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Uniqueness of Fixed Points for Multi-Valued Mappings in Orthogonal Ultrametric Spaces

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Abstract. This research aims to prove that multi-valued mappings in orthogonal ultrametric space (O-UMS) have only one fixed point (FP). We achieve this result using a variety of contraction conditions, without assuming spherical completeness. This allows us to state fixed-point problems exactly. Additionally, we explore the implications of these results for integral equations and nonlinear fractional integral-differential equations. By utilizing these contractions, our research contributes to a better understanding of O-UMS.

1. Introduction

Fixed-point is a well-known mathematical theory that has a wide range of applications. It uses contraction as a primary tool to establish a FP's existence and uniqueness. There are three primary subjects in the theory of FP: topological, metric, and discrete. Metric fixed-point theory studies the FPs of mappings in metric spaces, which are points that remain unchanged under the function. This theory provides an important framework for analyzing the existence, uniqueness,

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and properties of these FPs, and it has many applications across several disciplines, such as mathematics, physics, economics, computer science, and engineering. The Banach fixed-point theorem, also known as Banach's contraction principle, is one of the most well-known theorems about FPs. It is constructive and provides a solution to the existence and uniqueness of the operator equation $Tx = x$. It is also very useful for studying nonlinear equations. The Banach contraction principle suggests that if you have a contractive mapping on a complete metric space, then there is a FP. This can be used to solve many mathematical problems and prove that there are only one or more solutions.Several researchers have made significant contributions by developing, extending, and generalizing theorems in different areas.

The multi-valued mappings theory, combining analysis, topology, and geometry, has fascinated researchers in several mathematical disciplines.Applied mathematics problems often need to use multi-valued mappings instead of single-valued maps, which are used in traditional analysis. For instance, fixed-point methods for multi-valued mappings can solve stability and control theory. It's easy for learners to understand the idea of set-valued maps when they look at the equivalents of basic trigonometric functions. This shows the significance of multi-valued mappings for solving problems in mathematics. Multi-valued maps are interesting to mathematicians from many different fields because they are at the interface of analysis, topology, and geometry. Although classical analysis is concerned with mappings with a single value, numerous findings in applied mathematics involve multi-valued mappings. For example, fixed-point techniques for multi-valued mappings could be used to address problems in the fields of stability and control theory. While the researchers study the inverses of simple trigonometric functions, they first encounter a set-valued map. While they solve mathematical problems, it helps them understand multi-valued mappings.

The use of the Hausdorff metric to investigate FPs for multi-valued mappings was initially done by Markin [23]. In a subsequent work, Nadler [24] expanded the scope of the Banach contraction principle to include multi-valued contraction maps in complete metric spaces. Several researchers have extensively studied the suitability of the multi-valued version of the traditional Fixed Point Theorem (FPT) for certain mappings (see references [6, 20]). Aubin [30] was the first to investigate the notion of fixed points for multi-valued mappings, which is a significant concept that falls between single-valued maps and multi-valued maps. See [5, 29] for more information on FP theory in metric spaces.

The topic of non-Archimedean analysis was officially recognized in 1943 with the publication of Monna's collection of publications. Several authors have used Van Rooij's work on non-Archimedean Banach spaces (see [4], [27]), which has made significant effects on the success of researchers in this field. Gajic et al. [15] conducted a recent study where they utilized generalized contraction to obtain fixed-point results in an UMS. On the other hand, Rao et al. [28] established sufficient criteria for the existence of coincidence points in the case of three and four self-mappings. The criteria mentioned above are based on certain contractive conditions. For further information, see [21] articles and their cited references.

In 2016, Alaca et al. [1] discovered fixed-point outcomes for modular UMSs. Further, UMSs have distinct characteristics that differentiate them from other metric spaces. These spaces exhibit ultrametric inequality, resulting in unique geometric and topological features that lead to different outcomes when compared to standard metric spaces. In particular, we construct theorems for UMSs that take into account their unique characteristics. The existence of FPT in non-Archimedean normed and partially ordered ultrametric spaces was demonstrated by Mamghaderi [22] in 2017. His research focused on mappings that are both single-valued and strongly contracted. Additionally, Ramesh Kumar and Pitchaimani [27] examined precise-rich mappings in UMSs for contractions and set-valued contractions. Also, Almalki et al. [2] used different contractions to find common FP results in modular UMSs.

Initially, Gordji et al. introduced the concept of orthogonality in their work [19] and established FP theorems within the framework of orthogonally complete metric spaces. Researchers have been developing and expanding orthogonal metric spaces since their initial development. Some important progress has been made in this area by (see [9,16–18]). These researchers have expanded upon the original ideas, exploring new properties, applications, and FP theorems in orthogonal metric spaces. Their work greatly improved our understanding of orthogonal structures and their application in mathematical analysis.

Ultrametric spaces distinguish themselves from standard metric spaces due to a unique property known as ultrametric inequality. In these spaces, the distance between two points is always less than or equal to the maximum of their individual distances from a third point, contrasting with the typical triangle inequality found in standard metric spaces. The investigation of FPs in UMSs produces results that are both interesting. The Banach Fixed Point result for UMSs is a well-known result in this field. It says that if a contraction mapping is given to a complete O-UMS, it will have a single UFP. A contraction mapping is a function that reduces the distances between points, whereas completeness ensures that the space encompasses all its limit points.

1.1. **The frame work of this study.**

We have divided it into five sections. In Section 1, we discuss the motivation and background of this study. Section 2 discusses the preliminary results, and Section 3 covers the main results. Section 4 contains applications to integral equations. Section 5 presents the conclusion. The goal of this work is to find out if there are any unique FPs for multi-valued maps that meet more generalized contraction conditions in an O-UMS. Furthermore, we explore the application of integral equations for fixed-point problems.

2. Preliminaries

Here, we begin this section of our research by defining the following frequently used terms:

Definition 2.1. *[26] If all Cauchy sequences converge, we say that the space is complete in the ultrametric system. Assume that* $(\mho, \mathsf{B})$ *is an extreme geometry space. We say that* $CB(\mho)$ *is the* *collection of all closed, non-void subsets of $\mho$ that are bounded. i.e., Π is a Hausdorff metric.*

$$\Pi(\mathsf{A}^*,\mathsf{B}^*)=\max\left\{sup_{a\in\mathsf{A}}D(a,\mathsf{B}^*),sup_{b\in\mathsf{B}}D(b,\mathsf{A}^*)\right\},$$

for A^*, B^* in $CB(\mho)$, where $\mathsf{B}(x,\mathsf{B}^*)=\inf_{y\in\mathsf{B}^*}\mathsf{B}(x,y)$. It is evident that this is a UMS.

Definition 2.2. *[26] The map $\Pi:\mho\to CB(\mho)$ and $\mathsf{j}:\mho\to\mho$ so that, given a non-empty set $\mho$,*
(a) A point of coincidence between j and Π is defined as $\mathsf{u}\in\mho$ if $\mathsf{u}=\mathsf{j}(\gimel)\in\Pi(\gimel)$.
(b) A CFP of j and Π is said to be $\gimel\in\mho$ if $\gimel=\mathsf{j}(\gimel)\in\Pi(\gimel)$.
(c) If $\gimel\in\Pi(\gimel)$, then $\gimel\in\mho$ is said to be a FP of Π.

Remark 2.1. *Consider an UMS $(\mho,\mathsf{B})$, and $\mathcal{A},\mathcal{B}\in CB(\Omega)$. Then, there exist $\hbar\in\mathcal{B}$ such that, for any $\gimel\in\mathcal{A}$ and $\epsilon\geq 0$,*

$$\mathcal{B}(\gimel,\hbar)\leq\Pi(\mathcal{A},\mathcal{B})+\epsilon.$$

The idea of an O-set was first presented by Gordji et al. [19], who also provided examples and details on these sets.

