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Solving a Fractional Differential Equation via Extended Fuzzy Bipolar Metric Space

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Abstract. In this paper, we introduce notion of extended fuzzy bipolar metric space and prove some fixed point results in this space. Our results generalize and expand some of the literature's well-known results. We also explore some of the applications of our key results to integral equation and fractional differential equation.

1. Introduction and Preliminaries

Fixed point theory has been attracting the attention of many researchers lately and that is due to the fact that it has many applications in different fields [1–3]. In 1960, Schweizer and Sklar [4] introduced the notion of continuous triangular norm. After that in 1965, Zadeh [5] introduces the theory of fuzzy sets. Using the concept of fuzziness, in 1975, Kramosil and Michalek [6] defined the fuzzy metric space with the help of continuous t-norm. The fuzzy approach to the distance follows from the idea that it is not necessary that there always exist a real number to define the distance between any two points which we have to approximate or to find, but it is a fuzzy notion. In 1994, George and Veeramani [7] modified the definition of fuzzy metric spaces. Grabeic [8] extend the well known fixed point theorem of Banach to fuzzy metric spaces in the sense of Karamosil and Michalek [6]. After that, Gregori and Sapena [9] extended the fuzzy banach contraction theorem to fuzzy metric space in the sense George and Veeramani's [7]. Mutlu and Gurdal [10] generalized

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metric space which was called bipolar metric spaces and give a new concept of measurement of distance between the elements of two different sets. Bartwal, Dimri and Prasad, [11] introduced fuzzy bipolar metric space and proved fixed point theorems. Recently, Sezen [12] proved controlled fuzzy metric spaces and some related fixed point results. More details, one can see [14-24].

Now, we present some basic definitions, lemmas and propositions as follows:

Definition 1.1. *[7] Let Φ be a nonempty set. An ordered triple $(\Phi, \Theta, *)$ is called a fuzzy metric space if Θ is a fuzzy set on $\Phi^2 \times (0, \infty)$ and $*$ is a continuous σ-norm satisfying the following conditions for all $\pi, \upsilon, \mathfrak{z} \in \Phi$ and $\sigma, \varpi > 0$;*

(i) $\Theta(\pi, \upsilon, \sigma) > 0$;

(ii) $\Theta(\pi, \upsilon, \sigma) = 1$ *iff* $\pi = \upsilon$;

(iii) $\Theta(\pi, \upsilon, \sigma) = \Theta(\upsilon, \pi, \sigma)$;

(iv) $\Theta(\pi, \mathfrak{z}, \mathfrak{t} + \mathfrak{s}) \geq \Theta(\pi, \upsilon, \sigma) * \Theta(\upsilon, \mathfrak{z}, \varpi)$;

(v) $\Theta(\pi, \upsilon, .) : (0, \infty) \longrightarrow (0, 1]$ *is continuous.*

Definition 1.2. *[11] Let Δ and Φ be two non-empty sets. A quadruple $(\Delta, \Phi, \Theta_f, *)$ is said to be fuzzy bipolar metric space, where $*$ is continuous σ-norm and Θ_f is a fuzzy set on $\Delta \times \Phi \times (0, \infty)$, satisfying the following conditions for all $\sigma, \varpi, \mathfrak{r} > 0$:*

(FB1) $\Theta_f(\pi, \upsilon, \sigma) > 0$ *for all* $(\pi, \upsilon) \in \Delta \times \Phi$;

(FB2) $\Theta_f(\pi, \upsilon, \sigma) = 1$ *iff* $\pi = \upsilon$ *for* $\pi \in \Delta$ *and* $\upsilon \in \Phi$;

(FB3) $\Theta_f(\pi, \upsilon, \sigma) = \Theta_f(\upsilon, \pi, \sigma)$ *for all* $\pi, \upsilon \in \Delta \cap \Phi$;

(FB4) $\Theta_f(\pi_1, \upsilon_2, \mathfrak{t} + \mathfrak{s} + \mathfrak{r}) \geq \Theta_f(\pi_1, \upsilon_1, \sigma) * \Theta_f(\pi_2, \upsilon_1, \varpi) * \Theta_f(\pi_2, \upsilon_2, \mathfrak{r})$ *for all* $\pi_1, \pi_2 \in \Delta$ *and* $\upsilon_1, \upsilon_2 \in \Phi$;

(FB5) $\Theta_f(\pi, \upsilon, .) : [0, \infty) \longrightarrow [0, 1]$ *is left continuous;*

(FB6) $\Theta_f(\pi, \upsilon, .)$ *is non-decreasing for all* $\pi \in \Delta$ *and* $\upsilon \in \Phi$.

Definition 1.3. *Let Δ and Φ be two non-empty sets, $\zeta : \Delta \times \Phi \to [1, \infty)$. A quadruple $(\Delta, \Phi, \Theta_f, *)$ is said to be extended fuzzy bipolar metric space(EFBMS), where $*$ is continuous σ-norm and Θ_f is a fuzzy set on $\Delta \times \Phi \times (0, \infty)$, satisfying the following conditions for all $\sigma, \varpi, \mathfrak{r} > 0$:*

(FCB1) $\Theta_f(\pi, \upsilon, \sigma) > 0$ *for all* $(\pi, \upsilon) \in \Delta \times \Phi$;

(FCB2) $\Theta_f(\pi, \upsilon, \sigma) = 1$ *iff* $\pi = \upsilon$ *for* $\pi \in \Delta$ *and* $\upsilon \in \Phi$;

(FCB3) $\Theta_f(\pi, \upsilon, \sigma) = \Theta_f(\upsilon, \pi, \sigma)$ *for all* $\pi, \upsilon \in \Delta \cap \Phi$;

(FCB4) $\Theta_f(\pi_1, \upsilon_2, \zeta(\pi_1, \upsilon_2)(\mathfrak{t} + \mathfrak{s} + \mathfrak{r})) \geq \Theta_f(\pi_1, \upsilon_1, \sigma) * \Theta_f(\pi_2, \upsilon_1, \varpi)$ $* \Theta_f(\pi_2, \upsilon_2, \mathfrak{r})$ *for all* $\pi_1, \pi_2 \in \Delta$ *and* $\upsilon_1, \upsilon_2 \in \Phi$;

(FCB5) $\Theta_f(\pi, \upsilon, .) : [0, \infty) \longrightarrow [0, 1]$ *is left continuous;*

(FCB6) $\Theta_f(\pi, \upsilon, .)$ *is non-decreasing for all* $\pi \in \Delta$ *and* $\upsilon \in \Phi$.

Example 1.1. *Let $\Delta = \{1, 2, 3, 4\}$, $\Phi = \{2, 4, 5, 6\}$ and $\zeta : \Delta \times \Phi \to [1, \infty)$ be a mapping defined by $\zeta(\pi, \upsilon) = \pi + \upsilon + 1$. Define $\Theta_f : \Delta \times \Phi \times (0, \infty) \to [0, 1]$ defined by*

$$\Theta_f(\pi, \upsilon, \sigma) = \frac{\min\{\pi, \upsilon\} + \sigma}{\max\{\pi, \upsilon\} + \sigma},$$

for all $\pi \in \Delta$ *and* $\upsilon \in \Phi$*. Then* $(\Delta, \Phi, \Theta_f, \star)$ *is a EFBMS with the continuous* σ*-norm* $\star$ *such that* $\mathfrak{a}{*}f = \mathfrak{a}f$*. Now,* $\zeta(1,2) = 4, \zeta(1,4) = 6, \zeta(1,5) = 7, \zeta(1,6) = 8, \zeta(2,2) = 5, \zeta(2,4) = 7, \zeta(2,5) = 8, \zeta(2,6) = 9, \zeta(3,2) = 6, \zeta(3,4) = 8, \zeta(3,5) = 9, \zeta(3,6) = 10, \zeta(4,2) = 7, \zeta(4,4) = 9, \zeta(4,5) = 10$ *and* $\zeta(4,6) = 11$*. Axioms (FCB1) to (FCB3) and (FCB5), (FCB6) are easy to verify we only prove (FCB4). Let* $\pi_1 = 1, \upsilon_2 = 4, \upsilon_1 = 2$ *and* $\pi_2 = 3$*. Then*

$$\begin{aligned}\Theta_f(1,4,\sigma+\varpi+\mathfrak{r}) &= \frac{\min\{1,4\}+\sigma+\varpi+\mathfrak{r}}{\max\{1,4\}+\sigma+\varpi+\mathfrak{r}}\\ &= \frac{1+\sigma+\varpi+\mathfrak{r}}{4+\sigma+\varpi+\mathfrak{r}}.\end{aligned}$$

Now,

$$\begin{aligned}\Theta_f(1,2,\frac{\sigma}{\zeta(1,2)}) &= \frac{\min\{1,2\}+\frac{\sigma}{\zeta(1,2)}}{\max\{1,2\}+\frac{\sigma}{\zeta(1,2)}}\\ &= \frac{1+\frac{\sigma}{4}}{4+\frac{\sigma}{4}}\\ &= \frac{4+\sigma}{8+\sigma},\end{aligned}$$

$$\begin{aligned}\Theta_f(3,2,\frac{\varpi}{\zeta(1,2)}) &= \frac{\min\{3,2\}+\frac{\varpi}{\zeta(1,2)}}{\max\{3,2\}+\frac{\varpi}{\zeta(1,2)}}\\ &= \frac{2+\frac{\varpi}{4}}{3+\frac{\varpi}{4}}\\ &= \frac{8+\varpi}{12+\varpi}\end{aligned}$$

and

$$\begin{aligned}\Theta_f(3,4,\frac{\mathfrak{r}}{\zeta(1,2)}) &= \frac{\min\{3,4\}+\frac{\mathfrak{r}}{\zeta(1,2)}}{\max\{3,4\}+\frac{\mathfrak{r}}{\zeta(1,2)}}\\ &= \frac{3+\frac{\mathfrak{r}}{4}}{4+\frac{\mathfrak{r}}{4}}\\ &= \frac{12+\mathfrak{r}}{16+\mathfrak{r}}.\end{aligned}$$

Clearly,

$$\frac{1+\sigma+\varpi+\mathfrak{r}}{4+\sigma+\varpi+\mathfrak{r}} \geq \Big(\frac{4+\sigma}{8+\sigma}\Big)\Big(\frac{8+\varpi}{12+\varpi}\Big)\Big(\frac{12+\mathfrak{r}}{16+\mathfrak{r}}\Big),\ \forall\ \sigma,\varpi,\mathfrak{r} > 0.$$

So,

$$\Theta_f(1,4,\sigma+\varpi+\mathfrak{r}) \geq \Theta_f(1,2,\frac{\sigma}{\zeta(1,2)}) \star \Theta_f(3,2,\frac{\varpi}{\zeta(1,2)}) \star \Theta_f(3,4,\frac{\mathfrak{r}}{\zeta(1,2)}).$$

On the same steps, one can prove the remaining cases. Hence $(\Delta, \Phi, \Theta_f, \star)$ *is a EFBMS.*

Example 1.2. *If we take minimum σ-norm instead of product σ-norm in Example 1.1, then $(\Delta, \Phi, \Theta_f, \star)$ is not a EFBMS. For instance, let $\pi_1 = 1, \upsilon_2 = 4, \upsilon_1 = 2, \pi_2 = 3$ and $\sigma = 0.02, \varpi = 0.03, \mathfrak{r} = 0.04$ with $\zeta(\pi, \upsilon) = \pi + \upsilon + 1$, then*

$$\Theta_f(1, 4, 0.02 + 0.03 + 0.04) = \frac{1 + 0.09}{4 + 0.09} = 0.2665,$$

and

$$\Theta_f(1, 2, \frac{0.02}{\zeta(1,2)}) = \frac{1 + \frac{0.02}{4}}{2 + \frac{0.02}{4}} = 0.50124,$$

$$\Theta_f(3, 2, \frac{0.03}{\zeta(1,2)}) = \frac{2 + \frac{0.03}{4}}{3 + \frac{0.03}{4}} = 0.67250,$$

$$\Theta_f(3, 4, \frac{0.04}{\zeta(1,2)}) = \frac{3 + \frac{0.04}{4}}{4 + \frac{0.04}{4}} = 0.7581.$$

Clearly,

$$\Theta_f(1, 4, 0.02 + 0.03 + 0.04) \ngeq \Theta_f(1, 2, \frac{0.02}{\zeta(1,2)}) \star \Theta_f(3, 2, \frac{0.03}{\zeta(1,2)}) \star \Theta_f(3, 4, \frac{0.04}{\zeta(1,2)}).$$

Hence $(\Delta, \Phi, \Theta_f, \star)$ is not a EFBMS with minimum σ-norm.

Definition 1.4. *Let $(\Delta, \Phi, \Theta_f, *)$ be a EFBMS. The points belong to Δ, Φ and $\Delta \cap \Phi$ are called as Left, Right and Central points respectively and sequences belong to Δ, Φ and $\Delta \cap \Phi$ are named as Left, Right and Central sequences respectively.*

Lemma 1.1. *Let $(\Delta, \Phi, \Theta_f, *)$ be a EFBMS such that*

$$\Theta_f(\pi, \upsilon, \nu\sigma) \geq \Theta_f(\pi, \upsilon, \sigma)$$

for $\pi \in \Delta, \upsilon \in \Phi$ and $\nu \in (0, 1)$. Then $\pi = \upsilon$.

Proof. We have

$$\Theta_f(\pi, \upsilon, \nu\sigma) \geq \Theta_f(\pi, \upsilon, \sigma) \text{ for } \sigma > 0. \tag{1.1}$$

Since $\nu\sigma < \sigma$ for all $\sigma > 0$ and $\nu \in (0, 1)$, by (FCB6) we have

$$\Theta_f(\pi, \upsilon, \nu\sigma) \leq \Theta_f(\pi, \upsilon, \sigma). \tag{1.2}$$

From (1.1) and (1.2) and definition of EFBMS, we get $\pi = \upsilon$. □

Definition 1.5. *Let $(\Delta, \Phi, \Theta_f, *)$ be a EFBMS. A sequence $\{\pi_\aleph\} \in \Delta$ converges to a right point υ if and only if for every $\epsilon > 0$ and $\sigma > 0$, there exists $\aleph_0 \in \mathbb{N}$ such that $\Theta_f(\pi_\aleph, \upsilon, \sigma) > 1 - \epsilon$ for all $\aleph \geq \aleph_0$, i.e., $\Theta_f(\pi_\aleph, \upsilon, \sigma) \to 1$ as $\aleph \to \infty$ for all $\sigma > 0$ (symbolically $\{\pi_\aleph\} \to \upsilon$ or $\lim_{\aleph \to \infty} \pi_\aleph = \upsilon$ as $\aleph \to \infty$). Similarly, a right sequence $\{\upsilon_\aleph\}$ converges to a left point π if and only if for every $\epsilon > 0$ and $\sigma > 0$, there* *exists* $\aleph_0 \in \mathbb{N}$ *such that* $\Theta_f(\pi, \upsilon_\aleph, \sigma) > 1 - \epsilon$ *for all* $\aleph \geq \aleph_0$, *i.e.,* $\Theta_f(\pi, \upsilon_\aleph, \sigma) \to 1$ *as* $\aleph \to \infty$ *for all* $\sigma > 0$ *(symbolically* $\{\upsilon_\aleph\} \to \pi$ *or* $\lim_{\aleph\to\infty} \upsilon_\aleph = \pi$ *as* $\aleph \to \infty$*).*

Definition 1.6. *Let* $(\Delta, \Phi, \Theta_f, *)$ *be a EFBMS then:*

(i) *Sequence* $(\pi_\aleph, \upsilon_\aleph) \in \Delta \times \Phi$ *is named as a bisequence on* $(\Delta, \Phi, \Theta_f, *)$.

