a non-zero constant, meaning $\mathbb{M}_{+}^{11}$ is a tubular surface. The constants a and b must satisfy $b^2 + 4a = 0$. Note that $\mathbb{M}_{+}^{11}$ becomes a circular cylinder when $\kappa_1 = \kappa_2 = \kappa = 0$ and the condition $r^2 + br - a = 0$ is fulfilled. □

Theorem 5.6. *The timelike canal surface $\mathbb{M}_{+}^{2}$ satisfies the (κ_G, κ_M)-linear Weingarten condition if and only if it is one of the following types:*

(i) *A tube with radius $r = -\frac{b}{2a}$,*

(ii) *A surface of revolution of the form:*

$$\mathbf{\Psi}(s,\theta) = (r \sec\varphi \cos\theta, r \sec\varphi \sin\theta, r \tan\varphi + s),$$

where

$$s = c_2 \mp \int \sqrt{\frac{a - r^2 - br}{a - r^2 - br - c_1}} dr.$$

Proof. It is similar to (5.5).

□

6. Singularities of the canal surfaces in $\mathbb{E}_1^3$

In this section, we analyze the singularities of canal surfaces in the Minkowski 3-space $\mathbb{E}_1^3$. Singular points are characterized by the vanishing of the Lorentzian vector product of the partial derivatives of the surface. We derive the conditions under which singularities occur for both spacelike and timelike canal surfaces.

Definition 6.1. *Let $\mathbf{\Psi}(s,\theta)$ be a surface in $\mathbb{E}_1^3$. Then the singular points are the points on the surface $\mathbf{\Psi}(s,\theta)$ such that $\mathbf{\Psi}_s \times \mathbf{\Psi}_\theta = 0$ [9].*

Theorem 6.1. *The point $\mathbb{M}_{+}^{11}(\mathbb{M}_{+}^{12}) = \mathbf{\Psi}(s_0,\theta_0)$ of a spacelike surface $\mathbb{M}_{+}^{11}(\mathbb{M}_{+}^{12}) = \mathbf{\Psi}(s,\theta)$ is a singular point if and only if*

$$\sin^2\varphi - rr'' + r\sin\varphi w_1 = 0 \tag{6.1}$$

Proof.

$$\mathbf{\Psi}_s \times \mathbf{\Psi}_\theta = \left(\sin^2\varphi - rr'' + r\sin\varphi w_1\right)(r\cos\varphi \mathbf{T} + r\sin\varphi\cosh\theta \mathbf{N}_1 + \sinh\theta \mathbf{N}_2)$$

Then $\sin^2\varphi - rr'' + r\sin\varphi w_1 = 0$. □

Corollary 6.1. *If the vector $\mathbb{M}_{+}^{11}(\mathbb{M}_{+}^{12})$ of $\mathbf{\Psi}_s^1$ is on the normal plane spanned by $\mathbf{N}_1$ and $\mathbf{N}_2$ then all points on the surface are singular.*

Corollary 6.2. *If spacelike surface $\mathbb{M}_{+}^{11}(\mathbb{M}_{+}^{12})$ is cyclindrical cylinder or circular cone then it has no singular points on $\mathbb{M}_{+}^{11}(\mathbb{M}_{+}^{12})$.*

Theorem 6.2. *The point $\mathbb{M}^2_+ = \mathbf{\Psi}(s_0, \theta_0)$ of a timelike surface $\mathbb{M}^{11}_+(\mathbb{M}^2_+) = \mathbf{\Psi}(s, \theta)$ is a singular point if and only if*

$$rr'' + r\sec\varphi\, w_3 + \sec^2\varphi = 0 \tag{6.2}$$

Corollary 6.3. *If the vector $\mathbb{M}^2_+$ of $\mathbf{\Psi}^1_s$ is on the normal plane spanned by $\mathbf{N}_1$ and $\mathbf{N}_2$ then all points on the surface are singular.*

Corollary 6.4. *If a timelike surface $\mathbb{M}^2_+$ is a cylindrical cylinder or circular cone, then it has no singular points on $\mathbb{M}^2_+$.*

7. Computational example

Let $c(s)$ be the center curve timelike $\aleph$ given as:

$$\aleph(s) = \left(\sqrt{6}s, \cos\left(\sqrt{5}s\right), \sin\left(\sqrt{5}s\right)\right),$$

the Frenet frame

$$\begin{cases} \mathbf{T} = \left(\sqrt{6}, -\sqrt{5}\sin\left(\sqrt{5}s\right),\ \sqrt{5}\cos\left(\sqrt{5}s\right)\right), \\ \mathbf{N} = \left(0, -\cos\left(\sqrt{5}s\right), -\sin\left(\sqrt{5}s\right)\right), \\ \mathbf{B} = \left(\sqrt{5},\ \sqrt{6}\sin\left(\sqrt{5}s\right), -\sqrt{6}\cos\left(\sqrt{5}s\right)\right), \\ \kappa = 5, \quad \tau = \sqrt{30}, \\ \varphi = \int_0^s \sqrt{30}ds = \sqrt{30}s. \end{cases} \tag{7.1}$$

