Int. J. Anal. Appl. (2025), 23:233

International Journal of Analysis and Applications

Non-Bazilevič Functions Defined by Generalized M-Series Subordinating With Generalized Telephone Numbers

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Abstract. Compared to the class of Bazilevič functions, the so-called non-Bazilevič functions have not been investigated as thoroughly. A convex combination of the class of non-Bazilevič functions and its Alexander transform characterisation would be used to describe and analyze a new class of functions. The differential operator that is used to define the function class involves generalized M-series. In addition to unifying the generalized Gaussian hypergeometric function and the Mittag-Leffler function, the generalized M-series also generalizes a number of other well-known topics in *univalent function theory*. We focus on estimates involving the initial coefficients of the functions with Maclaurin series that are part of the defined function class. Additionally, we acquire the inverse and logarithmic coefficients for the specified function class.

1. Introduction and Definitions

In recent years, one of the most fascinating subjects has emerged: the study of the geometric behavior of analytic functions. Studying and describing the characteristics of analytic functions using geometrical and topological techniques is the primary goal of geometric function theory. One aims to link the analytical characteristics of functions with topological and geometrical insights, offering a more profound comprehension of the behavior of analytic functions. We now go over some fundamentals of geometric function theory as well as the analytic function subclasses that fall under this study's purview.

Received: Jul. 15, 2025.

2020 *Mathematics Subject Classification.* 30C45, 35A07, 35Q53.

Key words and phrases. generalized M-series; Mittag-Leffler function; generalized hypergeometric function; analytic function; univalent function; Schwarz function; non-Bazilevič functions; subordination; Fekete-Szegö inequality.

https://doi.org/10.28924/2291-8639-23-2025-233

ISSN: 2291-8639

1.1. **Generalized Telephone Numbers (GTN).** The usual as involution numbers, also known telephone numbers, are assumed by the recurrence relation

$$\mathcal{Q}(n) = \mathcal{Q}(n-1) + (n-1)\mathcal{Q}(n-2) \quad for \quad n \geq 2$$

with initial conditions

$$\mathcal{Q}(0) = \mathcal{Q}(1) = 1.$$

In1800, Heinrich August Rothe noted that $\Upsilon(n)$ is the number of involutions (self-inverse permutations)in the symmetric group (see,for example, [10, 19]).Relation between involution numbers and symmetric groups were observed for the first time in the year 1800. Since involutions correspond to standard Young tableaux, it is clear that the n^{th} involution number is also the number of Young tableaux on the set $1, 2, ..., n$ (for details,see [8]).According to John Riordan,the above recurrence relation,in fact,produces the number of connection patterns in a telephone system with n subscribers(see [33]).In 2017,Wlochand Wolowiec-Musial [45] introduced generalized telephone numbers $\mathcal{Q}(\wp, n)$ defined for integers $n \geq 0$and $\wp \geq 1$ by the following recursion:

$$\Upsilon(\wp, n) = \wp\Upsilon(\wp, n-1) + (n-1)\Upsilon(\wp, n-2)$$

with initial conditions

$$\Upsilon(\wp, 0) = 1, \Upsilon(\wp, 1) = \wp,$$

and studied some properties. In 2019, Bednarz and Wolowiec-Musial [7] introduced a new generalization of telephone numbers by

$$\Upsilon_\wp(n) = \Upsilon_\wp(n-1) + \wp(n-1)\Upsilon_\wp(n-2)$$

with initial conditions

$$\Upsilon_\wp(0) = \Upsilon_\wp(1) = 1$$

for integers $n \geq 2$ and $\wp \geq 1$. They gave the generating function,direct formula and matrix generators for these numbers. Moreover,they obtained interpretations and proved some properties of these numbers connected with congruences. Lately, they derived the exponential generating function and the summation formula for generalized telephone numbers $\Upsilon_\wp(n)$ as follows:

$$e^{x+\wp\frac{x^2}{2}} = \sum_{n=0}^{\infty} \Upsilon_\wp(n)\frac{x^n}{n!} \quad (\wp \geq 1)$$

As we can observe,if $\wp = 1$, then we obtain classical telephone numbers $\Upsilon(n)$. Clearly, $\Upsilon_\wp(n)$ is for some values of n as

(1) $\Upsilon_\wp(0) = \Upsilon_\wp = 1$,
(2) $\Upsilon_\wp(2) = 1 + \wp$,
(3) $\Upsilon_\wp(3) = 1 + 3\wp$
(4) $\Upsilon_\wp(4) = 1 + 6\wp + 3\wp^2$
(5) $\Upsilon_\wp(5) = 1 + 10\wp + 15\wp^2$
(6) $\Upsilon_\wp(6) = 1 + 15\wp + 45\wp^2 + 15\wp^3$.

We now consider the function

$$\Xi(z) := e^{\left(z+\wp\frac{z^2}{2}\right)} = 1 + z + \frac{1+\wp}{2}z^2 + \frac{1+3\wp}{6}z^3 + \frac{3\wp^2+6\wp+1}{24}z^4 + \frac{1+10\wp+15\wp^2}{120}z^5 + \cdots. \quad (1.1)$$

with its domain of definition as the open unit disk $\mathbb{U}$ studied for the class of analytic functions [11, 25, 29, 43].

