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Numerical Solution of a Singular Integral Equation of the First Kind with Hilbert Kernel

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Abstract. In solving practical problems in the fields of physics and engineering, singular integral equations are frequently encountered. Among these, singular integral equations with the Hilbert kernel constitute the periodic cases. In this article, we discuss the construction of an optimal quadrature formula for the numerical solution of Fredholm-type singular integral equations of the first kind with Hilbert kernels using the functional approach in the space $L_2^{(1)}(0,2\pi)$. Using the constructed optimal quadrature formula, the error between the exact solution and the approximate solution of the integral equation is demonstrated through examples. Graphs illustrate how the approximate value converges to the exact value as the number of nodes in the optimal quadrature formula increases.

1. Introduction

Singular integral equations are increasingly being applied to solve practical problems in various branches of physics, namely mechanics, electrodynamics, aerodynamics, and elasticity theory. It should be noted that in the aforementioned branches of physics, some problems have begun to be reduced to singular integrals, as a solid theoretical foundation has been established for them in the one-dimensional case ([4], [5], [10], [11], [12], [15]). These authors developed a theory for finding all solutions in the class of integrable functions for characteristic equations. From the theories presented in the above-mentioned literature, it is known that if the unknown function in the integral equation is periodic and its kernel is equal to $\cot \frac{x-t}{2}$, such an equation is called a

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singular integral equation with Hilbert kernel. We present a first-kind singular integral equation with Hilbert kernel of the following form

$$\frac{1}{2\pi}\int_{0}^{2\pi} g(x)\cot\frac{x-t}{2}dx = \varphi(t), \tag{1.1}$$

where $g(x)$, $\varphi(t)$ are 2π-periodic functions and $0 < t < 2\pi$, $0 \le x \le 2\pi$.

The solution of integral equation (1.1) when the condition

$$\int_{0}^{2\pi} \varphi(x)dx = 0 \tag{1.2}$$

is satisfied, will be equal to

$$g(t) = -\frac{1}{2\pi}\int_{0}^{2\pi} \varphi(x)\cot\frac{x-t}{2}dx + C, \tag{1.3}$$

where C is an arbitrary constant [4]. The solution of equation (1.1) in the form (1.3) obtained with condition (1.2) also consists of a singular integral with Hilbert kernel, and in many cases, finding the value of this definite integral is a challenging task.

For this reason, many scientists have provided methods for the approximate calculation of integrals of the form (1.3). These include interpolation methods [8], [9], discrete convolution method [10], and other numerical methods [31].

We also provide a new method for the approximate calculation of the integral (1.3) based on the functional approach. For this, the article is structured as follows. Section 2 of the article presents the problem statement, Section 3 shows how to obtain the expression for the upper bound of the quadrature formula error, Section 4 presents the finding of the conditional minimum of the norm for the error functional, Section 5 finds the optimal coefficients, and Section 6 presents numerical results.

2. Statement of the Problem

Thus, in this work, we consider the following quadrature formula for the solution of the integral equation (1.1) in the form (1.3) when condition (1.2) is satisfied:

$$\int_{0}^{2\pi} \varphi(x)\cot\frac{x-t}{2}dx \cong \sum_{\beta=0}^{N} C_\beta \varphi(x_\beta) \tag{2.1}$$

where C_β are coefficients, x_β are node points, $\varphi(x) \in L_2^{(1)}(0,2\pi)$. $L_2^{(1)}(0,2\pi)$ is a Hilbert space, which is defined as:

$$L_2^{(1)}(0,2\pi) = \{\varphi : [0,2\pi] \to \mathbb{R}, \varphi(x) - \text{absolutely continuous}, \varphi'(x) \in L_2(0,2\pi)\}.$$

In this $L_2^{(1)}(0,2\pi)$ space, the inner product of the functions φ and g is defined as follows

$$< \varphi, g >= \int_0^{2\pi} \varphi'(x) g'(x) dx \tag{2.2}$$

and the norm of the φ function is determined by the inner product (2.2) as follows

$$||\varphi|| = \sqrt{\int_0^{2\pi} (\varphi'(x))^2 dx} \tag{2.3}$$

In the quadrature formula (2.1) under consideration, the difference between the integral and the quadrature sum

$$\int_0^{2\pi} \varphi(x) \cot \frac{x-t}{2} dx - \sum_{\beta=0}^{N} C_\beta \varphi(x_\beta) = \int_{-\infty}^{\infty} (\varepsilon_{[0,2\pi]}(x) \cot \frac{x-t}{2} - \sum_{\beta=0}^{N} C_\beta \delta(x - x_\beta)) \varphi(x) dx =$$

$$= \int_{-\infty}^{\infty} \ell(x) \varphi(x) dx = (\ell, \varphi) \tag{2.4}$$

is called the error of the quadrature formula (2.1).