Definition 2.3. *[19] Consider a non-empty set $\mho$ and a binary relation $\perp$ that is a subset of $\mho\times\mho$. If a line is perpendicular and satisfies the following condition:*

$$\exists\gimel_o:(\forall\hbar,\hbar\perp\gimel_0)\ or\ (\forall\hbar,\gimel_0\perp\hbar)$$

it is known as an orthogonal set, often abbreviated as O-set. We can write this O-set as $(\mho,\perp)$.

Example 2.1. *[19] Let $\mho$ represent the collection that includes all persons globally. Let us consider the binary relation $\perp$ on $\mho$, where $\gimel\perp\hbar$ if $\gimel$ has the ability to donate blood to $\hbar$. According to the below table, if a person has blood type O-, then they are incompatible with all other blood types. This suggests that the combination of $(\mho,\perp)$ creates an O-set. It's worth mentioning that the selection of $\gimel_0$ in this O-set is not limited to just one option.*

In the given example, $\gimel_0$ could possibly represent an individual with blood type $AB+$. For all values of $\hbar$ in $\mho$, it is true that $\hbar\perp\gimel_0$ in this situation.

Definition 2.4. *[19] Consider $(\mho,\perp)$ as an O-set. A sequence $\gimel_n$ is considered an O-sequence if*

$$(\forall n,\gimel_n\perp\gimel_{n+1})\ or\ (\forall n,\gimel_{n+1}\perp\gimel_n).$$

Definition 2.5. *[19] Consider the O-metric space $(\mho,\mathsf{B})$. A mapping $\Pi:\mho\to\mho$ is considered to be O-continuous at $\gimel\in\mho$ if, for any O-sequence $\gimel_n\in\mho$ such that the distance between $\gimel_n$ and $\gimel$ approaches zero, the distance between $\Pi\gimel_n$ and $\Pi\gimel$ also approaches zero.*

Definition 2.6. *[19] Let $(\mho,\perp)$ be an O-set with the metric B. It is said that the triplet $(\mho,\mathsf{B},\perp)$ is O-complete if every O-Cauchy sequence converges.*

Table 1. Depiction of a blood types

Type	You can give blood to	You can receive blood from
$AB-$	$AB+, AB-$	$AB-, B-, O-, A-$
$B-$	$B+, B-, AB+, AB-$	$B-, O-$
$O-$	Every one	$O-$
$A-$	$A+, A-, AB+, AB-$	$A-, O-$
$AB+$	$AB+$	Every one
$B+$	$B+, AB+$	$B+, B-, O+, O-$
$O+$	$O+, A+, B+, AB+$	$O+, O-$
$A+$	$A+, AB+$	$A+, A-, O+, O-$

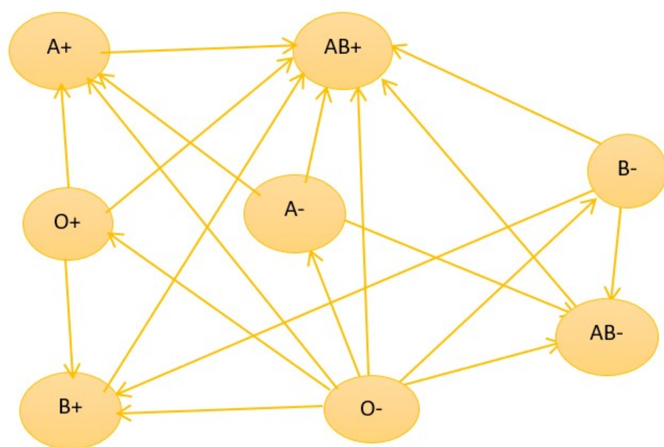

Figure 1. Visual depiction of a blood example.

Definition 2.7. *[19] Consider an O-metric space denoted by* $(\mho, \mathsf{B}, \perp)$*, where* $\mho$ *is the set of elements,* B *is the collection of subsets, and* $\perp$ *represents the distance function. Additionally, suppose that* λ *is a real number such that* $0 < \lambda < 1$*. A map* $\Pi : \mho \to \mho$ *is referred to as an O-contraction with a Lipschitz constant of* λ*. If, for every pair of elements* a *and* b *in* $\mho$ *such that* a *is perpendicular to* b*, the following condition is satisfied:*

$$\mathsf{B}(\Pi\daleth, \Pi\hbar) \leq \lambda \mathsf{B}(\daleth, \hbar).$$

Definition 2.8. *[19] Let* $(\mho, \perp)$ *be an O-set. A function* $\Pi : \mho \to \mho$ *is termed O-preserving if for any* $\daleth, \hbar \in \mho$*,* $\daleth \perp \hbar$ *implies* $\Pi\daleth \perp \Pi\hbar$*.*

Definition 2.9. *[31] Consider* $\mathbb{A}$ *and* $\mathbb{B}$ *are distinct non-void subsets of the O-set* $(\mho, \perp)$*.* $\perp$ *represents the relation stating that set* $\mathbb{A}$ *is O-related to set* $\mathbb{B}$*.* $\mathbb{A} \perp \mathbb{B}$ *if* $\daleth \in \mathbb{A}$ *and* $\hbar \in \mathbb{B}$*, then* $\daleth \perp \hbar$*.*

Theorem 2.1. *[31] Consider an O-complete O-metric space denoted by* $(\mho, \perp, \mathsf{B})$ *and* $\Pi : \mho \to \mathcal{CB}(\otimes)$ *be a multi-valued mapping on* $\mho$*. If these conditions hold:*

i) *there exist* $\gimel_0 \in \mho$ *such that* $\{\gimel_0\} \perp \Pi\gimel_0$ *or* $\Pi\gimel_0 \perp \{\gimel_0\}$.

ii) *for all* $\gimel, \hbar \in \mho$, $\gimel \perp \hbar$ *implies* $\Pi\gimel \perp \Pi\hbar$.

iii) *if* $\{\gimel_n\}$ *is an O-sequence in* $\mho$ *such that* $\gimel_n \to \gimel^* \in \mho$, *then* $\gimel_n \perp \gimel^*$ *or* $\gimel^* \perp \gimel_n$ *for all* $n \in \mathbb{N}$.

3. Main Results

This section uses a multi-valued O-contraction map to demonstrate the unique FPT in O-complete O-UMS. Our primary outcomes provide the following benefits:

1. Introduce the concept of an O-UMS.
2. The FP theorem for self-mappings defined on O-UMSs is given using extensions of O-multi-valued contractions.
3. We aim to determine whether the integral equation possesses a unique solution by applying our primary findings.

Definition 3.1. *Let* $(\Omega, \mathsf{B}, \perp)$ *be called an orthogonal ultrametric space (O- UMS) if* $(\Omega, \perp)$ *is an orthogonal set and* (Ω, B) *is an ultrametric space.*

Theorem 3.1. *Let* $(\mho, \mathsf{B}_\perp)$ *be a complete O-UMS and* $\Pi : \mho \to \mathcal{CB}(\otimes)$ *be a mapping that fulfills*

$$\Bbbk(\Pi\gimel, \Pi\hbar) \leq \alpha \mathsf{B}_\perp(\gimel, \hbar) + \beta \mathsf{B}_\perp(\gimel, \Pi\gimel) + \gamma \mathsf{B}_\perp(\hbar, \Pi\hbar) + \delta[\check{\mathsf{D}}(\gimel, \Pi\hbar) + \check{\mathsf{D}}(\hbar, \Pi\gimel)] \tag{3.1}$$

for all $\gimel, \hbar \in \mho$, *the conditions are* $\alpha, \beta, \gamma, \delta \geq 0$ *and* $2\delta + \alpha + \beta + \gamma < 1$. *Then* Π *has a UFP in* $\mho$.

