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Stability in Locally Convex Lattice Cones

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Abstract. In this work, we start by recalling the essential definitions of locally convex lattice cones and the concept of stability for functional equations. Based on these preliminary notions, we establish a main theorem that provides sufficient conditions for stability within the framework of locally convex cones. This theorem generalizes classical stability results and offers a deeper understanding of the stability behavior of mappings between locally convex cones under approximate conditions.

1. Introduction

The origins of the stability theory for functional equations trace back to a significant lecture delivered by Ulam [1] at the Mathematical Club of the University of Wisconsin in the fall of 1940. During this talk, Ulam posed a profound question that sparked a new line of mathematical inquiry: "If the assumptions of a theorem are approximately satisfied, does it follow that the conclusion is approximately valid as well?" This line of questioning laid the groundwork for what is now known as the stability problem of functional equations.

The central theme of this problem revolves around the following idea: given an approximate solution to a functional equation, under what conditions can this approximation be close to an exact solution? Whenever this is indeed the case, the corresponding functional equation is said to exhibit stability.

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The first substantial answer to Ulam's question was provided by Hyers in 1941 [2], marking the formal beginning of the field. His result demonstrated that if $f : E_1 \to E_2$ is a mapping between Banach spaces that satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

for some $\epsilon > 0$, then there exists a unique additive mapping $A : E_1 \to E_2$ and a constant $k > 0$ such that

$$\|f(x) - A(x)\| \leq \epsilon.$$

This method of establishing stability through additive approximation is often referred to as the direct method. When such stability results mirror Hyers' theorem and involve an error bound governed by ϵ, they are collectively known as Hyers-Ulam stability.

Building upon this, Aoki in 1950 extended Hyers' findings [3]. He considered mappings $f : E_1 \to E_2$ between Banach spaces satisfying

$$\|f(x+y) - f(x) - f(y)\| \leq \theta_0(\|x\|^p + \|y\|^p)$$

for some constants $\theta_0 \geq 0$ and $0 \leq p < 1$, and proved the existence of an additive mapping $A : E_1 \to E_2$ such that

$$\|f(x) - A(x)\| \leq \theta\|x\|^p$$

for some $\theta \geq 0$. This result generalized the classical stability concept by introducing a power-type control function.

For several decades following Aoki's contribution, the field saw limited progress until 1978, when Rassias reignited interest by significantly generalizing the stability framework [4]. His work led to what is now known as Hyers-Ulam-Rassias stability, a term acknowledging the broader context and influence of Rassias' formulation. Although Aoki's result preceded Rassias', the widespread adoption of the term "Hyers-Ulam-Rassias" reflects Rassias' impact on popularizing and expanding the theory.

Over time, many researchers have contributed to advancing this theory by adapting it to new types of functional equations, alternative control functions, and various topological or algebraic structures [5–13]. These efforts have extended the reach of Hyers-Ulam-type results well beyond their original scope, offering powerful tools for diverse applications across mathematics.

Locally convex cones were originally defined and systematically studied in [14] and [15]. A cone is a set $\mathcal{P}$ equipped with two operations: an addition $(a, b) \mapsto a + b$ and a scalar multiplication $(\lambda, a) \mapsto \lambda a$, where $\lambda \geq 0$ is a real number. The addition operation must be associative and commutative, and the set contains a distinguished element $0 \in \mathcal{P}$ serving as the additive identity.

Scalar multiplication is required to satisfy the classical compatibility rules, namely:

$$\lambda(\beta a) = (\lambda\beta)a, \quad (\lambda + \beta)a = \lambda a + \beta a, \quad \lambda(a + b) = \lambda a + \lambda b, \quad 1a = a, \quad 0a = 0,$$

for all $a, b \in \mathcal{P}$ and all $\lambda, \beta \geq 0$.

Notably, the cancellation property stating that $a + c = b + c$ implies $a = b$ is not assumed in general. This condition holds if and only if $\mathcal{P}$ can be embedded into a real vector space.

