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Construction of New Continuous K-g-Frames within Hilbert C^*-Modules

Sanae Touaiher[1,*], **Mohamed Rossafi**[2]

[1] *Laboratory Analysis, Geometry and Applications, University of Ibn Tofail, Kenitra, Morocco*
[2] *Laboratory Analysis, Geometry and Applications, Higher School of Education and Training, University of Ibn Tofail, Kenitra, Morocco*

**Corresponding author:* sanae.touaiher@uit.ac.ma

Abstract. In this paper, we investigate the construction of new continuous c-K-g-frames in Hilbert C*-modules, extending and generalizing existing frame theory results. Our main theorem establishes sufficient positivity conditions on an auxiliary operator T to ensure that the transformed family $\{\Lambda_\omega T\}_{\omega\in\Omega}$ forms a continuous c-K-g-frame. Through examples, we illustrate the necessity of these positivity conditions. We also present a method for combining multiple continuous c-K-g-frames into a single continuous c-$\sum_i K_i T_i$-g-frame. We prove associativity of continuous c-K-g-frames under product measure spaces, and explore exactness and stability under restriction to some measurable subsets for non-decreasing continuous K-g-frames with respect to an ordered measurable space. Additionally, we characterize dual frames in this setting, providing insight into their existence and construction. Our results extend and unify various notions of continuous frames in both Hilbert spaces and Hilbert C^*-modules.

1. Introduction

The theory of frames, initially developed by Duffin and Schaeffer [3] in 1952 to address issues related to nonharmonic Fourier series, has since undergone significant generalizations and refinements. A major advancement was the introduction of continuous frames by Ali, Antoine, and Gazeau [2] in 1993, and their interpretation as frames associated with measurable spaces by Gabardo and Han [6]. These developments paved the way for applications in signal processing, quantum physics, and other areas requiring continuous decompositions.

In parallel, the theory of Hilbert C^*-modules a generalization of Hilbert spaces in which the scalar field is replaced by a C^*-algebra has provided a rich framework for operator theory and noncommutative geometry. Frank and Larson [5] extended the concept of frames to this setting by introducing frames in Hilbert C^*-modules. Later, continuous frames were also generalized to this

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context [15,16], offering new tools for both theoretical exploration and practical applications. In recent studies, some authors have shown particular interest in the structure of K-frames for a given bounded operator K. This has led to the development of continuous K-g-frames (abbreviated as c-K-g-frames), which combine generalized frames (g-frames) with operator-based flexibility in a continuous setting. These structures are especially useful in handling perturbations, reconstruction algorithms, and duality theory within the framework of operator modules; For more detailed information on frames theory, readers are recommended to consult: [4,7,9–11,13,14,18–30].

Throughout the paper, (Ω, μ) denotes a measure space, and $\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2$ are separable Hilbert C^*-modules over a fixed C^*-algebra $\mathcal{A}$. The space $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_1, \mathcal{H}_2)$ denotes the set of all adjointable $\mathcal{A}$-linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$, with $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$ as shorthand for $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H})$. Additionally, $\mathrm{Ran}(K)$ denotes the range of the operator $K \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$, and $\{\mathcal{H}_\omega\}_{\omega\in\Omega}$ is a measurable field of Hilbert $\mathcal{A}$-modules indexed by Ω.

The paper is structured as follows. In Section 2, we provide the necessary background on Hilbert C^*-modules, adjointable operators, and Bochner integration. Section 3 presents the main results concerning c-K-g-frames, including sufficient and necessary conditions for their existence, and properties of the associated frame operator. One of our main theorems establishes sufficient positivity conditions on an auxiliary operator T to ensure that the transformed family $\{\Lambda_\omega T\}_{\omega\in\Omega}$ forms a c-K-g-frame.

We also investigate the construction of new c-K-g-frames in Hilbert C^*-modules, extending and generalizing existing results in frame theory. Given bounded adjointable operators K_1 and K_2 on Hilbert C^*-modules and a known continuous c-K_1-g-frame, we develop conditions under which new continuous c-K_2-g-frames can be constructed via operator transformations.

Through examples, we illustrate the necessity of these positivity conditions, demonstrating how their failure leads to the loss of frame properties. In the same section, we prove the associativity of c-K-g-frames under product measure spaces and explore exactness and stability under restriction to measurable subsets for non-decreasing c-K-g-frames.

Some examples and applications to operator-valued Hardy spaces and multiplication operators illustrate frame constructions via Hadamard products and functional sequences.

Finally, Section 4 is devoted to the existence and properties of the associated frame operators and duality relations. We also characterize dual frames in this setting, providing insights into their existence and construction.

2. Preliminaries

Let $\mathcal{A}$ be a unital C*-algebra, and let $\mathcal{H}$ be a left $\mathcal{A}$-module. We say that $\mathcal{H}$ is a *pre-Hilbert $\mathcal{A}$-module* if it is equipped with an $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle_{\mathcal{A}} : \mathcal{H}\times\mathcal{H} \to \mathcal{A}$, which satisfies the following properties: it is linear in the first argument, meaning that $\langle \alpha f + \beta g, h\rangle_{\mathcal{A}} = \alpha\langle f, h\rangle_{\mathcal{A}} + \beta\langle g, h\rangle_{\mathcal{A}}$ for all $f, g, h \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$; it is compatible with the $\mathcal{A}$-action, so that $\langle af, g\rangle_{\mathcal{A}} = a\langle f, g\rangle_{\mathcal{A}}$ for all $a \in \mathcal{A}$; it is adjoint symmetric, satisfying $\langle f, g\rangle_{\mathcal{A}} = \langle g, f\rangle^*_{\mathcal{A}}$; and it is positive-definite, meaning

$\langle f, f\rangle_{\mathcal{A}} \geq 0$ with equality if and only if $f = 0$. Given such an inner product, we define a norm on $\mathcal{H}$ by

$$\|f\| := \left\|\langle f, f\rangle_{\mathcal{A}}\right\|^{1/2}, \quad \text{for all } f \in \mathcal{H}.$$

If $\mathcal{H}$ is complete with respect to this norm, then it is called a *Hilbert $\mathcal{A}$-module*, or a *Hilbert C^*-module over $\mathcal{A}$*.

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert $\mathcal{A}$-modules. A map $T : \mathcal{H} \to \mathcal{K}$ is said to be *adjointable* if there exists a map $T^* : \mathcal{K} \to \mathcal{H}$ such that

$$\langle Tf, g\rangle_{\mathcal{A}} = \langle f, T^*g\rangle_{\mathcal{A}} \quad \text{for all } f \in \mathcal{H},\ g \in \mathcal{K}.$$

The following result is a Douglas's theorem version relative to C*-Hilbert modules.