(ii) *If both* $\pi_\aleph$ *and* $\upsilon_\aleph$ *converge, the sequence* $(\pi_\aleph, \upsilon_\aleph) \in \Delta \times \Phi$ *is said to be a convergent sequence. If both* $\pi_\aleph$ *and* $\upsilon_\aleph$ *converge to some center point, bisequence* $(\pi_\aleph, \upsilon_\aleph)$ *is said to be a biconvergent sequence.*

(iii) *A bisequence* $(\pi_\aleph, \upsilon_\aleph)$ *on EFBMS* $(\Delta, \Phi, \Theta_f, *)$ *is said to be a Cauchy bisequence if for each* $\epsilon > 0$, *there exists* $\aleph_0 \in \mathbb{N}$ *such that for all* $\aleph, \wp \geq \aleph_0 (\aleph, \wp \in \mathbb{N})$,*we have* $\Theta_f(\pi_\aleph, \upsilon_\wp, \sigma) > 1 - \epsilon$ *for each* $\sigma > 0$, *i.e., a bisequence* $(\pi_\aleph, \upsilon_\aleph)$ *is said to be a Cauchy bisequence if* $\Theta_f(\pi_\aleph, \upsilon_\wp, \sigma) \to 1$ *as* $\aleph, \wp \to \infty$ *for all* $\sigma > 0$.

Definition 1.7. *The EFBMS* $(\Delta, \Phi, \Theta_f, *)$ *is said to be complete if every Cauchy bisequence in* $\Delta \times \Phi$ *is convergent in it.*

Proposition 1.1. *In a EFBMS, every convergent Cauchy bisequence is biconvergent.*

Proof. Let $(\Delta, \Phi, \Theta_f, *)$ be a EFBMS and a bisequence $(\pi_\aleph, \upsilon_\aleph) \in \Delta \times \Phi$ such that $\{\pi_\aleph\} \to \upsilon \in \Phi$ and $\{\upsilon_\aleph\} \to \pi \in \Delta$. Since $(\pi_\aleph, \upsilon_\aleph)$ is convergent Cauchy bisequence, so for all $\sigma > 0$ we have

$$\Theta_f(\pi_\aleph, \upsilon_\wp, \sigma) \to 1 \text{ as } \aleph \to \infty,$$

which implies that

$$\Theta_f(\pi, \upsilon, \sigma) = 1 \text{ for all } \sigma > 0.$$

Hence by (FCB2) bisequence $(\pi_\aleph, \upsilon_\aleph)$ is biconvergent. □

Proposition 1.2. *In a EFBMS, every biconvergent bisequence is a Cauchy bisequence.*

Proof. Let $(\Delta, \Phi, \Theta_f, *)$ be a EFBMS and bisequence $(\pi_\aleph, \upsilon_\aleph) \in \Delta \times \Phi$ converges to a point $\pi_0 \in \Delta \cap \Phi$ for all $\aleph, \wp \in \mathbb{N}$ and $\sigma > 0$, by (FCB4), we have

$$\Theta_f(\pi_\aleph, \upsilon_\wp, \sigma) \geq \Theta_f(\pi_\aleph, \pi_0, \frac{\sigma}{3\zeta(\pi_\aleph, \pi_\wp)}) * \Theta_f(\pi_0, \pi_0, \frac{\sigma}{3\zeta(\pi_\aleph, \pi_\wp)})$$
$$* \Theta_f(\pi_0, \upsilon_\wp, \frac{\sigma}{3\zeta(\pi_\aleph, \pi_\wp)}).$$

Letting $\aleph, \wp \to \infty$, we get

$$\Theta_f(\pi_\aleph, \upsilon_\wp, \sigma) \geq 1 \text{ for all } \sigma > 0.$$

Which implies that $\Theta_f(\pi_\aleph, \upsilon_\wp, \sigma) \to 1$ for all $\sigma > 0$. Hence, $(\pi_\aleph, \upsilon_\aleph)$ is a Cauchy bisequence. □

Lemma 1.2. *Let* $(\Delta, \Phi, \Theta_f, *)$ *be a EFBMS and* $\kappa \in \Delta \cap \Phi$ *is a limit of a sequence then it is a unique limit of the sequence.*

Proof. Let $\{\pi_{\aleph}\} \in \Delta$ be a sequence. Suppose that $\{\pi_{\aleph}\} \to \upsilon \in \Phi$ and also $\{\pi_{\aleph}\} \to \kappa \in \Delta \cap \Phi$, then for $\sigma, \varpi, \mathfrak{r} > 0$, we have

$$\Theta_f(\kappa, \upsilon, \mathfrak{t} + \mathfrak{s} + \mathfrak{r}) \geq \Theta_f(\kappa, \kappa, \frac{\sigma}{\zeta(\kappa, \upsilon)}) * \Theta_f(\pi_{\aleph}, \kappa, \frac{\varpi}{\zeta(\kappa, \upsilon)}) * \Theta_f(\pi_{\aleph}, \upsilon, \frac{\mathfrak{r}}{\zeta(\kappa, \upsilon)}).$$

Letting $\aleph \to \infty$, we get

$$\Theta_f(\kappa, \upsilon, \mathfrak{t} + \mathfrak{s} + \mathfrak{r}) \geq 1,$$

which implies that $\kappa = \upsilon$, i.e., sequence $\{\pi_{\aleph}\}$ have a unique limit. □

Definition 1.8. *A point $\pi \in \Delta \cap \Phi$ is said to be fixed point for the mapping Ξ on $\pi \in \Delta \cap \Phi$ such that $\pi = \Xi\pi$.*

Motivated by Sezen [12], we prove fixed point theorems on EFBMS with an application.

2. Main Results

2.1. Fixed point theorems.

In this section, we prove the extension of some well known fixed point theorems to EFBMS.

Theorem 2.1. *Let $(\Delta, \Phi, \Theta_f, *)$ be a complete EFBMS with $\zeta : \Delta \times \Phi \to [1, \infty)$ such that*

$$\lim_{\sigma \to \infty} \Theta_f(\pi, \upsilon, \sigma) = 1 \text{ for all } \pi \in \Delta, \upsilon \in \Phi. \tag{2.1}$$

Let $\Xi : \Delta \cup \Phi \to \Delta \cup \Phi$ be mapping satisfying

(i) $\Xi(\Delta) \subseteq \Delta$ and $\Xi(\Phi) \subseteq \Phi$;

(ii) $\Theta_f(\Xi(\pi), \Xi(\upsilon), \nu\sigma) \geq \Theta_f(\pi, \upsilon, \sigma)$ for all $\pi \in \Delta, \upsilon \in \Phi$ and $\sigma > 0$, where $\nu \in (0, 1)$.

Further, suppose that for an arbitrary $\pi \in \Delta, \upsilon \in \Phi$ and $\aleph, \mathfrak{q} \in \mathbb{N}$, we have

$$\zeta(\pi_{\aleph}, \upsilon_{\aleph+\mathfrak{q}}) < \frac{1}{\nu}.$$

Then Ξ has a unique fixed point.

Proof. Fix $\pi_0 \in \Delta$ and $\upsilon_0 \in \Phi$ and assume that $\Xi(\pi_{\aleph}) = \pi_{\aleph+1}$ and $\Xi(\upsilon_{\aleph}) = \upsilon_{\aleph+1}$ for all $\aleph \in \mathbb{N} \cup \{0\}$. Then we get $(\pi_{\aleph}, \upsilon_{\aleph})$ as a bisequence on fuzzy controlled bipolar metric space $(\Delta, \Phi, \Theta_f, *)$. Now, we have

$$\Theta_f(\pi_1, \upsilon_1, \sigma) = \Theta_f(\Xi(\pi_0), \Xi(\upsilon_0), \sigma) \geq \Theta_f(\pi_0, \upsilon_0, \frac{\sigma}{\nu})$$

for all $\sigma > 0$ and $\aleph \in \mathbb{N}$. By simple induction we get

$$\Theta_f(\pi_{\aleph}, \upsilon_{\aleph}, \sigma) = \Theta_f(\Xi(\pi_{\aleph-1}), \Xi(\upsilon_{\aleph-1}), \sigma) \geq \Theta_f(\pi_0, \upsilon_0, \frac{\sigma}{\nu^{\aleph}}) \tag{2.2}$$

and

$$\Theta_f(\pi_{\aleph+1}, \upsilon_{\aleph}, \sigma) = \Theta_f(\Xi(\pi_{\aleph}), \Xi(\upsilon_{\aleph-1}), \sigma) \geq \Theta_f(\pi_1, \upsilon_0, \frac{\sigma}{\nu^{\aleph}}) \tag{2.3}$$

for all $\sigma > 0$ and $\aleph \in \mathbb{N}$.

Letting $\aleph < \wp$, for $\aleph, \wp \in \mathbb{N}$. Then, we get

$$\begin{aligned}\Theta_f(\pi_\aleph, \upsilon_\wp, \sigma) \geq &\Theta_f(\pi_\aleph, \upsilon_\aleph, \frac{\sigma}{3\zeta(\pi_\aleph, \upsilon_\wp)}) \star \Theta_f(\pi_{\aleph+1}, \upsilon_\aleph, \frac{\sigma}{3\zeta(\pi_\aleph, \upsilon_\wp)}) \\ &\star \Theta_f(\pi_{\aleph+1}, \upsilon_\wp, \frac{\sigma}{3\zeta(\pi_\aleph, \upsilon_\wp)}) \\ &\vdots \\ &\geq \Theta_f(\pi_\aleph, \upsilon_\aleph, \frac{\sigma}{3\zeta(\pi_\aleph, \upsilon_\wp)}) \star \Theta_f(\pi_{\aleph+1}, \upsilon_\aleph, \frac{\sigma}{3\zeta(\pi_\aleph, \upsilon_\wp)}) \star \cdots \\ &\star \Theta_f(\pi_{\wp-1}, \upsilon_{\wp-1}, \frac{\sigma}{3^{\wp-1}\zeta(\pi_\aleph, \upsilon_\wp)\zeta(\pi_{\aleph+1}, \upsilon_\wp)\cdots\zeta(\pi_{\wp-1}, \upsilon_\wp)}) \\ &\star \Theta_f(\pi_\wp, \upsilon_{\wp-1}, \frac{\sigma}{3^{\wp-1}\zeta(\pi_\aleph, \upsilon_\wp)\zeta(\pi_{\aleph+1}, \upsilon_\wp)\cdots\zeta(\pi_{\wp-1}, \upsilon_\wp)}) \\ &\star \Theta_f(\pi_\wp, \upsilon_\wp, \frac{\sigma}{3^{\wp-1}\zeta(\pi_\aleph, \upsilon_\wp)\zeta(\pi_{\aleph+1}, \upsilon_\wp)\cdots\zeta(\pi_{\wp-1}, \upsilon_\wp)}).\end{aligned}$$

Consequently,

$$\begin{aligned}\Theta_f(\pi_\aleph, \upsilon_\wp, \sigma) \geq \Theta_f(\pi_0, \upsilon_0, \frac{\sigma}{3\nu^\aleph\zeta(\pi_\aleph, \upsilon_\wp)}) \star \Theta_f(\pi_0, \upsilon_1, \frac{\sigma}{3\nu^\aleph\zeta(\pi_\aleph, \upsilon_\wp)}) \\ \star \cdots\cdots \star \Theta_f(\pi_0, \upsilon_0, \frac{\sigma}{3^{\wp-1}\nu^{\wp+1}\zeta(\pi_\aleph, \upsilon_\wp)\zeta(\pi_{\aleph+1}, \upsilon_\wp)\cdots\zeta(\pi_{\wp-1}, \upsilon_\wp)}).\end{aligned}$$

Letting $\aleph, \wp \to \infty$, we get

$$\Theta_f(\pi_\aleph, \upsilon_\wp, \sigma) \geq 1 \text{ for all } \sigma > 0.$$

Therefore, the bisequence $(\pi_\aleph, \upsilon_\aleph)$ is a Cauchy bisequence. Since $(\Delta, \Phi, \Theta_f, *)$ is a complete. So, bisequence $(\pi_\aleph, \upsilon_\aleph)$ is a convergent, then there exist a point $\kappa \in \Delta \cap \Phi$ which is a limit of the both sequence $\{\pi_\aleph\}$ and $\{\upsilon_\aleph\}$. Let we claim that $\kappa \in \Delta \cap \Phi$ is a fixed point of Ξ. Now,

$$\begin{aligned}\Theta_f(\Xi(\kappa), \kappa, \sigma) \geq \Theta_f(\Xi(\kappa), \Xi(\upsilon_\aleph), \frac{\sigma}{3\zeta(\Xi(\kappa), \kappa)}) * \Theta_f(\Xi(\pi_\aleph), \Xi(\upsilon_\aleph), \frac{\sigma}{3\zeta(\Xi(\kappa), \kappa)}) \\ * \Theta_f(\Xi(\pi_\aleph), \kappa, \frac{\sigma}{3\zeta(\Xi(\kappa), \kappa)}),\end{aligned}$$

for all $\aleph \in \mathbb{N}$ and $\sigma > 0$. As $\aleph \to \infty$, we have

$$\Theta_f(\Xi(\kappa), \kappa, \sigma) \to 1*1*1 = 1.$$

Therefore, $\Xi(\kappa) = \kappa$. Let $\mathfrak{v} \in \Delta \cap \Phi$ be a another fixed point of Ξ. Then,

$$\Theta_f(\kappa, \mathfrak{v}, \sigma) = \Theta_f(\Xi(\kappa), \Xi(\mathfrak{v}), \sigma) \geq \Theta_f(\kappa, \mathfrak{v}, \frac{\sigma}{\nu})$$

for $\nu \in (0, 1)$ and for all $\sigma > 0$. Hence, $\kappa = \mathfrak{v}$. □

Example 2.1. *Let* $\Delta = [0, 1]$, $\Phi = \{0\} \cup \mathbb{N} - \{1\}$ *and* $\zeta : \Delta \times \Phi \to [1, \infty)$ *be a mapping defined by* $\zeta(\pi, \upsilon) = \pi + \upsilon - 1$. *Define* $\Theta_f(\pi, \upsilon, \sigma) = \frac{\sigma}{\sigma + |\pi - \upsilon|^2}$ *for all* $\sigma > 0$ *and* $\pi \in \Delta$ *and* $\upsilon \in \Phi$. *Clearly,* $(\Delta, \Phi, \Theta_f, *)$ *is a complete EFBMS, where* $*$ *is a continuous σ-norm defined as* $\mathfrak{a} * f = \mathfrak{a} f$.