For the resulting expression (2.4), the following *error functional* corresponds to the dual space $L_2^{(1)*}(0,2\pi)$, and its form is as follows

$$\ell(x) = \varepsilon_{[0,2\pi]}(x) \cot \frac{x-t}{2} - \sum_{\beta=0}^{N} C_\beta \delta(x - x_\beta), \tag{2.5}$$

where $\varepsilon_{[0,2\pi]}(x)$ is the characteristic function of the segment $[0, 2\pi]$, and $\delta(x)$ is Dirac's delta function.

According to the definition of the norm of a linear continuous functional, the following is known

$$||\ell||_{L_2^{(1)*}} = \sup_{||\varphi|| \neq 0} \frac{|(\ell, \varphi)|}{||\varphi||}.$$

From the last equality, the Cauchy-Schwarz inequality follows

$$|(\ell, \varphi)| \leq ||\ell||_{L_2^{(1)*}} ||\varphi||_{L_2^{(1)}}. \tag{2.6}$$

According to the Cauchy-Schwarz inequality, the absolute value of expression (2.4) is estimated from above. For the $\varphi(x)$ function, according to the norm (2.3), the following condition holds

$$(\ell, 1) = 0. \tag{2.7}$$

Therefore, the upper bound of the error (2.4) of the quadrature formula (2.1) is estimated from above using the norm of the error functional $\ell(x)$ belonging to the conjugate space $L_2^{(1)^*}(0,2\pi)$.

Problem 2.1. *Find the norm of the error functional $\ell(x)$ of the quadrature formula* (2.1).

It can be seen that the error functional $\ell(x)$ depends on the coefficients C_β and the nodes x_β. Minimizing the norm of the error functional with respect to the coefficients and nodes is called the Nikolsky problem [13], [14]. When the x_β nodes are fixed, and only the norm of the error functional is minimized with respect to the coefficients C_β, it is called the Sard problem [16], [17], [18]. In this work, we solve the Sard problem at equally distributed fixed points $x_\beta = h\beta$ ($h = \frac{2\pi}{N}, N = 1, 2, ...$).

Minimizing the norm of the error functional with respect to the coefficients C_β, i.e.,

$$\|\mathring{\ell}\|_{L_2^{(1)*}(0,2\pi)} = \inf_{C_\beta} \|\ell\|_{L_2^{(1)*}(0,2\pi)} \tag{2.8}$$

coefficients satisfying this equality are called *optimal coefficients* and are denoted by $\mathring{C}_\beta$. The quadrature formula constructed using these optimal coefficients is called the *optimal quadrature formula*. Therefore, to construct an optimal quadrature formula corresponding to expression (2.1) in the space $L_2^{(1)}(0, 2\pi)$, the following problem must be solved.

Problem 2.2. *Find the $\mathring{C}_\beta$ optimal coefficients that achieve the value* (2.8).

The main purpose of this article is to construct an optimal quadrature formula that calculates the value of the integral (1.3) with high accuracy in the space $L_2^{(1)}(0, 2\pi)$.

In the space $L_2^{(m)}(-1, 1)$ space, singular integrals with Cauchy kernels were approximately calculated using the Sobolev method in works [21], [22], [23], [24]. Optimal quadrature formulas for regular integrals in various other spaces using the Sobolev method were constructed in works [7], [25], [26], [27].

3. Upper Bound for the Error of a Quadrature Formula

First, to solve Problem 2.1, we will use the concept of an extremal function introduced by Sobolev.

Definition 3.1. *The function $\psi_\ell(x)$ corresponding to the error functional $\ell(x)$, which transforms inequality* (2.6) *into equality, is called an extremal function.*

The form of the extremal function in the space $L_2^{(1)}(0, 2\pi)$ is defined as follows [29]

$$\psi_\ell(x) = -\ell(x) * G_1(x) + d, \tag{3.1}$$

where $G_1(x) = \frac{|x|}{2}$, d is an arbitrary constant.

Remark 3.1. *$*$ is a convolution operation, and the convolution of two functions is defined as follows*

$$\varphi(x) * \psi(x) = \int_{-\infty}^{\infty} \varphi(y)\psi(x-y)dy = \int_{-\infty}^{\infty} \varphi(x-y)\psi(y)dy.$$

According to this, we calculate the form of the extremal function

$$\psi_\ell(x) = -\ell(x) * G_1(x) + d = -\int_{-\infty}^{\infty} \ell(y)\frac{|x-y|}{2}dy + d = -\int_0^{2\pi} \frac{|x-y|}{2}\cot\frac{y-t}{2}dy + \sum_{\gamma=0}^{N} C_\gamma \frac{|x-h\gamma|}{2} + d.$$

Since $L_2^{(1)}(0,2\pi)$ is a Hilbert space, we present the Riesz theorem on the general form of a linear continuous functional.