Theorem 2.1. *[32] Consider $\mathcal{H}$, $\mathcal{H}_1$, $\mathcal{H}_2$ Hilbert C*-module over a C*-algebra $\mathcal{A}$. Suppose that $T \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_1, \mathcal{H})$, $S \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_2, \mathcal{H})$ and that the range of S, denoted $\mathrm{Ran}(S)$, is closed. Then, the following conditions are equivalent:*

1. $\mathrm{Ran}(T) \subseteq \mathrm{Ran}(S)$.
2. *There exists a non-negative $\lambda \geq 0$ such that $TT^* \leq \lambda^2 SS^*$.*
3. *There exists an operator $Q \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_1, \mathcal{H}_2)$ for which $T = SQ$.*

The next theorem characterize the operators on Hilbert C*-modules that possesses a Moore-Penrose inverse

Theorem 2.2. *[31, Theorem 2.2] Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules, and let $T \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$. Then the following statements are equivalent:*

1. *The range of T is closed.*
2. *The Moore-Penrose inverse $T^\dagger$ of T exists; this means that $T^\dagger$ is an element of $\mathrm{End}^*_{\mathcal{A}}(\mathcal{K}, \mathcal{H})$ which satisfies:*

$$TT^\dagger T = T, \quad T^\dagger TT^\dagger = T^\dagger, \quad (TT^\dagger)^* = TT^\dagger \text{ and } (T^\dagger T)^* = T^\dagger T.$$

In the context of Hilbert C^*-modules, integration theory plays a crucial role in defining continuous frames, synthesis and analysis operators, and related structures. To bridge classical and operator-valued settings, we begin by recalling the Bochner integral for Banach space-valued functions and its compatibility with bounded linear operators, see [12].

Let (Ω, Σ, μ) be a measure space, and let E be a Banach space. A function $f : \Omega \to E$ is said to be *strongly measurable* if it is the pointwise limit of simple functions and *Bochner integrable* if

$$\int_\Omega \|f(\omega)\|_E \, d\mu(\omega) < \infty.$$

The *Bochner integral* $\int_\Omega f(\omega)\, d\mu(\omega) \in E$ is then defined as the norm-limit of integrals of simple functions approximating f. A key property of the Bochner integral is its compatibility with bounded linear operators: if $T : E \to F$ is a bounded linear map between Banach spaces and $f : \Omega \to E$ is Bochner integrable, then

$$T\left(\int_\Omega f(\omega)\,d\mu(\omega)\right) = \int_\Omega T(f(\omega))\,d\mu(\omega).$$

Let $\mathcal{A}$ be a unital C^*-algebra, and let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module and $\{\mathcal{H}_\omega\}_\Omega$ a family of Hilbert $\mathcal{A}$-modules. Denote by $\left(\bigoplus_{\omega\in\Omega}\mathcal{H}_\omega,\mu\right)_{L^2}$ the space

$$\left\{F \in \prod_{\omega\in\Omega}\mathcal{H}_\omega \;:\; F \text{ strongly measurable and } \int_\Omega \|\langle F(\omega),F(\omega)\rangle\|_{\mathcal{A}}\,d\mu(\omega) < \infty\right\}.$$

It is known $\left(\bigoplus_{\omega\in\Omega}\mathcal{H}_\omega,\mu\right)_{L^2}$ equipped with $\mathcal{A}$-valued inner product

$$\langle F,G\rangle = \int_\Omega \langle F(\omega),G(\omega)\rangle_{\mathcal{A}}\,d\mu(\omega),$$

is a Hilbert C^*-module over $\mathcal{A}$. Now, consider a family $\{\Lambda_\omega\}_{\omega\in\Omega} \in \prod\limits_{\omega\in\Omega}\text{End}^*_{\mathcal{A}}(\mathcal{H},\mathcal{H}_\omega)$, such that $\{\Lambda_\omega\}_{\omega\in\Omega} \in \left(\bigoplus_{\omega\in\Omega}\mathcal{H}_\omega,\mu\right)_{L^2}$, for all $f \in \mathcal{H}$. Under these assumptions, we define the associated *analysis operator* T^* from $\mathcal{H}$ into $\left(\bigoplus_{\omega\in\Omega}\mathcal{H}_\omega,\mu\right)_{L^2}$ by

$$(T^*f)(\omega) := \{\Lambda_\omega(f)\}_{\omega\in\Omega}, \text{ for all } f \in \mathcal{H}$$

and the corresponding *synthesis operator* $T\colon \left(\bigoplus_{\omega\in\Omega}\mathcal{H}_\omega,\mu\right)_{L^2} \to \mathcal{H}$, defined weakly by

$$\langle TF,g\rangle_{\mathcal{A}} := \int_\Omega \langle \Lambda^*_\omega F(\omega),g\rangle_{\mathcal{A}}\,d\mu(\omega), \quad \text{for all } F \in \left(\bigoplus_{\omega\in\Omega}\mathcal{H}_\omega,\mu\right)_{L^2},\ g \in \mathcal{H}. \tag{2.1}$$

We now consider conditions under which the family $\{\Lambda_\omega\}_{\omega\in\Omega}$ forms a continuous K-g-frame (c-K-g-frame) for $\mathcal{H}$. Let $K \in \text{End}^*_{\mathcal{A}}(\mathcal{H})$, and suppose that the family $\{\Lambda_\omega f\}_{\omega\in\Omega}$ is strongly measurable for each $f \in \mathcal{H}$. Then, as shown in [8] (resp. in [1] for the Hilbert spaces case), $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_\omega\}_{\omega\in\Omega}$ if and only if the synthesis operator T is bounded, adjointable, and $\mathcal{A}$-linear, and there exists a constant $A > 0$ such that

$$AKK^* \le TT^*.$$

In this case, the analysis operator satisfies $T^*f = \{\Lambda_\omega f\}_{\omega\in\Omega}$, and $\|T\| \le \sqrt{B}$, where B is the upper c-K-g-frame bound of the family $\{\Lambda_\omega\}_{\omega\in\Omega}$.

We now turn to the *frame operator* associated with $\{\Lambda_\omega\}$, defined by

$$S(f) := \int_\Omega \Lambda^*_\omega\Lambda_\omega f\,d\mu(\omega), \quad f \in \mathcal{H},$$

where the integral is taken in the Bochner sense. Since each $\Lambda^*_\omega\Lambda_\omega f \in \mathcal{H}$, the integral yields a well-defined element of $\mathcal{H}$. The operator S is linear, bounded, positive, and self-adjoint on $\mathcal{H}$, due to the continuity of the adjointable maps $\Lambda^*_\omega\Lambda_\omega$ and the boundedness guaranteed by the frame inequality.

To analyze the action of S, we compute

$$\langle S(f), f\rangle_{\mathcal{A}} = \left\langle \int_{\Omega} \Lambda_{\omega}^{*}\Lambda_{\omega} f \, d\mu(\omega), f \right\rangle_{\mathcal{A}} = \int_{\Omega} \langle \Lambda_{\omega} f, \Lambda_{\omega} f\rangle_{\mathcal{A}} \, d\mu(\omega),$$

The following definition will serve as a basis in what follows; see [8, Definition 1.10] for an equivalent version.

Definition 2.3. Let (Ω, μ) be a measure space, $\mathcal{H}$ a Hilbert C^*-module over a C^*-algebra $\mathcal{A}$, and $K \in \mathrm{End}_{\mathcal{A}}^{*}(\mathcal{H})$. A continuous family of operators

$$\{\Lambda_{\omega}\}_{\omega\in\Omega} \subseteq \mathrm{End}_{\mathcal{A}}^{*}\left(\mathcal{H}, \left(\bigoplus_{\omega\in\Omega} \mathcal{H}_{\omega}, \mu\right)_{L^2}\right)$$

is called a c-K-g-frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that for all $f \in \mathcal{H}$,

$$A\,\langle K^{*}f, K^{*}f\rangle \;\leq\; \int_{\Omega} \langle \Lambda_{\omega} f, \Lambda_{\omega} f\rangle \, d\nu(\omega) \;\leq\; B\,\langle f, f\rangle.$$

3. Some Positivity and Stability Conditions for the Construction of c-K-g-Frames

Our first main theorem establishes sufficient positivity conditions on an auxiliary operator T to ensure that the transformed family $\{\Lambda_{\omega}T\}_{\omega\in\Omega}$ forms a c-K-g-frame.