Define $\Xi : \Delta \cup \Phi \to \Delta \cup \Phi$ *by*

$$\Xi(\kappa) = \begin{cases} \frac{\kappa}{2}, & \text{if } \kappa \in [0,1], \\ 0, & \text{if } \kappa \in \mathbb{N} - \{1\}, \end{cases}$$

for all $\kappa \in \Delta \cup \Phi$. Ξ *satisfies the condition (i) of Theorem 2.1. Now, suppose that* $\nu = \frac{1}{2}$ *then for all* $\sigma > 0$, *condition (ii) of Theorem 2.1 also satisfies by* Ξ. *We can construct the bisequences* $\Xi(\pi_{\aleph}) = \pi_{\aleph+1}$ *and* $\Xi(\upsilon_{\aleph}) = \upsilon_{\aleph+1}$ *for all* $\aleph \in \mathbb{N} \cup \{0\}$ *by assuming* $\pi_0 = 1$ *and* $\upsilon_0 = 2$, *we obtain a non-trivial sequence as* $(\pi_{\aleph}, \upsilon_{\aleph}) = \{(1,2), (\frac{1}{2}, 0), (\frac{1}{2^2}, 0), \cdots\}$. *By Theorem 2.1, we get* Ξ *have a unique fixed point, i.e.,*$\pi = 0$.

Theorem 2.2. *Let* $(\Delta, \Phi, \Theta_f, *)$ *be a complete EFBMS with* $\zeta : \Delta \times \Phi \to [1, \infty)$ *such that*

$$\lim_{\sigma \to \infty} \Theta_f(\pi, \upsilon, \sigma) = 1 \text{ for all } \pi \in \Delta, \upsilon \in \Phi. \tag{2.4}$$

Let $\Xi : \Delta \cup \Phi \to \Delta \cup \Phi$ *be mapping satisfying*

(i) $\Xi(\Delta) \subseteq \Phi$ *and* $\Xi(\Phi) \subseteq \Delta$;

(ii) $\Theta_f(\Xi(\upsilon), \Xi(\pi), \nu\sigma) \geq \Theta_f(\pi, \upsilon, \sigma)$, *for all* $\pi \in \Delta, \upsilon \in \Phi$ *and* $\sigma > 0$, *where* $\nu \in (0,1)$.

Further, suppose that for an arbitrary $\pi \in \Delta, \upsilon \in \Phi$ *and* $\aleph, \mathfrak{q} \in \mathbb{N}$, *we have*

$$\zeta(\pi_{\aleph}, \upsilon_{\aleph+\mathfrak{q}}) < \frac{1}{\nu}.$$

Then Ξ *has a unique fixed point.*

Proof. Fix $\pi_0 \in \Delta$ and assume that $\Xi(\pi_{\aleph}) = \upsilon_{\aleph}$ and $\Xi(\upsilon_{\aleph}) = \pi_{\aleph+1}$ for all $\aleph \in \mathbb{N} \cup \{0\}$. Then we get $(\pi_{\aleph}, \upsilon_{\aleph})$ as a bisequence on fuzzy controlled bipolar metric space $(\Delta, \Phi, \Theta_f, *)$. Now, we have

$$\Theta_f(\pi_1, \upsilon_0, \sigma) = \Theta_f(\Xi(\upsilon_0), \Xi(\pi_0), \sigma) \geq \Theta_f(\pi_0, \upsilon_0, \frac{\sigma}{\nu})$$

for all $\sigma > 0$ and $\aleph \in \mathbb{N}$. By simple induction we get

$$\Theta_f(\pi_{\aleph}, \upsilon_{\aleph}, \sigma) = \Theta_f(\Xi(\upsilon_{\aleph-1}), \Xi(\pi_{\aleph}), \sigma) \geq \Theta_f(\pi_0, \upsilon_0, \frac{\sigma}{\nu^{2\aleph}}) \tag{2.5}$$

and

$$\Theta_f(\pi_{\aleph+1}, \upsilon_{\aleph}, \sigma) = \Theta_f(\Xi(\upsilon_{\aleph}), \Xi(\pi_{\aleph}), \sigma) \geq \Theta_f(\pi_0, \upsilon_0, \frac{\sigma}{\nu^{2\aleph+1}}) \tag{2.6}$$

for all $\sigma > 0$ and $\aleph \in \mathbb{N}$.

Letting $\aleph < \wp$, for $\aleph, \wp \in \mathbb{N}$. Then, we get

$$\begin{aligned} \Theta_f(\pi_{\aleph}, \upsilon_{\wp}, \sigma) \geq & \Theta_f(\pi_{\aleph}, \upsilon_{\aleph}, \frac{\sigma}{3\zeta(\pi_{\aleph}, \upsilon_{\wp})}) \star \Theta_f(\pi_{\aleph}, \upsilon_{\aleph}, \frac{\sigma}{3\zeta(\pi_{\aleph}, \upsilon_{\wp})}) \\ & \star \Theta_f(\pi_{\aleph+1}, \upsilon_{\wp}, \frac{\sigma}{3\zeta(\pi_{\aleph}, \upsilon_{\wp})}) \\ & \vdots \\ & \geq \Theta_f(\pi_{\aleph}, \upsilon_{\aleph}, \frac{\sigma}{3\zeta(\pi_{\aleph}, \upsilon_{\wp})}) \star \Theta_f(\pi_{\aleph+1}, \upsilon_{\aleph}, \frac{\sigma}{3\zeta(\pi_{\aleph}, \upsilon_{\wp})}) \star \cdots \\ & \star \Theta_f(\pi_{\wp-1}, \upsilon_{\wp-1}, \frac{\sigma}{3^{\wp-1}\zeta(\pi_{\aleph}, \upsilon_{\wp})\zeta(\pi_{\aleph+1}, \upsilon_{\wp}) \cdots \zeta(\pi_{\wp-1}, \upsilon_{\wp})}) \end{aligned}$$

$$\star\, \Theta_f(\pi_\wp, \upsilon_{\wp-1}, \frac{\sigma}{3^{\wp-1}\zeta(\pi_\aleph, \upsilon_\wp)\zeta(\pi_{\aleph+1}, \upsilon_\wp)\cdots\zeta(\pi_{\wp-1}, \upsilon_\wp)})$$
$$\star\, \Theta_f(\pi_\wp, \upsilon_\wp, \frac{\sigma}{3^{\wp-1}\zeta(\pi_\aleph, \upsilon_\wp)\zeta(\pi_{\aleph+1}, \upsilon_\wp)\cdots\zeta(\pi_{\wp-1}, \upsilon_\wp)}).$$

Consequently,

$$\Theta_f(\pi_\aleph, \upsilon_\wp, \sigma) \geq \Theta_f(\pi_0, \upsilon_0, \frac{\sigma}{3\nu^{2\aleph}\zeta(\pi_\aleph, \upsilon_\wp)}) \star \Theta_f(\pi_0, \upsilon_0, \frac{\sigma}{3\nu^{2\aleph+1}\zeta(\pi_{\aleph+1}, \upsilon_\wp)})$$
$$\star \cdots\cdots \star\, \Theta_f(\pi_0, \upsilon_0, \frac{\sigma}{3^{\wp-1}\nu^{2\wp+1}\zeta(\pi_\aleph, \upsilon_\wp)\zeta(\pi_{\aleph+1}, \upsilon_\wp)\cdots\zeta(\pi_{\wp-1}, \upsilon_\wp)}).$$

Letting $\aleph, \wp \to \infty$, we get

$$\Theta_f(\pi_\aleph, \upsilon_\wp, \sigma) \geq 1 \text{ for all } \sigma > 0.$$

Therefore, the bisequence $(\pi_\aleph, \upsilon_\aleph)$ is a Cauchy bisequence. Since, $(\Delta, \Phi, \Theta_f, *)$ is a complete. So, bisequence $(\pi_\aleph, \upsilon_\aleph)$ is a convergent, then there exist a point $\kappa \in \Delta \cap \Phi$ which is a limit of the both sequences $\{\pi_\aleph\}$ and $\{\upsilon_\aleph\}$. Now, to prove $\kappa \in \Delta \cap \Phi$ is a fixed point of mapping Ξ. For this,

$$\Theta_f(\Xi(\kappa), \kappa, \sigma) \geq \Theta_f(\Xi(\kappa), \Xi(\pi_\aleph), \frac{\sigma}{3\zeta(\Xi(\kappa), \kappa)})$$
$$*\Theta_f(\Xi(\upsilon_\aleph), \Xi(\pi_\aleph), \frac{\sigma}{3\zeta(\Xi(\kappa), \kappa)})$$
$$*\Theta_f(\kappa, \Xi(\pi_\aleph), \frac{\sigma}{3\zeta(\Xi(\kappa), \kappa)}),$$

for all $\aleph \in \mathbb{N}$ and $\sigma > 0$. As $\aleph \to \infty$, we have

$$\Theta_f(\Xi(\kappa), \kappa, \sigma) \to 1*1*1 = 1.$$

Therefore, $\Xi(\kappa) = \kappa$. Let $\mathfrak{v} \in \Delta \cap \Phi$ be a another fixed point of Ξ. Then,

$$\Theta_f(\kappa, \mathfrak{v}, \sigma) = \Theta_f(\Xi(\mathfrak{v}), \Xi(\kappa), \sigma) \geq \Theta_f(\kappa, \mathfrak{v}, \frac{\sigma}{\nu})$$

for $\nu \in (0, 1)$ and for all $\sigma > 0$. Hence, $\kappa = \mathfrak{v}$. □

Example 2.2. *Let $\Delta = [0, 1]$, $\Phi = \{0\} \cup \mathbb{N} - \{1\}$ and $\zeta : \Delta \times \Phi \to [1, \infty)$ be a mapping defined by $\zeta(\pi, \upsilon) = 2(\pi + \upsilon) - 1$. Define*

$$\Theta_f(\pi, \upsilon, \sigma) = e^{-\frac{(\pi-\upsilon)^2}{\sigma}}, \ \forall\ \pi \in \Delta, \upsilon \in \Phi, \sigma > 0.$$

Then $(\Delta, \Phi, \Theta_f, *)$ is a complete EFBMS with product σ-norm. Define $\Xi : \Delta \cup \Phi \to \Delta \cup \Phi$ given by

$$\Xi(\kappa) = \begin{cases} \frac{1-3^{-\kappa}}{5}, & \text{if } \kappa \in [0, 1], \\ 0, & \text{if } \kappa \in \mathbb{N} - \{1\}, \end{cases}$$

for all $\kappa \in \Delta \cup \Phi$. Let $\pi \in [0,1]$ and $\upsilon \in \mathbb{N} - \{1\}$, then

$$\begin{aligned}\Theta_f(\Xi(\pi), \Xi(\upsilon), \nu\sigma) &= \Theta_f\left(\frac{1-3^{-\pi}}{5}, 0, \nu\sigma\right)\\ &= e^{-\frac{\left(\frac{1-3^{-\pi}}{5}\right)^2}{\nu\sigma}}\\ &\geq e^{-\frac{(\pi-\upsilon)^2}{\sigma}}\\ &= \Theta_f(\pi, \upsilon, \sigma).\end{aligned}$$

Therefore, all the hypotheses of Theorem 2.2 are satisfied. Hence Ξ has a unique fixed point, i.e., $\kappa = 0$.

For weak $\mathcal{B}$-contraction, Mihet [13] defined an increasing function $\Psi : (0,1] \to (0,1]$ such that $\lim_{\aleph\to\infty} \Psi^{\aleph}(\nu) = 1$ and $\Psi(\nu) \geq \nu$ for all $\nu \in (0,1]$. We are proving following fixed point theorem in EFBMS by using weak $\mathcal{B}$-contraction.

Theorem 2.3. *Let $(\Delta, \Phi, \Theta_f, *)$ be a complete EFBMS and $\Xi : \Delta \cup \Phi \to \Delta \cup \Phi$ a mapping with $\zeta : \Delta \times \Phi \to [1,\infty)$ satisfying*

(i) $\Xi(\Delta) \subseteq \Delta$ and $\Xi(\Phi) \subseteq \Phi$;

(ii) For $\pi \in \Delta, \upsilon \in \Phi$ and $\sigma > 0, \Theta_f(\pi, \upsilon, \sigma) > 0 \Rightarrow \Theta_f(\Xi(\pi), \Xi(\upsilon), \sigma) \geq \Psi(\Theta_f(\pi, \upsilon, \sigma))$.

Further, suppose that for an arbitrary $\pi \in \Delta, \upsilon \in \Phi$ and $\aleph, \mathfrak{q} \in \mathbb{N}$, we have

$$\zeta(\pi_\aleph, \upsilon_{\aleph+\mathfrak{q}}) < \frac{1}{\nu}.$$

Then Ξ has a fixed point.

Proof. Let $\pi_0 \in \Delta$ and $\upsilon_0 \in \Phi$ be such that $\Xi(\pi_\aleph) = \pi_{\aleph+1}$ and $\Xi(\upsilon_\aleph) = \upsilon_{\aleph+1}$ for all $\aleph \in \mathbb{N} \cup \{0\}$, then $(\pi_\aleph, \upsilon_\aleph)$ be a bisequence on fuzzy controlled bipolar metric space $(\Delta, \Phi, \Theta_f, *)$. By the definition of (FCB2) for all $\sigma > 0$ and condition (ii), we have

$$\Theta_f(\pi_\aleph, \upsilon_\aleph, \sigma) \geq \Psi^{\aleph}(\Theta_f(\pi_0, \upsilon_0, \sigma)) \tag{2.7}$$

and

$$\Theta_f(\pi_{\aleph+1}, \upsilon_\aleph, \sigma) \geq \Psi^{\aleph}(\Theta_f(\pi_1, \upsilon_0, \sigma)). \tag{2.8}$$

Letting $\aleph < \wp$, for $\aleph, \wp \in \mathbb{N}$. Then, we get

$$\begin{aligned}\Theta_f(\pi_\aleph, \upsilon_\wp, \sigma) \geq &\Theta_f(\pi_\aleph, \upsilon_\aleph, \frac{\sigma}{3\zeta(\pi_\aleph, \upsilon_\wp)}) \star \Theta_f(\pi_{\aleph+1}, \upsilon_\aleph, \frac{\sigma}{3\zeta(\pi_\aleph, \upsilon_\wp)})\\ &\star \Theta_f(\pi_{\aleph+1}, \upsilon_\wp, \frac{\sigma}{3\zeta(\pi_\aleph, \upsilon_\wp)})\\ &\vdots\\ &\geq \Theta_f(\pi_\aleph, \upsilon_\aleph, \frac{\sigma}{3\zeta(\pi_\aleph, \upsilon_\wp)}) \star \Theta_f(\pi_{\aleph+1}, \upsilon_\aleph, \frac{\sigma}{3\zeta(\pi_\aleph, \upsilon_\wp)}) \star \cdots\\ &\star \Theta_f(\pi_{\wp-1}, \upsilon_{\wp-1}, \frac{\sigma}{3^{\wp-1}\zeta(\pi_\aleph, \upsilon_\wp)\zeta(\pi_{\aleph+1}, \upsilon_\wp)\cdots\zeta(\pi_{\wp-1}, \upsilon_\wp)})\end{aligned}$$

$$\star\, \Theta_f(\pi_\wp, \upsilon_{\wp-1}, \frac{\sigma}{3^{\wp-1}\zeta(\pi_\aleph, \upsilon_\wp)\zeta(\pi_{\aleph+1}, \upsilon_\wp)\cdots\zeta(\pi_{\wp-1}, \upsilon_\wp)})$$
$$\star\, \Theta_f(\pi_\wp, \upsilon_\wp, \frac{\sigma}{3^{\wp-1}\zeta(\pi_\aleph, \upsilon_\wp)\zeta(\pi_{\aleph+1}, \upsilon_\wp)\cdots\zeta(\pi_{\wp-1}, \upsilon_\wp)}).$$