1.2. **Generalized M-series.** In the study of fractional differential equations, the Mittag-Leffler function has been widely utilized. Additionally, *Geometric Function Theory* has been studied in tandem with *Theory of Special Functions*, and the generalized Gaussian hypergeometric function, which is a solution to the famous Gaussian hypergeometric differential equation, has been crucial to this work. The M-series is one such generalization that has been shown to be a useful tool in investigations related to duality theory. Unifying the Mittag-Leffler function and the Gaussian hypergeometric function, recently Sharma and Jain in [37, Eq. 1] (also see [41]) defined the generalized M-series, which is given by

$${}_r\mathcal{M}_s^{\varrho,\varpi}(\kappa_1,\ldots,\kappa_r;\sigma_1,\ldots,\sigma_s;z) = \sum_{n=0}^{\infty} \frac{(\kappa_1)_n\ldots(\kappa_r)_n}{(\sigma_1)_n\ldots(\sigma_s)_n}\frac{z^n}{\Gamma(n\varrho+\varpi)}, \quad (1.2)$$

z, ϱ, $\varpi \in \mathbb{C}$, $\mathrm{Re}(\varrho) > 0$ and $(\kappa_i)_n$, $(\sigma_j)_n$ are the well-known Pochhammer symbol. Further the primary condition for the existence of the series (1.2) is that the denominator terms $\sigma_j's$, $(j = 1, 2, \ldots s)$ are never zero or negative integer. Whereas if any of the numerator terms $\kappa_j's$, $(j = 1, 2, \ldots r)$ is zero or negative integer, then the infinite series terminates to be polynomial in z. For formal definition and convergence pertaining to the generalized M-series, refer to [37, Eq. 1] (also see [41]). Also, note that the q-analogue of the generalized M-series was studied by Shimelis and Suthar [38–40].

Due to Karthikeyan et al. in [17] we consider Hadamard product and the generalized M-series

$$\begin{aligned} F(\kappa_1,\sigma_1;\varrho,\varpi)(z) &= D(\kappa_1,\sigma_1;\varrho,\varpi)(z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(\kappa_1)_{n-1}\ \ldots\ (\kappa_r)_{n-1}}{(\sigma_1)_{n-1}\ \ldots\ (\sigma_s)_{n-1}} \frac{\Gamma(\varpi)\, a_n z^n}{(n-1)!\Gamma(\varrho(n-1)+\varpi)}, \end{aligned} \quad (1.3)$$

where $m \geq 0$, $0 \leq \lambda \leq 1$ and the range of the parameters are same as in (1.2). Further $f(z)$ in (1.3) belongs to the class $\mathcal{A}$, the class of functions analytic in the unit disc $\mathbb{U} = \{z \in \mathbb{C}; |z| < 1\}$ which have an expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (a_n \in \mathbb{C}). \quad (1.4)$$

Karthikeyan et al. [17] discuss the particular instances and applications of the operator $F(\kappa_1,\sigma_1;\varrho,\varpi)(z)$. The significance and applications of the operator $F(\kappa_1,\sigma_1;\varrho,\varpi)(z)$ were covered in depth in [17] (see [5, 6, 15, 32]), so we have purposefully chosen not to repeat them here.

We let $\mathcal{F}$ be the class of Carathéodory's function (see [9]), a class of analytic functions (pz) with normalization $p(0) = 1$ and which satisfies $\mathrm{Re}\,(p(z)) > 0$, $(z \in \mathbb{U})$. Also, the classes of starlike and convex functions are known satisfy the inclusion $p(z) \in \mathcal{F}$ provided $p(z) = \frac{zf'(z)}{f(z)}$ and $p(z) = \frac{(zf'(z))'}{f'(z)}$ respectively. Here we will denote the classes of starlike and convex functions by $\mathcal{S}^*$ and C respectively. In [23], Miller et al. established the following differential inclusion

$$\operatorname{Re}\left\{(1-\vartheta)h(z) + \vartheta\left(h(z) + \frac{zh'(z)}{h(z)}\right)\right\} > 0, \quad \Longrightarrow \quad \operatorname{Re}(h(z)) > 0, \ (z \in \mathbb{U}), \tag{1.5}$$

where $h(z) = \frac{zf'(z)}{f(z)}$ for all real ϑ. For $0 \leq \vartheta \leq 1$, Mocanu class [24] is the class functions $f \in \mathcal{A}$ satisfies

$$\operatorname{Re}\left\{(1-\vartheta)\frac{zf'(z)}{f(z)} + \vartheta\frac{(zf'(z))'}{f'(z)}\right\} > 0, \ (z \in \mathbb{U}).$$

Several authors studied the geometrical implications when $h(z)$ is replaced with various analytic characterizations like $\frac{z^{1-\sigma}f'(z)}{[f(z)]^{1-\sigma}}$ or $\frac{z(f'(z))^{\delta}}{f(z)}$ or $\frac{zf'(z)}{g(z)}$ in (1.5). Here our main focus would be to study the implication if $h(z)$ in (1.5) is replaced with an analytic characterization associated with the class of Bazilevič functions.