Theorem 3.1. *Let $K, T \in \mathrm{End}_{\mathcal{A}}^{*}(\mathcal{H})$ and suppose that $\{\Lambda_{\omega}\}_{\omega\in\Omega}$ is a c-K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_{\omega}\}_{\omega\in\Omega}$ and*

$$(T^{*} - \overline{w}I)KK^{*}(T + wI) \geq 0 \tag{3.1}$$

for some non-zero complex number w. Then $\{\Lambda_{\omega}T\}_{\omega\in\Omega}$ is a c-K-g-frame for $\mathcal{H}$.

Proof. We know that there exist positive real constants A, B such that

$$A\,\langle K^{*}f, K^{*}f\rangle \leq \int_{\Omega} \langle \Lambda_{\omega} f, \Lambda_{\omega} f\rangle \mathrm{d}^{-}(!) \leq \mathrm{B}\langle \mathrm{f}, \mathrm{f}\rangle \tag{3.2}$$

for all $f \in \mathcal{H}$. Thus, for every $f \in \mathcal{H}$ we have

$$\begin{aligned} A\langle K^{*}Tf, K^{*}Tf\rangle &\leq \int_{\Omega} \langle \Lambda_{\omega} Tf, \Lambda_{\omega} Tf\rangle d\mu(\omega) \\ &\leq B\langle Tf, Tf\rangle \\ &\leq B\|T\|^{2}\langle f, f\rangle. \end{aligned}$$

Since $(T^{*} - \overline{w}I)KK^{*}(T + wI)$ and $(T^{*} - \overline{w}I)KK^{*}(T - wI)$ are positive and from the fact that

$$2w(T^{*} - \overline{w}I)KK^{*} = (T^{*} - \overline{w}I)KK^{*}(T + wI) - (T^{*} - \overline{w}I)KK^{*}(T - wI),$$

we have

$$w(T^{*} - \overline{w}I)KK^{*} = (w(T^{*} - \overline{w}I)KK^{*})^{*} = \overline{w}KK^{*}(T - wI).$$

Therefore

$$\begin{aligned}\langle K^*Tf, K^*Tf\rangle &= |w|^2\langle K^*f, K^*f\rangle + w\langle K^*f, K^*(T-wI)f\rangle \\ &\quad +\overline{w}\langle K^*(T-wI)f, K^*f\rangle + \langle K^*(T-wI)f, K^*(T-wI)f\rangle \\ &= |w|^2\langle K^*f, K^*f\rangle + 2w\langle (T-wI)^*KK^*f, f\rangle \\ &\quad +\langle K^*(T-wI)f, K^*(T-w)f\rangle \\ &= |w|^2\langle K^*f, K^*f\rangle + \langle (T^*-\overline{w}I)KK^*(T+wI)f, f\rangle \\ &\geq |w|^2\langle K^*f, K^*f\rangle\end{aligned}$$

for all $f \in \mathcal{H}$. So

$$|w|^2 A\langle K^*f, K^*f\rangle \leq \int_\Omega \langle \Lambda_\omega Tf, \Lambda_\omega Tf\rangle \mathrm{d}^-(!) \leq \mathrm{B}\|\mathrm{T}\|^2\langle \mathrm{f}, \mathrm{f}\rangle$$

for all $f \in \mathcal{H}$. Thus $\{\Lambda_\omega T\}_{\omega\in\Omega}$ is a $c-K-g$-frame for H. □

In the previous theorem, one can observe that the operator T satisfies condition (3.1) for a given nonzero $w \in \mathbb{C}$ if and only if, for every nonzero scalar λ, the operator λT satisfies (3.1) for $\frac{w}{\lambda}$. On the other hand, if an operator T satisfies $(T^* - uI)KK^* \geq 0$ for some positive scalar u, then T satisfies (3.1) for $w = u$. In particular, by choosing $T + I$ instead of T and setting $u = 1$, one can derive the following consequence of the theorem.

Corollary 3.2. *Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a c-K-g-frame for $\mathcal{H}$ and $T \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$. If T^*KK^* is a positive operator, then $\{\Lambda_\omega(I+T)\}_{\omega\in\Omega}$ is a c-K-g-frame for $\mathcal{H}$.*

The following example illustrates the necessity of the condition (3.1) in Theorem 1. It also shows that even when the original family $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K-g-frame, the transformed family $\{\Lambda_\omega T\}_{\omega\in\Omega}$ may fail to be a c-K-g-frame if the operator T does not satisfy the required positivity condition.

Example 3.3. Let $\mathcal{H} = \mathcal{A} \times \mathcal{A}$ be a Hilbert C*-module over the C*-algebra $\mathcal{A}$. Let $\Omega = [0,1]$, and for each $\omega \in \Omega$, let $\mathcal{H}_\omega = \mathcal{H}$. Let μ be the Lebesgue measure on Ω. Define the operators K, Λ_ω, T_0 and T on $\mathcal{H}$ by:

$$K(a,b) = \Lambda_\omega(a,b) = (a,0), \quad T_0(a,b) = (0,b), \quad T(a,b) = (b,a), \quad \text{for all } (a,b) \in \mathcal{H}.$$

Assume that $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K-g-frame. Then, there exists a positive scalar B such that:

$$Baa^* = B\langle K(a,b), K(a,b)\rangle = \int_\Omega \langle \Lambda_\omega T(a,b), \Lambda_\omega T(a,b)\rangle \, d\mu(\omega).$$

Now observe that:

$$\Lambda_\omega T(a,b) = KT(a,b) = K(b,a) = (b,0).$$

(1) Suppose that $\{\Lambda_\omega T\}_{\omega\in\Omega}$ is a c-K-g-frame. Then we must have:

$$\langle KT(a,b), KT(a,b)\rangle = \langle (b,0), (b,0)\rangle = bb^*.$$

So the frame inequality would imply:

$$Baa^* \leq bb^*,$$

for all $a, b \in \mathcal{A}$. However, this inequality cannot hold in general, leading to a contradiction. Hence, $\{\Lambda_\omega T\}_{\omega\in\Omega}$ is not a c-K-g-frame. Note that in this case, we also have:

$$K = K^*, \quad T = T^* = T^{-1}, \quad KK^* = K, \quad TKK^* = TK,$$

and

$$\langle TK(a,b), (a,b)\rangle = \langle (0,a), (a,b)\rangle = ab^*,$$

$$\langle K^*T(a,b), (a,b)\rangle = \langle (b,0), (a,b)\rangle = ba^*.$$

(2) Consider the operator $T_0(a,b) = (0,b)$. If $\{\Lambda_\omega T_0\}_{\omega\in\Omega}$ is a c-K-g-frame, then we would have:

$$T_0^* KK^* = 0, \quad \text{and} \quad KK^* T_0^* = 0.$$

Therefore, the positivity condition in Theorem 1 fails, and $\{\Lambda_\omega T_0\}_{\omega\in\Omega}$ is not a c-K-g-frame.

Let $(\Omega,\mu) = (\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2)$ be a product measure Euclidean space and let $\Lambda = \{\Lambda_{\omega_1,\omega_2}\}_{(\omega_1,\omega_2)\in\Omega_1\times\Omega_2}$ be a Bessel family for the Hilbert C*-module $\mathcal{H}$ with respect to the family $\{\mathcal{H}_{\omega_1,\omega_2}\}$ of Hilbert C*-modules. Based on Fubini Theorem for Banach spaces, see p. 93 in [12] The following theorem shows that we have the associativity property for c-K-g-frame.