Consequently,

$$\Theta_f(\pi_\aleph, \upsilon_\wp, \sigma) \geq \Psi^\aleph(\Theta_f(\pi_0, \upsilon_0, \frac{\sigma}{3\zeta(\pi_\aleph, \upsilon_\wp)})) \star \Psi^\aleph(\Theta_f(\pi_1, \upsilon_0, \frac{\sigma}{3\zeta(\pi_{\aleph+1}, \upsilon_\wp)}))$$
$$\star \cdots\cdots \star\, \Psi^\aleph(\Theta_f(\pi_0, \upsilon_0, \frac{\sigma}{3^{\wp-1}\zeta(\pi_\aleph, \upsilon_\wp)\zeta(\pi_{\aleph+1}, \upsilon_\wp)\cdots\zeta(\pi_{\wp-1}, \upsilon_\wp)})).$$

Letting $\aleph, \wp \to \infty$, we have $\Theta_f(\pi_\aleph, \upsilon_\wp, \sigma) \to 1$ for all $\sigma > 0$. Therefore, the bisequence $(\pi_\aleph, \upsilon_\aleph)$ is a Cauchy bisequence. Since, $(\Delta, \Phi, \Theta_f, *)$ is a complete. So, bisequence $(\pi_\aleph, \upsilon_\aleph)$ is a convergent, then there exist a point $\kappa \in \Delta \cap \Phi$ which is a limit of the both sequences $\{\pi_\aleph\}$ and $\{\upsilon_\aleph\}$. Now, to prove $\kappa \in \Delta \cap \Phi$ is a fixed point of mapping Ξ. Since, $\Theta_f(\pi_\aleph, \kappa, \sigma) \to \sigma$ for all $\sigma > 0$ and $\Theta_f(\pi_{\aleph+1}, \Xi(\kappa), \sigma) = \Theta_f(\Xi(\pi_\aleph), \Xi(\kappa), \sigma) \geq \Psi(\Theta_f(\pi_\aleph, (\kappa), \sigma)) \geq \Theta_f(\pi_\aleph, (\kappa), \sigma)$, and it follows that $\pi_{\aleph+1} \to \Xi(\kappa)$, which implies that $\Xi(\kappa) = \kappa$. □

Example 2.3. *Let $\Delta = \{2, 4, 5, 6\}, \Phi = \{1, 2\}, \mathfrak{a} * f = \mathfrak{a} f$ for all $\mathfrak{a}, f \in [0, 1]$ and $\zeta : \Delta \times \Phi \to [1, \infty)$ be a mapping defined by $\zeta(\pi, \upsilon) = \pi + \upsilon - 1$. Define*

$$\Theta_f(\pi, \upsilon, \sigma) = \frac{\min\{\pi, \upsilon\}^2 + \sigma}{\max\{\pi, \upsilon\}^2 + \sigma} \text{ for all } \pi \in \Delta, \upsilon \in \Phi \text{ and for all } \sigma > 0.$$

- Then $(\Delta, \Phi, \Theta_f, *)$ is an complete EFBMS. Now, define $\Psi : (0, 1] \to (0, 1]$ such that $\Psi(\nu) = \sqrt{\nu}$. Clearly, $\Psi(\nu) = \sqrt{\nu}$ satisfies conditions of Ψ function.

Let $\Xi : \Delta \cup \Phi \to \Delta \cup \Phi$ be a mapping such that $\Xi(2) = \Xi(4) = \Xi(1) = 2, \Xi(5) = \Xi(6) = 4$. Then all the conditions of Theorem 2.3 are satisfied. The fixed point of Ξ is $\pi = 2$.

Theorem 2.4. *Let $(\Delta, \Phi, \Theta_f, *)$ be a complete EFBMS and $\Xi : \Delta \cup \Phi \to \Delta \cup \Phi$ a mapping with $\zeta : \Delta \times \Phi \to [1, \infty)$ satisfying:*

(i) $\Xi(\Delta) \subseteq \Phi$ and $\Xi(\Phi) \subseteq \Delta$;

(ii) For $\pi \in \Delta, \upsilon \in \Phi$ and $\sigma > 0, \Theta_f(\pi, \upsilon, \sigma) > 0 \Longrightarrow \Theta_f(\Xi(\upsilon), \Xi(\pi), \sigma) \geq \Psi(\Theta_f(\pi, \upsilon, \sigma))$.

Further, suppose that for an arbitrary $\pi \in \Delta, \upsilon \in \Phi$ and $\aleph, \mathfrak{q} \in \mathbb{N}$, we have

$$\zeta(\pi_\aleph, \upsilon_{\aleph+\mathfrak{q}}) < \frac{1}{\nu}.$$

Then Ξ has a fixed point.

Proof. Proof of the Theorem follows on the lines of proof of the Theorem 2.3 and Theorem 2.2. □

3. Application

In this section, we study the existence and unique solution to an integral equations as an application of Theorem 2.1.

Theorem 3.1. *Let us consider the integral equation*

$$\pi(\varrho) = f(\varrho) + \int_{\Lambda_1 \cup \Lambda_2} \Omega(\varrho, \varpi, \pi(\varpi)) d\varpi, \ \ \varrho \in \Lambda_1 \cup \Lambda_2,$$

where $\Lambda_1 \cup \Lambda_2$ is a Lebesgue measurable set. Suppose

(T1) $\Omega : (\Lambda_1^2 \cup \Lambda_2^2) \times [0, \infty) \to [0, \infty)$ *and* $b \in L^\infty(\Lambda_1) \cup L^\infty(\Lambda_2)$,

(T2) there is a continuous function $\theta : \Lambda_1^2 \cup \Lambda_2^2 \to [0, \infty)$ *and* $\nu \in (0, 1)$ *such that*

$$|\Omega(\varrho, \varpi, \pi(\varpi)) - \Omega(\varrho, \varpi, \upsilon(\varpi))| \leq \sqrt{\nu\theta(\varrho, \varpi)}(|\pi(\varrho) - \upsilon(\varrho)|),$$

for $\varrho, \varpi \in \Lambda_1^2 \cup \Lambda_2^2$,

(T3) $\sup_{\varrho \in \Lambda_1 \cup \Lambda_2} \int_{\Lambda_1 \cup \Lambda_2} \theta(\varrho, \varpi) d\varpi \leq 1$.

Then the integral equation has a unique solution in $L^\infty(\Lambda_1) \cup L^\infty(\Lambda_2)$.

Proof. Let $\Delta = L^\infty(\Lambda_1)$ and $\Phi = L^\infty(\Lambda_2)$ be two normed linear spaces, where Λ_1, Λ_2 are Lebesgue measurable sets and $m(\Lambda_1 \cup \Lambda_2) < \infty$.

Consider $\Theta_f : \Delta \times \Phi \times (0, \infty) \to [0, 1]$ by

$$\Theta_f(\pi, \upsilon, \sigma) = e^{-\frac{\sup_{\varrho \in \Lambda_1 \cup \Lambda_2} |\pi(\varrho) - \upsilon(\varrho)|^2}{\sigma}}.$$

for all $\pi \in \Delta, \upsilon \in \Phi$. Define $\zeta : \Delta \times \Phi \to [1, \infty)$ given by $\zeta(\pi, \upsilon) = 1$. Then $(\Delta, \Phi, \Theta_f, \star)$ is a complete EFBMS. Define $\Xi : L^\infty(\Lambda_1) \cup L^\infty(\Lambda_2) \to L^\infty(\Lambda_1) \cup L^\infty(\Lambda_2)$ given by

$$\Xi(\pi(\varrho)) = f(\varrho) + \int_{\Lambda_1 \cup \Lambda_2} \Omega(\varrho, \varpi, \pi(\varpi)) d\varpi, \ \ \varrho \in \Lambda_1 \cup \Lambda_2.$$

Now,

$$\begin{aligned}
\Theta_f(\Xi\pi(\varrho), \Xi\upsilon(\varrho), \nu\sigma) &= e^{-\sup_{\varrho \in \Lambda_1 \cup \Lambda_2} \frac{|\Xi\pi(\varrho) - \Xi\upsilon(\varrho)|^2}{\nu\sigma}} \\
&= e^{-\sup_{\varrho \in \Lambda_1 \cup \Lambda_2} \frac{|f(\varrho) + \int_{\Lambda_1 \cup \Lambda_2} \Omega(\varrho, \varpi, \pi(\varpi)) d\varpi - f(\varrho) - \int_{\Lambda_1 \cup \Lambda_2} \Omega(\varrho, \varpi, \upsilon(\varpi)) d\varpi|^2}{\nu\sigma}} \\
&= e^{-\sup_{\varrho \in \Lambda_1 \cup \Lambda_2} \frac{\left| f(\varrho) + \int_{\Lambda_1 \cup \Lambda_2} \Omega(\sigma, \varpi, \pi(\varpi)) d\varpi - \left(f(\varrho) + \int_{\Lambda_1 \cup \Lambda_2} \Omega(\sigma, \varpi, \upsilon(\varpi)) d\varpi \right) \right|^2}{\nu\sigma}} \\
&\geq e^{-\sup_{\varrho \in \Lambda_1 \cup \Lambda_2} \frac{\int_{\Lambda_1 \cup \Lambda_2} |\Omega(\varrho, \varpi, \pi(\varpi)) - \Omega(\varrho, \varpi, \upsilon(\varpi))|^2 d\varpi}{\nu\sigma}} \\
&\geq e^{-\sup_{\varrho \in \Lambda_1 \cup \Lambda_2} \frac{\int_{\Lambda_1 \cup \Lambda_2} \nu\theta(\varrho, \varpi)(|\pi(\varrho) - \upsilon(\varrho)|^2) d\varpi}{\nu\sigma}} \\
&\geq e^{-\sup_{\varrho \in \Lambda_1 \cup \Lambda_2} \frac{\int_{\Lambda_1 \cup \Lambda_2} \nu\theta(\varrho, \varpi)(|\pi(\varrho) - \upsilon(\varrho)|^2) d\varpi}{\nu\sigma}} \\
&\geq e^{-\sup_{\varrho \in \Lambda_1 \cup \Lambda_2} \frac{|\pi(\varrho) - \upsilon(\varrho)|^2}{\sigma}} \\
&= \Theta_f(\pi, \upsilon, \sigma).
\end{aligned}$$

Hence all the hypotheses of a Theorem 2.1 are verified and consequently, the integral equation has a unique solution. □

4. Application to Fractional Differential Equations

In this section is devoted to prove the uniqueness of the solution of the following fractional differential equation consisting of Caputo fractional derivative. More details can be found in [20].

$$\mathcal{D}^{\ell}_{0+}\pi(\varrho) + \mathfrak{g}_1(\varrho, \pi(\varrho)) = 0, \quad 0 < \varrho < 1, \tag{4.1}$$

where, $1 < \ell \leq 2, \pi(0) + \pi'(0) = 0, \pi(1) + \pi'(1) = 0$ are the boundary conditions with $\mathfrak{g}_1 \colon [0,1] \times [0,\infty) \to [0,\infty)$ being continuous. Let $\varDelta = (C[0,1],[0,\infty)) = \{\mathfrak{g} : [0,1] \to [0,\infty) : \mathfrak{g}$ is a continuous function$\}$ and $\varPhi = (C[0,1],(-\infty,0]) = \{\mathfrak{g} : [0,1] \to (-\infty,0] : \mathfrak{g}$ is a continuous function$\}$. Consider $\Theta_f : \varDelta \times \varPhi \times (0,\infty) \to [0,1]$ by

$$\Theta_f(\pi, v, \sigma) = e^{-\frac{\sup_{\varrho\in\Lambda_1\cup\Lambda_2}|\pi(\varrho)-v(\varrho)|^2}{\sigma}},$$

where $\mathfrak{a} * f = \mathfrak{a}f$. Define $\zeta : \varDelta \times \varPhi \to [1,\infty)$ given by $\zeta(\pi, v) = 1$. Note that $\pi \in \varDelta \cup \varPhi$ solves (4.1) and whenever $\pi \in \varDelta \cup \varPhi$ is the solution of

$$\begin{aligned}\pi(\varrho) =& \frac{1}{\Gamma(\ell)}\int_0^1 (1-\rho)^{\ell-1}(1-\varrho)\mathfrak{g}(\rho,\pi(\rho))d\rho\\ &+ \frac{1}{\Gamma(\ell-1)}\int_0^1 (1-\rho)^{\ell-2}(1-\varrho)\mathfrak{g}(\rho,\pi(\rho))d\rho\\ &+ \frac{1}{\Gamma(\ell)}\int_0^{\varrho} (\varrho-\rho)^{\ell-1}\mathfrak{g}(\rho,\pi(\rho))d\rho.\end{aligned}$$

Theorem 4.1. *Consider the operator, define $\Xi : \varDelta \cup \varPhi \to \varDelta \cup \varPhi$ given by:*

$$\begin{aligned}\Xi\pi(\varrho) =& \frac{1}{\Gamma(\ell)}\int_0^1 (1-\rho)^{\ell-1}(1-\varrho)\mathfrak{g}(\rho,\pi(\rho))d\rho\\ &+ \frac{1}{\Gamma(\ell-1)}\int_0^1 (1-\rho)^{\ell-2}(1-\varrho)\mathfrak{g}(\rho,\pi(\rho))d\rho\\ &+ \frac{1}{\Gamma(\ell)}\int_0^{\varrho} (\varrho-\rho)^{\ell-1}\mathfrak{g}(\rho,\pi(\rho))d\rho.\end{aligned}$$

Suppose the conditions:

(i) for all $\pi \in \varDelta, v \in \varPhi$ and $\mathfrak{g} \colon [0,1] \times [0,\infty) \to [0,\infty)$, satisfies

$$|\mathfrak{g}(\rho,\pi(\rho)) - \mathfrak{g}(\rho,v(\rho))| \leq \nu^{\frac{1}{2}}|\pi(\rho) - v(\rho)|,$$