Theorem 3.1. *(Riesz theorem) Let H be a Hilbert space. Then for each linear continuous functional $\ell \in H^*$, there exists a unique function ψ_ℓ in the space H that satisfies the equality $(\ell,\varphi) =< \psi_\ell, \varphi >$ and for this function the following equality holds*

$$||\ell||_{H^*} = ||\psi_\ell||_H$$

and vice versa. For any $\psi_\ell \in H$, a unique functional $\ell \in H^*$ is found that satisfies the equalities $(\ell,\varphi) =< \psi_\ell, \varphi >$ and $||\ell||_{H^*} = ||\psi_\ell||_H$ [3].

According to the theorem, the extremal function $\psi_\ell(x)$ is a Riesz element, and the following holds

$$(\ell, \psi_\ell) = ||\ell||^2 = ||\psi_\ell||^2.$$

According to the theorem, we calculate the square of the norm of the error functional

$$||\ell||^2 = (\ell,\psi_\ell) = \int_{-\infty}^{\infty} \ell(x)\psi_\ell(x)dx = \int_{-\infty}^{\infty} \ell(x)(-\ell(x) * G_1(x) + d)dx =$$

$$= -\int_0^{2\pi}\int_0^{2\pi} \frac{|x-y|}{2}\cot\frac{x-t}{2}\cot\frac{y-t}{2}dxdy + 2\sum_{\beta=0}^{N} C_\beta \int_0^{2\pi} \frac{|x-h\beta|}{2}\cot\frac{x-t}{2}dx - \sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_\beta C_\gamma \frac{|h\beta-h\gamma|}{2}.$$

Therefore,

$$||\ell||^2 = -\int_0^{2\pi}\int_0^{2\pi} \frac{|x-y|}{2}\cot\frac{x-t}{2}\cot\frac{y-t}{2}dxdy + 2\sum_{\beta=0}^{N} C_\beta \int_0^{2\pi} \frac{|x-h\beta|}{2}\cot\frac{x-t}{2}dx - \sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_\beta C_\gamma \frac{|h\beta-h\gamma|}{2}. \quad (3.2)$$

Thus, Problem 2.1 is solved.

4. Finding the Conditional Minimum of the Norm for the Error Functional

Now we solve Problem 2.2. The square of the norm of the error functional obtained in (3.2) is a multivariable function with respect to the coefficients C_β. According to the theory of conditional extremum of a multivariable function, along with the condition (2.7), we find the local minimum of $||\ell||^2$. For this, we construct the Lagrange function

$$\Phi(C_\beta, \lambda) = ||\ell||^2 - 2\lambda(\ell, 1), \qquad \beta = 0, 1, ..., N.$$

Taking the partial derivatives of the resulting Φ function with respect to the coefficients C_β $(\beta = 0, 1, ..., N)$ and the unknown λ and setting them to zero, we obtain the following system of linear equations

$$\sum_{\gamma=0}^{N} C_\gamma \frac{|h\beta - h\gamma|}{2} + \lambda = f_1(h\beta), \ \ \beta = 0, 1, ..., N, \tag{4.1}$$

$$\sum_{\beta=0}^{N} C_\beta = g_0, \tag{4.2}$$

where

$$f_1(h\beta) = \int_0^{2\pi} \frac{|x - h\beta|}{2} \cot \frac{x-t}{2} dx =$$

$$= \sum_{k=0}^{\infty} \sum_{i=1}^{2k+1} (-1)^{3k+1+i} \frac{B_{2k} \cdot t^{2k+1-i}}{i! \cdot (2k+1-i)!} \left((2\pi)^i - 2(h\beta)^i \right) + 2(t - h\beta) \ln \left| \frac{\sin \frac{t}{2}}{\sin \frac{h\beta - t}{2}} \right|, \tag{4.3}$$

where the following formula was used [6]:

$$\int x^p \cot x dx = \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} B_{2k}}{(2k+p)(2k)!} x^{2k+p}, \qquad p \geq 1, \ \ |x| < \pi,$$

B_{2k} are Bernoulli numbers.

$$g_0 = \int_0^{2\pi} \cot \frac{x-t}{2} dx = 0.$$

To solve the system of linear equations (4.1)-(4.2), we will use the Sobolev method [29]. For this, we use the discrete analog $D_1(h\beta)$ of the differential operator d^2/dx^2. The operator $D_1(h\beta)$ is defined in the work [19] as follows

$$D_1(h\beta) = \frac{1}{h^2} \begin{cases} 0, & |\beta| \geq 2, \\ 1, & |\beta| = 1, \\ -2, & \beta = 0. \end{cases} \tag{4.4}$$

We use some properties of the discrete analog $D_1(h\beta)$. They are shown in the following theorem [19].