Theorem 3.4. *Let $(\Omega,\mu) = (\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2)$ be a product measure space and let $\Lambda = \{\Lambda_{\omega_1,\omega_2}\}_{(\omega_1,\omega_2)\in\Omega_1\times\Omega_2}$ be a Bessel family for the Hilbert C*-module $\mathcal{H}$ with respect to the family $\{\mathcal{H}_{\omega_1,\omega_2}\}$. Then:*

(1) *For each $\omega_1 \in \Omega_1$, the family $\{\Lambda_{\omega_1,\omega_2}\}_{\omega_2\in\Omega_2}$ is a Bessel family for $\mathcal{H}$ with respect to $\{\mathcal{H}_{\omega_1,\omega_2}\}_{\omega_2}$, and the associated analysis operator is given by*

$$\Lambda_{\omega_1} : \mathcal{H} \to \mathcal{H}_{\omega_1} := \left(\bigoplus_{\omega_2\in\Omega_2} \mathcal{H}_{\omega_1,\omega_2}, \mu_2 \right)_{L^2}, \quad f \mapsto (\Lambda_{\omega_1,\omega_2} f)_{\omega_2\in\Omega_2}.$$

(2) *Let $K \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$. Then the family Λ is a c-K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_{\omega_1,\omega_2}\}$ if and only if the family $\{\Lambda_{\omega_1}\}_{\omega_1\in\Omega_1}$ is a c-K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_{\omega_1}\}_{\omega_1\in\Omega_1}$.*

Proof. Since Λ is a Bessel family, there exists $B > 0$ such that for all $f \in \mathcal{H}$,

$$\int_{\Omega_1\times\Omega_2} \langle \Lambda_{\omega_1,\omega_2} f, \Lambda_{\omega_1,\omega_2} f\rangle \, d(\mu_1 \otimes \mu_2)(\omega_1,\omega_2) \le B\langle f, f\rangle.$$

Applying Fubini's theorem, we get

$$\int_{\Omega_1} \left(\int_{\Omega_2} \langle \Lambda_{\omega_1,\omega_2} f, \Lambda_{\omega_1,\omega_2} f\rangle \, d\mu_2(\omega_2) \right) d\mu_1(\omega_1) \le B\langle f, f\rangle.$$

Hence, for μ_1-almost every $\omega_1 \in \Omega_1$, the function $\omega_2 \mapsto \Lambda_{\omega_1,\omega_2} f$ belongs to $L^2(\Omega_2; \mathcal{H}_{\omega_1,\omega_2})$, and so the operator

$$\Lambda_{\omega_1} f := (\Lambda_{\omega_1,\omega_2} f)_{\omega_2}$$

defines a Bessel mapping from $\mathcal{H}$ into $\mathcal{H}_{\omega_1}$.

Now suppose that Λ is a c-K-g-frame, i.e., there exist constants $A, B > 0$ such that

$$A\langle K^*f, K^*f\rangle \leq \int_{\Omega_1\times\Omega_2} \langle \Lambda_{\omega_1,\omega_2} f, \Lambda_{\omega_1,\omega_2} f\rangle \, d(\mu_1\otimes\mu_2)(\omega_1,\omega_2) \leq B\langle f, f\rangle.$$

Using Fubini?s theorem again, we write:

$$\int_{\Omega_1}\left(\int_{\Omega_2} \langle \Lambda_{\omega_1,\omega_2} f, \Lambda_{\omega_1,\omega_2} f\rangle \, d\mu_2(\omega_2)\right) d\mu_1(\omega_1) = \int_{\Omega_1} \langle \Lambda_{\omega_1} f, \Lambda_{\omega_1} f\rangle \, d\mu_1(\omega_1).$$

Thus, the inequality becomes:

$$A\langle K^*f, K^*f\rangle \leq \int_{\Omega_1} \langle \Lambda_{\omega_1} f, \Lambda_{\omega_1} f\rangle \, d\mu_1(\omega_1) \leq B\langle f, f\rangle,$$

which shows that $\{\Lambda_{\omega_1}\}_{\omega_1\in\Omega_1}$ is a c-K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_{\omega_1}\}_{\omega_1}$.

Conversely, if $\{\Lambda_{\omega_1}\}_{\omega_1\in\Omega_1}$ is a c-K-g-frame, applying the same reasoning in reverse and using Fubini's theorem gives that Λ is a c-K-g-frame. □

A c-K-g-frame $(\Lambda_\omega)_{\omega\in\Omega}$ on Ω is called exact if, for every measurable subset $\Omega_1 \subseteq \Omega$ with $0 < \mu(\Omega_1) < +\infty$, the family $(\Lambda_\omega)_{\omega\in\Omega\setminus\Omega_1}$ is not a c-K-g-frame. The following result shows that a Bessel family that is not decreasing does not necessarily need to be exact.

Theorem 3.5. *Let $K \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$. Suppose that $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a continuous K-g-frame for $\mathcal{H}$ with respect to the family $\{\mathcal{H}_\omega\}_{\omega\in\Omega}$. Assume that $\{\Lambda_\omega\}_{\omega\in\Omega}$ is non-decreasing with respect to some order defined on Ω.*

$$\mu\left(\{\omega\in\Omega : \omega\leq\omega_0\}\right) < +\infty \quad \text{and} \quad \mu\left(\{\omega\in\Omega \,:\, \omega>\omega_0\}\right) \neq 0 \tag{3.3}$$

Then, the restricted family $\{\Lambda_\omega\}_{\omega\in\Omega\setminus\{\omega\in\Omega\,:\,\omega\leq\omega_0\}}$ is also a c-K-g-frame.

Proof. Since $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K-g-frame, there exist constants $A, B > 0$ such that

$$A\langle K^*f, K^*f\rangle \leq \int_{\Omega} \langle \Lambda_\omega f, \Lambda_\omega f\rangle \, d\mu(\omega) \leq B\langle f, f\rangle \quad \text{for all } f\in\mathcal{H}.$$

Denote $\Omega^- = \{\omega\in\Omega : \omega\leq\omega_0\}$, $\Omega^+ = \{\omega\in\Omega : \omega>\omega_0\}$ and $\Omega' = \Omega\setminus(\Omega^+\cup\Omega^-)$. Since the disjoint union $\Omega = \Omega^-\cup\Omega^+\cup\Omega'$ We estimate:

$$\int_{\Omega} \langle \Lambda_\omega f, \Lambda_\omega f\rangle \, d\mu(\omega) = \int_{\Omega^-} \langle \Lambda_\omega f, \Lambda_\omega f\rangle \, d\mu(\omega) + \int_{\Omega^+} \langle \Lambda_\omega f, \Lambda_\omega f\rangle \, d\mu(\omega) + \int_{\Omega'} \langle \Lambda_\omega f, \Lambda_\omega f\rangle \, d\mu(\omega).$$