Proof. For $\gimel_0 \in \gimel, \gimel_1 \in \Pi\gimel_0$,

$$[\gimel_0 \perp \gimel_1 \text{ or } \gimel_1 \perp \gimel_0], \text{ and } [\Pi\gimel_0 \perp \Pi\gimel_1 \text{ or } \Pi\gimel_1 \perp \Pi\gimel_0],$$

define $\mathsf{j} = \dfrac{\alpha + \beta + \delta}{1 - \gamma - \delta}$. Assume that $\mathsf{j} = 0$, then the proof is trivial. Consider $\mathsf{j} > 0$, then there exists $\gimel_2 \in \Pi\gimel_1$, $(\Pi\gimel_1 \perp \gimel_2)$ such that

$$\mathsf{B}_\perp(\gimel_1, \gimel_2) \leq \Bbbk(\Pi\gimel_0, \Pi\gimel_1) + \mathsf{j}.$$

Now, $\Pi\gimel_1, \Pi\gimel_2 \in CB(\mho)$ and $\gimel_2 \in \Pi\gimel_1$, there exist $\gimel_3 \in \Pi\gimel_2$ such that

$$\mathsf{B}_\perp(\gimel_2, \gimel_3) \leq \Bbbk(\Pi\gimel_1, \Pi\gimel_2) + \mathsf{j}^2.$$

To continue in this manner, for $\gimel_{n-1}, n > 1$, we obtain $\gimel_n \in \Pi\gimel_{n-1}$ satisfying the following

$$\mathsf{B}_\perp(\gimel_{n-1}, \gimel_n) \leq \Bbbk(\Pi\gimel_{n-2}, \Pi\gimel_{n-1}) + \mathsf{j}^{n-1}.$$

Now, using (3.1), for $n \in \mathbb{N}$, we have

$$\begin{aligned}
\mathsf{B}_\perp(\gimel_{n-1}, \gimel_n) &\leq \Bbbk(\Pi\gimel_{n-2}, \Pi\gimel_{n-1}) + \mathsf{j}^{n-1} \\
&\leq \alpha \mathsf{B}_\perp(\gimel_{n-2}, \gimel_{n-1}) + \beta \check{\mathsf{D}}(\gimel_{n-2}, \Pi\gimel_{n-2}) + \gamma \check{\mathsf{D}}(\gimel_{n-1}, \Pi\gimel_{n-1}) \\
&\quad + \delta[\check{\mathsf{D}}(\gimel_{n-2}, \Pi\gimel_{n-1}) + \check{\mathsf{D}}(\gimel_{n-1}, \Pi\gimel_{n-2})] \\
&\leq \alpha \mathsf{B}_\perp(\gimel_{n-2}, \gimel_{n-1}) + \beta \mathsf{B}_\perp(\gimel_{n-2}, \gimel_{n-1}) \\
&\quad + \gamma \mathsf{B}_\perp(\gimel_{n-1}, \gimel_n) + \delta[\mathsf{B}_\perp(\gimel_{n-2}, \gimel_n) + \mathsf{B}_\perp(\gimel_{n-1}, \gimel_{n-1})]
\end{aligned}$$

$$\begin{aligned}
&\le \alpha \mathsf{B}_\perp(\gimel_{n-2},\gimel_{n-1}) + \beta \mathsf{B}_\perp(\gimel_{n-2},\gimel_{n-1}) + \gamma \mathsf{B}_\perp(\gimel_{n-1},\gimel_n) + \delta \mathsf{B}_\perp(\gimel_{n-2},\gimel_n) \\
&\le \alpha \mathsf{B}_\perp(\gimel_{n-2},\gimel_{n-1}) + \beta \mathsf{B}_\perp(\gimel_{n-2},\gimel_{n-1}) + \gamma \mathsf{B}_\perp(\gimel_{n-1},\gimel_n) \\
&\quad + \delta\{\mathsf{B}_\perp(\gimel_{n-2},\gimel_{n-1}) + \mathsf{B}_\perp(\gimel_{n-1},\gimel_n)\} + \mathsf{j}^{n-1}.
\end{aligned}$$

Hence,

$$(1-\gamma-\delta)\mathsf{B}_\perp(\gimel_{n-1},\gimel_n) \le (\alpha+\beta+\delta)\mathsf{B}_\perp(\gimel_{n-2},\gimel_{n-1}) + \mathsf{j}^{n-1},$$

which implies

$$\begin{aligned}
\mathsf{B}_\perp(\gimel_{n-1},\gimel_n) &\le \frac{\alpha+\beta+\delta}{1-\gamma-\delta}\mathsf{B}_\perp(\gimel_{n-2},\gimel_{n-1}) + \frac{\mathsf{j}^{n-1}}{1-\gamma-\delta} \\
&\le \mathsf{j}\mathsf{B}_\perp(\gimel_{n-2},\gimel_{n-1}) + \frac{\mathsf{j}^{n-1}}{1-\gamma-\delta}.
\end{aligned}$$

Thus, we obtain

$$\mathsf{B}_\perp(\gimel_{n-1},\gimel_n) \le \mathsf{j}^{n-1}\mathsf{B}_\perp(\gimel_0,\gimel_1) + (n-1)\frac{\mathsf{j}^{n-1}}{1-\gamma-\delta}.$$

Observe that the sequence $\{\gimel_n\}$ is Cauchy in $\mho$ as $\mathsf{j} < 1$. Given that $\mho$ is complete, it implies that $\{\gimel_n\}$ converges to a point $\aleph \in \mho$. In other words,

$$\lim_{n\to\infty} \gimel_n = \aleph.$$

Now,

$$\begin{aligned}
\check{\mathsf{D}}(\aleph,\Pi\aleph) &\le \max\{\check{\mathsf{D}}(\aleph,\gimel_n), \check{\mathsf{D}}(\gimel_n,\Pi\aleph)\} \\
&\le \max\{\check{\mathsf{D}}(\aleph,\gimel_n), \Bbbk(\Pi\gimel_{n-1},\Pi\aleph)\} \\
&\le \max\{\mathsf{B}_\perp(\aleph,\gimel_n), \alpha\mathsf{B}_\perp(\gimel_{n-1},\aleph) + \beta\check{\mathsf{D}}(\gimel_{n-1},\Pi\gimel_{n-1}) \\
&\quad +\gamma\check{\mathsf{D}}(\aleph,\Pi\aleph) + \delta[\check{\mathsf{D}}(\gimel_{n-1},\Pi\aleph) + \check{\mathsf{D}}(\aleph,\gimel_n)]\} \\
&\le \max\{\mathsf{B}_\perp(\aleph,\gimel_n), \alpha\mathsf{B}_\perp(\gimel_{n-1},\aleph) + \beta\mathsf{B}_\perp(\gimel_{n-1},\gimel_n) \\
&\quad +\gamma\check{\mathsf{D}}(\aleph,\Pi\aleph) + \delta[\check{\mathsf{D}}(\gimel_{n-1},\Pi\aleph) + \check{\mathsf{D}}(\aleph,\gimel_n)]\}.
\end{aligned}$$

Allowing limit $n \to \infty$, we have

$$\check{\mathsf{D}}(\aleph,\Pi\aleph) \le (\gamma+\delta)\check{\mathsf{D}}(\aleph,\Pi\aleph),$$

which shows that $\check{\mathsf{D}}(\aleph,\Pi\aleph) = 0$, as $\gamma+\delta < 1$. Hence, Π has a FP $\aleph \in \mho$.
To prove Uniqueness:- Let us consider another FP $\aleph'$ of Π. Now using (3.1), we obtain

$$[\aleph' \perp \aleph \text{ or } \aleph \perp \aleph'] \text{ and } [\Pi\aleph' \perp \Pi\aleph \text{ or } \Pi\aleph \perp \Pi\aleph'].$$

Now,

$$\begin{aligned}
\mathsf{B}_\perp(\aleph,\aleph') &\le \Bbbk(\{\aleph\},\{\aleph'\}) = \Bbbk(\Pi\aleph,\Pi\aleph') \\
&\le \alpha\mathsf{B}_\perp(\aleph,\aleph') + \beta\check{\mathsf{D}}(\aleph,\Pi\aleph) + \gamma\check{\mathsf{D}}(\aleph',\Pi\aleph') + \delta[\check{\mathsf{D}}(\aleph,\Pi\aleph') + \check{\mathsf{D}}(\aleph',\Pi\aleph)] \\
&= (\alpha+2\delta)\mathsf{B}_\perp(\aleph,\aleph'),
\end{aligned}$$

which indicates $\aleph = \aleph'$ as $\alpha + 2\delta < 1$.