Now, we can find the timelike Bishop frame as:

$$\begin{aligned} \mathbf{T} &= \left(\sqrt{6}, -\sqrt{5}\sin\left(\sqrt{5}s\right),\ \sqrt{5}\cos\left(\sqrt{5}s\right)\right), \\ \mathbf{N_1} &= \{-\sqrt{5}\sin\left(\sqrt{30}s\right), -\cos\left(\sqrt{5}s\right)\cos\left(\sqrt{30}s\right) - \sqrt{6}\sin\left(\sqrt{5}s\right)\sin\left(\sqrt{30}s\right), \\ &\quad -\sin\left(\sqrt{5}s\right)\cos\left(\sqrt{30}s\right) + \sqrt{6}\cos\left(\sqrt{5}s\right)\sin\left(\sqrt{30}s\right)\}, \\ \mathbf{N_2} &= \{\sqrt{5}\cos\left(\sqrt{30}s\right), -\cos\left(\sqrt{5}s\right)\sin\left(\sqrt{30}s\right) + \sqrt{6}\sin\left(\sqrt{5}s\right)\cos\left(\sqrt{30}s\right), \\ &\quad -\sin\left(\sqrt{5}s\right)\sin\left(\sqrt{30}s\right) - \sqrt{6}\cos\left(\sqrt{5}s\right)\cos\left(\sqrt{30}s\right)\}. \end{aligned}$$

when the radius function $r(s) = \sqrt{3}s$, the timelike Bishop canal surface (see Figure 1);

$$\begin{aligned} \mathbf{\Psi}(s,\theta) = &\ \sqrt{6}s + 9\sqrt{2}s - 2\sqrt{15}s\cos\theta\sin\sqrt{30}s + 2\sqrt{15}s\sin\theta\cos\sqrt{30}s, \\ &\cos\sqrt{5}s - 3\sqrt{15}s\sin\sqrt{5}s + 2\sqrt{3}s\cos\theta(-\cos\sqrt{5}s\cos\sqrt{30}s - \sqrt{6}\sin\sqrt{5}s\sin\sqrt{30}s) + \\ &2\sqrt{3}s\sin\theta(-\cos\sqrt{5}s\sin\sqrt{30}s + \sqrt{6}\sin\sqrt{5}s\cos\sqrt{30}s), \\ &\sin\sqrt{5}s + 3\sqrt{15}s\cos\sqrt{5}s + 2\sqrt{3}s\cos\theta(-\sin\sqrt{5}s\cos\sqrt{30}s + \sqrt{6}\cos\sqrt{5}s\sin\sqrt{30}s) + \\ &2\sqrt{3}s\sin\theta(-\sin\sqrt{5}s\sin\sqrt{30}s - \sqrt{6}\cos\sqrt{5}s\cos\sqrt{30}s. \end{aligned}$$

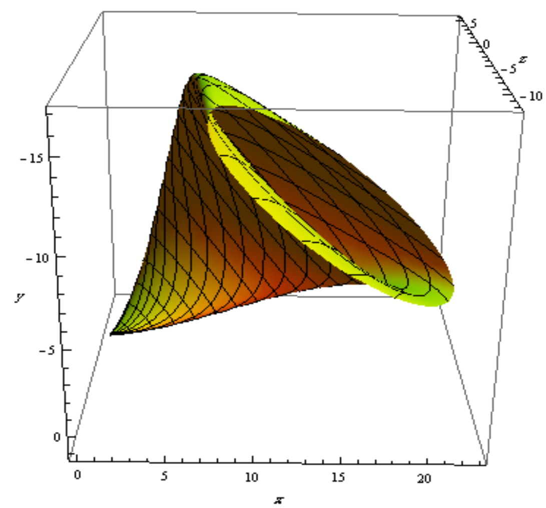

FIGURE 1. Timelike canal surface M_+^2 with $r(s) = \sqrt{3}s$

8. Conclusion

This study examines spacelike and timelike canal surfaces generated by $\mathbb{S}_1^2$ pseudo spheres in Minkowski 3-space utilizing the Bishop frame. $\mathbb{M}_+^{11}$, $\mathbb{M}_+^{12}$ represent spacelike Bishop canal surfaces, while $\mathbb{M}_+^2$ denotes a timelike Bishop canal surfaces. Linear Weingarten and Weingarten canal surfaces are categorized to display their geometric characteristics and singular points. Bishop canal surfaces can be examined in lightlike cone $\mathbb{Q}_1^2$ or hyperbolic space $\mathbb{H}_0^2$.

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