A preordered cone is a cone P together with a preorder relation $\leq$ (i.e., reflexive and transitive) that respects both addition and scalar multiplication. Specifically, if $x \leq y$, then for every $z \in P$ and $\lambda \in \mathbb{R}_+$, we have

$$x + z \leq y + z, \quad \text{and} \quad \lambda x \leq \lambda y.$$

Ordered vector spaces provide a typical example of ordered cones. Notably, the extended real line $\mathbb{R} \cup \{+\infty\}$ and its nonnegative subset $\mathbb{R}_+ \cup \{+\infty\}$, each endowed with the natural order and extended arithmetic (in particular, $0 \cdot (+\infty) = 0$), are ordered cones that cannot be embedded into any vector space.

Let $V \subseteq P$ be a subset of a preordered cone. We call V a 0-neighborhood system if the following conditions hold:

($v1$) For all $v \in V$, $0 < v$,
($v2$) For any $u, v \in V$, there exists $w \in V$ such that $w \leq u$ and $w \leq v$,
($v3$) If $u, v \in V$ and $\alpha > 0$, then both $u + v \in V$ and $\alpha v \in V$.

Given $a \in P$ and $v \in V$, we define the upper neighborhood of a as $v(a) := \{b \in P : b \leq a + v\}$, and the lower neighborhood as $(a)v := \{b \in P : a \leq b + v\}$. The symmetric topology on P is obtained by refining these two topologies, and the symmetric neighborhood of a is denoted by $v(a)v$.

A pair (P, V) is called a full locally convex cone if every element of P is bounded below in the sense that for each $a \in P$ and $v \in V$, there exists $\rho > 0$ such that $0 \leq a + \rho v$. Any subcone of P (not necessarily including V) satisfying this condition is referred to as a locally convex cone.

An element $a \in P$ is said to be upper bounded if for every $v \in V$, there exists $\lambda > 0$ with $a \leq \lambda v$. The element a is called bounded if it is both lower and upper bounded.

For example, considering $\xi := \{\varepsilon > 0 : \varepsilon \in \mathbb{R}\}$, the pairs $(\mathbb{R}, \xi)$ and $(\mathbb{R}_+, \xi)$ serve as examples of full locally convex cones.

Given a neighborhood $\nu \in V$ and $\epsilon > 0$, we define the corresponding upper and lower relative neighborhoods $\nu_\epsilon(a)$ and $(a)\nu_\epsilon$ for an element $a \in P$ by

$$\nu_\epsilon(a) = \{b \in P \mid b \leq a + \epsilon\nu\}$$
$$(a)\nu_\epsilon = \{b \in P \mid a \leq b + \epsilon\nu\}$$

The symmetric relative neighborhood is formed by the intersection of $\nu_\epsilon(a)$ and $(a)\nu_\epsilon$. In other words, $\nu^s_\epsilon(a) = \nu_\epsilon(a) \cap (a)\nu_\epsilon$. In fact, These neighborhoods are, convex subsets within P.
With varying $\nu \in V$ and $\epsilon > 0$, the neighborhoods $\nu_\epsilon(\cdot)$, $(\cdot)\nu_\epsilon$, and $\nu^s_\epsilon(\cdot)$ generate the upper, lower, and symmetric relative topologies on P, respectively.