We define the following new subclass in light of Shanmugam et al. [35,36] and the recent study on telephone numbers [11,25,27,29,43]:

Definition 1.1. *For parameters range mentioned as in the operator $F(\kappa_1, \sigma_1; \varrho, \varpi)f$ and $0 \leq \varsigma, \vartheta \leq 1$, a function $f(z) \in \mathcal{A}$ is said to be in $\mathcal{N}(\kappa_1, \sigma_1; \varsigma; \vartheta; \Xi)$ if and only if it satisfies the condition*

$$(1-\vartheta)\frac{z^{1+\varsigma}F'(\kappa_1, \sigma_1; \varrho, \varpi)(z)}{[F(\kappa_1, \sigma_1; \varrho, \varpi)(z)]^{1+\varsigma}} + \vartheta\frac{[zF'(\kappa_1, \sigma_1; \varrho, \varpi)(z)]'}{[F'(\kappa_1, \sigma_1; \varrho, \varpi)(z)]^{1+\varsigma}} \prec \Xi(z), \quad \forall\, z \in \mathbb{U}. \tag{1.6}$$

where $\Xi(z)$ is starlike symmetric with respect to horizontal axis and maps the unit disc onto a right-half plane which has an expansion of the form (1.1)

Remark 1.1. *The class $\mathcal{N}(\kappa_1, \sigma_1; \varsigma; \vartheta; \Xi)$ would reduce to*

$$\mathcal{S}(\kappa_1, \sigma_1; \varsigma; \vartheta; \Xi) = \left\{f \in \mathcal{A} : \frac{z^{1+\varsigma}F'(\kappa_1, \sigma_1; \varrho, \varpi)(z)}{[F(\kappa_1, \sigma_1; \varrho, \varpi)(z)]^{1+\varsigma}} \prec \Xi(z)\right\}. \tag{1.7}$$

and

$$C(\kappa_1, \sigma_1; \varsigma; \vartheta; \Xi) = \left\{f \in \mathcal{A} : \frac{[zF'(\kappa_1, \sigma_1; \varrho, \varpi)(z)]'}{[F'(\kappa_1, \sigma_1; \varrho, \varpi)(z)]^{1+\varsigma}} \prec \Xi(z)\right\} \tag{1.8}$$

by letting $\vartheta = 0$ and $\vartheta = 1$ in (1.6) *respectively.*

Note that $f(z) \in C(\kappa_1, \sigma_1; \varsigma; \vartheta; \Xi)$ if and only if $zf'(z) \in \mathcal{S}(\kappa_1, \sigma_1; \varsigma; \vartheta; \Xi)$. Hence, the class $\mathcal{N}(\kappa_1, \sigma_1; \varsigma; \vartheta; \Xi)$ is same as famous Mocanu class which unifies two classes related by the Alexander theorem.

Remark 1.2. *We will now discuss the significance of the defined function class.*

(1) *Letting $\varrho = 0 = \vartheta$ in (1.6), we let class $\mathcal{B}(\kappa_1, \sigma_1; \varsigma; \Xi)$ whose analytic characterization is given by*

$$(F'(\kappa_1, \sigma_1)(z))\left(\frac{z}{F(\kappa_1, \sigma_1)(z)}\right)^{1+\varsigma} \prec \Xi(z), \quad \forall z \in \mathbb{U},$$

where $F(\kappa_1, \sigma_1)f(z) = z + \sum_{n=2}^{\infty} \frac{(\kappa_1)_{n-1} \dots (\kappa_r)_{n-1}}{(\sigma_1)_{n-1} \dots (\sigma_s)_{n-1}} \frac{a_n z^n}{(n-1)!}$.

(2) *For the choice $\vartheta = \varrho = 0, r = 2, s = 1, \kappa_1 = \sigma_1, \kappa_2 = 1$, the class $\mathcal{N}(\kappa_1, \sigma_1; \varsigma; \vartheta; \Xi) \equiv \mathcal{N}(\varsigma, \Xi)$ which is defined by*

$$\left\{f'(z)\left(\frac{z}{f(z)}\right)^{1+\varsigma}\right\} \prec \Xi(z), \quad \forall z \in \mathbb{U}$$

called as Obradović type non-Bazilević functions linked telephone numbers(see Obradović in [28]).

(3) *Similarly, $\varsigma = \varrho = 0$, $r = 2$, $s = 1$, $\kappa_1 = \sigma_1$, $\kappa_2 = 1$, the class $\mathcal{N}(\kappa_1, \sigma_1; \varsigma; \vartheta; \Xi)$ reduces to Mocanu's ϑ-convex functions (see [24]) associated with telephone numbers.*

Further, various other new and well-known classes of univalent functions can be obtained by specializing the parameter involved in the operator (1.3) and the general function $\Xi(z)$ (see [4,12]).In this present investigation, we obtain Fekete-Szegö's inequality for certain non-Bazilevič functions $f(z)$ defined on the open unit disk. A similar results have been done for the function f^{-1}. Further application of our results to certain functions defined by convolution products with a normalized analytic functions is given, and in particular we obtain Fekete-Szegö inequalities for certain subclasses of non-Bazilevič functions.

2. Coefficient Inequalities

We will need the following lemmas to establish our main results.

Lemma 2.1. *[14, Theorem 1] If $L(z) = 1 + \sum_{r=1}^{\infty} \ell_r z^r \in \mathcal{F}$, and $\rho \in \mathbb{C}$, then*

$$\left|\ell_\varepsilon - \rho \ell_r \ell_{\varepsilon - r}\right| \leq 2 \max\{1; |2\rho - 1|\},$$

for all $1 \leq r \leq \varepsilon - 1$.

Motivated by the well-known results of Ma-Minda [21, p. 162] and Livingston [20, Lemma 1] in this section we obtain the initial coefficients and solution to the Fekete-Szegö [16]problem for $f \in \mathcal{N}(\kappa_1, \sigma_1; \varsigma; \vartheta; \Xi)$.