□

Theorem 3.2. *Let $(\mho, \mathsf{B}_\perp)$ be a complete O-UMS and $\Pi : \mho \to \mathcal{CB}(\otimes)$ be a mapping fulfills*

$$\Bbbk(\Pi\gimel, \Pi\hbar) \leq \alpha \mathsf{B}_\perp(\gimel, \hbar) + \beta \mathsf{B}_\perp(\gimel, \Pi\gimel) + \gamma \mathsf{B}_\perp(\hbar, \Pi\hbar) + \delta[\check{\mathsf{D}}(\gimel, \Pi\hbar) + \check{\mathsf{D}}(\hbar, \Pi\gimel)] + \lambda[\check{\mathsf{D}}(\gimel, \Pi\hbar) + \check{\mathsf{D}}(\hbar, \Pi\gimel)] \tag{3.2}$$

for all $\gimel, \hbar \in \mho$, wherein $2\delta + 2\lambda + \alpha + \beta + \gamma < 1$ and $\delta, \lambda, \alpha, \beta, \gamma \geq 0$. Then Π has a UFP in $\mho$.

Proof. For $\gimel_0 \in \gimel, \gimel_1 \in \Pi\gimel_0$,

$$[\gimel_0 \perp \gimel_1 \text{ or } \gimel_1 \perp \gimel_0], \text{ and } [\Pi\gimel_0 \perp \Pi\gimel_1 \text{ or } \Pi\gimel_1 \perp \Pi\gimel_0],$$

define $\mathsf{j} = \dfrac{\alpha + \beta + \delta + \lambda}{1 - (\gamma + \delta + \lambda)}$. Assume that $\mathsf{j} = 0$, then the proof is trivial. Consider $\mathsf{j} > 0$, then there exists $\gimel_2 \in \Pi\gimel_1$ such that

$$\mathsf{B}_\perp(\gimel_1, \gimel_2) \leq \Bbbk(\Pi\gimel_0, \Pi\gimel_1) + \mathsf{j}.$$

Now, $\Pi\gimel_1, \Pi\gimel_2 \in \mathcal{CB}(\otimes)$ and $\gimel_2 \in \Pi\gimel_1$, there exist $\gimel_3 \in \Pi\gimel_2$ such that

$$\mathsf{B}_\perp(\gimel_2, \gimel_3) \leq \Bbbk(\Pi\gimel_1, \Pi\gimel_2) + \mathsf{j}^2.$$

Continuing in a similar manner, for $\gimel_{n-1}, n > 1$, we obtain $\gimel_n \in \Pi\gimel_{n-1}$ satisfies the following

$$\mathsf{B}_\perp(\gimel_{n-1}, \gimel_n) \leq \Bbbk(\Pi\gimel_{n-2}, \Pi\gimel_{n-1}) + \mathsf{j}^{n-1}.$$

Now, using equation (3.2), for $n \in \mathbb{N}$, we have

$$\begin{aligned}
\mathsf{B}_\perp(\gimel_{n-1}, \gimel_n) \leq\ & \alpha \mathsf{B}_\perp(\gimel_{n-2}, \gimel_{n-1}) + \beta \check{\mathsf{D}}(\gimel_{n-2}, \Pi\gimel_{n-2}) + \gamma \check{\mathsf{D}}(\gimel_{n-1}, \Pi\gimel_{n-1}) \\
& + \delta[\check{\mathsf{D}}(\gimel_{n-2}, \Pi\gimel_{n-2}) + \check{\mathsf{D}}(\gimel_{n-1}, \Pi\gimel_{n-1})] \\
& + \lambda[\check{\mathsf{D}}(\gimel_{n-2}, \Pi\gimel_{n-1}) + \check{\mathsf{D}}(\gimel_{n-1}, \Pi\gimel_{n-2})] + \mathsf{j}^{n-1} \\
\leq\ & \alpha \mathsf{B}_\perp(\gimel_{n-2}, \gimel_{n-1}) + \beta \mathsf{B}_\perp(\gimel_{n-2}, \gimel_{n-1}) + \gamma \mathsf{B}_\perp(\gimel_{n-1}, \gimel_n) \\
& + \delta[\mathsf{B}_\perp(\gimel_{n-2}, \gimel_{n-1}) + \mathsf{B}_\perp(\gimel_{n-1}, \gimel_n)] \\
& + \lambda[\mathsf{B}_\perp(\gimel_{n-2}, \gimel_n) + \mathsf{B}_\perp(\gimel_{n-1}, \gimel_{n-1})] + \mathsf{j}^{n-1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
(1 - \gamma - \delta - \lambda)\mathsf{B}_\perp(\gimel_{n-1}, \gimel_n) &\leq (\alpha + \beta + \delta + \lambda)\mathsf{B}_\perp(\gimel_{n-2}, \gimel_{n-1}) + \mathsf{j}^{n-1} \\
\mathsf{B}_\perp(\gimel_{n-1}, \gimel_n) &\leq \mathsf{j}_1 \mathsf{B}_\perp(\gimel_{n-2}, \gimel_{n-1}) + \frac{\mathsf{j}^{n-1}}{1 - \gamma - \delta - \lambda} \\
\mathsf{B}_\perp(\gimel_{n-1}, \gimel_n) &\leq \mathsf{j}_1^{n-1} \mathsf{B}_\perp(\gimel_0, \gimel_1) + (n-1)\frac{\mathsf{j}^{n-1}}{1 - \gamma - \delta - \lambda}.
\end{aligned}$$

Observe that the sequence $\{\gimel_n\}$ is Cauchy in $\mho$ as $\mathsf{j} < 1$. Given that $\mho$ is complete, it implies that the sequence $\{\gimel_n\}$ converges to a point $\aleph \in \mho$. In other words,

$$\lim_{n \to \infty} \gimel_n = \aleph.$$

Now,

$$\begin{aligned}
\check{\mathsf{D}}(\aleph,\Pi\aleph) &\leq \max\{\check{\mathsf{D}}(\aleph,\gimel_n),\check{\mathsf{D}}(\gimel_n,\Pi\aleph)\}\\
&\leq \max\{\check{\mathsf{D}}(\aleph,\gimel_n),\Bbbk(\Pi\gimel_{n-1},\Pi\aleph)\}\\
&\leq \max\{\check{\mathsf{D}}(\aleph,\gimel_n),\alpha\mathsf{B}_\perp(\gimel_{n-1},\aleph)+\beta\check{\mathsf{D}}(\gimel_{n-1},\Pi\gimel_{n-1})+\gamma\check{\mathsf{D}}(\aleph,\Pi\aleph)\\
&\quad +\delta[\check{\mathsf{D}}(\gimel_{n-1},\Pi\gimel_{n-1})+\check{\mathsf{D}}(\aleph,\Pi\aleph)]+\lambda[\check{\mathsf{D}}(\gimel_{n-1},\Pi\aleph)+\check{\mathsf{D}}(\aleph,\Pi\gimel_{n-1})]\}\\
&\leq \max\{\check{\mathsf{D}}(\aleph,\gimel_n),\alpha\mathsf{B}_\perp(\gimel_{n-1},\aleph)+\beta\mathsf{B}_\perp(\gimel_{n-1},\gimel_n)+\gamma\check{\mathsf{D}}(\aleph,\Pi\aleph)\\
&\quad +\delta[\lceil(\gimel_{n-1},\gimel_n)+\check{\mathsf{D}}(\aleph,\Pi\aleph)]+\lambda[\lceil(\gimel_{n-1},\aleph)+\mathsf{B}_\perp(\aleph,\gimel_n)]\}.
\end{aligned}$$