Observe that if $\lambda = \sup\left\{1, \frac{\mu(\Omega^-)}{\mu(\Omega^+)}\right\}$, then

$$\begin{aligned} \lambda \int_{\Omega^+} \langle \Lambda_\omega f, \Lambda_\omega f\rangle \, d\mu(\omega) &\geq \mu(\Omega^-)\langle \Lambda_{\omega_0} f, \Lambda_{\omega_0} f\rangle \\ &\geq \int_{\Omega^-} \langle \Lambda_{\omega_0} f, \Lambda_{\omega_0} f\rangle \, d\mu(\omega) \\ &\geq \int_{\Omega^-} \langle \Lambda_\omega f, \Lambda_\omega f\rangle \, d\mu(\omega) \end{aligned}$$

for all $f\in\mathcal{H}$. Therefore,

$$\int_{\Omega\setminus\Omega^-} \langle \Lambda_\omega f, \Lambda_\omega f\rangle \, d\mu(\omega) \leq \int_{\Omega} \langle \Lambda_\omega f, \Lambda_\omega f\rangle \, d\mu(\omega) \leq (1+\lambda)\int_{\Omega\setminus\Omega^-} \langle \Lambda_\omega f, \Lambda_\omega f\rangle \, d\mu(\omega)$$

for all $f \in \mathcal{H}$. Thus,

$$\frac{A}{1+\lambda}\langle K^*f, K^*f\rangle \leq \int_{\Omega\setminus\Omega^-}\langle\Lambda_\omega f, \Lambda_\omega f\rangle\, d\mu(\omega) \leq B\langle f, f\rangle \quad \text{for all } f \in \mathcal{H}.$$

Therefore, $\{\Lambda_\omega\}_{\omega\in\Omega'}$ is also a c-K-g-frame. □

Example 3.6. Let $\mathcal{A}$ a C*-algebras and denote $\mathcal{Z}(\mathcal{A})$ its center. We denote by $H^2_{\mathcal{A}}(\mathbb{D})$ the $\mathcal{A}$-valued Hardy Hilbert C*-module over $\mathcal{A}$ defined by

$$H^2_{\mathcal{A}}(\mathbb{D}) := L^2(\partial\mathbb{D}, \mathcal{A}) \cap H(\mathbb{D}, \mathcal{A})$$

One can see that $f \in H^2_{\mathcal{A}}(\mathbb{D})$ if and only if there exist $(a_n)_{n\in\mathbb{N}_0} \in \ell^2(\mathcal{A})$ such that

$$f(z) = \sum_{n=0}^{\infty} z^n a_n \quad z \in \mathbb{D} \tag{3.4}$$

Now for let (Ω, μ) a measured space. Let $\phi := (\phi_n)_{n\in\mathbb{N}_0}$ a sequence of measurable functions $\phi_n : \Omega \to \mathbb{C}$, $n \in \mathbb{N}_0$, such that the function

$$M_\phi(f)(z) := \sum_{n=0}^{\infty} z^n \phi_n(\omega) a_n \quad z \in \mathbb{D}$$

belong to $H^2_{\mathcal{A}}(\mathbb{D})$ for all $f = \sum_{n=0}^{\infty} z^n a_n \in H^2_{\mathcal{A}}(\mathbb{D})$, and $\omega \in \Omega$. We define the operator Λ_ω on $H^2_{\mathcal{A}}(\mathbb{D})$ by

$$\Lambda_\omega(f)(z) = \sum_{n=0}^{\infty} z^n \phi_n(\omega) a_n \quad z \in \mathbb{D}$$

for all $f \in H^2_{\mathcal{A}}(\mathbb{D})$, with the expression (3.4). Note that $\Lambda_\omega(f)$ is the Hadamard product of ϕ and f. Thus,

$$\langle\Lambda_\omega(f), g\rangle = \sum_{n=0}^{\infty} \phi_n(\omega) a_n b_n^*,$$

for all $f, g \in H^2_{\mathcal{A}}(\mathbb{D})$. Hence

$$\langle\Lambda_\omega(f), \Lambda_\omega(f)\rangle = \sum_{n=0}^{\infty} |\phi_n(\omega)|^2 a_n a_n^*.$$

$$\int_\Omega \langle\Lambda_\omega(f), \Lambda_\omega(f)\rangle d\mu(\omega) = \sum_{n=0}^{\infty}\left(\int_\Omega |\phi_n(\omega)|^2 d\mu(\omega)\right) a_n a_n^*$$

Therefore, Λ is a Bessel c-g-frame if and only if $\left(\int_\Omega |\phi_n(\omega)|^2 d\mu(\omega)\right)_{n\in\mathbb{N}_0} \in \ell^\infty$.

(1) Let $\psi = \sum\limits_{n=1}^{\infty} \psi_n z^n \in H^2(\partial\mathbb{D})$, and K_ψ the operator defined on $\mathcal{H}$ by $K_\psi(f) = \sum\limits_{n=1}^{\infty} \psi_n z^n a^n$ for all $f = \sum\limits_{n=1}^{\infty} \psi_n a^n \in H^2_{\mathcal{A}}(\partial\mathbb{D})$. Thus $K_\psi \in End^*_{\mathcal{A}}(\mathcal{H})$ and

$$\langle K^*(f), K^*(f)\rangle = \sum_{n=0}^{\infty} |\psi_n|^2 a_n a_n^*.$$

Thus Λ is a c-K-g-frame if and only if there is a positive scalar B such that

$$B|\psi_n| \leq \left(\int_\Omega |\phi_n(\omega)|^2 d\mu(\omega)\right)^{\frac{1}{2}} \quad \text{for all } n \geq 0.$$

(2) Let $\Omega = (0,1)$ and μ the Lebesgue measure on Ω, $\phi_n(\omega) = \omega^n$ for all $n \geq 0$ and $\Lambda_\omega = M_\phi$. Then we have

$$\int_{\omega_0}^1 \langle \Lambda_\omega f, \Lambda_\omega f \rangle d\,\mu(\omega) = \sum_{n=0}^{\infty} \frac{1-\omega_0^{2n+1}}{2n+1} a_n a_n^*$$

for all $f = \sum_{n=0}^{\infty} z^n a_n$. Thus we have

$$(1-\omega_0) \sum_{n=0}^{\infty} \frac{1}{2n+1} a_n a_n^* \leq \int_{\omega_0}^1 \langle \Lambda_\omega f, \Lambda_\omega f \rangle d\,\mu(\omega)$$

for all $f = \sum_{n=0}^{\infty} z^n a_n$. This shows that

$$(1-\omega_0) \int_0^1 \langle \Lambda_\omega f, \Lambda_\omega f \rangle d\,\mu(\omega) \leq \int_{\omega_0}^1 \langle \Lambda_\omega f, \Lambda_\omega f \rangle d\,\mu(\omega) \leq \int_0^1 \langle \Lambda_\omega f, \Lambda_\omega f \rangle d\,\mu(\omega).$$

In fact in this case we can use Theorem 3.5 in order to prove that $\{\Lambda_\omega\}_{\omega \in (0,1)}$ is a c-K-g-frame for a given operator K is and only if $\{\Lambda_\omega\}_{\omega \in (\omega_0,1)}$ is a c-K-g-frame for a given $\omega_0 \in (0,1)$.