Theorem 2.1. *Let $f(z) \in \mathcal{N}(\kappa_1, \sigma_1; \varsigma; \vartheta; \Xi)$ and $0 \leq \varsigma < 1$, then we have*

$$|a_2| \leq \frac{1}{\left|(1+\vartheta)(1-\varsigma)\Gamma_2\right|} \tag{2.1}$$

$$|a_3| \leq \frac{1}{\left|(2-\varsigma)(1+2\vartheta)\Gamma_3\right|} \max\left\{1; \left|\frac{1+\wp}{2} + \frac{(1+3\vartheta)(\varsigma+1)(2-\varsigma)}{2(1+\vartheta)^2(1-\varsigma)^2}\right|\right\} \tag{2.2}$$

and for all $\rho \in \mathbb{C}$

$$\left|a_3-\rho a_2^2\right| \le \frac{1}{\left|(2-\varsigma)(1+2\vartheta)\Gamma_3\right|}\max\left\{1;\left|\frac{1+\wp}{2}+\frac{(1+3\vartheta)(\varsigma+1)(2-\varsigma)}{2(1+\vartheta)^2(1-\varsigma)^2}-\frac{\rho(1+2\vartheta)(2-\varsigma)\Gamma_3}{(1+\vartheta)^2(1-\varsigma)^2\Gamma_2^2}\right|\right\} \tag{2.3}$$

The inequalities are sharp.

Proof. As $f(z) \in \mathcal{N}(\kappa_1, \sigma_1; \varsigma; \vartheta; \Xi)$, by (1.6), we have

$$(1-\vartheta)\frac{z^{1+\varsigma}F'(\kappa_1,\, \sigma_1;\, \varrho,\, \varpi)(z)}{\left[F(\kappa_1,\, \sigma_1;\, \varrho,\, \varpi)(z)\right]^{1+\varsigma}}+\vartheta\frac{\left[zF'(\kappa_1,\, \sigma_1;\, \varrho,\, \varpi)(z)\right]'}{\left[F'(\kappa_1,\, \sigma_1;\, \varrho,\, \varpi)(z)\right]^{1+\varsigma}}=\Xi\left[w(z)\right]. \tag{2.4}$$

For function ℓ of the form $\ell(z)=1+\sum_{k=1}^{\infty}\ell_n z^n \in \mathcal{F}$, the right side of (2.4) will be of the form

$$1+\frac{\ell_1}{2}z+\left(\frac{\ell_2}{2}+\frac{(\wp-1)\ell_1^2}{8}\right)z^2+\left(\frac{\ell_3}{2}+(\wp-1)\frac{\ell_1\ell_2}{4}+\frac{(1-3\wp)}{48}\ell_1^3\right)z^3+\cdots., \tag{2.5}$$

The left hand side of (2.4) will be of the form

$$\begin{aligned}&(1-\vartheta)\frac{z^{1+\varsigma}F'(\kappa_1,\, \sigma_1;\, \varrho,\, \varpi)(z)}{\left[F(\kappa_1,\, \sigma_1;\, \varrho,\, \varpi)(z)\right]^{1+\varsigma}}+\vartheta\frac{\left[zF'(\kappa_1,\, \sigma_1;\, \varrho,\, \varpi)(z)\right]'}{\left[F'(\kappa_1,\, \sigma_1;\, \varrho,\, \varpi)(z)\right]^{1+\varsigma}}\\ &=1+(1+\vartheta)\,(1-\varsigma)\,a_2\Gamma_2 z+(2-\varsigma)\left[a_3(1+2\vartheta)\Gamma_3-\frac{(1+3\vartheta)(\varsigma+1)a_2^2\Gamma_2^2}{2}\right]z^2+\cdots. \end{aligned}\tag{2.6}$$

Equating coefficients of z from (2.6) and (2.5), we obtain

$$a_2=\frac{\ell_1}{2\,(1+\vartheta)\,(1-\varsigma)\,\Gamma_2} \tag{2.7}$$

Further by comparing coefficients of z^2 from (2.6) and (2.5), we get,

$$\begin{aligned}(2-\varsigma)\left[a_3(1+2\vartheta)\Gamma_3-\frac{(1+3\vartheta)(\varsigma+1)a_2^2\Gamma_2^2}{2}\right] &= \left(\frac{\ell_2}{2}+\frac{(\wp-1)\ell_1^2}{8}\right)\\ (2-\varsigma)\,(1+2\vartheta)a_3\Gamma_3 &= \frac{(2-\varsigma)\,(1+3\vartheta)(\varsigma+1)a_2^2\Gamma_2^2}{2}+\left(\frac{\ell_2}{2}+\frac{(\wp-1)\ell_1^2}{8}\right).\end{aligned}$$

Using (2.7) in above equation, we have

$$a_3=\frac{1}{2(2-\varsigma)(1+2\vartheta)\Gamma_3}\left[\ell_2-\frac{\ell_1^2}{2}\left(\frac{1-\wp}{2}-\frac{(1+3\vartheta)(\varsigma+1)(2-\varsigma)}{2(1+\vartheta)^2(1-\varsigma)^2}\right)\right]. \tag{2.8}$$

Using Lemma 2.1 in (2.8), we get (2.2).