Allowing limit $n\to\infty$, we get

$$\check{\mathsf{D}}(\aleph,\Pi\aleph)\leq(\gamma+\delta)\check{\mathsf{D}}(\aleph,\Pi\aleph),$$

which shows that $\check{\mathsf{D}}(\aleph,\Pi\aleph)=0$, as $\gamma+\delta<1$. Hence, Π has a FP $\aleph\in\mho$.
To prove Uniqueness:- Let us consider another FP $\aleph'$ of Π. Now using (3.2), we obtain

$$[\aleph'\perp\aleph \text{ or } \aleph\perp\aleph'] \text{ and } [\Pi\aleph'\perp\Pi\aleph \text{ or } \Pi\aleph\perp\Pi\aleph'].$$

Now,

$$\begin{aligned}
\mathsf{B}_\perp(\aleph,\aleph') &\leq \Bbbk(\{\aleph\},\{\aleph'\})=\Bbbk(\Pi\aleph,\Pi\aleph')\\
&\leq \alpha\mathsf{B}_\perp(\aleph,\aleph')+\beta\check{\mathsf{D}}(\aleph,\Pi\aleph)+\gamma\check{\mathsf{D}}(\aleph',\Pi\aleph')\\
&\quad +\delta[\check{\mathsf{D}}(\aleph,\Pi\aleph)+\check{\mathsf{D}}(\aleph',\Pi\aleph')]+\lambda[\check{\mathsf{D}}(\aleph,\Pi\aleph')+\check{\mathsf{D}}(\aleph',\Pi\aleph)]\\
&\leq (\alpha+2\lambda)\mathsf{B}_\perp(\aleph,\aleph'),
\end{aligned}$$

which indicates $\aleph=\aleph'$ as $\alpha+2\lambda<1$. □

Theorem 3.3. *Let $(\mho,\mathsf{B}_\perp)$ be a complete O-UMS and $\Pi:\mho\to\mathcal{CB}(\otimes)$ be a mapping fulfills*

$$\begin{aligned}
\Bbbk(\Pi\gimel,\Pi\hbar)\leq\alpha\mathsf{B}_\perp(\gimel,\hbar)+\beta\mathsf{B}_\perp(\gimel,\Pi\gimel)+\gamma\mathsf{B}_\perp(\hbar,\Pi\hbar)+\delta[\check{\mathsf{D}}(\gimel,\Pi\hbar)+\check{\mathsf{D}}(\hbar,\Pi\gimel)]\\
+\lambda[\check{\mathsf{D}}(\gimel,\Pi\hbar)+\check{\mathsf{D}}(\hbar,\Pi\gimel)]+\eta[\mathsf{B}_\perp(\gimel,\Pi\gimel)+\mathsf{B}_\perp(\gimel,\hbar)] \qquad (3.3)
\end{aligned}$$

for all $\gimel,\hbar\in\mho$, wherein $2\delta+2\lambda+2\eta+\alpha+\beta+\gamma<1$ and $\delta,\lambda,\eta,\alpha,\beta,\gamma\geq 0$. Then Π has a UFP in $\mho$.

Proof. For $\gimel_0\in\gimel,\gimel_1\in\Pi\gimel_0$,

$$[\gimel_0\perp\gimel_1 \text{ or } \gimel_1\perp\gimel_0], \text{ and } [\Pi\gimel_0\perp\Pi\gimel_1 \text{ or } \Pi\gimel_1\perp\Pi\gimel_0],$$

for $\gimel_0\in\gimel,\gimel_1\in\Pi\gimel_0$, define $\mathsf{j}=\dfrac{\alpha+\beta+\delta+\lambda+2\eta}{1-(\gamma+\delta+\lambda)}$. Assume that $\mathsf{j}=0$, then the proof is trivial. Consider $\mathsf{j}>0$, then there exists $\gimel_2\in\Pi\gimel_1$ such that

$$\mathsf{B}_\perp(\gimel_1,\gimel_2)\leq\Bbbk(\Pi\gimel_0,\Pi\gimel_1)+\mathsf{j}$$

Now, $\Pi\gimel_1,\Pi\gimel_2\in\mathcal{CB}(\otimes)$ and $\gimel_2\in\Pi\gimel_1$, there exist $\gimel_3\in\Pi\gimel_2$ such that

$$\mathsf{B}_\perp(\gimel_2,\gimel_3)\leq\Bbbk(\Pi\gimel_1,\Pi\gimel_2)+\mathsf{j}^2.$$

Continuing in a similar manner, for $\gimel_{n-1}, n > 1$, we get $\gimel_n \in \Pi\gimel_{n-1}$ satisfies the following

$$\mathsf{B}_{\perp}(\gimel_{n-1}, \gimel_n) \leq \Bbbk(\Pi\gimel_{n-2}, \Pi\gimel_{n-1}) + \mathsf{j}^{n-1}.$$

Now, using equation (3.3), for $n \in \mathbb{N}$, we have

$$\begin{aligned}
\mathsf{B}_{\perp}(\gimel_{n-1}, \gimel_n) &\leq \alpha\mathsf{B}_{\perp}(\gimel_{n-2}, \gimel_{n-1}) + \beta\check{\mathsf{D}}(\gimel_{n-2}, \Pi\gimel_{n-2}) + \gamma\check{\mathsf{D}}(\gimel_{n-1}, \Pi\gimel_{n-1}) \\
&\quad + \delta[\check{\mathsf{D}}(\gimel_{n-2}, \Pi\gimel_{n-2}) + \check{\mathsf{D}}(\gimel_{n-1}, \Pi\gimel_{n-1})] \\
&\quad + \lambda[\check{\mathsf{D}}(\gimel_{n-2}, \Pi\gimel_{n-1}) + \check{\mathsf{D}}(\gimel_{n-1}, \Pi\gimel_{n-2})] + \eta[\check{\mathsf{D}}(\gimel_{n-2}, \Pi\gimel_{n-2}, \gimel_{n-1})] + \mathsf{j}^{n-1} \\
&\leq \alpha\mathsf{B}_{\perp}(\gimel_{n-2}, \gimel_{n-1}) + \beta\mathsf{B}_{\perp}(\gimel_{n-2}, \gimel_{n-1}) + \gamma\mathsf{B}_{\perp}(\gimel_{n-1}, \gimel_n) \\
&\quad + \delta[\mathsf{B}_{\perp}(\gimel_{n-2}, \gimel_{n-1}) + \mathsf{B}_{\perp}(\gimel_{n-1}, \gimel_n)] \\
&\quad + \lambda[\mathsf{B}_{\perp}(\gimel_{n-2}, \gimel_n) + \mathsf{B}_{\perp}(\gimel_{n-1}, \gimel_{n-1})] \\
&\quad + \eta[\mathsf{B}_{\perp}(\gimel_{n-2}, \gimel_{n-1}) + \mathsf{B}_{\perp}(\gimel_{n-2}, \gimel_{n-1})] + \mathsf{j}^{n-1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
(1-\gamma-\delta-\lambda)\mathsf{B}_{\perp}(\gimel_{n-1}, \gimel_n) &\leq (\alpha+\beta+\delta+\lambda+2\eta)\mathsf{B}_{\perp}(\gimel_{n-2}, \gimel_{n-1}) + \mathsf{j}^{n-1} \\
\mathsf{B}_{\perp}(\gimel_{n-1}, \gimel_n) &\leq \mathsf{j}_2\mathsf{B}_{\perp}(\gimel_{n-2}, \gimel_{n-1}) + \frac{\mathsf{j}^{n-1}}{1-\gamma-\delta-\lambda} \\
\mathsf{B}_{\perp}(\gimel_{n-1}, \gimel_n) &\leq \mathsf{j}_2^n\mathsf{B}_{\perp}(\gimel_0, \gimel_1) + (n-1)\frac{\mathsf{j}^{n-1}}{1-\gamma-\delta-\lambda}.
\end{aligned}$$

