International Journal of Analysis and Applications
ISSN 2291-8639
Volume 7, Number 2 (2015), 104-128
http://www.etamaths.com

A SPECTRAL ANALYSIS OF LINEAR OPERATOR PENCILS ON BANACH SPACES WITH APPLICATION TO QUOTIENT OF BOUNDED OPERATORS

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Abstract. Let $\mathcal{X}$ and $\mathcal{Y}$ two complex Banach spaces and (A, B) a pair of bounded linear operators acting on $\mathcal{X}$ with value on $\mathcal{Y}$. This paper is concerned with spectral analysis of the pair (A, B). We establish some properties concerning the spectrum of the linear operator pencils $A - \lambda B$ when B is not necessarily invertible and $\lambda \in \mathbb{C}$. Also, we use the functional calculus for the pair (A, B) to prove the corresponding spectral mapping theorem for (A, B). In addition, we define the generalized Kato essential spectrum and the closed range spectra of the pair (A, B) and we give some relationships between this spectrums. As application, we describe a spectral analysis of quotient operators.

1. Introduction

Let $\mathcal{L}(X, Y)$ be the Banach algebra of all bounded linear operators from one complex Banach space X to another Y. If $X = Y$, then $\mathcal{L}(X, X) = \mathcal{L}(X)$. For $A \in \mathcal{L}(X, Y)$ we denote by $R(A)$ its range, $N(A)$ its null space and $\sigma(A)$ its spectrum. If $A \in \mathcal{L}(X)$, we denote by $\varrho(A)$ the resolvent set of A. Let I_X (respectively I_Y) denotes the identity operator in X (respectively in Y). Recall that an operator $A \in \mathcal{L}(X)$ is called nilpotent if $A^p = 0$ for some $p \in \mathbb{N}^*$ and A is said to be quasi-nilpotent if $\sigma(A) = \{0\}$. For a set M, let ∂M, $\overline{M}$ denote the boundary and the closure of M, respectively. Let $A - \lambda B$ be a linear operator pencil, where A and B are in $\mathcal{L}(X, Y)$ and $\lambda \in \mathbb{C}$. The operator B is not considered injective or surjective.

For the study of spectral properties of the quotient operators

$$A/B : Bx \longrightarrow Ax, \text{defined by } A/B(Bx) = Ax, \text{ where } N(B) \subset N(A),$$

we need to consider the spectrum of the operator pencil $A - \lambda B$ where $\lambda \in \mathbb{C}$. Furthermore, many authors consider the generalized eigenvalue problems $Ax = \lambda Bx$ and discussed the spectra of quadratic operator pencils, see [2, 13, 22]. Note that, in the finite dimensional case the generalized eigenvalue problems is one of the basic problems in the control theory of linear systems with finite dimensional state space. The solution of this problem is well-known as Rosenbrok's theorem [29]. However, in the infinite dimensional case, a complete description of the spectra of operators $A - \lambda B$ is known when the pair (A, B) is exactly controllable, that is the matrix operator $[B, AB, ..., A^{p-1}B] \in \mathcal{L}(X^p, Y)$ is right invertible for some integer p. If

2010 *Mathematics Subject Classification.* 47A05, 47B33, 47A10.

Key words and phrases. Operator pencils; Functional calculus; Spectral mapping theorem; Browder spectrum; Generalized Kato type spectrum; Quotient of bounded operators.

B is self-adjoint, positive and invertible then the eigenvalue problem $Ax = \lambda Bx$ is equivalent to $B^{-1}Ax = \lambda x$ or to $B^{-1/2}AB^{-1/2}y = \lambda y$ with $y = B^{1/2}x$, and the problem is also equivalent to a standard one for a self-adjoint operator where the spectrum is real. Thus, the interesting case is when both A and B are not sign-definite, the pencil spectrum can be non-real. In particular, if neither A nor B is invertible, then the problem poses major difficulties. Typical problems include: characterization of the spectrum of $A - \lambda B$, localization of non-real eigenvalues, asymptotic of real eigenvalues, dependence on parameters and often the use of complex analysis. Similar problems, as well as some other related questions, have been studied in a variety of situations in mathematical literature, see [9, 16, 32]. In physical literature, our problem appears in the study of electron waveguides in graphene, see [18, 31] and many references there.

The objective of this paper is to investigate the spectrum of linear operator pencils of type $A - \lambda B$. Our work generalizes initially some results of [28] to the case of operators defined on a Banach space X with values in another Banach space Y which is not necessarily equal to X. Thereafter, we extend our study to some different essential spectra. We state some basic results for linear operator pencils with non-empty resolvent set. We present particularly a simple demonstration than that obtained by Ditkin in [14] on the spectrum of $(A - \lambda B)$ when B is assumed to be compact. The originality of our technique allows us to operate a functional calculus on linear operator pencils. We also got a spectral characterization on quotient operators through that we have established on linear operator pencils. The obtained results bring quite information for the investigation of joint spectra and in particular the spectra of quotients operators. The present work is organized as follow: After the second section where several basic definitions and facts will be recalled, in section 3, we study some basic spectral properties for linear operator pencils. The fourth section is consecrated to the functional calculus of a pair of bounded operators. In section five, we investigate the isolated points of the spectrum of a pair of bounded linear operators. We define various essential spectra of linear operator pencils on a Banach space. We define the generalized Kato essential spectrum of a pair of bounded operators, and we also give some relationships between this spectrums and the closed range spectra. The obtained results are finally used in the last section to describe a spectral analysis of quotient operators.

2. Preliminaries

We begin this section by the following definitions.

Definition 2.1. For a pair (A, B) of operators in $\mathcal{L}(X, Y)$, the spectrum $\sigma(A, B)$ of the linear operator pencil $(A - \lambda B)$, or of the pair (A, B), is defined by:

$$\begin{aligned} \sigma(A, B) &= \{\lambda \in \mathbb{C} \text{ such that } (A - \lambda B) \text{ is not invertible } \} \\ &= \{\lambda \in \mathbb{C} \text{ such that } 0 \in \sigma(A - \lambda B)\}. \end{aligned}$$

The resolvent set $\varrho(A, B)$ of the pair (A, B) is the complement of the set $\sigma(A, B)$ in $\mathbb{C}$.