Now to prove (5), we consider

$$\begin{aligned}\left|a_3-\rho a_2^2\right| &= \left|\frac{1}{2(2-\varsigma)(1+2\vartheta)\Gamma_3}\left[\ell_2-\frac{\ell_1^2}{2}\left(\frac{1-\wp}{2}-\frac{(1+3\vartheta)(\varsigma+1)(2-\varsigma)}{2(1+\vartheta)^2(1-\varsigma)^2}\right)\right]\right.\\ &\quad\left.-\ \frac{\rho\ell_1^2}{4\,(1+\vartheta)^2\,(1-\varsigma)^2\,\Gamma_2^2}\right|\\ &= \left|\frac{1}{2(2-\varsigma)(1+2\vartheta)\Gamma_3}\left[\ell_2-\frac{\ell_1^2}{2}\left(\frac{1-\wp}{2}-\frac{(1+3\vartheta)(\varsigma+1)(2-\varsigma)}{2(1+\vartheta)^2(1-\varsigma)^2}\right.\right.\right.\end{aligned}$$

$$+ \quad \frac{\rho(2-\varsigma)(1+2\vartheta)\Gamma_3}{(1+\vartheta)^2(1-\varsigma)^2\Gamma_2^2}\Bigg)\Bigg]\Bigg|.$$

$$= \quad \left|\frac{1}{2(2-\varsigma)(1+2\vartheta)\Gamma_3}\left[\ell_2 - W\ell_1^2\right]\right|. \tag{2.9}$$

where

$$W = \frac{1}{2}\left(\frac{1-\wp}{2} - \frac{(1+3\vartheta)(\varsigma+1)(2-\varsigma)}{2(1+\vartheta)^2(1-\varsigma)^2} + \frac{\rho(2-\varsigma)(1+2\vartheta)\Gamma_3}{(1+\vartheta)^2(1-\varsigma)^2\Gamma_2^2}\right)$$

Further, by Lemma 2.1 we deduce

$$\left|a_3 - \rho a_2^2\right| \leq \frac{1}{\left|(2-\varsigma)(1+2\vartheta)\Gamma_3\right|}\max\left\{1; \left|\frac{1+\wp}{2} + \frac{(1+3\vartheta)(\varsigma+1)(2-\varsigma)}{2(1+\vartheta)^2(1-\varsigma)^2} - \frac{\rho(1+2\vartheta)(2-\varsigma)\Gamma_3}{(1+\vartheta)^2(1-\varsigma)^2\Gamma_2^2}\right|\right\}. \tag{2.10}$$

An examination of the proof shows that the equality for (2.3) holds if $\ell_1 = 0, \ell_2 = 2$. Equivalently, by Lemma 2.1 we have

$$\Xi\left(\frac{\vartheta(z)-1}{\vartheta(z)+1}\right) = \Xi\left(\frac{\frac{1+z^2}{1-z^2}-1}{\frac{1+z^2}{1-z^2}+1}\right) = \Xi\left(\frac{2z^2}{1+z^2}\right) = \Xi_2(z).$$

Therefore, the extremal function of the class $\mathcal{N}(\kappa_1, \sigma_1; \varsigma; \vartheta; \Xi)$ is given by

$$(1-\vartheta)\frac{z^{1+\varsigma}F'(\kappa_1,\, \sigma_1;\, \varrho,\, \varpi)(z)}{[F(\kappa_1,\, \sigma_1;\, \varrho,\, \varpi)(z)]^{1+\varsigma}} + \vartheta\frac{[zF'(\kappa_1,\, \sigma_1;\, \varrho,\, \varpi)(z)]'}{[F'(\kappa_1,\, \sigma_1;\, \varrho,\, \varpi)(z)]^{1+\varsigma}} = \Xi_2(z).$$

Similarly, the equality for (2.3) holds if $\ell_2 = 2$. Equivalently, by Lemma 2.1 we have

$$\Xi\left(\frac{\vartheta(z)-1}{\vartheta(z)+1}\right) = \Xi\left(\frac{\frac{1+z}{1-z}-1}{\frac{1+z}{1-z}+1}\right) = \Xi\left(\frac{2z}{1+z}\right) = \Xi_1(z).$$

Therefore, the extremal function in $\mathcal{N}(\kappa_1, \sigma_1; \varsigma; \vartheta; \Xi)$ is given by

$$(1-\vartheta)\frac{z^{1+\varsigma}F'(\kappa_1,\, \sigma_1;\, \varrho,\, \varpi)(z)}{[F(\kappa_1,\, \sigma_1;\, \varrho,\, \varpi)(z)]^{1+\varsigma}} + \vartheta\frac{[zF'(\kappa_1,\, \sigma_1;\, \varrho,\, \varpi)(z)]'}{[F'(\kappa_1,\, \sigma_1;\, \varrho,\, \varpi)(z)]^{1+\varsigma}} = \Xi_1(z),$$

and the proof of the theorem is complete. □

Let $\vartheta = 1$ in Theorem 2.1, we get the following.