Proposition 1.1. *[14] Suppose that (P, V) denotes a locally convex cone. Then the following three conditions are logically equivalent:*

(1) *The symmetric relative topology on P is Hausdorff.*

(2) *The weak topology on P is Hausdorff.*
(3) *The weak preorder on P is antisymmetric.*

Example 1.1. *Consider $P = \overline{\mathbb{R}}$ with the neighborhood system $V = \{\epsilon \in \mathbb{R} \mid \epsilon > 0\}$. For a fixed neighborhood $v = 1$ and any $\epsilon > 0$, the relative neighborhoods of an element $a \in \overline{\mathbb{R}}$ are defined as*

$$v_\epsilon(a) = (-\infty, a + \epsilon].$$

Thus, the weak preorder matches the natural order of $\overline{\mathbb{R}}$. The lower relative neighborhoods are defined as

$$(a)v_\epsilon = [a - \epsilon, +\infty].$$

This leads to,

$$v^s_\epsilon(a) = [a - \epsilon, a + \epsilon]$$

Definition 1.1. *[14] We say that a pair (P, V) is a* locally convex upward lattice cone *if the preorder on P is antisymmetric, the supremum $a \vee b$ exists for all $a, b \in P$, and the following two conditions hold:*

(UL1) *$(a + c) \vee (b + c) = (a \vee b) + c$ for all $a, b, c \in P$;*
(UL2) *if $a \le c + v$ and $b \le c + w$ for some $a, b, c \in P$ and $v, w \in V$, then $a \vee b \le c + (v + w)$.*

Similarly, (P, V) is called a locally convex downward lattice cone *if the order is antisymmetric, the infimum $a \wedge b$ exists for all $a, b \in P$, and the following two conditions are satisfied:*

(DL1) *$(a + c) \wedge (b + c) = (a \wedge b) + c$ for all $a, b, c \in P$;*
(DL2) *if $c \le a + v$ and $c \le b + w$ for some $a, b, c \in P$ and $v, w \in V$, then $c \le (a \wedge b) + (v + w)$.*

Whenever both the upward and downward lattice conditions hold simultaneously, the pair (P, V) is referred to as a locally convex lattice cone. *That is, a* locally convex lattice cone *is defined as a locally convex cone (P, V) in which P is partially ordered with an antisymmetric order, admits suprema and infima for all pairs of elements, and satisfies the compatibility conditions (UL1), (UL2), (DL1), and (DL2).*

Definition 1.2. *[14] A locally convex lattice cone (P, V) is said to be Archimedean if for every $a \in P$,*

$$\lim_{n\to\infty} \frac{1}{n} a = 0$$

with respect to the symmetric relative topology.

Proposition 1.2. *[14] Suppose (P, V) is a locally convex lattice cone of either upward or downward type. Then, the binary lattice operation $(a, b) \mapsto a \vee b$ (respectively, $(a, b) \mapsto a \wedge b$) defines a continuous mapping from $P \times P$ to P, provided that P carries the symmetric relative topology.*

Definition 1.3. *[18] Let P and Q be lattice cones. A mapping $T : P \to Q$ is called a* lattice homomorphism *if for all $a, b \in P$, it holds that*

$$T(a \vee b) = T(a) \vee T(b).$$

For more information on lattices and their properties, the readers are referred to references [17–19].

2. Main results

In this section, we investigate the stability of lattice homomorphisms in the setting of locally convex lattice cones. This extends the study conducted in [16], where the stability of linear operators was examined in the framework of locally convex cones. The presence of the lattice structure allows for stronger conclusions and a more refined analysis in this context.

Theorem 2.1. *Let (P_1, V_1) be a locally convex lattice cone and (P_2, V_2) be a locally convex Archimedean lattice cone. Suppose that (P_2, V_2) is complete with respect to the symmetric relative topology. If function $f : P_1 \to P_2$ satisfies*

$$f(\tau x \vee \mu y) \in \nu(\tau f(x) \vee \mu f(y))\nu, \tag{2.1}$$

for all $\nu \in V_2$, $x, y \in P_1$ and $\tau, \mu > 1$ with $f(0) = 0$, then there exists a unique lattice homomorphism $T : P_1 \to P_2$ such that

$$T(x) \in (\nu\gamma)f(x)(\nu\gamma) \tag{2.2}$$

where $\gamma = \frac{1}{\tau-1}$, and

$$T(\tau x) = \tau T(x) \tag{2.3}$$

for all $\nu \in V_2$ and $x \in P_1$.