Theorem 4.1. *Discrete analogue $D_1(h\beta)$ of the differential operator d^2/dx^2 satisfies the following equalities:*
1) $D_1(h\beta) * G_1(h\beta) = \delta(h\beta)$
2) $D_1(h\beta) * 1 = 0$,

where $\delta(h\beta)$ is the discrete delta function, i.e., $\delta(h\beta) = 0$ for $\beta \neq 0$ and $\delta(0) = 1$.

Since we will now use functions with discrete arguments and operations on them, we will present some concepts about them:

Definition 4.1. *The function $\varphi(h\beta)$ is a function of discrete argument, if it is given on some set of integer values of β.*

Definition 4.2. *The inner product of two discrete argument functions $\varphi(h\beta)$ and $\psi(h\beta)$ is given by*

$$[\varphi(h\beta), \psi(h\beta)] = \sum_{\beta=-\infty}^{\infty} \varphi(h\beta) \cdot \psi(h\beta),$$

if the series on the right hand side converges absolutely.

Definition 4.3. *The convolution of two functions $\varphi(h\beta)$ and $\psi(h\beta)$ is the inner product*

$$\varphi(h\beta) * \psi(h\beta) = [\varphi(h\gamma), \psi(h\beta - h\gamma)] = \sum_{\gamma=-\infty}^{\infty} \varphi(h\gamma) \cdot \psi(h\beta - h\gamma).$$

According to the Sobolev method, assuming that $\beta < 0$ and $\beta > N$, $C_\beta = 0$, we convert equation (4.1) to convolutional form, where because the coefficients C_β are discrete, we set $C_\beta = C[\beta]$ as follows

$$C[\beta] * G_1[\beta] + \lambda = f_1[\beta], \qquad \beta = 0, 1, ..., N.$$

From here, we introduce the following notation

$$v[\beta] = C[\beta] * G_1[\beta], \tag{4.5}$$

$$u[\beta] = v[\beta] + \lambda,$$

then the following holds:

$$u[\beta] = f_1[\beta], \;\; \beta = 0, 1, ..., N.$$

$D_1(h\beta)$ according to the properties of the differential operator

$$C[\beta] = h \cdot D_1[\beta] * u[\beta]. \tag{4.6}$$

Thus, according to expression (4.6), to find the optimal coefficients of quadrature formula (2.1), we need to determine the complete form of $u[\beta]$ for $\beta < 0$ and $\beta > N$. For this, we rewrite expression (4.5):

$$v[\beta] = C[\beta] * G_1[\beta] = \sum_{\gamma=0}^{N} C[\gamma] \cdot \frac{|h\beta - h\gamma|}{2}.$$

Hence for $\beta < 0$

$$v[\beta] = -\sum_{\gamma=0}^{N} C[\gamma] \cdot \frac{[\beta - \gamma]}{2} = -\frac{[\beta]}{2} g_0 + p,$$

where $p = \frac{1}{2} \sum\limits_{\gamma=0}^{N} C[\gamma][\gamma]$.

Then for $\beta > N$

$$v[\beta] = \sum_{\gamma=0}^{N} C[\gamma] \cdot \frac{[\beta - \gamma]}{2} = \frac{[\beta]}{2} g_0 - p.$$

Considering that $g_0 = 0$, we rewrite the form of $u[\beta]$

$$u[\beta] = \begin{cases} p + \lambda & , \quad \beta < 0, \\ f_1[\beta] & , \quad 0 \le \beta \le N, \\ -p + \lambda & , \quad \beta > N. \end{cases}$$

Here we introduce the following notations

$$p + \lambda = a_0^-, \qquad -p + \lambda = a_0^+,$$

from this we can determine the following

$$p = \frac{a_0^- - a_0^+}{2}, \qquad \lambda = \frac{a_0^- + a_0^+}{2}.$$

Then $u[\beta]$ will have the following form:

$$u[\beta] = \begin{cases} a_0^- & , \quad \beta < 0, \\ f_1[\beta] & , \quad 0 \le \beta \le N, \\ a_0^+ & , \quad \beta > N. \end{cases} \tag{4.7}$$

In (4.7), a_0^- and a_0^+ are unknowns. If we find these unknowns, the form of $u[\beta]$ will be fully determined. To find these unknowns, from (4.7) we determine that for $\beta = 0$, $a_0^- = f_1[0]$ and for $\beta = N$, $a_0^+ = f_1[N]$. Thus, the form of $u[\beta]$ is fully determined

$$u[\beta] = \begin{cases} f_1[0] & , \quad \beta < 0, \\ f_1[\beta] & , \quad 0 \le \beta \le N, \\ f_1[N] & , \quad \beta > N. \end{cases} \tag{4.8}$$

5. Finding the Coefficients of the Optimal Quadrature Formula

For the optimal coefficients of the quadrature formula (2.1), the following theorem holds.