Corollary 2.1. *If $f(z) \in C(\kappa_1, \sigma_1; \varsigma; \vartheta; \Xi)$(see Remark1.1) and $0 \leq \varsigma < 1$, then we have*

$$|a_2| \leq \frac{1}{2\left|(1-\varsigma)\,\Gamma_2\right|}, \qquad |a_3| \leq \frac{1}{3\left|(2-\varsigma)\Gamma_3\right|}\max\left\{1; \left|\frac{1+\wp}{2} + \frac{(\varsigma+1)(2-\varsigma)}{2(1-\varsigma)^2}\right|\right\}$$

and for all $\rho \in \mathbb{C}$

$$\left|a_3 - \rho a_2^2\right| \leq \frac{1}{3\left|(2-\varsigma)\Gamma_3\right|}\max\left\{1; \left|\frac{1+\wp}{2} + \frac{(\varsigma+1)(2-\varsigma)\psi_1}{2(1-\varsigma)^2} - \frac{3\rho(2-\varsigma)}{4(1-\varsigma)^2}\right|\right\}.$$

Letting $\vartheta = \varsigma = \varrho = 0, r = 2, s = 1$, $\kappa_1 = \sigma_1$, and $\kappa_2 = 1$ in Theorem 2.1, we get the following result.

Corollary 2.2. *[42, Theorem 3.1.] If $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{S}^*(\psi)$, then for all $\rho \in \mathbb{C}$ we have*

$$\left|a_3 - \rho a_2^2\right| \leq \frac{1}{2}\max\left\{1; \left|\frac{3+\wp}{2} - 2\rho\right|\right\}.$$

The inequality is sharp for the function f_* given by

$$f_*(z) = \begin{cases} z\exp\displaystyle\int_0^z \frac{\Xi(t)-1}{t}dt, & \text{if } \left|\frac{3+\wp}{2} - 2\rho\right| \geq 1, \\ z\exp\displaystyle\int_0^z \frac{\Xi(t^2)-1}{t}dt, & \text{if } \left|\frac{3+\wp}{2} - 2\rho\right| \leq 1. \end{cases} \tag{2.11}$$

Proof. In Theorem 2.1, taking $\vartheta = \varsigma = \varrho = m = 0, r = 2, s = 1$, $\kappa_1 = \sigma_1$, and $\kappa_2 = 1$, we get the inequality

$$\left|a_3 - \rho a_2^2\right| \leq \begin{cases} \dfrac{1}{2}, & \text{if } \left|\frac{3+\wp}{2} - 2\rho\right| \leq 1, \\ \dfrac{1}{2}\left|\dfrac{3+\wp}{2} - 2\rho\right|, & \text{if } \left|\frac{3+\wp}{2} - 2\rho\right| \geq 1. \end{cases}$$

Finally, following a similar technique to that for the sharpness of Theorem 3.1 of [42], we obtain (2.11). □

3. Coefficient Estimates of $f^{-1}(z)$

The inverse f^{-1}, defined by $f^{-1}(f(z)) = z, z \in \mathbb{U}$ and $f(f^{-1}(t)) = t$, $(|t| < r; r \geq 1/4)$. The coefficient inequalities of the inverse functions $\mathcal{N}(\kappa_1, \sigma_1; \varsigma; \vartheta; \Xi)$ are valid only for the functions which are univalent. From [18, Lemma 2.1.], for $f^{-1}(t) = w + \sum_{k=2}^{\infty} b_k w^k$, $(|t| < r; r \geq 1/4)$, we have

$$b_2 = -a_2, \quad b_3 = 2a_2^2 - a_3 \quad \text{and} \quad b_3 - \tau b_2^2 = 2a_2^2 - a_3 - \tau a_2^2.$$

Taking modulus in the above equality and using the inequalities (2.1) and (2.3), we get the following result.

Theorem 3.1. *Let $f \in \mathcal{N}(\kappa_1, \sigma_1; \varsigma; \vartheta; \Xi)$ and let f^{-1} be the inverse of f defined by*

$$f^{-1}(t) = w + \sum_{k=2}^{\infty} b_k t^k, \quad (|t| < r; r \geq 1/4),$$

then we have

$$|b_2| \leq \frac{1}{\left|(1+\vartheta)(1-\varsigma)\Gamma_2\right|}$$

and

$$|b_3| \leq \frac{1}{\left|(2-\varsigma)\Gamma_3\right|}\max\left\{1; \left|\frac{1+\wp}{2} + \frac{(1+3\vartheta)(\varsigma+1)(2-\varsigma)}{2(1+\vartheta)^2(1-\varsigma)^2} - \frac{2(2-\varsigma)\Gamma_3}{(1+\vartheta)^2(1-\varsigma)^2\Gamma_2^2}\right|\right\}$$

Also, for all $\tau \in \mathbb{C}$

$$\left|b_3-\tau b_2^2\right| \le \frac{1}{\left|(2-\varsigma)\Gamma_3\right|}\max\left\{1;\left|\frac{1+\wp}{2}+\frac{(1+3\vartheta)(\varsigma+1)(2-\varsigma)}{2(1+\vartheta)^2(1-\varsigma)^2}\right.\right.$$
$$\left.\left.-\frac{(\tau-2)(2-\varsigma)\Gamma_3}{(1+\vartheta)^2(1-\varsigma)^2\Gamma_2^2}\right|\right\},$$

where $\varsigma \neq 1$.

4. Logarithmic Coefficients for Functions Belonging to $\mathcal{N}(\kappa_1,\sigma_1;\, \varsigma;\, \vartheta;\, \Xi)$

Logarithmic coefficients took the spotlight when Milin in [22] studied its properties which would imply the bounds of the Taylor coefficients of univalent functions. For detailed study, refer to [2,3].