Proof. Substituting $x = y$ and $\tau = \eta$ in (2.1), The following relationship is obtained.

$$f(\tau x) \in \nu(\tau f(x))\nu, \tag{2.4}$$

for all $x \in P$, $\tau > 1$, and $\nu \in V_2$. As a result, the bellow inequalities hold.

$$\begin{cases} f(\tau x) \le \tau f(x) + \epsilon\nu, & \text{for } \epsilon \ge 0, \\ \\ \tau f(x) \le f(\tau x) + \epsilon\nu, & \text{for } \epsilon \ge 0. \end{cases} \tag{2.5}$$

We simplify the above expression and divide both sides by τ, we get

$$\begin{cases} \frac{1}{\tau} f(\tau x) \le f(x) + \frac{\epsilon\nu}{\tau}, \\ \\ f(x) \le \frac{1}{\tau} f(\tau x) + \frac{\epsilon\nu}{\tau}. \end{cases} \tag{2.6}$$

Next, replacing x with τx in (2.6) yields:

$$\begin{cases} \frac{1}{\tau} f(\tau^2 x) \le f(\tau x) + \frac{\epsilon\nu}{\tau}, \\ \\ f(\tau x) \le \frac{1}{\tau} f(\tau^2 x) + \frac{\epsilon\nu}{\tau}. \end{cases} \tag{2.7}$$

By comparing results (2.5) and (2.7), we obtain at the following expressions.

$$\begin{cases} \frac{1}{\tau} f(\tau^2 x) \leq \tau f(x) + \epsilon\nu + \frac{\epsilon\nu}{\tau}, \\ \\ \tau f(x) \leq \frac{1}{\tau} f(\tau^2 x) + \epsilon\nu + \frac{\epsilon\nu}{\tau}. \end{cases} \tag{2.8}$$

Then we divide both sides (2.8) by τ again, which gives,

$$\begin{cases} \frac{1}{\tau^2} f(\tau^2 x) \leq f(x) + \frac{\epsilon\nu}{\tau} + \frac{\epsilon\nu}{\tau^2}, \\ \\ f(x) \leq \frac{1}{\tau^2} f(\tau^2 x) + \frac{\epsilon\nu}{\tau} + \frac{\epsilon\nu}{\tau^2}. \end{cases} \tag{2.9}$$

Finally, substituting x with τx in (2.9), we obtain

$$\begin{cases} \frac{1}{\tau^2} f(\tau^3 x) \leq f(\tau x) + \frac{\epsilon\nu}{\tau} + \frac{\epsilon\nu}{\tau^2}, \\ \\ f(\tau x) \leq \frac{1}{\tau^2} f(\tau^3 x) + \frac{\epsilon\nu}{\tau} + \frac{\epsilon\nu}{\tau^2}. \end{cases} \tag{2.10}$$

From (2.5) and (2.10), it can be deduced that

$$\begin{cases} \frac{1}{\tau^2} f\left(\tau^3 x\right) \leqslant \tau f(x) + \varepsilon\nu + \frac{\varepsilon\nu}{\tau} + \frac{\varepsilon\nu}{\tau^2} \\ \\ \tau f(x) \leqslant \frac{1}{\tau^2} f\left(\tau^3 x\right) + \varepsilon\nu + \frac{\varepsilon\nu}{\tau} + \frac{\varepsilon\nu}{\tau^2} \end{cases}$$

So that

$$\begin{cases} \frac{1}{\tau^3} f\left(\tau^3 x\right) \leqslant f(x) + \frac{\varepsilon\nu}{\tau} + \frac{\varepsilon\nu}{\tau^2} + \frac{\varepsilon\nu}{\tau^3} \\ \\ f(x) \leqslant \frac{1}{\tau^3} f\left(\tau^3 x\right) + \frac{\varepsilon\nu}{\tau} + \frac{\varepsilon\nu}{\tau^2} + \frac{\varepsilon\nu}{\tau^3} \end{cases}$$