(2.1)

$$\varrho(A, B) = \mathbb{C}\backslash\sigma(A, B) = \left\{\lambda \in \mathbb{C} \text{ such that } R_\lambda(A, B) = (A - \lambda B)^{-1} \text{ exists in } \mathcal{L}(X, Y)\right\}$$

$R_\lambda(A, B)$ is called the resolvent of $(A - \lambda B)$. So

$$\sigma(A, B) = \{\lambda \in \mathbb{C} \text{ such that } N(A - \lambda B) \neq \{0\} \text{ or } R(A - \lambda B) \neq Y\}.$$

Thus, the spectrum $\sigma(A,B)$ of $(A-\lambda B)$ is the set of all scalars λ in $\mathbb{C}$ for which the operator $(A-\lambda B)$ fails to be an invertible element of the Banach algebra $\mathcal{L}(X,Y)$. From [33, Theorem 3.2.4], $\sigma(A-\lambda B)$ can be an unbounded set. Besides that the spectrum $\sigma(A,B)$ can be an empty set. According to the nature of such a failure, $\sigma(A,B)$ can be split into many disjoint parts. A classical partition comprises three parts. The point spectrum of (A,B) defined by

$$\sigma_p(A,B) = \{\lambda \in \mathbb{C} : \, (A-\lambda B) \text{ is not injective}\}\,. \tag{2.2}$$

A complex number $\lambda \in \mathbb{C}$ is an eigenvalue of $(A-\lambda B)$ if there exists a nonzero vector x in X such that $Ax = \lambda Bx$, then $N(A-\lambda B) \neq \{0\}$. The algebraic multiplicity of an eigenvalue λ is the dimension of the respective eigenspace $N(A-\lambda B)$. The second parts is the set $\sigma_c(A,B)$ of those λ for which $(A-\lambda B)$ has a densely defined but unbounded inverse on its range;

$$\sigma_c(A,B) = \left\{\lambda \in \mathbb{C} : N(A-\lambda B) = \{0\}\,, \overline{R(A-\lambda B)} = Y \text{ and } R(A-\lambda B) \neq Y\right\} \tag{2.3}$$

which is referred to as the continuous spectrum of $(A-\lambda B)$. The third parts is the residual spectrum of $(A-\lambda B)$ is the set $\sigma_r(A,B)$ of all scalars λ such that $(A-\lambda B)$ has an inverse on its range that is not densely defined;

$$\sigma_r(A,B) = \left\{\lambda \in \mathbb{C} : \, N(A-\lambda B) = \{0\} \text{ and } \overline{R(A-\lambda B)} \subsetneq Y\right\}. \tag{2.4}$$

The collection $\{\sigma_p(A,B), \sigma_c(A,B), \sigma_r(A,B)\}$ forms a partition of $\sigma(A,B)$, which means that they are pairwise disjoint and $\sigma(A,B) = \sigma_p(A,B) \cup \sigma_c(A,B) \cup \sigma_r(A,B)$.

Remark 2.2. 1) If $X = Y$ and $B = I_X$, the spectrum of the linear operator pencil $A-\lambda I_X$ is the spectrum of A, ie $\sigma(A, I_X) = \sigma(A)$. $\varrho(A, I_X) = \varrho(A)$ and $R_\lambda(A, I_X) = R_\lambda(A) = (A-\lambda I_X)^{-1}$ if $\lambda \in \varrho(A)$.
2) If $X = Y$ is a finite dimensional vector space, $\dim X < \infty$, the spectrum $\sigma(A-\lambda B)$ coincides with the complex plane or it contains no more than n points.

Example 2.3. Let $X = Y = L^2([0,1])$ and define the multiplication operators A and B in $L^2([0,1])$ by $Af(x) = (x+1)f(x)$ and $Bf(x) = xf(x)$. Then A and B are bounded with $\|A\| = 2$, $\|B\| = 1$. If $(A-\lambda B)f(x) = [(1-\lambda)x+1]\,f(x) = 0$, then $f = 0$ in $L^2([0,1])$ when $\lambda \in \mathbb{C}\backslash\{1\}$. Thus, $(A-\lambda B)$ has no eigenvalues if $\lambda \in \mathbb{C}\backslash\{1\}$. However, if $\lambda = 1$, $(A-B)f(x) = f(x)$, thus $1 \in \sigma_p(A,B)$. Consequently, $\sigma_p(A,B) = \{1\}$.
If $x_\lambda = \frac{1}{\lambda-1}$ or else $\lambda \in \mathbb{C}\backslash\,[2,+\infty[$, then $[(1-\lambda)x+1]^{-1}\,f(x) \in L^2([0,1])$ for any $f \in L^2([0,1])$ because $[(1-\lambda)x+1]^{-1}$ is bounded on $[0,1]$. Thus, $\mathbb{C}\backslash\,(\{1\} \cup [2,+\infty[)$ is in $\varrho(A,B)$. If $\lambda \in [2,+\infty[$, then $(A-\lambda B)$ is not onto, because $c\,[(1-\lambda)x+1]^{-1} \notin L^2([0,1])$ for $c \neq 0$, so the nonzero constant functions c do not belong to the range of $(A-\lambda B)$. However, The range of $(A-\lambda B)$ is dense. Indeed, for any $f \in L^2([0,1])$, let

$$f_n(x) = \begin{cases} f(x) & \text{if } \ |x-x_\lambda| \geq \frac{1}{n} \\ 0 \ \text{ if } & |x-x_\lambda| < \frac{1}{n}. \end{cases}$$

Then, $\lim\limits_{n\to+\infty} f_n = f$ in $L^2([0,1])$ and $f_n \in R(A-\lambda B)$, since $[(1-\lambda)x+1]^{-1}\,f_n \in L^2([0,1])$, then it follows that $\sigma_p(A,B) = \{1\}$, $\sigma_c(A,B) = [2,+\infty[$, $\sigma_r(A,B) = \emptyset$ and $\sigma(A,B) = \{1\} \cup [2,+\infty[$.

Remark 2.4. If X^* and Y^* are respectively the dual spaces of X and Y, and $A^*, B^* : Y^* \longrightarrow X^*$ are the adjoint of A and B respectively, then $\sigma(A,B) = \sigma(A^*, B^*)$.

The spectra and sub-spectra of the pair (A,B) and its adjoint (A^*, B^*) are related by the following relations:

Theorem 2.5. *Let (A,B) a pair of operators in $\mathcal{L}(X,Y)$, then the following hold:*
(1) $\sigma_r(A,B) \subset \sigma_p(A^*,B^*) \subset \sigma_r(A,B) \cup \sigma_p(A,B)$.
(2) $\sigma_p(A,B) \subset \sigma_r(A^*,B^*) \cup \sigma_p(A^*,B^*)$.
(3) $\sigma_c(A,B) \subset \sigma_r(A^*,B^*) \cup \sigma_c(A^*,B^*)$.
(4) $\sigma_c(A^*,B^*) \subset \sigma_c(A,B)$.
(5) $\sigma_r(A^*,B^*) \subset \sigma_p(A,B) \cup \sigma_c(A,B)$.