(ii)

$$\sup_{\varrho\in(0,1)}\left|\frac{1-\varrho}{\Gamma(\ell+1)} + \frac{1-\varrho}{\Gamma(\ell)} + \frac{\varrho^{\ell}}{\Gamma(\ell+1)}\right|^2 = \eta < 1,$$

holds. Then eqution (4.1) *have a unique solution.*

Proof. Let $\pi \in \Delta, \upsilon \in \Phi$ and consider

$$\begin{aligned}
|\Xi\pi(\varrho) - \Xi\upsilon(\varrho)|^2 =& \Bigg| \frac{1}{\Gamma(\ell)} \int_0^1 (1-\rho)^{\ell-1}(1-\varrho)(\mathfrak{g}(\rho,\pi(\rho)) - \mathfrak{g}(\rho,\upsilon(\rho)))d\rho \\
&+ \frac{1}{\Gamma(\ell-1)} \int_0^1 (1-\rho)^{\ell-2}(1-\varrho)(\mathfrak{g}(\rho,\pi(\rho)) - \mathfrak{g}(\rho,\upsilon(\rho)))d\rho \\
&+ \frac{1}{\Gamma(\ell)} \int_0^\varrho (\varrho-\rho)^{\ell-1}(\mathfrak{g}(\rho,\pi(\rho)) - \mathfrak{g}(\rho,\upsilon(\rho)))d\rho \Bigg|^2 \\
&\leq \Bigg(\frac{1}{\Gamma(\ell)} \int_0^1 (1-\rho)^{\ell-1}(1-\varrho)\Big|\mathfrak{g}(\rho,\pi(\rho)) - \mathfrak{g}(\rho,\upsilon(\rho))\Big|d\rho \\
&+ \frac{1}{\Gamma(\ell-1)} \int_0^1 (1-\rho)^{\ell-2}(1-\varrho)\Big|\mathfrak{g}(\rho,\pi(\rho)) - \mathfrak{g}(\rho,\upsilon(\rho))\Big|d\rho \\
&+ \frac{1}{\Gamma(\ell)} \int_0^\varrho (\varrho-\rho)^{\ell-1}\Big|\mathfrak{g}(\rho,\pi(\rho)) - \mathfrak{g}(\rho,\upsilon(\rho))\Big|d\rho \Bigg)^2 \\
&\leq \Bigg(\frac{1}{\Gamma(\ell)} \int_0^1 (1-\rho)^{\ell-1}(1-\varrho)\nu^{\frac{1}{\aleph}}|\pi(\rho) - \upsilon(\rho)|d\rho \\
&+ \frac{1}{\Gamma(\ell-1)} \int_0^1 (1-\rho)^{\ell-2}(1-\varrho)\nu^{\frac{1}{\aleph}}|\pi(\rho) - \upsilon(\rho)|d\rho \\
&+ \frac{1}{\Gamma(\ell)} \int_0^\varrho (\varrho-\rho)^{\ell-1}\nu^{\frac{1}{\aleph}}|\pi(\rho) - \upsilon(\rho)|d\rho \Bigg)^2 \\
&= \nu|\pi(\varrho) - \upsilon(\varrho)|^2 \Bigg(\frac{1}{\Gamma(\ell)} \int_0^1 (1-\rho)^{\ell-1}(1-\varrho)d\rho \\
&+ \frac{1}{\Gamma(\ell-1)} \int_0^1 (1-\rho)^{\ell-2}(1-\varrho)d\rho + \frac{1}{\Gamma(\ell)} \int_0^\varrho (\varrho-\rho)^{\ell-1}d\rho \Bigg)^2 \\
&= \nu|\pi(\varrho) - \upsilon(\varrho)|^2 \left(\frac{1-\varrho}{\Gamma(\ell+1)} + \frac{1-\varrho}{\Gamma(\ell)} + \frac{\varrho^\ell}{\Gamma(\ell+1)} \right)^2 \\
&\leq \nu|\pi(\varrho) - \upsilon(\varrho)|^2 \sup_{\varrho\in(0,1)} \left(\frac{1-\varrho}{\Gamma(\ell+1)} + \frac{1-\varrho}{\Gamma(\ell)} + \frac{\varrho^\ell}{\Gamma(\ell+1)} \right)^2 \\
&= \eta\nu|\pi(\varrho) - \upsilon(\varrho)|^2 \\
&\leq \nu|\pi(\varrho) - \upsilon(\varrho)|^2,
\end{aligned}$$

so, we have

$$\Big|\Xi\pi(\varrho) - \Xi\upsilon(\varrho)\Big|^2 \leq \nu|\pi(\varrho) - \upsilon(\varrho)|^2,$$

i.e.,

$$-\frac{\sup_{\varrho\in[0,1]} \Big|\Xi\pi(\varrho) - \Xi\upsilon(\varrho)\Big|^2}{\nu\sigma} \geq -\frac{\sup_{\varrho\in[0,1]} |\pi(\varrho) - \upsilon(\varrho)|^2}{\sigma}$$

$$\exp\left(-\frac{\sup_{\varrho\in[0,1]}\left|\Xi\pi(\varrho)-\Xi\upsilon(\varrho)\right|^2}{\nu\sigma}\right)\geq\exp\left(-\frac{\sup_{\varrho\in[0,1]}|\pi(\varrho)-\upsilon(\varrho)|^2}{\sigma}\right),$$

thus, we have

$$\Theta_f(\Xi\pi(\varrho),\Xi\upsilon(\varrho),\nu\sigma)\geq\Theta_f(\pi(\varrho),\upsilon(\varrho),\sigma).$$

Hence, all the hypothesis of a Theorem 2.1 are verified and consequently, the equation (4.1) have a unique solution. □

4.1. **Common fixed point theorems.** In this section, we prove the extension of some well known common fixed point theorems to EFBMS.

Theorem 4.2. *Let $(\Delta,\Phi,\Theta_f,*)$ be a complete EFBMS with $\zeta:\Delta\times\Phi\to[1,\infty)$ s.t.*

$$\lim_{\sigma\to\infty}\Theta_f(\pi,\upsilon,\sigma)=1\;\forall\;\pi\in\Delta,\upsilon\in\Phi. \tag{4.2}$$

Let $\Xi,\mathcal{S}:\Delta\cup\Phi\to\Delta\cup\Phi$ be mapping satisfying

(i) $\Xi(\Delta)\subseteq\Delta$, $\mathcal{S}(\Delta)\subseteq\Delta$ and $\Xi(\Phi)\subseteq\Phi$, $\mathcal{S}(\Phi)\subseteq\Phi$;

(ii) $\Theta_f(\Xi(\pi),\mathcal{S}(\upsilon),\nu\sigma)\geq\Theta_f(\pi,\upsilon,\sigma)\;\forall\;\pi\in\Delta,\upsilon\in\Phi$ and $\sigma>0$, where $\nu\in(0,1)$.

Further, suppose that for an arbitrary $\upsilon\in\Delta,\pi\in\Phi$ and $\aleph,\mathfrak{q}\in\mathbb{N}$, we have

$$\zeta(\upsilon_\aleph,\pi_{\aleph+\mathfrak{q}})<\frac{1}{\nu}.$$

Then Ξ and $\mathcal{S}$ have a unique CFP (UCFP).

Proof. Fix $\pi_0\in\Delta$ and $\upsilon_0\in\Phi$ and assume that $\Xi(\pi_{2\aleph})=\pi_{2\aleph+1}$, $\mathcal{S}(\pi_{2\aleph+1})=\pi_{2\aleph+2}$ and $\Xi(\upsilon_{2\aleph})=\upsilon_{2\aleph+1}$, $\mathcal{S}(\upsilon_{2\aleph+1})=\upsilon_{2\aleph+2}\;\forall\;\aleph\in\mathbb{N}\cup\{0\}$. Then $(\{\pi_\aleph\},\{\upsilon_\aleph\})$ is a bisequence on EFBMS $(\Delta,\Phi,\Theta_f,*)$. Now,

$$\Theta_f(\pi_1,\upsilon_1,\sigma)=\Theta_f(\Xi(\pi_0),\mathcal{S}(\upsilon_0),\sigma)\geq\Theta_f(\pi_0,\upsilon_0,\frac{\sigma}{\nu})$$

for all $\sigma>0$ and $\aleph\in\mathbb{N}$. Then

$$\Theta_f(\pi_{2\aleph+1},\upsilon_{2\aleph+1},\sigma)=\Theta_f(\Xi(\pi_{2\aleph}),\mathcal{S}(\upsilon_{2\aleph}),\sigma)\geq\Theta_f(\pi_0,\upsilon_0,\frac{\sigma}{\nu^{2\aleph+1}}) \tag{4.3}$$

and

$$\Theta_f(\pi_{2\aleph+1},\upsilon_{2\aleph+2},\sigma)=\Theta_f(\Xi(\pi_{2\aleph}),\mathcal{S}(\upsilon_{2\aleph+1}),\sigma)\geq\Theta_f(\pi_0,\upsilon_1,\frac{\sigma}{\nu^{2\aleph+1}}) \tag{4.4}$$

for all $\sigma>0$ and $\aleph\in\mathbb{N}$. Letting $\aleph,\wp\to\infty$, we get

$$\Theta_f(\upsilon_\aleph,\pi_\wp,\sigma)\geq 1 \text{ for all } \sigma>0.$$

Letting $\aleph<\wp$, for $\aleph,\wp\in\mathbb{N}$. Then

$$\begin{aligned}\Theta_f(\pi_\aleph,\upsilon_\wp,\sigma)\geq&\Theta_f(\pi_\aleph,\upsilon_{\aleph+1},\frac{\sigma}{3\zeta(\pi_\aleph,\upsilon_\wp)})\star\Theta_f(\pi_{\aleph+1},\upsilon_{\aleph+1},\frac{\sigma}{3\zeta(\pi_\aleph,\upsilon_\wp)})\\&\star\Theta_f(\pi_{\aleph+1},\upsilon_\wp,\frac{\sigma}{3\zeta(\pi_\aleph,\upsilon_\wp)})\end{aligned}$$

$$\begin{aligned}
&\vdots\\
&\geq \Theta_f(\pi_{\aleph}, \upsilon_{\aleph+1}, \frac{\sigma}{3\zeta(\pi_{\aleph}, \upsilon_{\wp})}) \star \Theta_f(\pi_{\aleph+1}, \upsilon_{\aleph+1}, \frac{\sigma}{3\zeta(\pi_{\aleph}, \upsilon_{\wp})})\\
&\star \cdots \star \Theta_f(\pi_{\wp-2}, \upsilon_{\wp-1}, \frac{\sigma}{3^{\wp-2}\zeta(\pi_{\aleph}, \upsilon_{\wp})\zeta(\pi_{\aleph+1}, \upsilon_{\wp}) \cdots \zeta(\pi_{\wp-1}, \upsilon_{\wp})})\\
&\star \Theta_f(\pi_{\wp-1}, \upsilon_{\wp-1}, \frac{\sigma}{3^{\wp-1}\zeta(\pi_{\aleph}, \upsilon_{\wp})\zeta(\pi_{\aleph+1}, \upsilon_{\wp}) \cdots \zeta(\pi_{\wp-1}, \upsilon_{\wp})})\\
&\star \Theta_f(\pi_{\wp-1}, \upsilon_{\wp}, \frac{\sigma}{3^{\wp-1}\zeta(\pi_{\aleph}, \upsilon_{\wp})\zeta(\pi_{\aleph+1}, \upsilon_{\wp}) \cdots \zeta(\pi_{\wp-1}, \upsilon_{\wp})}).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\Theta_f(\pi_{\aleph}, \upsilon_{\wp}, \sigma) \geq \Theta_f(\pi_0, \upsilon_1, \frac{\sigma}{3\nu^{\aleph}\zeta(\pi_{\aleph}, \upsilon_{\wp})}) \star \Theta_f(\pi_0, \upsilon_0, \frac{\sigma}{3\nu^{\aleph+1}\zeta(\pi_{\aleph}, \upsilon_{\wp})})\\
\star \cdots\cdots \star \Theta_f(\pi_0, \upsilon_1, \frac{\sigma}{3^{\wp-1}\nu^{\wp-1}\zeta(\pi_{\aleph}, \upsilon_{\wp})\zeta(\pi_{\aleph+1}, \upsilon_{\wp}) \cdots \zeta(\pi_{\wp-1}, \upsilon_{\wp})}).
\end{aligned}$$

Letting $\aleph, \wp \to \infty$, we get

$$\Theta_f(\pi_{\aleph}, \upsilon_{\wp}, \sigma) \geq 1 \ \forall \ \sigma > 0.$$

Therefore, the bisequence $(\{\pi_{\aleph}\}, \{\upsilon_{\aleph}\})$ is a Cauchy bisequence. Since $(\Delta, \Phi, \Theta_f, *)$ is a complete, the bisequence $(\{\pi_{\aleph}\}, \{\upsilon_{\aleph}\})$ is a convergent. Therefore, bisequence $(\{\pi_{\aleph}\}, \{\upsilon_{\aleph}\})$ is biconvergent then

$$\{\pi_{\aleph}\} \to \pi \text{ and } \{\upsilon_{\aleph}\} \to \pi \ \forall \ \pi \in \Delta \cap \Phi.$$

From (FCB4),

$$\begin{aligned}
\Theta_f(\Xi(\pi), \pi, \sigma) &\geq \Theta_f(\Xi(\pi), \upsilon_{\aleph+1}, \frac{\sigma}{3\zeta(\Xi(\pi), \pi)}) \star \Theta_f(\pi_{\aleph+1}, \upsilon_{\aleph+1}, \frac{\sigma}{3\zeta(\Xi(\pi), \pi)})\\
&* \Theta_f(\pi_{\aleph+1}, \pi, \frac{\sigma}{3\zeta(\Xi(\pi), \pi)})\\
&= \Theta_f(\Xi(\pi), \mathcal{S}(\upsilon_{\aleph}), \frac{\sigma}{3\zeta(\Xi(\pi), \pi)}) \star \Theta_f(\pi_{\aleph+1}, \upsilon_{\aleph+1}, \frac{\sigma}{3\zeta(\Xi(\pi), \pi)})\\
&* \Theta_f(\pi_{\aleph+1}, \pi, \frac{\sigma}{3\zeta(\Xi(\pi), \pi)})\\
&\geq \Theta_f(\pi, \upsilon_{\aleph}, \frac{\sigma}{3\zeta(\Xi(\pi), \pi)}) \star \Theta_f(\pi_{\aleph+1}, \upsilon_{\aleph+1}, \frac{\sigma}{3\zeta(\Xi(\pi), \pi)})\\
&* \Theta_f(\pi_{\aleph+1}, \pi, \frac{\sigma}{3\zeta(\Xi(\pi), \pi)}),
\end{aligned}$$

for all $\aleph \in \mathbb{N}$ and $\sigma > 0$ and as $\aleph \to \infty$, we have

$$\Theta_f(\Xi(\pi), \pi, \sigma) \to 1*1*1 = 1.$$

From (FCB2), we get $\Xi(\pi) = \pi$. Again,

$$\begin{aligned}
\Theta_f(\pi, \mathcal{S}(\pi), \sigma) &= \Theta_f(\Xi(\pi), \mathcal{S}(\pi), \sigma)\\
&\geq \Theta_f(\pi, \pi, \frac{\sigma}{\nu}) = 1.
\end{aligned}$$

Therefore, $\mathcal{S}(\pi) = \pi$. Hence π is CFP of Ξ and $\mathcal{S}$. Let $\rho \in \varDelta \cap \varPhi$ be another fixed point of Ξ and $\mathcal{S}$. Then

$$\varTheta_f(\pi, \rho, \sigma) = \varTheta_f(\Xi(\pi), \mathcal{S}(\rho), \sigma) \geq \varTheta_f(\pi, \rho, \frac{\sigma}{\nu})$$

for $\nu \in (0, 1)$ and $\forall\, \sigma > 0$. Therefore $\pi = \rho$. □

Example 4.1. *Let $\varDelta = [0, 2]$, $\varPhi = \{0\} \cup \mathbb{N} - \{1, 2\}$ and $\zeta : \varDelta \times \varPhi \to [1, \infty)$ be a mapping defined by $\zeta(\pi, \upsilon) = \pi + \upsilon - 1$. Define $\varTheta_f = \frac{\sigma}{\sigma + |\pi - \upsilon|^2}$ $\forall\, \sigma > 0$ and $\pi \in \varDelta$ and $\upsilon \in \varPhi$. Clearly, $(\varDelta, \varPhi, \varTheta_f, *)$ is a complete EFBMS, where $*$ is a continuous σ-norm given by $\mathfrak{i} * \mathfrak{b} = \mathfrak{i}\mathfrak{b}$. Define $\Xi, \mathcal{S} : \varDelta \cup \varPhi \to \varDelta \cup \varPhi$ given by*

$$\Xi(\pi) = \begin{cases} 2 - \pi, & \text{if } \pi \in [0, 2], \\ 2, & \text{if } \pi \in \mathbb{N} - \{1, 2\}, \end{cases}$$

and

$$\mathcal{S}(\pi) = \begin{cases} \pi, & \text{if } \pi \in [0, 2], \\ 2, & \text{if } \pi \in \mathbb{N} - \{1, 2\}, \end{cases}$$

for all $\pi \in \varDelta \cup \Phi$. Clearly, Ξ and $\mathcal{S}$ are satisfied the axiom (i) of Theorem 4.2. Let $\nu = \frac{1}{2}$, then $\forall\, \sigma > 0$, we obtain