Using induction on n, we obtain

$$\begin{cases} \frac{1}{\tau^n} f\left(\tau^n x\right) \leqslant f(x) + \varepsilon\nu \sum_{i=1}^{n} \frac{1}{\tau^i} = f(x) + \frac{\varepsilon\nu}{\tau-1}\left[1 - \frac{1}{\tau^n}\right] \\ \\ f(x) \leqslant \frac{1}{\tau^n} f\left(\tau^n x\right) + \varepsilon\nu \sum_{i=1}^{n} \frac{1}{\tau^i} = \frac{1}{\tau^n} f\left(\tau^n x\right) + \frac{\varepsilon\nu}{\tau-1}\left[1 - \frac{1}{\tau^n}\right]. \end{cases} \tag{2.11}$$

Replacing x by $\tau^m x$ in (2.11), we get

$$\begin{cases} \frac{1}{\tau^n} f\left(\tau^n \tau^m x\right) \leqslant f\left(\tau^m x\right) + \frac{\varepsilon\nu}{\tau-1}\left[1 - \frac{1}{\tau^n}\right] \\ \\ f\left(\tau^m x\right) \leqslant \frac{1}{\tau^n} f\left(\tau^n \tau^m x\right) + \frac{\varepsilon\nu}{\tau-1}\left[1 - \frac{1}{\tau^n}\right]. \end{cases} \tag{2.12}$$

Multiplying $\frac{1}{\tau^m}$ on both sides of (2.12), we have

$$\begin{cases} \frac{1}{\tau^{n+m}} f\left(\tau^{n+m} x\right) \leqslant \frac{1}{\tau^m} f\left(\tau^m x\right) + \frac{\varepsilon\nu}{\tau^m(\tau-1)}\left[1 - \frac{1}{\tau^n}\right] \\ \\ \frac{1}{\tau^m} f\left(\tau^m x\right) \leqslant \frac{1}{\tau^{n+m}} f\left(\tau^{n+m} x\right) + \frac{\varepsilon\nu}{\tau^m(\tau-1)}\left[1 - \frac{1}{\tau^n}\right]. \end{cases} \tag{2.13}$$

These two inequalities show that the sequence $\left\{\frac{1}{\tau^n} f\left(\tau^n x\right)\right\}$ is a Cauchy sequence on (P_2, V_2) under the symmetric relative topology. As (P_2, V_2) is complete, the sequence converges in this topology. Since (P_2, V_2) is a locally convex Archimedean lattice cone, the preorder on P_2 is necessarily antisymmetric. It then follows from Proposition 1.1 that the symmetric relative topology on (P_2, V_2) is Hausdorff, which guarantees the uniqueness of limits of sequences. Define

$$T(x) = \lim_{n\to\infty} \frac{1}{\tau^n} f\left(\tau^n x\right).$$

Since (P_2, V_2) is a locally convex Archimedean lattice cone, we have

$$\lim_{n\to\infty} \frac{1}{\tau^n} \varepsilon\gamma\nu = 0,$$

where $\gamma = \frac{1}{\tau-1}$. Thus, taking the limit as $n \to \infty$ in (2.11), we obtain have

$$\begin{cases} T(x) \leqslant f(x) + \varepsilon\gamma\nu \\ f(x) \leqslant T(x) + \varepsilon\gamma\nu. \end{cases} \tag{2.14}$$

It means that

$$T(x) \in (\gamma\nu) f(x) (\gamma\nu).$$

Putting $\tau = \eta = \tau^n$ in (2.1), we obtain

$$f\left(\tau^n x \vee \tau^n y\right) \in \nu\left(\tau^n f(x) \vee \tau^n f(y)\right)\nu.$$