Proof. (1) Let $\lambda \in \sigma_r(A,B)$, then $\overline{R(A-\lambda B)}$ is not dense in Y. By the Hahn–Banach theorem, there exists a non-zero $y^* \in Y^*$ that vanishes on $R(A-\lambda B)$. Thus, for all $x \in X$,

$$\langle (A-\lambda B)x, y^* \rangle = \langle x, (A^* - \lambda B^*)y^* \rangle = 0.$$

Therefore $(A^* - \lambda B^*)y^* = 0$ and $\lambda \in \sigma_p(A^*, B^*)$. Next suppose that

$$(A^* - \lambda B^*)z^* = 0 \text{ where } z^* \neq 0,$$

that is $\langle x, (A^* - \lambda B^*)z^* \rangle = \langle (A-\lambda B)x, z^* \rangle = 0$ for all $x \in X$. If $R(A-\lambda B)$ is dense, then z^* must be the zero functional, which is a contradiction. The claim is proved. In particular, when X and Y are reflexive Banach spaces, we have $\sigma_r(A^*,B^*) \subset \sigma_p(A^{**},B^{**}) = \sigma_p(A,B)$. One shows (2) to (5) by the same argument. ∎

There are some overlapping parts of the spectrum of linear operator pencils which are commonly used. For instance, the compression spectrum $\sigma_{CP}(A,B)$ and the approximate point spectrum $\sigma_{AP}(A,B)$, which are defined respectively by:

$$\begin{aligned} \sigma_{CP}(A,B) &= \{\lambda \in \mathbb{C} :\ R(A-\lambda B) \text{ is not dense in } Y\} \\ \sigma_{AP}(A,B) &= \{\lambda \in \mathbb{C} :\ (A-\lambda B) \text{ is not bounded below}\}. \end{aligned} \tag{2.5}$$

Let (A,B) a pair of operators in $\mathcal{L}(X,Y)$, we list below some classical results concerning $\sigma_{CP}(A,B)$ and $\sigma_{AP}(A,B)$.

Theorem 2.6. *(1) The following assertions are pairwise equivalent.*
(i) For every $\varepsilon > 0$, there is a unit vector $x_\varepsilon \in X$ such that $\|(A-\lambda B)x_\varepsilon\|_Y < \varepsilon$.
(ii) There is a sequence $(x_n)_{n\in\mathbb{N}}$ of unit vectors in X such that $\lim\limits_{n\to+\infty} \|(A-\lambda B)x_n\|_Y = 0$.
(iii) $\lambda \in \sigma_{AP}(A,B)$.
(2) The approximate point spectrum $\sigma_{AP}(A,B)$ is a closed subset of $\mathbb{C}$ and that includes the boundary $\partial\sigma(A,B)$ of the spectrum $\sigma(A,B)$.
(3) If X and Y are reflexive Banach spaces, we have

$$\sigma_{CP}(A,B) = \sigma_p(A^*,B^*) \text{ and } \sigma_r(A,B) = \sigma_{CP}(A,B) \backslash \sigma_p(A,B).$$

Proof. (1) Clearly (i) implies (ii). If (ii) holds, then there is no constant $\delta > 0$ such that $\delta = \delta \|x_n\|_X \leq \|(A-\lambda B)x_n\|_Y$ for all $n \in \mathbb{N}$. Thus, $(A-\lambda B)$ is not bounded below, and so (ii) implies (iii). Conversely, if $(A-\lambda B)$ is not bounded below, then there is no constant $\delta > 0$ such that $\delta \|x\|_X \leq \|(A-\lambda B)x\|_Y$ for all $x \in X$ or, equivalently, for every $\varepsilon > 0$ there exists a nonzero t_ε in X such that

$\|(A-\lambda B)t_\varepsilon\|_Y \leq \varepsilon \|t_\varepsilon\|_X$. Set $x_\varepsilon = \frac{t_\varepsilon}{\|t_\varepsilon\|_X}$, hence (iii) implies (ii).
(2) The quantity $j(A-\lambda B) = \inf\limits_{\|x\|_X} \|(A-\lambda B)x\|_Y$ is called the injective modulus of the pair (A,B) at λ, and obviously by virtue of 1) we have $j(A-\lambda B)=0$ if and only if $\lambda \in \sigma_{AP}(A,B)$. Moreover, it is easy to show that

$$\text{(2.6)} \qquad \begin{cases} |j(A-\lambda B) - j(A-\mu B)| \leq |\lambda - \mu| \, \|B\| \\ j(A-\lambda B) = \|R_\lambda(A,B)\|^{-1} \ \text{; for all } \lambda \in \varrho(A,B). \end{cases}$$

Since the function $j(A-\lambda B)$ is continuous at λ and $\sigma_{AP}(A,B)$ is the inverse image by j of 0, it follows that $\sigma_{AP}(A,B)$ is closed. Now, let $\lambda \in \varrho(A,B)$, then $j(A-\lambda B) = \|R_\lambda(A,B)\|^{-1} > 0$ and $\lambda \notin \sigma_{AP}(A,B)$. Hence $\sigma_{AP}(A,B) \subset \sigma(A,B)$. The case $\partial\sigma(A,B) = \emptyset$ is obvious. If $\lambda \in \partial\sigma(A,B) = \sigma(A,B) \cap \overline{\varrho(A,B)}$, then there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $\varrho(A,B)$ such that $\lim\limits_{n\to+\infty} \lambda_n = \lambda$. Since $(A-\lambda B)$ is not bounded invertible, then there exists a subsequence of $(\lambda_n)_{n\in\mathbb{N}}$ for which $\lim\limits_{n\to+\infty} \|R_{\lambda_n}(A,B)\| = +\infty$. Thus, $\lim\limits_{n\to+\infty} \|R_{\lambda_n}(A,B)\|^{-1} = \lim\limits_{n\to+\infty} j(A-\lambda_n B) = 0$. By continuity of $j(.)$, we deduce that $j(A-\lambda B)=0$ and then $\lambda \in \sigma_{AP}(A,B)$.
(3) The proof of (3) is similar to the proof of (1) in the previous theorem. ∎

The condition $0 \in \varrho(A,B)$ is understood as the continuous reversibility of the operator A, furthermore it is quite simple to see that

$$\sigma(A,B)\setminus\{0\} = \left\{ \frac{1}{\lambda} : \ \lambda \in \sigma(B,A) \right\}.$$

This result can be extended to 0 and ∞ by introducing the concept of extended spectrum of a pair (A,B) of bounded operators from X to Y. Let $\widetilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denote the Riemann sphere. $\widetilde{\mathbb{C}}$ is equipped with the following topology: $U \subseteq \widetilde{\mathbb{C}}$ is open if and only if $U \subseteq \mathbb{C}$ and U is open in $\mathbb{C}$ or if $U = V \cup \{\infty\}$ where $V \subseteq \mathbb{C}$ such that $\mathbb{C}\setminus V$, the complement of V in $\mathbb{C}$, is compact in $\mathbb{C}$. Then $\widetilde{\mathbb{C}}$ is a compact Hausdorff space.

Definition 2.7. The extended spectrum $\widetilde{\sigma}(A,B)$ of a pair (A,B) of bounded operators from X to Y is a subset of $\widetilde{\mathbb{C}}$ which coincides with $\sigma(A,B)$ if both functions $\varrho(A,B) \ni \lambda \longrightarrow BR_\lambda(A,B) : \widetilde{\mathbb{C}} \longrightarrow \mathcal{L}(Y)$ and $\varrho(A,B) \ni \lambda \longrightarrow R_\lambda(A,B)B : \widetilde{\mathbb{C}} \longrightarrow \mathcal{L}(X)$ are holomorphic at the point ∞ and coincide with $\sigma(A,B) \cup \{\infty\}$ otherwise. The set $\widetilde{\varrho}(A,B) = \widetilde{\mathbb{C}}\setminus\widetilde{\sigma}(A,B)$ is called the extended resolvent set of the pair (A,B). We set $(A-\infty B)^{-1} = 0$.
For $\lambda \in \varrho(A,B)$ the two operators $R_{\lambda,l}(A,B) = BR_\lambda(A,B) \in \mathcal{L}(Y)$ and $R_{\lambda,r}(A,B) = R_\lambda(A,B)B \in \mathcal{L}(X)$ are called the left and the right resolvent of the pair (A,B), respectively. Note that they are also called pseudo resolvent (see [34]).