Theorem 5.1. *The coefficients* $\mathring{C}_\beta$, $\beta = 0, 1, ..., N$ *of the optimal quadrature formula* (2.1) *in the space* $L_2^{(1)}(0, 2\pi)$ *have the fpllowing forms*

$$C[0] = \frac{1}{h}\Big(f_1[1] - f_1[0]\Big);$$

$$C[\beta] = \frac{1}{h}\Big(f_1[\beta - 1] - 2f_1[\beta] + f_1[\beta + 1]\Big), \quad \beta = 1, 2, ..., N - 1;$$

$$C[N] = \frac{1}{h}\Big(f_1[N - 1] - f_1[N]\Big); \tag{5.1}$$

here $f_1[\beta]$ is defined by (4.3).

Proof. To find the optimal coefficients of the quadrature formula (2.1), we use formula (4.6). In this case, the discrete analog $D_1[\beta]$ of the differential operator d^2/dx^2 is defined by formula (4.4), and the function $u[\beta]$ is determined by expression (4.8)

$$C[\beta] = hD_1[\beta] * u[\beta] =$$

$$= h\left(\sum_{\gamma=-\infty}^{-1} D_1[\beta-\gamma]u[\gamma] + \sum_{\gamma=0}^{N} D_1[\beta-\gamma]u[\gamma] + \sum_{\gamma=N+1}^{\infty} D_1[\beta-\gamma]u[\gamma]\right) =$$

$$= h\left(\sum_{\gamma=1}^{\infty} D_1[\beta+\gamma]u[-\gamma] + \sum_{\gamma=0}^{N} D_1[\beta-\gamma]u[\gamma] + \sum_{\gamma=1}^{\infty} D_1[\beta-(\gamma+N)]u[\gamma+N]\right) =$$

$$= h\left(\sum_{\gamma=1}^{\infty} D_1[\beta+\gamma]f_1[0] + \sum_{\gamma=0}^{N} D_1[\beta-\gamma]f_1[\gamma] + \sum_{\gamma=1}^{\infty} D_1[\beta-(\gamma+N)]f_1[N]\right).$$

for $\beta = 0$

$$C[0] = h\left(\sum_{\gamma=1}^{\infty} D_1[\gamma]f_1[0] + \sum_{\gamma=0}^{N} D_1[-\gamma]f_1[\gamma] + \sum_{\gamma=1}^{\infty} D_1[-(\gamma+N)]f_1[N]\right) =$$

$$= h\left(\frac{1}{h^2}f_1[0] + \frac{1}{h^2}(-2f_1[0] + f_1[1])\right) = \frac{1}{h}\left(f_1[1] - f_1[0]\right);$$

for $\beta = 1, 2, ..., N-1$

$$C[\beta] = h\left(\sum_{\gamma=1}^{\infty} D_1[\beta+\gamma]f_1[0] + \sum_{\gamma=0}^{N} D_1[\beta-\gamma]f_1[\gamma] + \sum_{\gamma=1}^{\infty} D_1[\beta-(\gamma+N)]f_1[N]\right) =$$

$$= h\left(\sum_{\gamma=0}^{\beta-2} D_1[\beta-\gamma]f_1[\gamma] + D_1[1]f_1[\beta-1] + D_1[0]f_1[\beta] + D_1[-1]f_1[\beta+1] + \sum_{\gamma=\beta+2}^{N} D_1[\beta-\gamma]f_1[\gamma]\right) =$$

$$= \frac{1}{h}\left(f_1[\beta-1] - 2f_1[\beta] + f_1[\beta+1]\right);$$

for $\beta = N$

$$C[N] = h\left(\sum_{\gamma=1}^{\infty} D_1[N+\gamma]f_1[0] + \sum_{\gamma=0}^{N} D_1[N-\gamma]f_1[\gamma] + \sum_{\gamma=1}^{\infty} D_1[-\gamma]f_1[N]\right) =$$

$$= h\left(\frac{1}{h^2}(-2f_1[N] + f_1[N-1]) + \frac{1}{h^2}f_1[N]\right) = \frac{1}{h}\left(f_1[N-1] - f_1[N]\right).$$

□

The theorem is proven.

Problem 2.2 is solved.

In the next section, we will use the optimal coefficients found to approximately calculate singular integral equations with a Hilbert kernel.

6. Numerical Results

In this section, we conduct numerical comparisons with the exact solutions of several Fredholm-type singular integral equations of the first kind with Hilbert kernels to determine the order of accuracy of the optimal quadrature formula. The purpose is to demonstrate the effectiveness of the optimal quadrature formula. For each example, the errors in the approximation of the integral's value to the exact solution are numerically analyzed in tabular and graphical form as the number of node points increases.