Hence,

$$\begin{cases} f\left(\tau^n x \vee \tau^n y\right) \leqslant \tau^n f(x) \vee \tau^n f(y) + \varepsilon\nu \\ \tau^n f(x) \vee \tau^n f(y) \leqslant f\left(\tau^n x \vee \tau^n y\right) + \varepsilon\nu \end{cases} \tag{2.15}$$

Substituting x with $\tau^n x$ and y with $\tau^n y$, we have

$$\begin{cases} f\left(\tau^n\left(\tau^n x \vee \tau^n y\right)\right) \leqslant \tau^n f\left(\tau^n x\right) \vee \tau^n f\left(\tau^n y\right) + \varepsilon\nu \\ \tau^n f\left(\tau^n x\right) \vee \tau^n f(y) \leqslant f\left(\tau^n\left(\tau^n x \vee \tau^n y\right)\right) + \varepsilon\nu \end{cases} \tag{2.16}$$

Dividing τ^{2n} in above inequality, therefore

$$\begin{cases} \frac{1}{\tau^{2n}} f\left(\tau^{2n}(x \vee y)\right) \leqslant \frac{1}{\tau^n}\left[f\left(\tau^n x\right) \vee f\left(\tau^n y\right)\right] + \frac{1}{\tau^{2n}}\varepsilon\nu \\ \frac{1}{\tau^n}\left[f\left(\tau^n x\right) \vee f\left(\tau^n y\right)\right] \leqslant \frac{1}{\tau^{2n}} f\left(\tau^{2n}(x \vee y)\right) + \frac{1}{\tau^{2n}}\varepsilon\nu \end{cases} \tag{2.17}$$

Since (P_2, V_2) is a locally convex Archimedean lattice cone, then lattice operations are continuous and $\frac{1}{\tau^n}\nu \to 0$ as $n \to \infty$. So, by letting $n \to \infty$, we reach

$$\begin{cases} T(x \vee y) \leqslant T(x) \vee T(y) \\ \\ T(x) \vee T(y) \leqslant T(x \vee y). \end{cases} \tag{2.18}$$

Since weak preorder P_2 is antisymmetric, we get

$$T(x \vee y) = T(x) \vee T(y)$$

therefore, T is a lattice homomorphism.
Next, we show that $T(\tau x) = \tau T(x)$ for all $x \in P_1$ and $\tau > 1$. From (2.1) we have

$$\begin{cases} f(\tau x \vee \eta y) \leqslant \tau f(x) \vee \eta f(y) + \varepsilon \nu \\ \\ \tau f(x) \vee \eta f(y) \leqslant f(\tau x \vee \eta y) + \varepsilon \nu \end{cases} \tag{2.19}$$

Choose $\eta = \tau, y = 0$ and substitute $2^n \tau$ for τ, we obtain

$$\begin{cases} f\left(2^n \tau x \vee 0\right) \leqslant 2^n \tau f(x) \vee 0 + \varepsilon \nu \\ \\ 2^n \tau f(x) \vee 0 \leqslant f\left(2^n \tau x \vee 0\right) + \varepsilon \nu. \end{cases} \tag{2.20}$$

Now, we replace x with $2^n x$. Consequently,

$$\begin{cases} f\left(4^n \tau x\right) \leqslant 2^n \tau f\left(2^n x\right) + \varepsilon \nu \\ \\ 2^n \tau f\left(2^n x\right) \leqslant f\left(4^n \tau x\right) + \varepsilon \nu \end{cases} \tag{2.21}$$

Dividing both sides of 4^n, we get

$$\begin{cases} \frac{1}{4^n} f\left(4^n \tau x\right) \leqslant \frac{1}{2^n} \tau f\left(2^n x\right) + \frac{\varepsilon \nu}{4^n} \\ \\ \frac{1}{2^n} \tau f\left(2^n x\right) \leqslant \frac{1}{4^n} f\left(4^n \tau x\right) + \frac{\varepsilon \nu}{4^n} \end{cases} \tag{2.22}$$