(3) $K = M_z$, we have $K(f) = \sum_{n=0}^{\infty} a_n z^{n+1}$, thus

$$\langle K(f), g \rangle := \sum_{n=0}^{\infty} a_n b_{n+1}^*.$$

Hence $K^*(f) = \sum_{n=0}^{\infty} a_{n+1} z^n$

$$\langle K^*(f), K^*(f) \rangle := \sum_{n=0}^{\infty} a_{n+1} a_{n+1}^*.$$

Therefore $\mathrm{Ran}(K^*) = ZH^2_{\mathcal{A}}(\mathbb{D})$. Thus Λ is a c-K-g-frame if and only if there is a positive scalar B such that

$$B \leq \left(\int_\Omega |\phi_n(\omega)|^2 d\mu(\omega)\right)^{\frac{1}{2}} \quad \text{for all } n \geq 1.$$

Theorem 3.7. *Let $K_1, K_2, ..., K_n, T_1, T_2, ..., T_n \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$, where n is a positive integer, and suppose that $\{\Lambda_\omega\}_{\omega \in \Omega}$ is a c-K_i-g-frame, for $i \in \{1, ..., n\}$. Then $\{\Lambda_\omega\}_{\omega \in \Omega}$ is a c-$\left(\sum_{i=1}^{n} K_i T_i\right)$-g-frame for $\mathcal{H}$.*

Proof. We will prove the theorem for $n = 2$, the general case can be deduced easily by induction. By assumption we know that there exist positive real constants A_1, A_2, B such that

$$A_i \left\langle K_i^* f, K_i^* f \right\rangle \leq \int_\Omega \langle \Lambda_\omega f, \Lambda_\omega f \rangle \mathrm{d}\bar{\ }(!) \leq \mathrm{B}\langle \mathrm{f}, \mathrm{f} \rangle \tag{3.5}$$

for all $f \in \mathcal{H}$ and $i \in \{1,2\}$. Observe that

$$\begin{aligned}\langle (K_1T_1+K_2T_2)^* f, (K_1T_1+K_2T_2)^* f\rangle &= \langle T_1^*K_1^*f + T_2^*K_2^*f, T_1^*K_1^*f + T_2^*K_2^*f\rangle \\ &= \langle T_1^*K_1^*f, T_1^*K_1^*f\rangle + \langle T_2^*K_2^*f, T_2^*K_2^*f\rangle \\ &\quad + \langle T_1^*K_1^*f, T_2^*K_2^*f\rangle + \langle T_2^*K_2^*f, T_1^*K_1^*f\rangle\end{aligned}$$

Using the fact that for all $(h,g) \in \mathcal{H}$ we have

$$\langle h,g\rangle + \langle g,h\rangle \leq \langle h,h\rangle + \langle g,g\rangle$$

This follows from the fact $0 \leq \langle h-g, h-g\rangle = \langle h,h\rangle + \langle g,g\rangle - \langle h,g\rangle - \langle g,h\rangle$. Thus, we obtain

$$\begin{aligned}\langle (K_1T_1+K_2T_2)^* f, (K_1T_1+K_2T_2)^* f\rangle &\leq 2\langle T_1^*K_1^*f, T_1^*K_1^*f\rangle + 2\langle T_2^*K_2^*f, T_2^*K_2^*f\rangle \\ &\leq 2\|T_1^*\|^2\langle K_1^*f, K_1^*f\rangle + 2\|T_2^*\|^2\langle K_2^*f, K_2^*f\rangle \\ &\leq 2\left(\frac{\|T_1^*\|^2}{A_1} + \frac{\|T_2^*\|^2}{A_2}\right)\int_\Omega \langle \Lambda_\omega f, \Lambda_\omega f\rangle \mathrm{d}^{-}(!)\end{aligned}$$

for any $f \in \mathcal{H}$. It follows that $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-$(K_1T_1+K_2T_2)$-g-frame. □

In the following, according two bounded adjointable operators K_1 and K_2 between Hilbert C^*-modules and a given c-K_1-g-frame, we construct some related c-K_2-g-frames.

Theorem 3.8. *Suppose that $K_1 \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_1)$ and $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K_1-g-frame for $\mathcal{H}_1$. Let $K_2 \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_2)$, $T \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_1, \mathcal{H}_2)$ has closed range and $TK_1 = K_2T$. If $\mathrm{Ran}\left(K_2^*\right) \subset \mathrm{Ran}(T)$, then $\{\Lambda_\omega T^*\}_{\omega\in\Omega}$ is a c-K_2-g-frame for $\mathcal{H}_2$.*

Proof. Since $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K_1-g-frame for $\mathcal{H}_1$, there exist positive constants A, B such that

$$A\langle K_1{}^*f, K_1{}^*f\rangle \leq \int_\Omega \langle \Lambda_\omega h, \Lambda_\omega h\rangle \mathrm{d}^{-}(!) \leq \mathrm{B}\langle \mathrm{h}, \mathrm{h}\rangle$$

for all $h \in \mathcal{H}_1$. Therefore

$$\begin{aligned}A\langle K_1^*T^*h, K_1^*T^*h\rangle &\leq \int_\Omega \langle \Lambda_\omega T^*h, \Lambda_\omega T^*h\rangle \mathrm{d}^{-}(!) \\ &\leq B\langle T^*h, T^*h\rangle \\ &\leq B\|T\|^2\langle h,h\rangle\end{aligned}$$

for all $h \in \mathcal{H}_2$. Since $TK_1 = K_2T, \mathrm{Ran}\left(K_2^*\right) \subseteq \mathrm{Ran}(T)$ and $\mathrm{Ran}(T)$ is closed. The Pseudo-inverse of the operator $T^\dagger$ of T exists and we have $(T^\dagger)^*T^*K_2^* = K_2^*$. This follows from the fact that for each $h \in \mathcal{H}_2$, $K_2^*h = Tf$ for some $f \in \mathcal{H}_1$. Thus

$$(T^\dagger)^*T^*K_2^*h = (TT^\dagger)^*Tf = TT^\dagger Tf = Tf = K^*h.$$

$$\begin{aligned}\left\|T^\dagger\right\|^{-2}\langle K_2^*h, K_2^*h\rangle &= \left\|(T^\dagger)^*\right\|^{-2}\langle (T^\dagger)^*T^*K_2^*h, (T^\dagger)^*T^*K_2^*h\rangle \\ &\leq \langle T^*K_2^*h, T^*K_2^*h\rangle \\ &= \langle K_1^*T^*h, K_1^*T^*h\rangle\end{aligned}$$

This implies that

$$A\left\|T^\dagger\right\|^{-2}\langle K_2{}^*h, K_2{}^*h\rangle \leq \int_\Omega \langle \Lambda_\omega T^*h, \Lambda_\omega T^*h\rangle \mathrm{d}^{-}(!) \leq \mathrm{B}\|\mathrm{T}\|^2\langle \mathrm{h}, \mathrm{h}\rangle.$$

We conclude that $\{\Lambda_\omega T^*\}_{\omega\in\Omega}$ is a c-K_2-g-frame for $\mathcal{H}_2$. □

As a simple consequence of the previous theorem, we can state

Corollary 3.9. *Let* $K \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$ *and* $\{\Lambda_\omega\}_{\omega\in\Omega}$ *be a c-K-g-frame on the Hilbert* C^**-module* $\mathcal{H}$*. Let* $T \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$ *with closed range and* $TK = KT$*. If* $Ran(K^*) \subseteq Ran(T)$*, then* $\{\Lambda_\omega T^*\}_{\omega\in\Omega}$ *is a c-K-g-frame on* $\mathcal{H}$*.*

Recall that an operator $T \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_1, \mathcal{H}_2)$ is bounded below if there exists a constant $C > 0$ such that $T^*T \geq C\,\mathrm{Id}_{\mathcal{H}_1}$ i.e., $\langle Th, Th\rangle \geq C\langle h, h\rangle$, for all $h \in \mathcal{H}_1$.