Case 1: Let $\pi \in [0, 2]$ and $\upsilon \in [0, 2]$, then

$$\begin{aligned} \varTheta_f(\Xi(\pi), \mathcal{S}(\upsilon), \nu\sigma) &= \varTheta_f(2 - \pi, \upsilon, \nu\sigma) \\ &= \frac{\nu\sigma}{\nu\sigma + |2 - \pi - \upsilon|^2} \\ &\geq \frac{\sigma}{\sigma + |\pi - \upsilon|^2} \\ &= \varTheta_f(\pi, \upsilon, \sigma). \end{aligned}$$

Case 2: Let $\pi \in [0, 2]$ and $\upsilon \in \mathbb{N} - \{1, 2\}$, then

$$\begin{aligned} \varTheta_f(\Xi(\pi), \mathcal{S}(\upsilon), \nu\sigma) &= \varTheta_f(2 - \pi, 2, \nu\sigma) \\ &= \frac{\nu\sigma}{\nu\sigma + |2 - \pi - 2|^2} \\ &\geq \frac{\sigma}{\sigma + |\pi - \upsilon|^2} \\ &= \varTheta_f(\pi, \upsilon, \sigma). \end{aligned}$$

Case 3: Let $\pi \in \mathbb{N} - \{1, 2\}$ and $\upsilon \in [0, 2]$, then

$$\begin{aligned} \varTheta_f(\Xi(\pi), \mathcal{S}(\upsilon), \nu\sigma) &= \varTheta_f(2, \upsilon, \nu\sigma) \\ &= \frac{\nu\sigma}{\nu\sigma + |2 - \upsilon|^2} \\ &\geq \frac{\sigma}{\sigma + |\pi - \upsilon|^2} \\ &= \varTheta_f(\pi, \upsilon, \sigma). \end{aligned}$$

Case 4: Let $\pi \in \mathbb{N} - \{1,2\}$ and $\upsilon \in \mathbb{N} - \{1,2\}$, then

$$\begin{aligned}\Theta_f(\Xi(\pi), \mathcal{S}(\upsilon), \nu\sigma) &= \Theta_f(2,2,\nu\sigma) \\ &= 1 \\ &\geq \frac{\sigma}{\sigma + |\pi - \upsilon|^2} \\ &= \Theta_f(\pi, \upsilon, \sigma).\end{aligned}$$

Therefore, axiom (ii) of Theorem 4.2 also fulfills by Ξ and $\mathcal{S}$. By Theorem 4.2, we get Ξ and $\mathcal{S}$ have a UCFP, i.e., $\pi = 1$.

We present our second result.

Theorem 4.3. *Let $(\Delta, \Phi, \Theta_f, *)$ be a complete EFBMS with $\zeta : \Delta \times \Phi \to [1, \infty)$ such that*

$$\lim_{\sigma\to\infty} \Theta_f(\pi, \upsilon, \sigma) = 1 \ \forall\ \pi \in \Delta, \upsilon \in \Phi. \tag{4.5}$$

Let $\Xi, \mathcal{S} : \Delta \cup \Phi \to \Delta \cup \Phi$ be mapping satisfying

(i) $\Xi(\Delta) \subseteq \Phi$, $\Xi(\Phi) \subseteq \Delta$ and $\mathcal{S}(\Delta) \subseteq \Phi$, $\mathcal{S}(\Phi) \subseteq \Delta$;

(ii) $\Theta_f(\Xi(\upsilon), \mathcal{S}(\pi), \nu\sigma) \geq \Theta_f(\pi, \upsilon, \sigma)\ \forall\ \pi \in \Delta, \upsilon \in \Phi$ and $\sigma > 0$, where $\nu \in (0, 1)$.

Further, suppose that for an arbitrary $\upsilon \in \Delta, \pi \in \Phi$ and $\aleph, \mathfrak{q} \in \mathbb{N}$, we have

$$\zeta(\upsilon_\aleph, \pi_{\aleph+\mathfrak{q}}) < \frac{1}{\nu}.$$

Then Ξ and $\mathcal{S}$ have a UCFP.

Proof. Fix $\pi_0 \in \Delta$ and $\upsilon_0 \in \Phi$ and assume that $\Xi(\pi_{2\aleph}) = \upsilon_{2\aleph}$, $\mathcal{S}(\pi_{2\aleph+1}) = \upsilon_{2\aleph+1}$ and $\Xi(\upsilon_{2\aleph}) = \mathfrak{x}'_{2\aleph+1}$, $\mathcal{S}(\upsilon_{2\aleph+1}) = \pi_{2\aleph+2}\ \forall\ \aleph \in \mathbb{N} \cup \{0\}$. Then $(\{\pi_\aleph\}, \{\upsilon_\aleph\})$ is a bisequence on EFBMS $(\Delta, \Phi, \Theta_f, *)$. Now,

$$\Theta_f(\pi_1, \upsilon_0, \sigma) = \Theta_f(\Xi(\upsilon_0), \mathcal{S}(\pi_0), \sigma) \geq \Theta_f(\pi_0, \upsilon_0, \frac{\sigma}{\nu})$$

for all $\sigma > 0$ and $\aleph \in \mathbb{N}$. Then

$$\Theta_f(\pi_{2\aleph+1}, \upsilon_{2\aleph+1}, \sigma) = \Theta_f(\Xi(\upsilon_{2\aleph}), \mathcal{S}(\pi_{2\aleph+1}), \sigma) \geq \Theta_f(\pi_0, \upsilon_0, \frac{\sigma}{\nu^{4\aleph+1}}) \tag{4.6}$$

and

$$\Theta_f(\pi_{2\aleph+1}, \upsilon_{2\aleph}, \sigma) = \Theta_f(\Xi(\upsilon_{2\aleph}), \mathcal{S}(\pi_{2\aleph}), \sigma) \geq \Theta_f(\pi_0, \upsilon_0, \frac{\sigma}{\nu^{4\aleph}}) \tag{4.7}$$

for all $\sigma > 0$ and $\aleph \in \mathbb{N}$. Letting $\aleph < \wp$, for $\aleph, \wp \in \mathbb{N}$. Then

$$\begin{aligned}\Theta_f(\pi_\aleph, \upsilon_\wp, \sigma) \geq &\Theta_f(\pi_\aleph, \upsilon_\aleph, \frac{\sigma}{3\zeta(\pi_\aleph, \upsilon_\wp)}) \star \Theta_f(\pi_{\aleph+1}, \upsilon_\aleph, \frac{\sigma}{3\zeta(\pi_\aleph, \upsilon_\wp)}) \\ &\star \Theta_f(\pi_{\aleph+1}, \upsilon_\wp, \frac{\sigma}{3\zeta(\pi_\aleph, \upsilon_\wp)}) \\ &\vdots \\ &\geq \Theta_f(\pi_\aleph, \upsilon_\aleph, \frac{\sigma}{3\zeta(\pi_\aleph, \upsilon_\wp)}) \star \Theta_f(\pi_{\aleph+1}, \upsilon_\aleph, \frac{\sigma}{3\zeta(\pi_\aleph, \upsilon_\wp)}) \star \cdots\end{aligned}$$

$$\star\, \Theta_f(\pi_{\wp-1}, \upsilon_{\wp-1}, \frac{\sigma}{3^{\wp-1}\zeta(\pi_{\aleph}, \upsilon_{\wp})\zeta(\pi_{\aleph+1}, \upsilon_{\wp})\cdots\zeta(\pi_{\wp-1}, \upsilon_{\wp})})$$
$$\star\, \Theta_f(\pi_{\wp}, \upsilon_{\wp-1}, \frac{\sigma}{3^{\wp-1}\zeta(\pi_{\aleph}, \upsilon_{\wp})\zeta(\pi_{\aleph+1}, \upsilon_{\wp})\cdots\zeta(\pi_{\wp-1}, \upsilon_{\wp})})$$
$$\star\, \Theta_f(\pi_{\wp}, \upsilon_{\wp}, \frac{\sigma}{3^{\wp-1}\zeta(\pi_{\aleph}, \upsilon_{\wp})\zeta(\pi_{\aleph+1}, \upsilon_{\wp})\cdots\zeta(\pi_{\wp-1}, \upsilon_{\wp})}).$$

Consequently,

$$\Theta_f(\pi_{\aleph}, \upsilon_{\wp}, \sigma) \geq \Theta_f(\pi_0, \upsilon_0, \frac{\sigma}{3\nu^{2\aleph+1}\zeta(\pi_{\aleph}, \upsilon_{\wp})}) \star \Theta_f(\pi_0, \upsilon_0, \frac{\sigma}{3\nu^{2\aleph}\zeta(\pi_{\aleph}, \upsilon_{\wp})}) \star \cdots$$
$$\cdots \star \Theta_f(\pi_0, \upsilon_0, \frac{\sigma}{3^{\wp-1}\nu^{2\wp+1}\zeta(\pi_{\aleph}, \upsilon_{\wp})\zeta(\pi_{\aleph+1}, \upsilon_{\wp})\cdots\zeta(\pi_{\wp-1}, \upsilon_{\wp})}).$$

Letting $\aleph, \wp \to \infty$, we get

$$\Theta_f(\pi_{\aleph}, \upsilon_{\wp}, \sigma) \geq 1 \ \forall\ \sigma > 0.$$

Thus, the bisequence $(\{\pi_{\aleph}\}, \{\upsilon_{\aleph}\})$ is a Cauchy bisequence. Since $(\Delta, \Phi, \Theta_f, *)$ is a complete, the bisequence $(\{\pi_{\aleph}\}, \{\upsilon_{\aleph}\})$ is a convergent. Therefore, the bisequence $(\{\pi_{\aleph}\}, \{\upsilon_{\aleph}\})$ is biconvergent then

$$\{\pi_{\aleph}\} \to \pi \text{ and } \{\upsilon_{\aleph}\} \to \pi \ \forall\ \pi \in \Delta \cap \Phi.$$

From (FCB4),

$$\begin{aligned}
\Theta_f(\Xi(\pi), \pi, \sigma) \geq\ & \Theta_f(\Xi(\pi), \upsilon_{\aleph+1}, \frac{\sigma}{3\zeta(\Xi(\pi), \pi)}) \star \Theta_f(\pi_{\aleph+1}, \upsilon_{\aleph+1}, \frac{\sigma}{3\zeta(\Xi(\pi), \pi)}) \\
& * \Theta_f(\pi_{\aleph+1}, \pi, \frac{\sigma}{3\zeta(\Xi(\pi), \pi)}) \\
& = \Theta_f(\Xi(\pi), \mathcal{S}(\pi_{\aleph+1}), \frac{\sigma}{3\zeta(\Xi(\pi), \pi)}) \star \Theta_f(\Xi\upsilon_{\aleph}, \mathcal{S}\pi_{\aleph+1}, \frac{\sigma}{3\zeta(\Xi(\pi), \pi)}) \\
& \star\, \Theta_f(\pi_{\aleph+1}, \pi, \frac{\sigma}{3\zeta(\Xi(\pi), \pi)}) \\
& \geq \Theta_f(\pi, \pi_{\aleph+1}, \frac{\sigma}{3\nu\zeta(\Xi(\pi), \pi)}) \star \Theta_f(\upsilon_{\aleph}, \pi_{\aleph+1}, \frac{\sigma}{3\nu\zeta(\Xi(\pi), \pi)}) \\
& * \Theta_f(\pi_{\aleph+1}, \pi, \frac{\sigma}{3\zeta(\Xi(\pi), \pi)}),
\end{aligned}$$

for all $\aleph \in \mathbb{N}$ and $\sigma > 0$ and as $\aleph \to \infty$, we have

$$\Theta_f(\Xi(\pi), \pi, \sigma) \to 1*1*1 = 1.$$

From (FCB2), we get $\Xi(\pi) = \pi$. Again,

$$\begin{aligned}
\Theta_f(\pi, \mathcal{S}(\pi), \sigma) &= \Theta_f(\Xi(\pi), \mathcal{S}(\pi), \sigma) \\
&\geq \Theta_f(\pi, \pi, \frac{\sigma}{\nu}) = 1.
\end{aligned}$$

Therefore, $\mathcal{S}(\pi) = \pi$. Hence π is CFP of Ξ and $\mathcal{S}$. Let $\rho \in \Delta \cap \Phi$ be another fixed point of Ξ and $\mathcal{S}$. Then

$$\Theta_f(\pi, \rho, \sigma) = \Theta_f(\Xi(\pi), \mathcal{S}(\rho), \sigma) \geq \Theta_f(\pi, \rho, \frac{\sigma}{\nu})$$

for $\nu \in (0,1)$ and $\forall\, \sigma > 0$. Therefore $\pi = \rho$. □

Example 4.2. *Let $\Delta = \{0,1,2,7\}$ and $\Phi = \{0, \frac{1}{3}, \frac{1}{2}, 3\}$ and define a continuous σ-norm as $\mathfrak{r} * \mho = \min\{\mathfrak{r}, \mho\}$. Let $\zeta : \Delta \times \Phi \to [1,\infty)$ be a mapping defined by $\zeta(\pi, \upsilon) = \pi + \upsilon - 1$. Now, define $\Theta_f(\pi, \upsilon, \sigma) = \exp^{-\frac{|\pi-\upsilon|^2}{\sigma}}$ $\forall\, \sigma > 0, \pi \in \Delta$ and $\upsilon \in \Phi$. Then $(\Delta, \Phi, \Theta_f, *)$ is a complete EFBMS. Suppose we define a mapping $\Xi, \mathcal{S} : \Delta \cup \Phi \to \Delta \cup \Phi$ s.t.*

$$\Xi(\pi) = \begin{cases} \frac{1}{3}, & \text{if } \pi \in \{7,2\}, \\ 0, & \text{if } \pi \in \{0, \frac{1}{3}, \frac{1}{2}, 1, 3\} \end{cases}$$

and

$$\mathcal{S}(\pi) = \begin{cases} \frac{1}{2}, & \text{if } \pi \in \{7,2\}, \\ 0, & \text{if } \pi \in \{0, \frac{1}{3}, \frac{1}{2}, 1, 3\} \end{cases}$$

Now, suppose that $\nu = \frac{1}{2}$, then $\forall\, \sigma > 0$, we obtain
Case 1: Let $\pi \in \{7,2\}$ and $\upsilon \in \{7,2\}$, then

$$\begin{aligned} \Theta_f(\Xi(\pi), \mathcal{S}(\upsilon), \nu\sigma) &= \Theta_f\left(\frac{1}{3}, \frac{1}{2}, \nu\sigma\right) \\ &= \frac{\nu\sigma}{\nu\sigma + |\frac{1}{3} - \frac{1}{2}|^2} \\ &\geq \frac{\sigma}{\sigma + |\pi - \upsilon|^2} \\ &= \Theta_f(\pi, \upsilon, \sigma). \end{aligned}$$