Example 6.1. *Solve the singular integral equation with the Hilbert kernel*

$$\frac{1}{2\pi}\int_0^{2\pi} g(x)\cot\frac{x-t}{2}dx = \cos t. \tag{6.1}$$

Solution. *Since the integral equation* (6.1) *satisfies the condition* (1.2)*, this equation has the following solution according to* (1.3)

$$g(t) = -\frac{1}{2\pi}\int_0^{2\pi} \cos x\cot\frac{x-t}{2}dx. \tag{6.2}$$

The integral (6.2) *has an exact solution* $g(t) = \sin t$ *(see [5]). Thus, the integral equation* (6.1) *has the solution* $g(x) = \sin x$*. We will approximate the exact solution of this integral* (6.2) *using the optimal quadrature formula* (2.1).

Table 1 and Table 2 show the exact value of the integral calculated using the Maple mathematical package, the value of the integral calculated using the optimal quadrature formula (OQF), and the absolute error between them when $N = 10$ and $N = 100$, respectively.

These numerical results were calculated when the number of nodes was $N = 10$.

t	$g(t)$	OQF	$\lvert g(t) - OQF\rvert$
$\frac{2\pi}{9}$	0.642787609686538	0.634477720875035	8.3098888115030E-03
$\frac{4\pi}{9}$	0.984807753012207	0.955978845267640	2.8828907744567E-02
$\frac{6\pi}{9}$	0.866025403784440	0.832944977510745	3.3080426273695E-02
$\frac{8\pi}{9}$	0.342020143325668	0.327538128272579	1.4482015053089E-02
$\frac{10\pi}{9}$	-0.342020143325668	-0.327538128272584	1.4482015053084E-02
$\frac{12\pi}{9}$	-0.866025403784440	-0.832944977510745	3.3080426273695E-02
$\frac{14\pi}{9}$	-0.984807753012207	-0.955978845267635	2.8828907744572E-02
$\frac{16\pi}{9}$	-0.642787609686538	-0.634477720875055	8.3098888114830E-03

Table 1. Here $g(t) = \sin t$, $OQF = -\frac{1}{2\pi}\sum_{\beta=0}^{N} C[\beta]\cos(h\beta)$

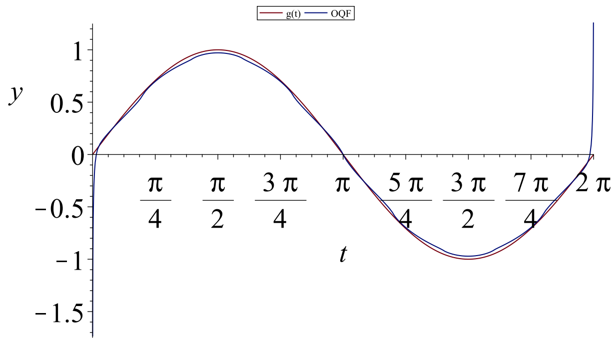

FIGURE 1. This figure shows graphs of $g(t) = \sin t$ and $OQF = -\frac{1}{2\pi}\sum_{\beta=0}^{N} C[\beta]\cos(h\beta), N = 10$;

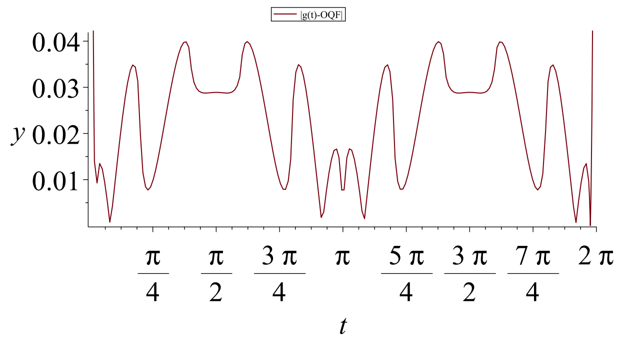

FIGURE 2. This figure shows the error graph of $|g(t) - OQF|, N = 10$;

These numerical results were calculated when the number of nodes was $N = 100$.

t	$g(t)$	OQF	$\lvert g(t) - OQF\rvert$
$\frac{2\pi}{9}$	0.642787609686538	0.642720351959590	6.72577269480E-05
$\frac{4\pi}{9}$	0.984807753012207	0.984517232639225	2.90520372982E-04
$\frac{6\pi}{9}$	0.866025403784440	0.865674715834655	3.50687949785E-04
$\frac{8\pi}{9}$	0.342020143325668	0.341863672689138	1.56470636530E-04
$\frac{10\pi}{9}$	-0.342020143325668	-0.341863672689141	1.56470636527E-04
$\frac{12\pi}{9}$	-0.866025403784440	-0.865674715834685	3.50687949755E-04
$\frac{14\pi}{9}$	-0.984807753012207	-0.984517232639215	2.90520372992E-04
$\frac{16\pi}{9}$	-0.642787609686538	-0.642720351959060	6.72577274780E-05