Observe that the sequence $\{\gimel_n\}$ is Cauchy in $\mho$ as $\mathsf{j} < 1$. Given that $\mho$ is complete, it implies that the sequence $\{\gimel_n\}$ converges to $\aleph \in \mho$. In other words,

$$\lim_{n\to\infty} \gimel_n = \aleph.$$

Now,

$$\begin{aligned}
\check{\mathsf{D}}(\aleph, \Pi\aleph) &\leq \max\{\check{\mathsf{D}}(\aleph, \gimel_n), \check{\mathsf{D}}(\gimel_n, \Pi\aleph)\} \\
&\leq \max\{\check{\mathsf{D}}(\aleph, \gimel_n), \Bbbk(\Pi\gimel_{n-1}, \Pi\aleph)\} \\
&\leq \max\{\check{\mathsf{D}}(\aleph, \gimel_n), \alpha\mathsf{B}_{\perp}(\gimel_{n-1}, \aleph) + \beta\check{\mathsf{D}}(\gimel_{n-1}, \Pi\gimel_{n-1}) + \gamma\check{\mathsf{D}}(\aleph, \Pi\aleph) \\
&\quad + \delta[\check{\mathsf{D}}(\gimel_{n-1}, \Pi\gimel_{n-1}) + \check{\mathsf{D}}(\aleph, \Pi\aleph)] \\
&\quad + \lambda[\check{\mathsf{D}}(\gimel_{n-1}, \Pi\aleph) + \check{\mathsf{D}}(\aleph, \Pi\gimel_{n-1})] + \eta[\check{\mathsf{D}}(\gimel_{n-1}, \Pi\gimel_{n-1}) + \mathsf{B}_{\perp}(\gimel_{n+1}, \aleph)]\} \\
&\leq \max\{\check{\mathsf{D}}(\aleph, \aleph), \alpha\mathsf{B}_{\perp}(\aleph, \aleph) + \beta\mathsf{B}_{\perp}(\aleph, \Pi\aleph) + \gamma\check{\mathsf{D}}(\aleph, \Pi\aleph) + \delta[\check{\mathsf{D}}(\aleph, \Pi\aleph) + \check{\mathsf{D}}(\aleph, \Pi\aleph)] \\
&\quad + \lambda[\check{\mathsf{D}}(\aleph, \Pi\aleph) + \check{\mathsf{D}}(\aleph, \Pi\aleph)] + \eta[\check{\mathsf{D}}(\aleph, \Pi\aleph)]\} \\
\check{\mathsf{D}}(\aleph, \Pi\aleph) &\leq (\beta+\gamma+2\delta+2\lambda+\eta)\check{\mathsf{D}}(\aleph, \Pi\aleph)
\end{aligned}$$

which shows that $\check{\mathsf{D}}(\aleph, \Pi\aleph) = 0$, as $\beta+\gamma+2\delta+2\lambda+\eta < 1$. Hence, Π has a FP $\aleph \in \mho$.
To prove Uniqueness:- Let us consider another FP $\aleph'$ of Π. Now using (3), we obtain

$$[\aleph' \perp \aleph \text{ or } \aleph \perp \aleph'] \text{ and } [\Pi\aleph' \perp \Pi\aleph \text{ or } \Pi\aleph \perp \Pi\aleph'].$$

Now,

$$\begin{aligned}\mathsf{B}_\perp(\aleph,\aleph') &\leq \Bbbk(\{\aleph\},\{\aleph'\}) = \Bbbk(\Pi\aleph,\Pi\aleph')\\ &\leq \alpha\mathsf{B}_\perp(\aleph,\aleph') + \beta\check{\mathsf{D}}(\aleph,\Pi\aleph) + \gamma\check{\mathsf{D}}(\aleph',\Pi\aleph') + \delta[\check{\mathsf{D}}(\aleph,\Pi\aleph) + \check{\mathsf{D}}(\aleph',\Pi\aleph')]\\ &+ \lambda[\check{\mathsf{D}}(\aleph,\Pi\aleph') + \check{\mathsf{D}}(\aleph',\Pi\aleph)] + \eta[\check{\mathsf{D}}(\aleph,\Pi\aleph) + \mathsf{B}_\perp(\aleph,\aleph')]\\ &\leq (\alpha + 2\lambda + \eta)\mathsf{B}_\perp(\aleph,\aleph'),\end{aligned}$$

which indicates $\aleph = \aleph'$ as $\alpha + 2\lambda + \eta < 1$.

□

Example 3.1. *The space $\mho = [0,1]$ is a complete O-UMS with discrete metric. Let $\Pi : \mho \to \mho$ be defined by*

$$\Pi\daleth = \frac{1}{4} \ \forall\daleth \in \mho.$$

Note that

$$\Bbbk(\Pi\daleth, \Pi\hbar) = 0,$$

and

$$\begin{aligned}&\alpha\mathsf{B}_\perp(\daleth,\hbar) + \beta\mathsf{B}_\perp(\daleth,\frac{1}{4}) + \gamma\mathsf{B}_\perp(\hbar,\frac{1}{4}) + \delta[\check{\mathsf{D}}(\daleth,\frac{1}{4}) + \check{\mathsf{D}}(\hbar,\frac{1}{4})]\\ &\quad + \lambda[\check{\mathsf{D}}(\daleth,\frac{1}{4}) + \check{\mathsf{D}}(\hbar,\frac{1}{4})] + \eta[\mathsf{B}_\perp(\daleth,\frac{1}{4}) + \mathsf{B}_\perp(\daleth,\hbar)]\\ &= \begin{cases} 2\delta + 2\lambda + 2\eta + \alpha + \beta + \gamma, & \text{if } \daleth \neq \hbar \neq \frac{1}{4}\\ \lambda + \eta + \alpha + \gamma + \delta, & \text{if } \hbar \neq \daleth = \frac{1}{4}\\ \alpha + \beta + \delta + \lambda + 2\eta, & \text{if } \daleth \neq \hbar = \frac{1}{4}\\ 0, & \text{if } \daleth = \hbar = \frac{1}{4}.\end{cases}\end{aligned}$$

Since (3.3) *holds, for all $\daleth, \hbar \in \mho$, where $\daleth, \hbar, \mathsf{c} \geq 0$ and $\alpha + \beta + \gamma + 2\delta + 2\lambda + 2\eta < 1$, by Theorem 3.3, Π has a UFP $\frac{1}{4}$ in $\mho$.*

4. Applications

4.1. Application to integral equations.

Theorem 4.1. *Let $(\mho, \mathsf{B}, \perp)$ be a complete O-UMS and $\Pi : \mho \to \mathcal{CB}(\otimes)$ be a continuous map to ensure*

$$\begin{aligned}\Bbbk(\Pi\daleth, \Pi\hbar) &\leq \alpha\mathsf{B}_\perp(\daleth,\hbar) + \beta\mathsf{B}_\perp(\daleth,\Pi\daleth) + \gamma\mathsf{B}_\perp(\hbar,\Pi\hbar) + \delta[\check{\mathsf{D}}(\daleth,\Pi\hbar) + \check{\mathsf{D}}(\hbar,\Pi\daleth)]\\ &\quad + \lambda[\check{\mathsf{D}}(\daleth,\Pi\hbar) + \check{\mathsf{D}}(\hbar,\Pi\daleth)] + \eta[\mathsf{B}_\perp(\daleth,\Pi\daleth) + \mathsf{B}_\perp(\daleth,\hbar)]\end{aligned}$$

for all elements $\delta, \lambda, \eta, \alpha, \beta, \gamma$ in the interval $[0,1)$, and let the sum $2\delta + 2\lambda + 2\eta + \alpha + \beta + \gamma$ be less than 1. Then Π admit a UFP.