Through this definition we have then immediately $\widetilde{\sigma}(A,B)$ and $\widetilde{\sigma}(B,A)$ are compact subsets of $\widetilde{\mathbb{C}}$ and

$$\text{(2.7)} \qquad \widetilde{\sigma}(A,B) = \left\{ \frac{1}{\lambda} \ : \ \lambda \in \widetilde{\sigma}(B,A) \right\}.$$

For more details on the spectrum $\widetilde{\sigma}(A,B)$, let λ_0 be a fixed point of $\varrho(A,B)$ and define $\Phi_0 : \widetilde{\mathbb{C}} \longrightarrow \widetilde{\mathbb{C}}$ by:

$$\Phi_0(\lambda) = \begin{cases} \frac{1}{\lambda-\lambda_0} \text{ if } \lambda \neq \lambda_0,\ \lambda \neq \infty \\ \infty \text{ if } \lambda = \lambda_0 \\ 0 \text{ if } \lambda = \infty. \end{cases} \tag{2.8}$$

Then, Φ_0 is an homeomorphism, its inverse mapping is given by

$$\lambda = \Phi_0^{-1}(\mu) = \begin{cases} \frac{1}{\mu} + \lambda_0 \text{ if } \mu \neq \lambda_0,\ \mu \neq \infty \\ \infty \text{ if } \mu = 0 \\ 0 \text{ if } \mu = \infty. \end{cases} \tag{2.9}$$

However,

$$\begin{aligned} A - \lambda B &= -\mu^{-1}(A - \lambda_0 B)[R_{\lambda_0,r}(A,B) - \mu I_X] \\ &= -\mu^{-1}[R_{\lambda_0,l}(A,B) - \mu I_Y](A - \lambda_0 B) \end{aligned}$$

where $\mu = (\lambda-\lambda_0)^{-1} \neq 0$. So, $\lambda \in \varrho(A,B)$ if and only if $\mu = \Phi_0(\lambda) \in \varrho(R_{\lambda_0,j}(A,B))$, $j = r, l$, then

$$\begin{aligned} \Phi_0(\widetilde{\sigma}(A,B)) &= \sigma(R_{\lambda_0,r}(A,B)) = \sigma(R_{\lambda_0,l}(A,B)) \\ \widetilde{\sigma}(A,B) &= \Phi_0^{-1}(\sigma(R_{\lambda_0,j}(A,B))),\ j = r, l. \end{aligned} \tag{2.10}$$

We can also directly deduce that if $\lambda \in \varrho(A,B)$ then

$$dist(\lambda, \widetilde{\sigma}(A,B)) \geq \frac{1}{\|R_{\lambda,j}(A,B)\|}\ ;\ j = l, r. \tag{2.11}$$

3. Some basic spectral properties of linear operator pencils

In this section we give some spectral properties of the operator pencils $(A-\lambda B)$. We begin by the following theorem.

Theorem 3.1. *Let $A, B \in \mathcal{L}(X,Y)$. Then the following assertions hold:*
(1) $\sigma(A,B)$ is a closed set in $\mathbb{C}$.
(2) If $\varrho(A,B) \neq \emptyset$, then $A(N(B))$ is closed in Y.
(3) If A is invertible, then $(A - \lambda B)$ is equivalent to the linear pencil $I_X - \lambda A^{-1}B$ and hence $\varrho(A,B) = \varrho(I_X - \lambda A^{-1}B)$, $\sigma(A,B) = \sigma(I_X - \lambda A^{-1}B)$, $\sigma_i(A,B) = \sigma_i(I_X - \lambda A^{-1}B)$, $i = p, c, r$, and $\lambda \in \varrho(A,B)$ for sufficiently small $|\lambda|$.
(4) The resolvent operator $R_\lambda(A,B)$ for $\lambda \in \varrho(A,B)$ is holomorphic function on $\varrho(A,B)$ with values in $\mathcal{L}(Y,X)$ and

$$\begin{aligned} \frac{d^n}{d\lambda^n} R_\lambda(A,B) &= n! R_\lambda(A,B) B^n (R_\lambda(A,B))^n \\ &= n!(R_\lambda(A,B))^n B^n R_\lambda(A,B). \end{aligned} \tag{3.1}$$

(5) If $\lambda, \mu \in \varrho(A,B)$, then we have the equalities

$$\begin{aligned} R_\lambda(A,B) - R_\mu(A,B) &= (\lambda-\mu) R_\lambda(A,B) B R_\mu(A,B) \\ R_{\lambda,j}(A,B) - R_{\mu,j}(A,B) &= (\lambda-\mu) R_{\lambda,j}(A,B) R_{\mu,j}(A,B)\ ;\ j = l, r. \end{aligned} \tag{3.2}$$

(6) If $\lambda, \mu \in \varrho(A,B)$, then

$$R_\lambda(A,B) B R_\mu(A,B) = R_\mu(A,B) B R_\lambda(A,B). \tag{3.3}$$

We say that the operators $R_\lambda(A,B)$ and $R_\mu(A,B)$ commute modulo B.
(7) For all $\lambda \in \varrho(A,B)$, $R_{\lambda,l}(A,B)$ and $R_{\lambda,r}(A,B)$ have the same spectrum that is, $\sigma(R_{\lambda,l}(A,B)) = \sigma(R_{\lambda,r}(A,B))$.

(8) $\sigma(A,B)=\emptyset$ if and only if A is continuously invertible and $A^{-1}B$ is quasi-nilpotent on X.

Proof. (1) Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of elements in $\sigma(A,B)$ such that $(\lambda_n)_{n\in\mathbb{N}}$ converges to $\alpha\in\mathbb{C}$. Then, $(A-\lambda_n B)_{n\in\mathbb{N}}$ is a sequence of non-invertible operators in $\mathcal{L}(X,Y)$ which converges strongly to the operator $(A-\alpha B)$. We deduce that $(A-\alpha B)$ can not be invertible in $\mathcal{L}(X,Y)$ since the set of all non-invertible operators in $\mathcal{L}(X,Y)$ is closed. Consequently, $\alpha\in\sigma(A,B)$ and this shows that $\sigma(A,B)$ is closed set in $\mathbb{C}$.
(2) Observe that for any $\lambda\in\mathbb{C}$,

$$A(N(B))=(A-\lambda B)(N(B)).$$

Now, if $\lambda\in\varrho(A,B)$, the operator $(A-\lambda B)$ has a continuous inverse, so it maps closed subspaces to closed subspaces. The claim is now proved since $N(B)$ is closed in X. The statements in (3) follow from the equality,

$$(A-\lambda B)=A(I_X-\lambda A^{-1}B)$$

and the fact that the set of all invertible operators in $\mathcal{L}(X,Y)$ is open. Indeed, select λ sufficiently small so that $|\lambda|\left\|A^{-1}B\right\|<1$. Then $(I_X-\lambda A^{-1}B)$ is invertible and $(I_X-\lambda A^{-1}B)^{-1}=\sum_{n=0}^{\infty}\lambda^n A^{-n}B^n\in\mathcal{L}(X)$. Therefore

$$R_\lambda(A,B)=\sum_{n=0}^{\infty}\lambda^n A^{-n}B^n A^{-1}. \tag{3.4}$$

(4) Observe that

$$\begin{aligned} R_\lambda(A,B) &= \left[(A-\lambda_0 B)-(\lambda-\lambda_0)B\right]^{-1} \\ &= R_{\lambda_0}(A,B)\left[I_Y-(\lambda-\lambda_0)BR_{\lambda_0}(A,B)\right]^{-1}. \end{aligned}$$