Theorem 3.10. *Let* $K_1 \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_1)$*, and* $\{\Lambda_\omega\}_{\omega\in\Omega}$ *be a tight c-*K_1*-g-frame on* $\mathcal{H}_1$*. Let* $K_2 \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_2)$ *be such that* K^* *is surjective, and* $T \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_1, \mathcal{H}_2)$ *such that* $\mathrm{Ran}(T)$ *is closed and* $TK_1 = K_2T$*. Then* $\{\Lambda_\omega T^*\}_{\omega\in\Omega}$ *is a c-*K_2*-g-frame on* $\mathcal{H}_2$ *if and only if T is surjective.*

Proof. If T is surjective, then $\mathrm{Ran}(K_2^*) \subseteq \mathrm{Ran}(T)$ and by Theorem 3.8 $\{\Lambda_\omega T^*\}_{\omega\in\Omega}$ is a c-K_2-g-frame on $\mathcal{H}_2$. Conversely, let us assume that $\{\Lambda_\omega T^*\}_{\omega\in\Omega}$ is a c-K_2-g-frame on $\mathcal{H}_2$ with lower bound A_1, then for each $h \in \mathcal{H}_2$ we have

$$A_1\langle K_2^*h, K_2^*h\rangle \leq \int_\Omega \langle \Lambda_\omega T^*h, \Lambda_\omega T^*h\rangle \, d\mu(\omega)$$

Since $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a tight c-K_1-g-frame for $\mathcal{H}_1$ with bound A, then

$$A\langle K_1^*f, K_1^*f\rangle = \int_\Omega \langle \Lambda_\omega f, \Lambda_\omega f\rangle \, d\mu(\omega), \quad f \in \mathcal{H}_1$$

Moreover since $K_1^*T^* = T^*K_2^*$, for any $h \in \mathcal{H}_2$ we have

$$\begin{aligned} A\langle T^*K_2^*h, T^*K_2^*h\rangle &= A\langle K_1^*T^*h, K_1^*T^*h\rangle \\ &= \int_\Omega \langle \Lambda_\omega T^*h, \Lambda_\omega T^*h\rangle \, d\mu(\omega) \\ &\geq A_1\langle K_2^*h, K_2^*h\rangle \end{aligned}$$

Therefore,

$$\langle T^*K_2^*h, T^*K_2^*h\rangle \geq \frac{A_1}{A}\langle K_2^*h, K_2^*h\rangle, \quad h \in \mathcal{H}_2$$

Since K_2^* is surjective, the above implies that $\frac{A_1}{A}\mathrm{Id}_{\mathcal{H}_2} \leq TT^*$ thus by Theorem [32] T is surjective. □

4. On Continuous K-g-Duals

In this section, it is introduced the c-K-g-dual frame of a given c-K-g-frame and we obtain c-K-g-dual of Parseval c-K-g-frames. Then we obtain synthesis operator related to c-K-g-duals. We first recall the definition of c-K-frame dual.

Definition 4.1. Let $K \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$ and $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_\omega\}_{\omega\in\Omega}$. A dual c-$K$-g-frame of $\{\Lambda_\omega\}_{\omega\in\Omega}$ for $\mathcal{H}$ with respect to $\{\mathcal{H}_\omega\}_{\omega\in\Omega}$ is a c-g-Bessel family $\{\Gamma_\omega\}_{\omega\in\Omega}$ for $\mathcal{H}$ with respect to $\{\mathcal{H}_\omega\}_{\omega\in\Omega}$, that satisfies

$$\langle Kf, h\rangle = \int_\Omega \langle \Lambda_\omega^*\Gamma_\omega f, h\rangle \, d\mu(\omega) \quad \text{for all } f, h \in \mathcal{H} \tag{4.1}$$

Theorem 4.2. *Let $K \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$ and $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ a c-K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_\omega\}_{\omega\in\Omega}$ such that its operator frame S_Λ has a closed rang. Then $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ possesses a dual c-K-g-frame $\{\Gamma_\omega\}_{\omega\in\Omega}$. Furthermore, $\{\Gamma_\omega\}_{\omega\in\Omega}$ is a c-K^*-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_\omega\}_{\omega\in\Omega}$ and possesses $\{\Lambda_\omega\}_{\omega\in\Omega}$ as a dual c-K^*-g-frame.*

Proof. There exists a positive real scalar A such that $AKK^* \leq S_\Lambda = T_\Lambda T^*_\Lambda$, where T_Λ is the synthesis operator of Λ. Thus by Theorem 2.1 there exists an operator

$$Q \in End^*_{\mathcal{A}}\left(\mathcal{H}, \left(\bigoplus_{\omega\in\omega} H_\omega, \mu\right)_{L^2}\right); \quad f \mapsto (\Gamma_\omega(f))_{\omega\in\Omega}$$

such that $K = T_\Lambda Q$. Observe that for every $f \in \mathcal{H}$ we have

$$\begin{aligned}\int_\Omega \langle \Gamma_\omega f, \Gamma_\omega f\rangle \, d\mu(\omega) &= \int_\Omega \langle \Gamma^*_\omega \Gamma_\omega f, f\rangle \, d\mu(\omega)\\ &\leq \int_\Omega \|\Gamma_\omega\|^2 \langle f, f\rangle \, d\mu(\omega)\\ &= \left(\int_\Omega \|\Gamma_\omega\|^2 \, d\mu(\omega)\right)\langle f, f\rangle.\end{aligned}$$

Thus $(\Gamma_\omega)_{\omega\in\Omega}$ is a Bessel g-frame where over $\mathcal{H}$. Furthermore,

$$\begin{aligned}\langle Kf, g\rangle &= \langle T_\Lambda Q f, g\rangle\\ &= \langle T_\Lambda \left(\Gamma_\omega(f)\right)_{\omega\in\Omega}, g\rangle\\ &= \langle \int_\Omega \Lambda^*_\omega \Gamma_\omega(f) \mathrm{d}^-(!), \mathrm{g}\rangle\\ &= \int_\Omega \langle \Lambda^*_\omega \Gamma_\omega(f), g\rangle \mathrm{d}^-(!)\end{aligned}$$

Thus $\{\Gamma_\omega\}_{\omega\in\Omega}$ is a dual c-K-g-frame of $\{\Lambda_\omega\}_{\omega\in\Omega}$. Now, from the equality $K = T_\Lambda Q$ and $T^*_\Lambda T_\Lambda \leq \|T^*_\Lambda T_\Lambda\| I$ we obtain that $K^*K = Q^* T^*_\Lambda T Q \leq \|T^*_\Lambda T_\Lambda\| Q^* Q$. Hence for every $f \in \mathcal{H}$, we have

$$\begin{aligned}\langle Kf, Kf\rangle &= \langle K^*Kf, f\rangle\\ &\leq \|T^*_\Lambda T_\Lambda\| \langle Q^*Qf, f\rangle\\ &= \|T^*_\Lambda T_\Lambda\| \int_\Omega \langle \Gamma^*_\omega \Gamma_\omega(f), f\rangle \mathrm{d}^-(!)\\ &= \|T^*_\Lambda T_\Lambda\| \int_\Omega \langle \Gamma_\omega(f), \Gamma_\omega(f)\rangle \mathrm{d}^-(!)\end{aligned}$$

Theretofore $\{\Gamma_\omega\}_{\omega\in\Omega}$ is a c-K^*-g-frame.