If f is analytic in $\mathbb{U}$, with $\frac{f(z)}{z} \neq 0$ for all $z \in \mathbb{U}$, then the well-known logarithmic coefficients $c_n := c_n(f), n \in \mathbb{N}$, of f are given by

$$\log\frac{f(z)}{z} = 2\sum_{n=1}^{\infty} c_n z^n, z \in \mathbb{U}, \quad \log 1 = 0. \tag{4.1}$$

Now we will add additional criterion to the class $\mathcal{N}(\kappa_1,\sigma_1;\, \varsigma;\, \vartheta;\, \Xi)$, so that logarithmic coefficients of $\mathcal{N}(\kappa_1,\sigma_1;\, \varsigma;\, \vartheta;\, \Xi)$ is well-defined. That is, we let
$\mathcal{LN}(\kappa_1,\sigma_1;\, \varsigma;\, \vartheta;\, \Xi) = \mathcal{N}(\kappa_1,\sigma_1;\, \varsigma;\, \vartheta;\, \Xi) \cap \left\{f \text{ is analytic in } \mathbb{U} : \frac{f(z)}{z} \neq 0,\, z \in \mathbb{U}\right\}$. Note that for all functions $\mathcal{LN}(\kappa_1,\sigma_1;\, \varsigma;\, \vartheta;\, \Xi)$, the relation (4.1) is well-defined.

Theorem 4.1. *If $f(z) \in \mathcal{LN}(\kappa_1,\sigma_1;\, \varsigma;\, \vartheta;\, \Xi)$ with the logarithmic coefficients given by (4.1), then we have for $\varsigma \neq 1$*

$$|c_1| \le \frac{1}{2\left|(1+\vartheta)\,(1-\varsigma)\,\Gamma_2\right|}, \tag{4.2}$$

$$|c_2| \le \frac{1}{\left|(2-\varsigma)\Gamma_3\right|}\max\left\{1;\left|\frac{1+\wp}{2}+\frac{(1+3\vartheta)(\varsigma+1)(2-\varsigma)}{2(1+\vartheta)^2(1-\varsigma)^2}-\frac{(2-\varsigma)\Gamma_3}{2(1+\vartheta)^2(1-\varsigma)^2\Gamma_2^2}\right|\right\}, \tag{4.3}$$

and

$$\left|c_2-\mu c_1^2\right| \le \frac{1}{\left|(2-\varsigma)\Gamma_3\right|}\max\left\{1;\left|\frac{1+\wp}{2}+\frac{(1+3\vartheta)(\varsigma+1)(2-\varsigma)}{2(1+\vartheta)^2(1-\varsigma)^2}-\frac{(1+\mu)(2-\varsigma)\Gamma_3}{(1+\vartheta)^2(1-\varsigma)^2\Gamma_2^2}\right|\right\}. \tag{4.4}$$

Proof. From $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and equating the first two coefficients of relation (4.1), we get

$$c_1 = \frac{a_2}{2},\ c_2 = \frac{1}{2}\left(a_3 - \frac{a_2^2}{2}\right).$$

Using (2.7)) and (2.8) in the above equation and applying Lemma 2.1, we obtain (4.2) and (4.3). To obtain (4.4), consider

$$\left|c_2-\mu c_1^2\right| = \frac{1}{2}\left[a_3 - \frac{(1+\mu)}{2}a_2^2\right].$$

Changing $\rho = \frac{1+\mu}{2}$ in (5), we get the desired result. □

5. Applications to Functions Defined by Certain Distribution

Let consider a given function $\psi \in \mathcal{A}$ of the form

$$\psi(z) = z + \sum_{n=2}^{\infty} \psi_n z^n, \; z \in \mathbb{U}. \tag{5.1}$$

In this section we will define a new function class $\mathcal{M}^{\psi}_{\mathfrak{R}}(\varsigma, \vartheta, \lambda)$ based on the *convolution (Hadamard) product* and we discuss an application of the Poisson distribution series to this function class.

If $f \in \mathcal{A}$ has the form (1.4), then the convolution product of f and ψ is given by

$$(f * \psi)(z) := z + \sum_{n=2}^{\infty} \psi_n a_n z^n, \; z \in \mathbb{U},$$

and let define the class

$$\mathcal{M}^{\psi}_{\mathfrak{R}}(\varsigma, \vartheta, \lambda) := \{f \in \mathcal{A} : f * \psi \in \mathcal{M}_{\mathfrak{R}}(\varsigma, \vartheta, \lambda)\}.$$

We will obtain an upper bound for the Fekete-Szegő functional for the class $\mathcal{M}^{\psi}_{\mathfrak{R}}(\varsigma, \vartheta, \lambda)$, corresponding to the Theorem 3.1 to 4.1. Since the proofs are similar with these previous results, we will omit them.

Theorem 5.1. *Suppose that $\psi \in \mathcal{A}$ has the form* (5.1)*, such that $\psi_2 \cdot \psi_3 \neq 0$. If f given by* (1.4) *belongs to $\mathcal{M}^{\psi}_{\mathfrak{R}}(\varsigma, \vartheta, \lambda)$, then for any ρ complex number we have*

$$\left|a_3 - \rho a_2^2\right| \leq \frac{1}{\left|(2-\varsigma)(1+2\vartheta)\psi_3\Gamma_3\right|} \max\left\{1; \left|\frac{1+\wp}{2} + \frac{(1+3\vartheta)(\varsigma+1)(2-\varsigma)}{2(1+\vartheta)^2(1-\varsigma)^2} - \frac{\rho(1+2\vartheta)(2-\varsigma)\psi_3\Gamma_3}{(1+\vartheta)^2(1-\varsigma)^2\psi_2^2\Gamma_2^2}\right|\right\},$$

where the notations are the same like in Theorem 2.1.