If $|\lambda-\lambda_0|<\frac{1}{\left\|R_{\lambda_0,l}(A,B)\right\|}$, then the second inverse above is given by a convergent Neumann series:

$$R_\lambda(A,B)=R_{\lambda_0}(A,B)\sum_{n=0}^{\infty}B^n(R_{\lambda_0}(A,B))^n(\lambda-\lambda_0)^n. \tag{3.5}$$

Thus, $R_\lambda(A,B)$ is given by a convergent power series about any point $\lambda_0\in\varrho(A,B)$ that is, the resolvent set $\varrho(A,B)$ is open, so $R_\lambda(A,B)$ defines an $\mathcal{L}(Y,X)$ -valued holomorphic function on the resolvent set $\varrho(A,B)$ of $(A-\lambda B)$. Note that from the series one obtains that

$$\begin{aligned} \frac{d^n}{d\lambda^n}R_\lambda(A,B)\mid_{\lambda=\lambda_0} &= n!(R_{\lambda_0}(A,B)B^n(R_{\lambda_0}(A,B))^n \\ &= n!(R_{\lambda_0}(A,B))^n B^n R_{\lambda_0}(A,B). \end{aligned}$$

Hence, we obtain (3.1) for any $\lambda\in\varrho(A,B)$.
(5)

$$\begin{aligned} (A-\lambda B)^{-1}-(A-\mu B)^{-1} &= (A-\lambda B)^{-1}[(A-\mu B)-(A-\lambda B)](A-\mu B)^{-1} \\ &= (\lambda-\mu)R_\lambda(A,B)BR_\mu(A,B). \end{aligned}$$

By the same method and (2.11) we obtain the identities for the left and the right resolvent of the pair (A,B). Indeed if for example $j=l$,

$$\begin{aligned} R_{\lambda,l}(A,B)-R_{\mu,l}(A,B) &= B[R_{\lambda}(A,B)-R_{\mu}(A,B)] \\ &= (\lambda-\mu)BR_{\lambda}(A,B)BR_{\mu}(A,B) \\ &= (\lambda-\mu)R_{\lambda,l}(A,B)R_{\mu,l}(A,B). \end{aligned}$$

(6) By using (2.11) we have

$$(\lambda-\mu)\left[R_{\lambda}(A,B)BR_{\mu}(A,B)-R_{\mu}(A,B)BR_{\lambda}(A,B)\right]=0,$$

this proves the result.
(7) For all $\lambda\in\varrho(A,B)$, $R_{\lambda,l}(A,B)$ and $R_{\lambda,r}(A,B)$ are similar :

$$R_{\lambda}(A,B)R_{\lambda,l}(A,B)=R_{\lambda,r}(A,B)R_{\lambda}(A,B).$$

It is clear, that similar operators $R_{\lambda,l}(A,B)$ and $R_{\lambda,r}(A,B)$ have the same spectral properties and, particularly, $\sigma(R_{\lambda,l}(A,B))=\sigma(R_{\lambda,r}(A,B))$.
(8) If $\sigma(A,B)=\emptyset$, then $R_{\lambda}(A,B)\in\mathcal{L}(Y,X)$ for all $\lambda\in\mathbb{C}$. In particular, A is invertible with bounded inverse $A^{-1}\in\mathcal{L}(Y,X)$ and hence

$$\sigma(A,B)=\sigma(A(I_X-\lambda A^{-1}B))=\sigma((I_X-\lambda A^{-1}B)).$$

If $\lambda_0\in\sigma(A^{-1}B))$ and $\lambda_0\neq 0$, then $(\lambda_0 I_X-A^{-1}B)$ and $(I_X-\frac{1}{\lambda_0}A^{-1}B)$ are not invertible in $\mathcal{L}(X)$, this gives $\frac{1}{\lambda_0}\in\sigma(A,B)$, this is a contradiction. Since $A^{-1}B\in\mathcal{L}(X)$, then $\sigma(A^{-1}B)\neq\emptyset$, so $\lambda_0=0$ and hence $\sigma(A^{-1}B)=\{0\}$. Conversely, suppose that $\lambda_0\in\sigma(A,B)$ and A is invertible in $\mathcal{L}(X,Y)$. Thus $\lambda_0\neq 0$ and $(A-\lambda_0 B)=\lambda_0 A(\frac{1}{\lambda_0}I_X-A^{-1}B)$. As $(A-\lambda_0 B)$ is not invertible in $\mathcal{L}(X,Y)$, we deduce that $(\frac{1}{\lambda_0}I_X-A^{-1}B)$ is not invertible in $\mathcal{L}(X)$. Hence, $\frac{1}{\lambda_0}\in\sigma(A^{-1}B)$. ∎

Note that $\sigma(A,B)$ is not necessarily bounded, see [33, Theorem 3.2.4]. The following examples shows that $\sigma(A,B)$ can be the whole complex plane and it can be discrete. Moreover, it may be empty.

Example 3.2. 1) Let $A=\begin{pmatrix} 2 & 2 \\ 0 & 3 \end{pmatrix}$ and $B=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $\det(A-\lambda B)=3(2-\lambda)$ and $\sigma(A,B)=\{2\}$.

2) If $A=\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ and $B=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $\det(A-\lambda B)=3, \sigma(A,B)=\emptyset$ and $\varrho(A,B)=\mathbb{C}$.

3) If $A=\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ and $B=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then $\det(A-\lambda B)=0, \sigma(A,B)=\mathbb{C}$ and $\varrho(A,B)=\emptyset$.