Case 2: Let $\pi \in \{7,2\}$ and $\upsilon \in \{0, \frac{1}{3}, \frac{1}{2}, 1, 3\}$, then

$$\begin{aligned} \Theta_f(\Xi(\pi), \mathcal{S}(\upsilon), \nu\sigma) &= \Theta_f\left(\frac{1}{3}, 0, \nu\sigma\right) \\ &= \frac{\nu\sigma}{\nu\sigma + |\frac{1}{3}|^2} \\ &\geq \frac{\sigma}{\sigma + |\pi - \upsilon|^2} \\ &= \Theta_f(\pi, \upsilon, \sigma). \end{aligned}$$

Case 3: Let $\pi \in \{0, \frac{1}{3}, \frac{1}{2}, 1, 3\}$ and $\upsilon \in \{7,2\}$, then

$$\begin{aligned} \Theta_f(\Xi(\pi), \mathcal{S}(\upsilon), \nu\sigma) &= \Theta_f(0, \frac{1}{2}, \nu\sigma) \\ &= \frac{\nu\sigma}{\nu\sigma + |\frac{1}{2}|^2} \\ &\geq \frac{\sigma}{\sigma + |\pi - \upsilon|^2} \\ &= \Theta_f(\pi, \upsilon, \sigma). \end{aligned}$$

Case 4: Let $\pi \in \{0, \frac{1}{3}, \frac{1}{2}, 1, 3\}$ *and* $\upsilon \in \{0, \frac{1}{3}, \frac{1}{2}, 1, 3\}$*, then*

$$\begin{aligned}\Theta_f(\Xi(\pi), \mathcal{S}(\upsilon), \nu\sigma) &= \Theta_f(0, 0, \nu\sigma)\\ &= 1\\ &\geq \frac{\sigma}{\sigma + |\pi - \upsilon|^2}\\ &= \Theta_f(\pi, \upsilon, \sigma).\end{aligned}$$

Therefore, axiom (ii) of Theorem 4.3 also fulfills by Ξ *and* $\mathcal{S}$*. By Theorem 4.3, we get* Ξ *and* $\mathcal{S}$ *have a UCFP, i.e.,* $\pi = 0$*.*

For weak $\mathcal{B}$-contraction, Mihet [13] defined an increasing function $\Omega : (0, 1] \to (0, 1]$ s.t. $\lim_{\aleph\to\infty} \Omega^{\aleph}(\nu) = 1$ and $\Omega(\nu) \geq \nu \ \forall \ \nu \in (0, 1]$. Now, we present our result.

Theorem 4.4. *Let* $(\Delta, \Phi, \Theta_f, *)$ *be a complete EFBMS and* $\Xi, \mathcal{S} : \Delta \cup \Phi \to \Delta \cup \Phi$ *be a mappings with* $\zeta : \Delta \times \Phi \to [1, \infty)$ *satisfying*

(i) $\Xi(\Delta) \subseteq \Delta, \mathcal{S}(\Delta) \subseteq \Delta$ *and* $\Xi(\Phi) \subseteq \Phi, \mathcal{S}(\Phi) \subseteq \Phi$*;*

(ii) For $\pi \in \Delta, \upsilon \in \Phi$ *and* $\sigma > 0, \Theta_f(\pi, \upsilon, \sigma) > 0 \Rightarrow \Theta_f(\Xi(\pi), \mathcal{S}(\upsilon), \sigma) \geq \Omega(\Theta_f(\pi, \upsilon, \sigma))$*.*

Further, suppose that for an arbitrary $\upsilon \in \Delta, \pi \in \Phi$ *and* $\aleph, \mathfrak{q} \in \mathbb{N}$*, we have*

$$\zeta(\upsilon_{\aleph}, \pi_{\aleph+\mathfrak{q}}) < \frac{1}{\nu}.$$

Then Ξ *and* $\mathcal{S}$ *have a CFP.*

Proof. Fix $\pi_0 \in \Delta$ and $\upsilon_0 \in \Phi$ and assume that $\Xi(\pi_{\aleph}) = \pi_{\aleph+1}, \mathcal{S}(\pi_{\aleph+1}) = \pi_{\aleph+2}$ and $\Xi(\upsilon_{\aleph}) = \upsilon_{\aleph+1}$, $\mathcal{S}(\upsilon_{\aleph+1}) = \upsilon_{\aleph+2} \ \forall \ \aleph \in \mathbb{N} \cup \{0\}$. Then $(\{\pi_{\aleph}\}, \{\upsilon_{\aleph}\})$ is a bisequence on EFBMS $(\Delta, \Phi, \Theta_f, *)$. Now,

$$\Theta_f(\pi_{\aleph}, \upsilon_{\aleph}, \sigma) \geq \Omega^{\aleph}(\Theta_f(\pi_0, \upsilon_0, \sigma)) \tag{4.8}$$

and

$$\Theta_f(\pi_{\aleph+1}, \upsilon_{\aleph}, \sigma) \geq \Omega^{\aleph}(\Theta_f(\pi_1, \upsilon_0, \sigma)). \tag{4.9}$$

Letting $\aleph < \wp$, for $\aleph, \wp \in \mathbb{N}$. Then,

$$\begin{aligned}\Theta_f(\pi_{\aleph}, \upsilon_{\wp}, \sigma) \geq & \Theta_f(\pi_{\aleph}, \upsilon_{\aleph}, \frac{\sigma}{3\zeta(\pi_{\aleph}, \upsilon_{\wp})}) \star \Theta_f(\pi_{\aleph+1}, \upsilon_{\aleph}, \frac{\sigma}{3\zeta(\pi_{\aleph}, \upsilon_{\wp})})\\ & \star \Theta_f(\pi_{\aleph+1}, \upsilon_{\wp}, \frac{\sigma}{3\zeta(\pi_{\aleph}, \upsilon_{\wp})})\\ & \vdots\\ & \geq \Theta_f(\pi_{\aleph}, \upsilon_{\aleph}, \frac{\sigma}{3\zeta(\pi_{\aleph}, \upsilon_{\wp})}) \star \Theta_f(\pi_{\aleph+1}, \upsilon_{\aleph}, \frac{\sigma}{3\zeta(\pi_{\aleph}, \upsilon_{\wp})})\\ & \star \cdots \star \Theta_f(\pi_{\wp-1}, \upsilon_{\wp-1}, \frac{\sigma}{3^{\wp-1}\zeta(\pi_{\aleph}, \upsilon_{\wp})\zeta(\pi_{\aleph+1}, \upsilon_{\wp}) \cdots \zeta(\pi_{\wp-1}, \upsilon_{\wp})})\\ & \star \Theta_f(\pi_{\wp}, \upsilon_{\wp-1}, \frac{\sigma}{3^{\wp-1}\zeta(\pi_{\aleph}, \upsilon_{\wp})\zeta(\pi_{\aleph+1}, \upsilon_{\wp}) \cdots \zeta(\pi_{\wp-1}, \upsilon_{\wp})})\end{aligned}$$

$$\star\, \Theta_f(\pi_\wp, \upsilon_\wp, \frac{\sigma}{3^{\wp-1}\zeta(\pi_\aleph, \upsilon_\wp)\zeta(\pi_{\aleph+1}, \upsilon_\wp)\cdots\zeta(\pi_{\wp-1}, \upsilon_\wp)}).$$

Consequently,

$$\Theta_f(\pi_\aleph, \upsilon_\wp, \sigma) \geq \Omega^\aleph(\Theta_f(\pi_0, \upsilon_0, \frac{\sigma}{3\zeta(\pi_\aleph, \upsilon_\wp)})) \star \Omega^\aleph(\Theta_f(\pi_1, \upsilon_0, \frac{\sigma}{3\zeta(\pi_\aleph, \upsilon_\wp)}))$$
$$\star \cdots \star\, \Omega^\aleph(\Theta_f(\pi_0, \upsilon_0, \frac{\sigma}{3^{\wp-1}\zeta(\pi_\aleph, \upsilon_\wp)\zeta(\pi_{\aleph+1}, \upsilon_\wp)\cdots\zeta(\pi_{\wp-1}, \upsilon_\wp)})).$$

Letting $\aleph, \wp \to \infty$, we have $\Theta_f(\pi_\aleph, \upsilon_\wp, \sigma) \to 1\ \forall\, \sigma > 0$. Thus, the bisequence $(\{\pi_\aleph\}, \{\upsilon_\aleph\})$ is a Cauchy bisequence. Since $(\Delta, \Phi, \Theta_f, *)$ is a complete, the bisequence $(\{\pi_\aleph\}, \{\upsilon_\aleph\})$ is a convergent. Therefore, the bisequence $(\{\pi_\aleph\}, \{\upsilon_\aleph\})$ is biconvergent then

$$\{\pi_\aleph\} \to \pi \text{ and } \{\upsilon_\aleph\} \to \pi\ \forall\, \pi \in \Delta \cap \Phi.$$

From (FCB4),

$$\begin{aligned}
\Theta_f(\Xi(\pi), \pi, \sigma) \geq\ & \Theta_f(\Xi(\pi), \upsilon_{\aleph+1}, \frac{\sigma}{3\zeta(\Xi(\pi), \pi)}) \star \Theta_f(\pi_{\aleph+1}, \upsilon_{\aleph+1}, \frac{\sigma}{3\zeta(\Xi(\pi), \pi)}) \\
& *\Theta_f(\pi_{\aleph+1}, \pi, \frac{\sigma}{3\zeta(\Xi(\pi), \pi)}) \\
=\ & \Theta_f(\Xi(\pi), \mathcal{S}(\upsilon_\aleph), \frac{\sigma}{3\zeta(\Xi(\pi), \pi)}) \star \Theta_f(\pi_{\aleph+1}, \upsilon_{\aleph+1}, \frac{\sigma}{3\zeta(\Xi(\pi), \pi)}) \\
& *\Theta_f(\pi_{\aleph+1}, \pi, \frac{\sigma}{3\zeta(\Xi(\pi), \pi)}) \\
\geq\ & \Omega(\Theta_f(\pi, \upsilon_\aleph, \frac{\sigma}{3\zeta(\Xi(\pi), \pi)})) \star \Theta_f(\pi_{\aleph+1}, \upsilon_{\aleph+1}, \frac{\sigma}{3\zeta(\Xi(\pi), \pi)}) \\
& *\Theta_f(\pi_{\aleph+1}, \pi, \frac{\sigma}{3\zeta(\Xi(\pi), \pi)}),
\end{aligned}$$

for all $\aleph \in \mathbb{N}$ and $\sigma > 0$. Letting $\aleph \to \infty$, we have

$$\Theta_f(\Xi(\pi), \pi, \sigma) \to 1*1*1 = 1.$$

From (FCB2), we get $\Xi(\pi) = \pi$. Again,

$$\begin{aligned}
\Theta_f(\pi, \mathcal{S}(\pi), \sigma) &= \Theta_f(\Xi(\pi), \mathcal{S}(\pi), \sigma) \\
&\geq \Omega(\Theta_f(\pi, \pi, \sigma)).
\end{aligned}$$

Therefore, $\mathcal{S}(\pi) = \pi$. Hence π is CFP of Ξ and $\mathcal{S}$. □

Example 4.3. *Let $\Delta = \{2, 4, 5, 6\}$, $\Phi = \{1, 2\}$, $i*\mathfrak{b} = i\mathfrak{b}\ \forall\, i, \mathfrak{b} \in [0, 1]$ and $\zeta : \Delta \times \Phi \to [1, \infty)$ be a mapping defined by $\zeta(\pi, \upsilon) = \pi + \upsilon - 1$. Define*

$$\Theta_f(\pi, \upsilon, \sigma) = \frac{(\min\{\pi, \upsilon\})^2 + \sigma}{(\max\{\pi, \upsilon\})^2 + \sigma}\ \forall\, \pi \in \Delta, \upsilon \in \Phi \text{ and } \forall\, \sigma > 0.$$

Then $(\Delta, \Phi, \Theta_f, *)$ is a complete EFBMS. Now, define $\Omega : (0, 1] \to (0, 1]$ s.t. $\Omega(\nu) = \sqrt{\nu}$. Clearly, $\Omega(\nu) = \sqrt{\nu}$ satisfies conditions of Ω function.

Define $\Xi, \mathcal{S} : \Delta \cup \Phi \to \Delta \cup \Phi$ *given by* $\Xi(2) = \Xi(4) = \Xi(1) = 2, \Xi(5) = \Xi(6) = 4$ *and* $\mathcal{S}(2) = \mathcal{S}(5) = \mathcal{S}(6) = 2, \mathcal{S}(4) = \mathcal{S}(1) = 4$. *Then all the conditions of Theorem 4.4 are satisfied. The CFP of* Ξ *and* $\mathcal{S}$ *is* $\pi = 2$.

Theorem 4.5. *Let* $(\Delta, \Phi, \Theta_f, *)$ *be a EFBMS and* $\Xi, \mathcal{S} : \Delta \cup \Phi \to \Delta \cup \Phi$ *be a mappings with* $\zeta : \Delta \times \Phi \to [1, \infty)$ *satisfying*

(i) $\Xi(\Delta) \subseteq \Phi$, $\Xi(\Phi) \subseteq \Delta$ *and* $\mathcal{S}(\Delta) \subseteq \Phi$, $\mathcal{S}(\Phi) \subseteq \Delta$;

(ii) For $\pi \in \Delta, \upsilon \in \Phi$ *and* $\sigma > 0, \Theta_f(\pi, \upsilon, \sigma) > 0 \Longrightarrow \Theta_f(\Xi(\upsilon), \Xi(\pi), \sigma) \geq \Omega(\Theta_f(\pi, \upsilon, \sigma))$.

Further, suppose that for an arbitrary $\upsilon \in \Delta, \pi \in \Phi$ *and* $\aleph, \mathfrak{q} \in \mathbb{N}$, *we have*

$$\zeta(\upsilon_{\aleph}, \pi_{\aleph+\mathfrak{q}}) < \frac{1}{\nu}.$$

Then Ξ *and* $\mathcal{S}$ *have a CFP.*

Proof. Combining Theorem 4.4 and Theorem 4.3 to get our result. □

5. Application

In this section, we present an integral equations as an application of Theorem 4.2.

Theorem 5.1. *Let us consider the integral equation*

$$\pi(\omega) = \mathfrak{b}(\omega) + \int_{\Lambda_1 \cup \Lambda_2} \mathcal{G}_1(\omega, \mho, \pi(\mho)) d\mho, \ \omega \in \Lambda_1 \cup \Lambda_2,$$

$$\pi(\omega) = \mathfrak{b}(\omega) + \int_{\Lambda_1 \cup \Lambda_2} \mathcal{G}_2(\omega, \mho, \pi(\mho)) d\mho, \ \omega \in \Lambda_1 \cup \Lambda_2,$$

where $\Lambda_1 \cup \Lambda_2$ *is a Lebesgue measurable set. Suppose*

(T1) $\mathcal{G}_1, \mathcal{G}_2 : (\Lambda_1^2 \cup \Lambda_2^2) \times [0, \infty) \to [0, \infty)$ *and* $b \in L^\infty(\Lambda_1) \cup L^\infty(\Lambda_2)$,

(T2) there is a continuous function $\theta : \Lambda_1^2 \cup \Lambda_2^2 \to [0, \infty)$ *and* $\nu \in (0, 1)$ *s.t.*

$$|\mathcal{G}_1(\omega, \mho, \pi(\mho)) - \mathcal{G}_2(\omega, \mho, \upsilon(\mho))| \leq \sqrt{\nu \theta(\omega, \mho)}(|\pi(\omega) - \upsilon(\omega)|),$$

for $\omega, \mho \in \Lambda_1^2 \cup \Lambda_2^2$,

(T3) $\sup_{\omega \in \Lambda_1 \cup \Lambda_2} \int_{\Lambda_1 \cup \Lambda_2} \theta(\omega, \mho) d\mho \leq 1$.