Consider $\mathsf{W} = C([0,1], \mathbb{R}^+)$ as the set of continuous functions defined on $[0,1]$, with values in the non-negative real numbers. The following equation is an integral equation:

$$\pi(\mathsf{e}) = \int_0^{\mathsf{e}} \Xi(\mathsf{e}, \mathsf{j}, \pi(\mathsf{j}))d\mathsf{j} \tag{4.1}$$

for every $\mathsf{e} \in [0,1]$, where $\Xi : [0,1] \times [0,1] \times \mathsf{W} \to \mathbb{R}$. For $\pi \in C([0,1], \mathbb{R}^+)$, supremum norm as assuming $\|\pi\|_\epsilon = \sup_{s\in[0,1]}\{|\pi(s)|\mathsf{e}^s\}$ and for each $\pi, \tau \in C([0,1], \mathbb{R}^+)$, define

$$\begin{aligned}\pi \perp \tau \Longleftrightarrow \mathsf{B}_\perp(\pi, \tau) &= \frac{1}{2} \sup_{s\in[0,1]} \{|\pi(s) + \tau(s)|\mathsf{e}^s\} \\ &= \frac{1}{2} \|\pi + \tau\|_\epsilon .\end{aligned}$$

It is evident that $\mathsf{C}([0,1], \mathbb{R}^+, \mathsf{B}_\perp)$ is a complete O-UMS. So, we obtained the consequent:

Theorem 4.2. *Assume that*

(i) $\Xi : [0,1] \times [0,1] \times W \to \mathbb{R}$*;*
(ii) *Define*

$$\Pi\pi(\mathsf{e}) = \int_0^{\mathsf{e}} \Xi(\mathsf{e}, \mathsf{j}, \pi(\mathsf{j}))d\mathsf{j},$$

such that

$$\left|\Xi(\mathsf{e}, \mathsf{j}, \pi(\mathsf{j})) + \Xi(\mathsf{e}, \mathsf{j}, \tau(\mathsf{j}))\right| \leq \frac{\Omega(\pi, \tau)}{\Omega(\pi, \tau) + 1}$$

for each $\mathsf{e}, \mathsf{j} \in [0,1]$ *and* $\pi, \tau \in C([0,1], \mathbb{R}^+)$*, where*

$$\begin{aligned}\Omega(\pi, \tau) = \alpha\mathsf{B}_\perp(\daleth, \hbar) + \beta\mathsf{B}_\perp(\daleth, \Pi\daleth) + \gamma\mathsf{B}_\perp(\hbar, \Pi\hbar) + \delta[\check{\mathsf{D}}(\daleth, \Pi\hbar) + \check{\mathsf{D}}(\hbar, \Pi\daleth)] \\ + \lambda[\check{\mathsf{D}}(\daleth, \Pi\hbar) + \check{\mathsf{D}}(\hbar, \Pi\daleth)] + \eta[\mathsf{B}_\perp(\daleth, \Pi\daleth) + \mathsf{B}_\perp(\daleth, \hbar)].\end{aligned}$$

Then (4.1) *has a unique solution.*

Proof. By (ii),

$$\begin{aligned}|\Pi\pi + \Pi\tau| &= \int_0^{\mathsf{e}} \left|\Xi(\mathsf{e}, \mathsf{j}, \pi(\mathsf{j})) + \Xi(\mathsf{e}, \mathsf{j}, \tau(\mathsf{j}))\right| d\mathsf{j} \\ &\leq \int_0^{\mathsf{e}} \frac{\Omega(\pi, \tau)}{\Omega(\pi, \tau) + 1} \mathsf{e}^{\mathsf{j}} d\mathsf{j} \\ &\leq \frac{\Omega(\pi, \tau)}{\Omega(\pi, \tau) + 1} \int_0^{\mathsf{e}} \mathsf{e}^{\mathsf{j}} d\mathsf{j} \\ &\leq \frac{\Omega(\pi, \tau)}{\Omega(\pi, \tau) + 1} \mathsf{e}^{\mathsf{e}}.\end{aligned}$$

This implies

$$|\Pi\pi + \Pi\tau| \leq \frac{\Omega(\pi, \tau)}{\Omega(\pi, \tau) + 1},$$

$$\|\Pi\pi + \Pi\tau\|_\epsilon \leq \frac{\Omega(\pi,\tau)}{\Omega(\pi,\tau)+1},$$
$$\frac{\Omega(\pi,\tau)+1}{\Omega(\pi,\tau)} \leq \frac{1}{\|\Pi\pi + \Pi\tau\|_\epsilon},$$
$$1 + \frac{1}{\Omega(\pi,\tau)} \leq \frac{1}{\|\Pi\pi + \Pi\tau\|_\epsilon},$$
$$1 - \frac{1}{\left\|\Pi\pi(e) + \Pi\tau(e)\right\|_\epsilon} \leq \frac{-1}{\Omega(\pi,\tau)}.$$

The conditions of Theorem 4.1 are satisfied in all respects, and $\mathsf{B}_\perp(\pi,\tau) = \frac{1}{2}\|\pi+\tau\|_\epsilon$. Therefore, the integral equation (4.1) has a unique solution. □

4.2. **Application to non-linear fractional integro-differential equation.**

The Caputo derivative for a continuous mapping $\Lambda : [0,\infty) \to \mathbb{R}$ of order $\Theta > 0$ is defined as follows:

$$^{c}\mathcal{D}^{\Theta}\Lambda(\mathfrak{b}) = \frac{1}{\Gamma(n-\Theta)}\int_0^1 \frac{\Lambda^(n)(\mathbb{c}\mathfrak{b}\mathbb{c})}{(\mathfrak{b}-\mathbb{c})^{\Theta-n+1}}, \quad n-1 \leq \Theta < n, n = |\Theta|+1, \tag{4.2}$$

the symbol Γ represents the gamma function and $|\Theta|$ represents the integer component of the positive real number Θ.

Here, we examine the non-linear fractional integro-differential equation of Caputo type:

$$f(z) = \begin{cases} {}^{c}\mathcal{D}^{\Theta}\pi(\mathfrak{b}) = \Lambda(\mathfrak{b},\pi(\mathfrak{b})), & \mathfrak{b} \in \mathcal{I} = [0,1], 1 < \Theta \leq 2, \\ \pi(0) = 0, & \pi(1) = \int_0^\theta \pi\mathbb{c}d\mathbb{c} \end{cases} \tag{4.3}$$

where $\pi \in (C[0,1],\mathbb{R}), \theta \in (0,1)$, and $F : \mathcal{I}\times\mathbb{R} \to \mathbb{R}$ is a continuous function (see, [7]).

Denote $\mho = \{\pi : \pi \in (C[0,1],\mathbb{R})\}$ with norm supremum $\|\pi\| = sup_{\mathfrak{b}\in[0,1]}|\pi(t)|$. So $(\mho, \|.\|)$ is a Banach space.

The space $C([0,1],\mathbb{R})$ has an ultrametric denoted by $\mathsf{B} : C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R}) \to [0,\infty)$. $\mathsf{B}(\pi,\tau) = \|\pi-\tau\| = sup_{\mathfrak{b}\in[0,1]}|\pi(t)-\tau(t)|$ and denote an orthogonal relation $\pi \perp \tau$ iff $\pi\tau \leq 0$, $\forall\ \pi,\tau \in \mho$. Then $(\mho,\perp,\mathsf{B})$ is any O-UMS.

A solution of equation (4.3) is clearly a FP of the integral equation.

$$\Pi\pi(\mathfrak{b}) = \frac{1}{\Gamma(\Theta)}\int_0^{\mathfrak{b}} (\mathfrak{b}-\mathbb{c})^{\Theta-1}F((\mathbb{c},\pi(\mathbb{c})))ds - \frac{2\mathfrak{b}}{(2-\theta^2)\Gamma(\Theta)}\int_0^1 (1-\mathbb{c})^{\Theta-1}F((\mathbb{c},\pi(\mathbb{c})))ds$$
$$+ \frac{2\mathfrak{b}}{(2-\theta^2)\Gamma(\Theta)}\int_0^\theta \left(\int_0^s (\mathbb{c}-\mathfrak{m})^{\Theta-1}F((\mathbb{c},\pi(\mathfrak{m})))d\mathfrak{m}\right)ds \tag{4.4}$$

Theorem 4.3. *Assume that $F : \mathcal{I}\times\mathbb{R} \to \mathbb{R}$ is continuous functions fulfilling*

$$|F(\mathbb{c},\pi(\mathbb{c})) - F(\mathbb{c},\tau(\mathbb{c}))| \leq \frac{\Gamma(\Theta+1)}{5}e^{-\tau}|\pi(\mathbb{c})-\tau(\mathbb{c})| \tag{4.5}$$

For every $\mathbb{c} \in [0,1]$ for each $\tau > 0$ and $\forall\ \pi,\tau \in C([0,1],\mathbb{R})$, the FDE (4.3) with the specified boundary conditions possesses a solution.