4) Let $X=H^2\left(]0,1[\right)\cap H_0^1\left(]0,1[\right)$ be the Hilbert space of complex measurable functions f on $]0,1[$ such that

$$\int_0^1(|f(t)|^2+|f^{'}(t)|^2+|f^{''}(t)|^2)dt<\infty,$$

$$<f,g>_X=\int_0^1(f(t)\overline{g(t)}+f^{'}(t)\overline{g^{'}(t)}+f^{''}(t)\overline{g^{''}(t)})dt\ ;\ f,g\in X$$

and $f(0)=f(1)=0$, where the derivatives are taken in the distribution sense. Let $Y=L^2\left(]0,1[\right)$ be the Hilbert space of complex measurable functions on $]0,1[$ such that

$$< f,g >_Y = \int_a^b f(t)\overline{g(t)}dt \text{ and } \int_0^1 |f(t)|^2 dt < \infty \ .$$

Define $A \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}(X,Y)$ by setting

$$\begin{cases} Af(t) = f^{''}(t) + \pi^2 f(t) \\ Bf(t) = f(t). \end{cases}$$

For each $g \in Y$ and $\lambda \in \mathbb{C}$ we wish to find $f \in X$ to solve the differential equation

$$f^{''}(t) + (\pi^2 - \lambda)f(t) = g(t).$$

With the above notations this equation can be written in the form $(A-\lambda B)f(t) = g(t)$ and hence the solution is given, if $\lambda \notin \sigma(A,B)$, by $f(t) = (A-\lambda B)^{-1}g(t)$. Set $e_k(t) = \sqrt{2}\sin(k\pi t)$, $k \in \mathbb{N}^*$, $t \in]0,1[$. It is well known that each $f \in X$ can be written as

$$f = \sum_{k=1}^{\infty} f_k e_k \text{ such that } \sum_{k=1}^{\infty} \left(1 + k^2\pi^2\right)^2 |f_k|^2 < \infty$$

and each $g \in Y$ can be written as $g = \sum_{k=1}^{\infty} g_k e_k$ such that $\sum_{k=1}^{\infty} |g_k|^2 < \infty$. It is easy to see that $(A-\lambda B)$ is invertible in $\mathcal{L}(X,Y)$. Indeed, by the equation of coefficients in the respective Fourier series we obtain the solution of the differential equation:

$$f_1 = -\frac{g_1}{\lambda} \text{ and } f_k = -\frac{g_k}{\lambda + (k^2-1)\pi^2} \ , \ k = 2,3,...$$

Thus,

$$f = -\frac{g_1}{\lambda}e_1 - \sum_{k=2}^{\infty} \frac{g_k e_k}{(k^2-1)\pi^2}\left(1 - \frac{\lambda}{(k^2-1)\pi^2} + ...\right)$$

The expansion is a Laurent series of f with a pole of order 1 at 0. Hence, $\sigma(A,B) = \sigma_p(A,B) = \{0\}$.

Even if A and B are self-adjoint operators on Banach spaces, the spectrum of the pencil $(A-\lambda B)$ is often complex. For the finite-dimensional example consider the hermitian matrices $A = \begin{pmatrix} 1 & i\sqrt{2} \\ -i\sqrt{2} & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$\det(A-\lambda B) = -\lambda^2 - \lambda - 1 \text{ and } \sigma(A,B) = \left\{\frac{-1-i\sqrt{3}}{2}, \frac{-1+i\sqrt{3}}{2}\right\}.$$

For the infinite-dimensional case, this follows from the fact that the operator $T = \begin{pmatrix} B & I_X \\ -A & 0 \end{pmatrix}$ is not self-adjoint on $\mathcal{L}(X) \oplus \mathcal{L}(X)$, knowing that under certain conditions the linear operator pencils $(A-\lambda B)$ is equivalent to the quadratic operator pencils $M_\lambda = A - \lambda B + \lambda^2 I_X$ (see e.g. [12], [4]).

Theorem 3.3. *Let $A, B \in \mathcal{L}(X)$. If $\varrho(A,B) \neq \emptyset$ and B is invertible, then*

$$\begin{aligned} \varrho(A,B) &= \varrho(AB^{-1}) = \varrho(B^{-1}A), \\ \sigma(A,B) &= \sigma(AB^{-1}) = \sigma(B^{-1}A). \end{aligned} \tag{3.6}$$

Proof. Let $\lambda \in \varrho(A, B)$, then for all $y \in Y$,

$$B(A - \lambda B)^{-1})(AB^{-1} - \lambda I_Y)y = B(A - \lambda B)^{-1})(A - \lambda B)B^{-1}y = y.$$

Thus, $(AB^{-1} - \lambda I_Y)R_{\lambda,l}(A, B) = R_{\lambda,l}(A, B)(AB^{-1} - \lambda I_Y)$, $\lambda \in \varrho(AB^{-1})$ and $R_{\lambda,l}(A, B) = (AB^{-1} - \lambda I_Y)^{-1}$. By the same argument we have also $\lambda \in \varrho(B^{-1}A)$ and $R_{\lambda,r}(A, B) = (B^{-1}A - \lambda I_X)^{-1}$. This imply that $\varrho(A, B) \subset \varrho(AB^{-1}) \cap \varrho(B^{-1}A)$. Conversely, since B is invertible $\lambda \in \varrho(A, B)$ once $\lambda \in \varrho(B^{-1}A)$ or $\lambda \in \varrho(AB^{-1})$. Thus, $\varrho(A, B) = \varrho(AB^{-1}) \cap \varrho(B^{-1}A)$. Equality (3.6) follows from $\sigma(R_{\lambda,l}(A, B)) = \sigma(R_{\lambda,r}(A, B))$. ∎

Corollary 3.4. *If B is invertible, then $R_{\lambda,l}(A, B)$ and $R_{\lambda,r}(A, B)$ are resolvent operators at λ respectively of AB^{-1} and $B^{-1}A$.*

Now it can be shown that under certain conditions any closed complex subspace is the spectrum of a linear operator pencils (see [28]).

Theorem 3.5. *There exists a pair of bounded operators (A, B) on a separable Banach space X such that for every closed subspace M of $\mathbb{C}$ we have*

$$\widetilde{\sigma}(A, B) = M.$$

Proof. Let $(z_j)_{j\in\mathbb{N}}$ be a dense subset of M. If $0 \notin M$, then there exists $\delta > 0$ such that $|z_j| \geq \delta$ or each $j \in \mathbb{N}$. Consider an arbitrary bounded invertible operator C on H where H is assumed to be a separable Banach space, then

$$\widetilde{\sigma}(C - (\frac{1}{z_j})\lambda C) = \{z_j\}.$$

Denote $X = \bigoplus_{j=0}^{\infty} H$, $A = \bigoplus_{j=0}^{\infty} C$ and $B = \bigoplus_{j=0}^{\infty}(\frac{1}{z_j})C$. Thus,

$$\widetilde{\sigma}(A, B) = \overline{\bigcup_{j=0}^{\infty} \widetilde{\sigma}(C - (\frac{1}{z_j})\lambda C)} = \overline{(z_j)_{j\in\mathbb{N}}} = M.$$