Then the integral equations have a unique common solution in $L^\infty(\Lambda_1) \cup L^\infty(\Lambda_2)$.

Proof. Let $\Delta = L^\infty(\Lambda_1)$ and $\Phi = L^\infty(\Lambda_2)$ be two normed linear spaces, where Λ_1, Λ_2 are Lebesgue measurable sets and $m(\Lambda_1 \cup \Lambda_2) < \infty$.
Consider $\Theta_f : \Delta \times \Phi \times (0, \infty) \to [0, 1]$ by

$$\Theta_f(\pi, \upsilon, \sigma) = e^{-\frac{\sup_{\omega \in \Lambda_1 \cup \Lambda_2} |\pi(\omega) - \upsilon(\omega)|^2}{\sigma}}.$$

for all $\pi \in \Delta, \upsilon \in \Phi$. Define $\zeta : \Delta \times \Phi \to [1, \infty)$ given by $\zeta(\pi, \upsilon) = 1$. Then $(\Delta, \Phi, \Theta_f, \star)$ is a complete EFBMS. Define $\Xi, \mathcal{S} : L^\infty(\Lambda_1) \cup L^\infty(\Lambda_2) \to L^\infty(\Lambda_1) \cup L^\infty(\Lambda_2)$ given by

$$\Xi(\pi(\omega)) = \mathfrak{b}(\omega) + \int_{\Lambda_1 \cup \Lambda_2} \mathcal{G}_1(\omega, \mho, \pi(\mho)) d\mho, \ \omega \in \Lambda_1 \cup \Lambda_2,$$

$$\mathcal{S}(\pi(\omega)) = \mathfrak{b}(\omega) + \int_{\Lambda_1 \cup \Lambda_2} \mathcal{G}_2(\omega, \mho, \pi(\mho)) d\mho, \ \omega \in \Lambda_1 \cup \Lambda_2.$$

Now,

$$\begin{aligned}
\Theta_f(\Xi\pi(\omega), \mathcal{S}\upsilon(\omega), \nu\sigma) &= e^{-\sup_{\omega \in \Lambda_1 \cup \Lambda_2} \frac{|\Xi\pi(\omega) - \mathcal{S}\upsilon(\omega)|^2}{\nu\sigma}} \\
&= e^{-\sup_{\omega \in \Lambda_1 \cup \Lambda_2} \frac{|\mathfrak{b}(\omega) + \int_{\Lambda_1 \cup \Lambda_2} \mathcal{G}_1(\omega, \mho, \pi(\mho)) d\mho - \mathfrak{b}(\omega) - \int_{\Lambda_1 \cup \Lambda_2} \mathcal{G}_2(\omega, \mho, \pi(\mho)) d\mho|^2}{\nu\sigma}} \\
&\geq e^{-\sup_{\omega \in \Lambda_1 \cup \Lambda_2} \frac{\int_{\Lambda_1 \cup \Lambda_2} |\mathcal{G}_1(\omega, \mho, \pi(\mho)) - \mathcal{G}_2(\omega, \mho, \upsilon(\mho))|^2 d\mho}{\nu\sigma}} \\
&\geq e^{-\sup_{\omega \in \Lambda_1 \cup \Lambda_2} \frac{\int_{\Lambda_1 \cup \Lambda_2} \nu\theta(\omega, \mho)(|\pi(\omega) - \upsilon(\omega)|^2) d\mho}{\nu\sigma}} \\
&\geq e^{-\sup_{\omega \in \Lambda_1 \cup \Lambda_2} \frac{\int_{\Lambda_1 \cup \Lambda_2} \nu\theta(\omega, \mho)(|\pi(\omega) - \upsilon(\omega)|^2) d\mho}{\nu\sigma}} \\
&\geq e^{-\sup_{\omega \in \Lambda_1 \cup \Lambda_2} \frac{|\pi(\omega) - \upsilon(\omega)|^2}{\sigma}} \\
&= \Theta_f(\pi, \upsilon, \sigma).
\end{aligned}$$

Hence, all the hypothesis of a Theorem 4.2 are verified and consequently, the integral equations have a unique common solution. □

6. Application to Fractional Differential Equations

In this section is devoted to prove the uniqueness of the solution of the following fractional differential equation consisting of Caputo fractional derivative. More details can be found in ([20]- [23]).

$$\begin{aligned}
&\mathcal{D}^{\delta}_{0+}\pi(\omega) + \mathfrak{g}_1(\omega, \pi(\omega)) = 0, \quad 0 < \omega < 1, \\
&\mathcal{D}^{\delta}_{0+}\pi(\omega) + \mathfrak{g}_2(\omega, \pi(\omega)) = 0, \quad 0 < \omega < 1,
\end{aligned} \tag{6.1}$$

where, $1 < \delta \leq 2, \pi(0) + \pi'(0) = 0, \pi(1) + \pi'(1) = 0$ are the boundary conditions with $\mathfrak{g}$: $[0,1] \times [0, \infty) \to [0, \infty)$ being continuous. Let $\Delta = (C[0,1], [0, \infty)) = \{\mathfrak{g} : [0,1] \to [0, \infty) : \mathfrak{g}$ is a continuous function$\}$ and $\Phi = (C[0,1], (-\infty, 0]) = \{\mathfrak{g} : [0,1] \to (-\infty, 0] : \mathfrak{g}$ is a continuous function$\}$. Consider $\Theta_f : \Delta \times \Phi \times (0, \infty) \to [0,1]$ by

$$\Theta_f(\pi, \upsilon, \sigma) = e^{-\frac{\sup_{\omega \in \Lambda_1 \cup \Lambda_2} |\pi(\omega) - \upsilon(\omega)|^2}{\sigma}},$$

where $i * \mathfrak{b} = i\mathfrak{b}$. Define $\zeta : \Delta \times \Phi \to [1, \infty)$ given by $\zeta(\pi, \upsilon) = 1$. Note that $\pi \in \Delta \cup \Phi$ solves (6.1) and whenever $\pi \in \Delta \cup \Phi$ is the solution of

$$\begin{aligned}\pi(\omega) =& \frac{1}{\Gamma(\delta)}\int_0^1 (1-\lambda)^{\delta-1}(1-\omega)\mathfrak{g}_1(\lambda, \pi(\lambda))d\lambda \\ &+ \frac{1}{\Gamma(\delta-1)}\int_0^1 (1-\lambda)^{\delta-2}(1-\omega)\mathfrak{g}_1(\lambda, \pi(\lambda))d\lambda \\ &+ \frac{1}{\Gamma(\delta)}\int_0^\omega (\omega-\lambda)^{\delta-1}\mathfrak{g}_2(\lambda, \pi(\lambda))d\lambda, \\ \pi(\omega) =& \frac{1}{\Gamma(\delta)}\int_0^1 (1-\lambda)^{\delta-1}(1-\omega)\mathfrak{g}_2(\lambda, \pi(\lambda))d\lambda \\ &+ \frac{1}{\Gamma(\delta-1)}\int_0^1 (1-\lambda)^{\delta-2}(1-\omega)\mathfrak{g}_2(\lambda, \pi(\lambda))d\lambda \\ &+ \frac{1}{\Gamma(\delta)}\int_0^\omega (\omega-\lambda)^{\delta-1}\mathfrak{g}_2(\lambda, \pi(\lambda))d\lambda.\end{aligned}$$

Theorem 6.1. *Consider the operator Define $\Xi, \mathcal{S} : \Delta \cup \Phi \to \Delta \cup \Phi$ given by:*

$$\begin{aligned}\Xi\pi(\omega) =& \frac{1}{\Gamma(\delta)}\int_0^1 (1-\lambda)^{\delta-1}(1-\omega)\mathfrak{g}_1(\lambda, \pi(\lambda))d\lambda \\ &+ \frac{1}{\Gamma(\delta-1)}\int_0^1 (1-\lambda)^{\delta-2}(1-\omega)\mathfrak{g}_1(\lambda, \pi(\lambda))d\lambda \\ &+ \frac{1}{\Gamma(\delta)}\int_0^\omega (\omega-\lambda)^{\delta-1}\mathfrak{g}_1(\lambda, \pi(\lambda))d\lambda, \\ \mathcal{S}\pi(\omega) =& \frac{1}{\Gamma(\delta)}\int_0^1 (1-\lambda)^{\delta-1}(1-\omega)\mathfrak{g}_2(\lambda, \pi(\lambda))d\lambda \\ &+ \frac{1}{\Gamma(\delta-1)}\int_0^1 (1-\lambda)^{\delta-2}(1-\omega)\mathfrak{g}_2(\lambda, \pi(\lambda))d\lambda \\ &+ \frac{1}{\Gamma(\delta)}\int_0^\omega (\omega-\lambda)^{\delta-1}\mathfrak{g}_2(\lambda, \pi(\lambda))d\lambda.\end{aligned}$$

Suppose the conditions:

(i) for all $\pi \in \Delta, \upsilon \in \Phi$ and $\mathfrak{g}_1, \mathfrak{g}_2 \colon [0,1] \times [0, \infty) \to [0, \infty)$, satisfies

$$|\mathfrak{g}_1(\lambda, \pi(\lambda)) - \mathfrak{g}_2(\lambda, \upsilon(\lambda))| \le \nu^{\frac{1}{2}}|\pi(\lambda) - \upsilon(\lambda)|,$$

(ii)

$$\sup_{\omega\in(0,1)} \left|\frac{1-\omega}{\Gamma(\delta+1)} + \frac{1-\omega}{\Gamma(\delta)} + \frac{\omega^\delta}{\Gamma(\delta+1)}\right|^2 = \eta < 1,$$

holds. Then eqution (4.1) *have a unique common solution.*

Proof. Let $\pi \in \Delta, \upsilon \in \Phi$ and consider

$$|\Xi\pi(\omega) - \mathcal{S}\upsilon(\omega)|^2 = \left|\frac{1}{\Gamma(\delta)}\int_0^1 (1-\lambda)^{\delta-1}(1-\omega)(\mathfrak{g}_1(\lambda, \pi(\lambda) - \mathfrak{g}_2(\lambda, \upsilon(\lambda))))d\lambda\right.$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\delta-1)}\int_0^1(1-\lambda)^{\delta-2}(1-\omega)(\mathfrak{g}_1(\lambda,\pi(\lambda)-\mathfrak{g}_2(\lambda,v(\lambda))))d\lambda\\
&+\frac{1}{\Gamma(\delta)}\int_0^\omega(\omega-\lambda)^{\delta-1}(\mathfrak{g}_1(\lambda,\pi(\lambda)-\mathfrak{g}_2(\lambda,v(\lambda))))d\lambda\Bigg|^2\\
&\le\Bigg(\frac{1}{\Gamma(\delta)}\int_0^1(1-\lambda)^{\delta-1}(1-\omega)\Big|(\mathfrak{g}_1(\lambda,\pi(\lambda)-\mathfrak{g}_2(\lambda,v(\lambda))))\Big|d\lambda\\
&+\frac{1}{\Gamma(\delta-1)}\int_0^1(1-\lambda)^{\delta-2}(1-\omega)\Big|(\mathfrak{g}_1(\lambda,\pi(\lambda)-\mathfrak{g}_2(\lambda,v(\lambda))))\Big|d\lambda\\
&+\frac{1}{\Gamma(\delta)}\int_0^\omega(\omega-\lambda)^{\delta-1}\Big|(\mathfrak{g}_1(\lambda,\pi(\lambda)-\mathfrak{g}_2(\lambda,v(\lambda))))\Big|d\lambda\Bigg)^2\\
&\le\Bigg(\frac{1}{\Gamma(\delta)}\int_0^1(1-\lambda)^{\delta-1}(1-\omega)\nu^{\frac{1}{2}}|\pi(\lambda)-v(\lambda)|d\lambda\\
&+\frac{1}{\Gamma(\delta-1)}\int_0^1(1-\lambda)^{\delta-2}(1-\omega)\nu^{\frac{1}{2}}|\pi(\lambda)-v(\lambda)|d\lambda\\
&+\frac{1}{\Gamma(\delta)}\int_0^\omega(\omega-\lambda)^{\delta-1}\nu^{\frac{1}{2}}|\pi(\lambda)-v(\lambda)|d\lambda\Bigg)^2\\
&=\nu|\pi(\omega)-v(\omega)|^2\Bigg(\frac{1}{\Gamma(\delta)}\int_0^1(1-\lambda)^{\delta-1}(1-\omega)d\lambda\\
&+\frac{1}{\Gamma(\delta-1)}\int_0^1(1-\lambda)^{\delta-2}(1-\omega)d\lambda+\frac{1}{\Gamma(\delta)}\int_0^\omega(\omega-\lambda)^{\delta-1}d\lambda\Bigg)^2\\
&=\nu|\pi(\omega)-v(\omega)|^2\left(\frac{1-\omega}{\Gamma(\delta+1)}+\frac{1-\omega}{\Gamma(\delta)}+\frac{\omega^\delta}{\Gamma(\delta+1)}\right)^2\\
&\le\nu|\pi(\omega)-v(\omega)|^2\sup_{\omega\in(0,1)}\left(\frac{1-\omega}{\Gamma(\delta+1)}+\frac{1-\omega}{\Gamma(\delta)}+\frac{\omega^\delta}{\Gamma(\delta+1)}\right)^2\\
&=\eta\nu|\pi(\omega)-v(\omega)|^2\\
&\le\nu|\pi(\omega)-v(\omega)|^2,
\end{aligned}
$$

so, we have

$$
\Big|\Xi\pi(\omega)-\mathcal{S}v(\omega)\Big|^2\le\nu|\pi(\omega)-v(\omega)|^2,
$$

i.e.,

$$
-\frac{\sup_{\omega\in[0,1]}\Big|\Xi\pi(\omega)-\mathcal{S}v(\omega)\Big|^2}{\nu\sigma}\ge-\frac{\sup_{\omega\in[0,1]}|\pi(\omega)-v(\omega)|^2}{\sigma}
$$

$$
\exp\left(-\frac{\sup_{\omega\in[0,1]}\Big|\Xi\pi(\omega)-\mathcal{S}v(\omega)\Big|^2}{\nu\sigma}\right)\ge\exp\left(-\frac{\sup_{\omega\in[0,1]}|\pi(\omega)-v(\omega)|^2}{\sigma}\right),
$$

thus, we have

$$\Theta_f(\Xi\pi(\omega), \mathcal{S}\upsilon(\omega), \nu\sigma) \geq \Theta_f(\pi(\omega), \upsilon(\omega), \sigma)$$

Hence, all the hypothesis of a Theorem 4.2 are verified and consequently, the equation (6.1) have a unique common solution. □

7. Conclusion

In this paper, we proved fixed point theorems and common fixed point theorems on EFBMS with an application. Beg, G. Arul Joseph and Gunaseelan [24], proved common coupled fixed point theorem on fuzzy bipolar metric space. It is an open problem to prove common coupled fixed point theorem on EFBMS.

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