□

Theorem 4.3. *Let $K \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$ such that K^* is surjective and $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_\omega\}_{\omega\in\Omega}$. Denote its synthesis operator by T_Λ, then the synthesis operator T_Γ related to $\Gamma = \left\{\Lambda_\omega (K^*)^\dagger\right\}_{\omega\in\Omega}$ is the bounded adjointable $\mathcal{A}$-linear operator $T_\Gamma = K^\dagger T$. Furthermore, if $\{\Lambda_\omega\}_{\omega\in\Omega}$ is Parseval c-K-g-frame for $\mathcal{H}$, then $\left\{\Lambda_\omega \left(K^*\right)^\dagger\right\}_{\omega\in\Omega}$ is a c-K-g-dual of $\{\Lambda_\omega\}_{\omega\in\Omega}$ and it is both a Parseval c-K^*K-g-frame and Parseval c-g-frame.*

Proof. Given that Γ is a c-K^*-g-frame for $\mathcal{H}$, it follows that for every $F \in \left(\bigoplus_{\omega\in\Omega} \mathcal{H}_\omega, \mu\right)_{L^2}$ and $g \in \mathcal{H}$ we have

$$\begin{aligned}\langle T_\Gamma F, g\rangle &= \int_\Omega \left\langle \left(\Lambda_\omega \left(K^\dagger\right)^*\right)^* F(\omega), g\right\rangle d\mu(\omega) \\ &= \int_\Omega \left\langle K^\dagger \Lambda_\omega{}^* F(\omega), g\right\rangle d\mu(\omega) \\ &= \left\langle K^\dagger T_\Lambda F, g\right\rangle\end{aligned}$$

Therefore $T_\Gamma = K^\dagger T_\Lambda$ is a bounded adjointable adjointable $\mathcal{A}$-linear operator.

To prove the rest of the theorem, assume that $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a Parseval c-K-g-frame. Since K^* is surjective, we have

$$\langle Kf, g\rangle = \langle KK^*\left((K^*)^\dagger f\right), g\rangle = \int_\Omega \langle \Lambda_\omega^* \Lambda_\omega (K^*)^\dagger f, g\rangle d\mu(\omega)$$

for all $f, g \in \mathcal{H}$. Therefore $\left\{\Lambda_\omega\left(K^*\right)^\dagger\right\}_{\omega\in\Omega}$ is a c-K-g-dual of $\{\Lambda_\omega\}_{\omega\in\Omega}$. Now, from the fact that $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K-g-frame , we deduce that

$$\begin{aligned}\int_\Omega \langle \Lambda_\omega\left((K^*)^\dagger f\right), \Lambda_\omega\left((K^*)^\dagger f\right)\rangle \mathrm{d}^-(!) &= \langle K^*(K^*)^\dagger f, K^*(K^*)^\dagger f\rangle \\ &= \langle \left(K^\dagger K\right)^* f, \left(K^\dagger K\right)^* f\rangle\end{aligned} \tag{4.2}$$

for all $f \in \mathcal{H}$. Thus $\left\{\Lambda_\omega\left(K^\dagger\right)^*\right\}_{\omega\in\Omega}$ is a Parseval c-$K^\dagger K$-g-frame for $\mathcal{H}$. To complete the proof observe that the surjectivity of K^* implies that $\left(K^\dagger K\right)^* = K^*(K^*)^\dagger f\rangle$ is the identity operator on $\mathcal{H}$, thus the formula (4.2) tells us that $\left\{\Lambda_\omega\left(K^*\right)^\dagger\right\}_{\omega\in\Omega}$ is a Parseval c-g-frame. □

For the following result, we need to keep in mind that a c-g-Bessel sequence $\{\Lambda_\omega\}_{\omega\in\Omega}$ in $\mathcal{H}$ is said to be independent if, for any c-g-Bessel sequence $\{\Gamma_\omega\}_{\omega\in\Omega}$ in $\mathcal{H}$ we have

$$\int_\Omega \Lambda_\omega{}^* \Gamma_\omega f d\mu(\omega) = 0 \quad \forall f \in \mathcal{H} \implies \Gamma_\omega = 0 \quad \text{a.e.} \tag{4.3}$$

Theorem 4.4. *Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a Parseval c-K-g-frame for $\mathcal{H}$, where $K \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$ such that its adjoint K^* is surjective. Then $\{\Lambda_\omega\}_{\omega\in\Omega}$ is independent if and only if its c-K-g-dual $\left\{\Lambda_\omega(K^*)^\dagger\right\}_{\omega\in\omega}$ is independent.*

Proof. By hypothesis $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a Parseval c-K-g-frame for $\mathcal{H}$, then we have

$$\int_\Omega KK^* f d\mu(\omega) = \int_\Omega \Lambda_\omega{}^* \Lambda_\omega f d\mu(\omega)$$

for all $f \in \mathcal{H}$. Since K^* is surjective, its pseudo-inverse $(K^*)^\dagger$ is well defined on $\mathcal{H}$ and $KK^*(K^*)^\dagger K^* = KK^*$. Thus

$$\int_\Omega \Lambda_\omega{}^* \Lambda_\omega f d\mu(\omega) = \int_\Omega \Lambda_\omega^* \Lambda_\omega (K^*)^\dagger K^* f d\mu(\omega)$$

for all $f \in \mathcal{H}$. Therefore

$$\int_\Omega \Lambda_\omega^* \left(\Lambda_\omega - \Lambda_\omega (K^*)^\dagger K^*\right) f d\mu(\omega) = 0,$$

for all $f \in \mathcal{H}$. Thus, if $\{\Lambda_\omega\}_{\omega\in\Omega}$ is independent, then by (4.3) $\Lambda_\omega - \Lambda_\omega(K^*)^\dagger K^*$ a.e. Thus we conclude that

$$\Lambda_\omega{}^* = KK^\dagger \Lambda_\omega^* \quad \text{a.e.}$$

Now let us assume that for every $f \in \mathcal{H}$, $\int_\Omega \left(\Lambda_\omega (K^*)^\dagger\right)^* \Gamma_\omega f d\mu(\omega) = 0$, for a given c-g-Bessel sequence $\{\Gamma_\omega\}_{\omega\in\Omega}$. Then, one can check that

$$\begin{aligned}\int_\Omega \Lambda_\omega{}^* \Gamma_\omega f d\mu(\omega) &= \int_\Omega KK^\dagger \Lambda_\omega{}^* \Gamma_\omega f d\mu(\omega)\\ &= K\int_\Omega K^\dagger \Lambda_\omega{}^* \Gamma_\omega f d\mu(\omega) = 0\end{aligned}$$

So by assumption, $\Gamma_\omega = 0$, a.e.

For the reverse implication, let us assume that $\left\{\Lambda_\omega \left(K^\dagger\right)^*\right\}_{\omega\in\Omega}$ is independent. Suppose that for a c-g-Bessel sequence $\{\Gamma_\omega\}_{\omega\in\Omega}$, we have $\int_\Omega \Lambda_\omega{}^* \Gamma_\omega f d\mu(\omega) = 0$, for all $f \in \mathcal{H}$. Then

$$\int_\Omega \left(\Lambda_\omega (K^*)^\dagger\right)^* \Gamma_\omega f d\mu(\omega) = K^\dagger \int_\Omega \Lambda_\omega{}^* \Gamma_\omega f d\mu(\omega) = 0$$

this implies that $\Gamma_\omega = 0$, a.e. So $\{\Lambda_\omega\}_{\omega\in\Omega}$ is independent. □

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