TABLE 2. Here $g(t) = \sin t$ and $OQF = -\frac{1}{2\pi}\sum_{\beta=0}^{N} C[\beta]\cos(h\beta)$

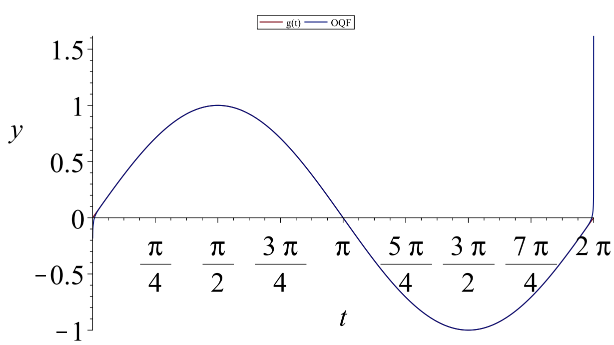

FIGURE 3. This figure shows graphs of $g(t) = \sin t$ and $OQF = -\frac{1}{2\pi}\sum_{\beta=0}^{N} C[\beta]\cos(h\beta), N = 100$;

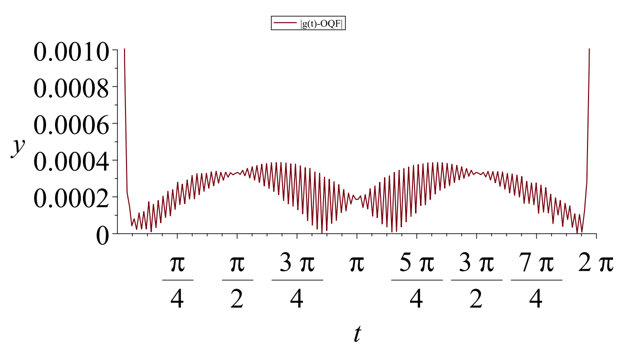

FIGURE 4. This figure shows the error graph of $|g(t) - OQF|, N = 100$;

Remark 6.1. *As can be seen from the Figures 1-4, the error decreases as the number of node points N increases.*

Example 6.2. *Solve the singular integral equation with a Hilbert kernel*

$$\frac{1}{2\pi}\int_{0}^{2\pi} g(x)\cot\frac{x-t}{2}dx = \sin t. \tag{6.3}$$

Solution. *Since the integral equation* (6.1) *satisfies the condition* (1.2), *this equation has the following solution according to* (1.3)

$$g(t) = -\frac{1}{2\pi}\int_{0}^{2\pi} \sin x \cot\frac{x-t}{2}dx. \tag{6.4}$$

The integral (6.4) *has an exact solution* $g(t) = -\cos t$ *(see [5]). Thus, the integral equation* (6.3) *has the solution* $g(x) = -\cos x$. *We will approximate the exact solution of this integral* (6.4) *using the optimal quadrature formula* (2.1).

Table 3 and Table 4 show the exact value of the integral calculated using the Maple mathematical package, the value of the integral calculated using the optimal quadrature formula (OQF), and the absolute error between them when $N = 10$ and $N = 100$, respectively.

These numerical results were calculated when the number of nodes was $N = 10$.

t	$g(t)$	OQF	$\lvert g(t) - OQF\rvert$
$\frac{2\pi}{9}$	0.766044443118979	0.726773910587135	3.9270532531844E-02
$\frac{4\pi}{9}$	0.173648177666934	0.148941121134408	2.4707056532526E-02
$\frac{6\pi}{9}$	-0.500000000000000	-0.496575061480595	3.4249385194050E-03
$\frac{8\pi}{9}$	-0.939692620785909	-0.914027554549930	2.5665066235979E-02
$\frac{10\pi}{9}$	-0.939692620785909	-0.914027554549935	2.5665066235974E-02
$\frac{12\pi}{9}$	-0.500000000000000	-0.496575061480587	3.4249385194130E-03
$\frac{14\pi}{9}$	0.173648177666934	0.148941121134406	2.4707056532528E-02
$\frac{16\pi}{9}$	0.766044443118979	0.726773910587185	3.9270532531794E-02

TABLE 3. Here $g(t) = \cos t$, $OQF = \frac{1}{2\pi}\sum_{\beta=0}^{N} C[\beta]\sin(h\beta)$

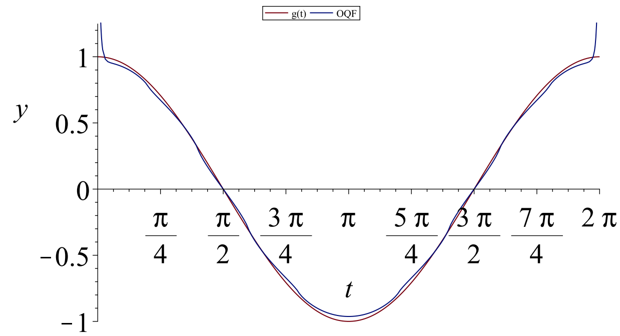

FIGURE 5. This figure shows graphs of $g(t) = \cos t$ and $OQF = \frac{1}{2\pi}\sum_{\beta=0}^{N} C[\beta]\sin(h\beta), N = 10;$