Proof. The space $\mho = C([0,1],\mathbb{R})$ has an equipped with ultrametric $\mathsf{B} : \mho \times \mho \to [0,\infty)$ describe as $\mathsf{B}(\pi,\tau) = sup_{\flat\in[0,1]}|\pi(\flat) - \tau(\flat)|$. for every $\pi,\tau \in \mho$. Denote orthogonal relation $\pi \perp \tau$ iff $\pi,\tau \geq 0$, for each $\pi,\tau \in \mho$. Then $(\mho,\perp,\flat)$ is an O- UMS. Denote $\Pi : \mho \to \mho$ as in (4.4). So Π is $\perp$-continuous. First, we prove that Π is $\perp$-preserving, let $\pi(\flat) \perp \tau(\flat)$ for every $\flat \in [0,1]$. Now, from (4.4), we obtain

$$\Pi\pi(\flat) = \frac{1}{\Gamma(\Theta)}\int_0^{\flat}(\flat-\mathbb{c})^{\Theta-1}F((\mathbb{c},\pi(\mathbb{c})))ds - \frac{2\flat}{(2-\theta^2)\Gamma(\Theta)}\int_0^1(1-\mathbb{c})^{\Theta-1}F((\mathbb{c},\pi(\mathbb{c})))ds$$
$$+\frac{2\flat}{(2-\theta^2)\Gamma(\Theta)}\int_0^{\theta}\left(\int_0^s(\mathbb{c}-\mathfrak{m})^{\Theta-1}F((\mathbb{c},\pi(\mathfrak{m})))d\mathfrak{m}\right)ds > 0.$$

This implies that $\Pi\pi \perp \Pi\tau$. Given any $\flat \in [0,1]$ such that $\pi(\flat) \perp \tau\flat)$, we have:

$$\begin{aligned}
|\Pi\pi(\flat) - \Pi\tau(\flat)| =& \left|\frac{1}{\Gamma(\Theta)}\int_0^{\flat}(\flat-\mathbb{c})^{\Theta-1}F((\mathbb{c},\pi(\mathbb{c})))ds - \right.\\
&\frac{2\flat}{(2-\theta^2)\Gamma(\Theta)}\int_0^1(1-\mathbb{c})^{\Theta-1}F((\mathbb{c},\pi(\mathbb{c})))ds\\
&+\frac{2\flat}{(2-\theta^2)\Gamma(\Theta)}\int_0^{\theta}\left(\int_0^s(\mathbb{c}-\mathfrak{m})^{\Theta-1}F((\mathbb{c},\pi(\mathfrak{m})))d\mathfrak{m}\right)ds\\
&-\left(\frac{1}{\Gamma(\Theta)}\int_0^{\flat}(\flat-\mathbb{c})^{\Theta-1}F((\mathbb{c},\tau(\mathbb{c})))ds\right.\\
&-\frac{2\flat}{(2-\theta^2)\Gamma(\Theta)}\int_0^1(1-\mathbb{c})^{\Theta-1}F((\mathbb{c},\tau(\mathbb{c})))ds\\
&\left.\left.+\frac{2\flat}{(2-\theta^2)\Gamma(\Theta)}\int_0^{\theta}\left(\int_0^s(\mathbb{c}-\mathfrak{m})^{\Theta-1}F((\mathbb{c},\tau(\mathfrak{m})))d\mathfrak{m}\right)ds\right)\right|.\\
\leq& \frac{1}{\Gamma(\Theta)}\int_0^{\flat}(\flat-\mathbb{c})^{\Theta-1}|F(\mathbb{c},\tau(\mathbb{c}))-F(\mathbb{c},\tau(\mathbb{c}))|ds-\\
&\frac{2\flat}{(2-\theta^2)\Gamma(\Theta)}\int_0^1(1-\mathbb{c})^{\Theta-1}|F(\mathbb{c},\pi(\mathbb{c}))-F(\mathbb{c},\tau(\mathbb{c}))|ds\\
&+\frac{2\flat}{(2-\theta^2)\Gamma(\Theta)}\int_0^{\theta}\left(\int_0^s(\mathbb{c}-\mathfrak{m})^{\Theta-1}|F(\mathbb{c},\pi(\mathfrak{m}))-F(\mathbb{c},\tau(\mathfrak{m}))|d\mathfrak{m}\right)ds\\
\leq& \frac{1}{\Gamma(\Theta)}\int_0^{\flat}(\flat-\mathbb{c})^{\Theta-1}\left[\frac{\Gamma(\Theta+1)}{5}e^{-\tau}sup_{\mathbb{c}\in[0,1]}|\pi(\mathbb{c})-\tau(\mathbb{c})|\right]ds\\
&-\frac{2\flat}{(2-\theta^2)\Gamma(\Theta)}\int_0^1(1-\mathbb{c})^{\Theta-1}\left[\frac{\Gamma(\Theta+1)}{5}e^{-\tau}sup_{\mathbb{c}\in[0,1]}|\pi(\mathbb{c})-\tau(\mathbb{c})|\right]ds\\
&+\frac{2\flat}{(2-\theta^2)\Gamma(\Theta)}\int_0^{\theta}\left(\int_0^s(\mathbb{c}-\mathfrak{m})^{\Theta-1}\left[\frac{\Gamma(\Theta+1)}{5}e^{-\tau}\sup_{\mathbb{c}\in[0,1]}|\pi(\mathbb{c})-\tau(\mathbb{c})|\right]d\mathfrak{m}\right)ds\\
\leq& \left[\frac{\Gamma(\Theta+1)}{5}e^{-\tau}\sup_{\mathbb{c}\in[0,1]}|\pi(\mathbb{c})-\tau(\mathbb{c})|\right]\times\sup_{\flat\in[0,1]}\left(\frac{1}{\Gamma(\Theta)}\int_0^{\flat}(\flat-\mathbb{c})^{\Theta-1}ds\right.
\end{aligned}$$

$$- \frac{2\flat}{(2-\theta^2)\Gamma(\Theta)} \int_0^1 (1-\mathbb{c})^{\Theta-1} ds + \frac{2\flat}{(2-\theta^2)\Gamma(\Theta)} \int_0^\theta \left(\int_0^s (\mathbb{c}-\mathfrak{m})^{\Theta-1} d\mathfrak{m} \right) ds \Bigg)$$
$$\leq e^{-\tau} \sup_{\mathbb{c}\in[0,1]} |\pi(\mathbb{c}) - \tau(\mathbb{c})| = e^{-\tau} \mathsf{B}(\pi, \tau),$$

for all $\pi, \tau \in \mho$. Therefore, Π has a FP. The Caputo-type nonlinear FDE (4.3) possesses a solution is yielded. □

5. Conclusion

Numerous researchers accomplished interesting findings by examining fixed-point theorems in O-UMSs utilizing various mathematical methods and contraction mappings. To accomplish this, we have specifically concentrated on examining fixed points in all O-UMSs and different contraction mappings in our analytical methodology. To improve our results, we utilized a specific instance to illustrate the complicated proofs and logical processes underlying these theorems. In the future, we will commit to expanding the scope of fixed-point results through our research activities and novel and significant findings in the field of mathematical analysis.

Availability of Data and Material: The data used in this study are within the article.

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