If $0 \in M$, there exists $\gamma > 0$, such that $(z_j)_{j\in\mathbb{N}} = (u_k)_{k\in K} \cup (v_l)_{l\in L}$ with $K \cup L = \mathbb{N}$, $K \cap L = \emptyset$, $(u_k)_k \cap (v_l)_l = \emptyset$, $|u_k| \leq \gamma$ for all $k \in K$ and $|v_l| > \gamma$ for all $l \in L$. Take now with the same previous considerations, $\widetilde{A}_1 = \bigoplus_{j=0}^{\infty} C$, $\widetilde{B}_1 = \bigoplus_{k\in K} u_k C$, $\widetilde{A}_2 = \bigoplus_{l\in L} \left(\frac{1}{v_l}\right) C$ and $\widetilde{B}_2 = \bigoplus_{j=0}^{\infty} C$. Then $A = \widetilde{A}_1 \oplus \widetilde{A}_2$, $B = \widetilde{B}_1 \oplus \widetilde{B}_2$ are bounded on X and

$$\widetilde{\sigma}(A, B) = \widetilde{\sigma}(\widetilde{A}_1, \widetilde{B}_1) \cup \widetilde{\sigma}(\widetilde{A}_2, \widetilde{B}_2) = \overline{(u_k)_{k\in K}} \cup \overline{(v_l)_{l\in L}} = \overline{(z_j)_{j\in\mathbb{N}}} = M.$$

∎

4. Functional calculus on a pair of bounded operators

The functional calculus under consideration in this article is of Riesz-Dunford type, but extended to unbounded spectra. Since $\sigma(A, B)$ can be unbounded, it is necessary to make some assumptions on A and $B \in \mathcal{L}(X, Y)$. The first such functional calculus was defined by Bade [5] for operators with spectrum in a strip. But there are now several other classes of operators with similar functional calculus. Let $A, B \in \mathcal{L}(X, Y)$, where B is not necessarily invertible and let Ω an open set of the extended complex plane $\widetilde{\mathbb{C}}$ containing the extended spectrum $\widetilde{\sigma}(A, B)$ of the pair (A, B). Denoted by symbol $\mathcal{H}(\Omega)$ the set of holomorphic functions on Ω with topology of uniform convergent on compact subsets from Ω. $\mathcal{H}(\Omega)$ is a commutative algebra. More precisely, let $\widetilde{\mathcal{H}}(\Omega)$ be the set of pairs $(f, \widetilde{D})$, where $\widetilde{D}$ is an open subset of $\widetilde{\mathbb{C}}$ containing Ω and f is an analytic function on $\widetilde{D}$. We introduce the relation $(f_1, \widetilde{D}_1) \sim (f_2, \widetilde{D}_2)$ if and only if $f_1 = f_2$ in a neighborhood of Ω contained in $\widetilde{D}_1 \cap \widetilde{D}_2$. We set $\mathcal{H}(\Omega) = \widetilde{\mathcal{H}}(\Omega) / \sim$. Let γ be a contour in a domain of $f \in \mathcal{H}(\Omega)$ that encircles $\widetilde{\sigma}(A, B)$ and consists of a finite number of rectifiable Jordan curves with a positive orientation. Then similar to Dunford's operator calculus we define an operator-function $f(A, B)$ of the pair of bounded operators (A, B) as follows:

$$f(A,B) = \frac{1}{2\pi i} \int_{\gamma} f(\lambda) R_{\lambda}(A,B) d\lambda \tag{4.1}$$

where $f(A, B)$ is a well-defined continuous linear operator from $\mathcal{H}(\Omega)$ to $\mathcal{L}(Y, X)$. Note that $f(A, B)$ is a bounded operator, by the de notion of the spectrum and the properties of integration. It is also useful for our study to introduce the two following operators $f_l(A, B)$ from $\mathcal{H}(\Omega)$ to $\mathcal{L}(Y)$ and $f_r(A, B)$ from $\mathcal{H}(\Omega)$ to $\mathcal{L}(X)$ by the formula:

$$f_j(A,B) = \frac{1}{2\pi i} \int_{\gamma} f(\lambda) R_{\lambda,j}(A,B) d\lambda, \ j = r, l. \tag{4.2}$$

Note that if $\widetilde{\sigma}(A, B)$ is the whole Riemann sphere, then the functional calculus is trivial, since $\mathcal{H}(\widetilde{\mathbb{C}})$ coincides with constant functions. The first main result of this section is the following:

Theorem 4.1. *Let $A, B \in \mathcal{L}(X, Y)$. For $f, g \in \mathcal{H}(\Omega)$ and $\lambda_0 \in \varrho(A, B)$ we have the following assertions :*
(1) If $f^*(\lambda) = \frac{1}{(\lambda - \lambda_0)}$, then $R_{\lambda_0}(A, B) = f^*(A, B)$.
(2) $f_l(A, B)$ and $f_r(A, B)$ are continuous homomorphisms of algebra $\mathcal{H}(\Omega)$ and we have the following properties:
(i) $f_l(A, B) = Bf(A, B)$ and $f_r(A, B) = f(A, B)B$.
(ii) $f_r(A, B)g(A, B) = g(A, B)f_l(A, B)$.
(iii) $f_l(A, B)(A - \mu B) = (A - \mu B)f_r(A, B)$.
(iv) $f_l^*(A, B) = BR_{\lambda_0}(A, B)$ and $f_r^*(A, B) = R_{\lambda_0}(A, B)B$.
(3) If $X = Y$, $Cf(A, B)C^{-1} = f(CAC^{-1}, CBC^{-1})$ holds for any bounded invertible operator C in $\mathcal{L}(X)$.

Proof. (1) $f^* \in \mathcal{H}(\Omega)$ since f^* is holomorphic on $\mathbb{C} \backslash \{\lambda_0\}$. Thus,

$$\begin{aligned}
(A - \lambda_0 B) f^*(A, B) &= \frac{1}{2\pi i} \int_{\gamma} \frac{(A - \lambda_0 B)}{(\lambda - \lambda_0)} R_{\lambda}(A, B) d\lambda \\
&= \frac{1}{2\pi i} \int_{\gamma} \frac{1}{(\lambda - \lambda_0)} [I_Y + BR_{\lambda}(A, B)] d\lambda \\
&= \frac{1}{2\pi i} \int_{\gamma_{\lambda_0}} \frac{d\lambda}{(\lambda - \lambda_0)} I_Y + \frac{1}{2\pi i} int_{\Gamma} BR_{\lambda}(A, B) d\lambda = I_Y
\end{aligned}$$

where γ_{λ_0} is a closed curve having λ_0 in its interior. Similarly we obtain the equality $f^*(A, B)(A - \lambda_0 B) = I_X$. Thus, $(A - \lambda_0 B)$ is invertible in $\mathcal{L}(X, Y)$ and $(A - \lambda_0 B)^{-1} = R_{\lambda_0}(A, B) = f^*(A, B)$. (2) It is clear that the maps $f(A, B)$, $f_l(A, B)$ and $f_r(A, B)$ are linear on $\mathcal{H}(\Omega)$. Let us show that $f_l(A, B)$ and $f_r(A, B)$

are multiplicative. Let $f, g \in \mathcal{H}(\Omega)$. Choose a contour γ_2 around γ_1 both in Ω.