Therefore, by passing n to in infinity, we have

$$\begin{cases} T(\tau x) \leqslant \tau T(x) \\ \\ \tau T(x) \leqslant T(\tau x) \end{cases} \tag{2.23}$$

Note that $\lim_{n \to \infty} \frac{\varepsilon \nu}{4^n} = 0$, since P_2 has Archemidean property. Hence,

$$T(\tau x) = \tau T(x),$$

for all $x \in P_1$. □

Corollary 2.1. *Let (P, V) be a locally convex lattice cone and (R, ξ) (or $(\mathbb{R}_+, \xi)$) be a locally convex Archimedean lattice cone which is complete with respect to the symmetric relative topology. Assume that a function $f : (P, V) \to (R, \xi)$ (or $f : (P, V) \to (\mathbb{R}_+, \xi)$) satisfies*

$$f(x \vee y) \in \varepsilon(f(x) \vee f(y))\varepsilon$$

for all $x, y \in P$, and $\varepsilon \in \xi$, with $f(0) = 0$. Then there exists a unique lattice homomorphism $T : (P, V) \to (R, \xi)$ (or $T : (P, V) \to (\mathbb{R}_+, \xi)$) such that

$$T(x) \in \varepsilon(f(x))\varepsilon$$

for all $x \in P$ and $\varepsilon \in \xi$.

Proof. Consider the function $f : (P, V) \to (R, \xi)$ (or $f : (P, V) \to (\mathbb{R}_+, \xi)$) satisfying

$$f(x \vee y) \in \varepsilon(f(x) \vee f(y))\varepsilon$$

for all $x, y \in P$, and $\varepsilon \in \xi$, with $f(0) = 0$. Since (R, ξ) (or $(\mathbb{R}_+, \xi)$) is a locally convex Archimedean lattice cone which is complete with respect to its symmetric relative topology, all the assumptions of Theorem 2.1 are satisfied (with $(P_1, V_1) = (P, V)$ and $(P_2, V_2) = (R, \xi)$). Hence, by Theorem 2.1, there exists a unique lattice homomorphism $T : P \to R$ such that

$$T(x) \in \varepsilon(f(x))\varepsilon \quad \text{for all } x \in P \text{ and } \varepsilon \in \xi.$$

This completes the proof. □

Theorem 2.2. *Let (P_1, V_1) and (P_2, V_2) be locally convex lattice cones, where (P_2, V_2) is an Archimedean cone that is complete with respect to the symmetric relative topology. Suppose $p : [0, \infty) \to [0, \infty)$ is a continuous function. Let $f : P_1 \to P_2$ be a mapping such that for all $x, y \in P_1$, $\tau, \mu > 0$ and $\nu \in V_2$,*

$$f(\tau x \vee \mu y) \in \nu\left(\frac{\tau p(\tau) f(x) \vee \mu p(\mu) f(y)}{p(\tau) \vee p(\mu)}\right)\nu. \tag{2.24}$$

Then, there exists a unique mapping $T : P_1 \to P_2$ satisfying the following:

(1) *T is a lattice homomorphism, i.e., $T(x \vee y) = T(x) \vee T(y)$ and*

$$T(x) \in \nu(f(x))\nu$$

for all $x \in P_1$ and $\nu \in V_2$.

(2) *$T(\tau x) = \tau T(x)$ for all $\tau > 0$ and $x \in P_1$.*

Proof. The proof is analogous to that of Theorem 2.1, with only minor adjustments required to account for the presence of the continuous function p. Hence, we omit the details. □

3. Conclusion

We introduced the notion of locally convex lattice cones as a natural extension of lattice structures in topological vector spaces. Building upon this framework, we investigated the stability of lattice homomorphisms within these cones. The results presented in this paper not only generalize existing stability theorems but also provide new insights into the behavior of approximate lattice homomorphisms in a locally convex setting. These findings contribute to the broader understanding of stability theory in ordered topological structures.

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