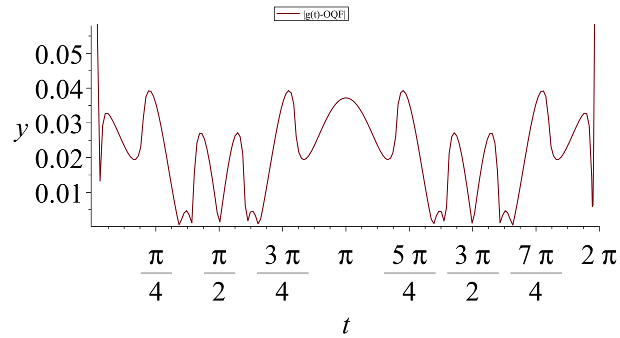

FIGURE 6. This figure shows the error graph of $|g(t) - OQF|, N = 10;$

These numerical results were calculated when the number of nodes was $N = 100$.

t	$g(t)$	OQF	$\lvert g(t) - OQF\rvert$
$\frac{2\pi}{9}$	0.766044443118979	0.765667567866665	3.76875252314E-04
$\frac{4\pi}{9}$	0.173648177666934	0.173400292491196	2.47885175738E-04
$\frac{6\pi}{9}$	-0.500000000000000	-0.499953795693014	4.62043069860E-05
$\frac{8\pi}{9}$	-0.939692620785909	-0.939403175141790	2.89445644119E-04
$\frac{10\pi}{9}$	-0.939692620785909	-0.939403175141755	2.89445644154E-04
$\frac{12\pi}{9}$	-0.500000000000000	-0.499953795693042	4.62043069580E-05
$\frac{14\pi}{9}$	0.173648177666934	0.173400292491074	2.47885175860E-04
$\frac{16\pi}{9}$	0.766044443118979	0.765667567867245	3.76875251734E-04

TABLE 4. Here $g(t) = \cos t$ and $OQF = \frac{1}{2\pi}\sum_{\beta=0}^{N} C[\beta]\sin(h\beta)$

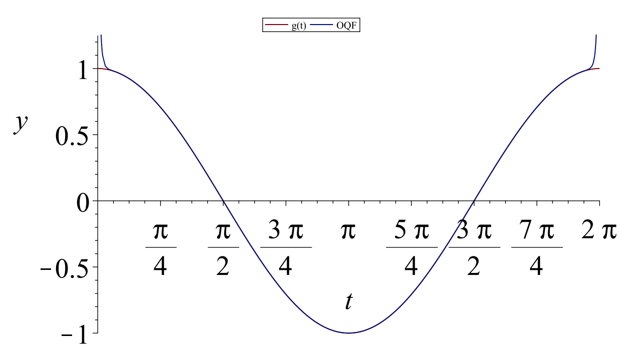

FIGURE 7. This figure shows graphs of $g(t) = \cos t$ and $OQF = \frac{1}{2\pi}\sum_{\beta=0}^{N} C[\beta]\sin(h\beta), N = 100;$

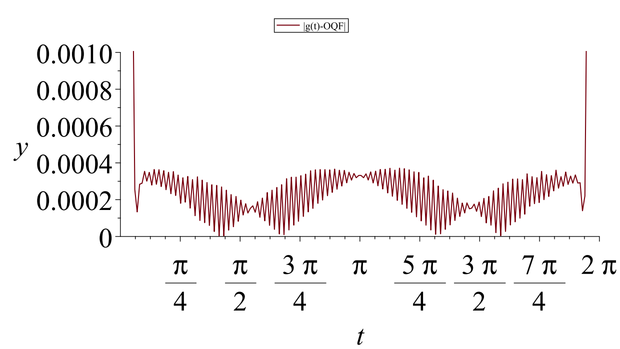

FIGURE 8. This figure shows the error graph of $|g(t) - OQF|, N = 100;$

Remark 6.2. *As can be seen from the Figures 5-8, the error decreases as the number of node points N increases.*

7. Conclution

In this article, an optimal quadrature formula was constructed using the Sobolev method for high-precision approximate calculations of singular integral equations with Hilbert kernel, and an analytical representations of the corresponding optimal coefficients were found. Using these optimal coefficients, the exact and approximate solutions of two singular integral equations with Hilbert kernels were compared. Their results are presented in tables, and the errors between the exact and approximate solutions for the case $0 < t < 2\pi$ are illustrated using graphs. As can be seen from the tables and graphs, the accuracy of the constructed optimal quadrature formula increases as the number of nodal points N increases, resulting in a decrease in error.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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