Proof. The left hand side of (2.4) will be of the form

$$(1-\vartheta)\frac{z^{1+\varsigma}F'(\kappa_1, \sigma_1; \varrho, \varpi)(z)}{[F(\kappa_1, \sigma_1; \varrho, \varpi)(z)]^{1+\varsigma}} + \vartheta\frac{[zF'(\kappa_1, \sigma_1; \varrho, \varpi)(z)]'}{[F'(\kappa_1, \sigma_1; \varrho, \varpi)(z)]^{1+\varsigma}}$$

$$= 1 + (1+\vartheta)(1-\varsigma) a_2\psi_n\Gamma_2 z + (2-\varsigma)\left[a_3(1+2\vartheta)\psi_3\Gamma_3 - \frac{(1+3\vartheta)(\varsigma+1)a_2^2\psi_2^2\Gamma_2^2}{2}\right]z^2 + \cdots. \tag{5.2}$$

□

Employing the techniques as in Theorem 2.1 , we get

$$a_2 = \frac{\ell_1}{2(1+\vartheta)(1-\varsigma)\psi_2\Gamma_2} \tag{5.3}$$

$$(2-\varsigma)(1+2\vartheta)a_3\psi_3\Gamma_3 = \frac{(2-\varsigma)(1+3\vartheta)(\varsigma+1)a_2^2\psi_2^2\Gamma_2^2}{2} + \left(\frac{\ell_2}{2} + \frac{(\wp-1)\ell_1^2}{8}\right).$$

Using (5.3) in above equation, we have

$$a_3 = \frac{1}{2(2-\varsigma)(1+2\vartheta)\psi_3\Gamma_3}\left[\ell_2 - \frac{\ell_1^2}{2}\left(\frac{1-\wp}{2} - \frac{(1+3\vartheta)(\varsigma+1)(2-\varsigma)}{2(1+\vartheta)^2(1-\varsigma)^2}\right)\right]. \tag{5.4}$$

Employing the techniques as in lines of Theorem 2.1 , we get

$$\left|a_3 - \rho a_2^2\right| = \left|\frac{1}{2(2-\varsigma)(1+2\vartheta)\psi_3\Gamma_3}\left[\ell_2 - W\ell_1^2\right]\right|. \tag{5.5}$$

where

$$W = \frac{1}{2}\left(\frac{1-\wp}{2} - \frac{(1+3\vartheta)(\varsigma+1)(2-\varsigma)}{2(1+\vartheta)^2(1-\varsigma)^2} + \frac{\rho(2-\varsigma)(1+2\vartheta)\psi_3\Gamma_3}{(1+\vartheta)^2(1-\varsigma)^2\psi_2^2\Gamma_2^2}\right)$$

further, by Lemma 2.1 we deduce

$$\left|a_3 - \rho a_2^2\right| \leq \frac{1}{\left|(2-\varsigma)(1+2\vartheta)\psi_3\Gamma_3\right|}$$
$$\times \max\left\{1; \left|\frac{1+\wp}{2} + \frac{(1+3\vartheta)(\varsigma+1)(2-\varsigma)}{2(1+\vartheta)^2(1-\varsigma)^2} - \frac{\rho(1+2\vartheta)(2-\varsigma)\psi_3\Gamma_3}{(1+\vartheta)^2(1-\varsigma)^2\psi_2^2\Gamma_2^2}\right|\right\}.$$

Remark 5.1. *1. For* $\psi(z) = z + \sum\limits_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!}e^{-m}a_n z^n$ *power series whose coefficients are probabilities of the Poisson distribution introduced and studied by Porwal [31] (see also, [26]).*

2. For $\psi(z) = z + \sum_{n=2}^{\infty} \frac{(m_N)^{n-1}}{(n-1)!}e^{-m_N}a_n z^n$ *whose coefficients are probabilities of neutrosophic Poisson distribution defined and investigated in [1].*

3. For $\psi(z) = z + \sum_{k=2}^{\infty} \frac{[\lambda(k-1)]^{k-2}e^{-\lambda(k-1)}}{(k-1)!}z^k$, $(0 < \lambda \leq 1)$ *whose coefficients are probabilities of the Borel distribution [44]*

4. For $\psi(z) = z + \sum\limits_{n=2}^{\infty} \binom{n+m-2}{m-1} \cdot q^{n-1}(1-q)^m z^n$,*, whose coefficients are probabilities of the Pascal distribution [13].*

Further by following the steps on lines similar to Theorem 2.1 and 5.1 after an obvious change of the coefficients of parameter ρ one can deduce the results analogues to Theorems3.1 to 4.1 based on various probability distribution listed in Remark 5.1

6. Conclusions

Using the defined operator, we have defined a subclass of analytic functions whose analytic characterization is associated with the class of non-Bazilevič functions subordinated with telephone numbers. Though one has to be content with the parameters involved, but it helps in specializing most of the subclass of the univalent function theory. Some bounds of the initial coefficients are our main results.The defined class is not only new but the results obtained here is also new. That is, for a class of non-Bazilevič functions only conditions for starlikeness or univalence have been found but the coefficient bounds have not been established based on telephone numbers.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding authors.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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