$$\begin{aligned}
& -\frac{1}{4\pi^2}\int_{\gamma_1} f(\lambda)R_{\lambda,l}(A,B)d\lambda \int_{\gamma_2} g(\mu)R_{\mu,l}(A,B)d\mu \\
= & -\frac{1}{4\pi^2}\int_{\gamma_1}\int_{\gamma_2} \frac{f(\lambda)g(\mu)}{(\lambda-\mu)}[R_{\lambda,l}(A,B)-R_{\mu,l}(A,B)]d\lambda d\mu \\
= & -\frac{1}{4\pi^2}\int_{\gamma_1} f(\lambda)R_{\lambda,l}(A,B)d\lambda \int_{\gamma_2}\frac{g(\mu)}{(\lambda-\mu)}d\mu + \frac{1}{4\pi^2}\int_{\gamma_2} g(\mu)R_{\mu,l}(A,B)d\mu \int_{\gamma_1}\frac{f(\lambda)}{(\lambda-\mu)}d\lambda.
\end{aligned}$$

But

$$\int_{\gamma_1}\frac{f(\lambda)}{(\lambda-\mu)}d\lambda = 0 \text{ and } \int_{\gamma_2}\frac{g(\mu)}{(\lambda-\mu)}d\mu = -2\pi i g(\lambda).$$

Thus

$$f_l(A,B)g_l(A,B) = (fg)_l(A,B).$$

By a similar calculation we obtain $f_r(A,B)g_r(A,B) = (fg)_r(A,B)$. Consequently, $f_l(A,B)$ and $g_l(A,B)$ (resp. $f_r(A,B)$ and $g_r(A,B)$) commute. Equalities in (i) follow directly from commutation of bounded operators with integration. For (ii),

$$\begin{aligned}
f_r(A,B)g(A,B) &= f(A,B)Bg(A,B) = f(A,B)g_l(A,B) \\
&= -\frac{1}{4\pi^2}\int_{\gamma_1}\int_{\gamma_2} f(\lambda)g(\mu)R_\lambda(A,B)BR_\mu(A,B)d\lambda d\mu \\
&= -\frac{1}{4\pi^2}\int_{\gamma_1}\int_{\gamma_2} f(\lambda)g(\mu)R_\mu(A,B)BR_\lambda(A,B)d\lambda d\mu \\
&= g(A,B)f_l(A,B).
\end{aligned}$$

This, since $R_\lambda(A,B)$ and $R_\mu(A,B)$ commute modulo B (see formula (3.3)).
(iii)

$$\begin{aligned}
(A-\mu B)f_r(A,B) &= \frac{1}{2\pi i}\int_\gamma f(\lambda)(A-\mu B)R_\lambda(A,B)Bd\lambda \\
&= \frac{1}{2\pi i}\int_\gamma (\lambda-\mu)f(\lambda)BR_\lambda(A,B)Bd\lambda \\
&= \frac{1}{2\pi i}\int_\gamma f(\lambda)BR_\lambda(A,B)Ad\lambda - \mu f_l(A,B)B \\
&= f_l(A,B)A - \mu f_l(A,B)B = f_l(A,B)(A-\mu B).
\end{aligned}$$

(iv) By virtue of (i), we obtain:

$$\begin{aligned}
f_l^*(A,B) &= \frac{1}{2\pi i}\int_\gamma \frac{1}{(\lambda-\lambda_0)}BR_\lambda(A,B)d\lambda = Bf^*(A,B) = BR_{\lambda_0}(A,B), \\
f_r^*(A,B) &= \frac{1}{2\pi i}\int_\gamma \frac{1}{(\lambda-\lambda_0)}R_\lambda(A,B)Bd\lambda = f^*(A,B)B = R_{\lambda_0}(A,B)B.
\end{aligned}$$

(3) Let $X = Y$,

$$\begin{aligned}
Cf(A,B)C^{-1} &= \frac{1}{2\pi i}\int_\gamma f(\lambda)[C(A-\lambda B)C^{-1}]^{-1}d\lambda \\
&= \frac{1}{2\pi i}\int_\gamma f(\lambda)[(CAC^{-1}-\lambda CBC^{-1})]^{-1}d\lambda \\
&= f(CAC^{-1}, CBC^{-1}).
\end{aligned}$$

∎

Sahin and Ragimov gave in [30] a result on the absence of the point ∞ in the extended spectrum $\widetilde{\sigma}(A,B)$ of a pair (A,B) of bounded linear operators in different Banach spaces by considering a reducing decomposition of the pair (A,B). Precisely, they showed that $\infty \notin \widetilde{\sigma}(A,B)$ if and only if (A,B) has the following reducing decomposition : $X = X_1 \oplus X_2$, $Y = Y_1 \oplus Y_2$ in direct sums of their respectively closed subspaces X_j, Y_j such that $AX_j \subset Y_j$, $BX_j \subset Y_j$, $A_j = A_{|X_j}$, $B_j = B_{|X_j}$, $j = 1, 2$,

$$(A,B) = (A_1 \oplus A_2, B_1 \oplus B_2) = (A_1, B_1) \oplus (A_2, B_2), \tag{4.3}$$

where the operators A_2 and B_1 are continuously invertible,

$$(A_2^{-1}B_2)^2 = 0 \text{ and } \widetilde{\sigma}(A,B) = \sigma(A,B) = \sigma(A_1,B_1).$$

Now we will prove the second main result of this section:

Theorem 4.2. *Let $A, B \in \mathcal{L}(X,Y)$, $f \in \mathcal{H}(\Omega)$ and $\lambda_0 \in \varrho(A,B)$. Then,*
(1)

$$\sigma(R_{\lambda_0,l}(A,B)) = \sigma(R_{\lambda_0,r}(A,B)) = \{\frac{1}{\lambda - \lambda_0} \ : \ \lambda \in \widetilde{\sigma}(A,B)\}. \tag{4.4}$$

(2) Spectral mapping theorem of a pair of bounded linear operators:

$$\sigma(f_r(A,B)) = \sigma(f_l(A,B)) = f(\widetilde{\sigma}(A,B)) = \{f(\lambda) \ : \ \lambda \in \widetilde{\sigma}(A,B)\}. \tag{4.5}$$

Proof. (1) If $\infty \notin \widetilde{\sigma}(A,B)$, by virtue of the reduction (4.3) we can consider $B = B_1$ invertible in $\mathcal{L}(X,Y)$ and $A = A_1$. By Theorem 3.1, we also have $\sigma(R_{\lambda_0,l}(A,B)) = \sigma(R_{\lambda_0,r}(A,B))$. Therefore,

$$\begin{aligned} A - \lambda B &= (A - \lambda_0 B)[I_X - (\lambda - \lambda_0)R_{\lambda_0,r}(A,B)] \\ &= [I_Y - (\lambda - \lambda_0)R_{\lambda_0,l}(A,B)](A - \lambda_0 B). \end{aligned}$$

Thus, $\lambda_0 \neq \lambda \in \varrho(A,B)$ if and only if $\frac{1}{(\lambda-\lambda_0)} \in \varrho(R_{\lambda_0,r}(A,B))$ (or $\frac{1}{(\lambda-\lambda_0)} \in \varrho(R_{\lambda_0,l}(A,B))$) which gives the equality (4.3). $\infty \in \widetilde{\sigma}(A,B)$ means that $R_{\lambda_0,l}(A,B)$ and $R_{\lambda_0,r}(A,B)$ are not invertible.
(2) Here we take the same constructs used by Sahin and Ragimov given through the Gelfand representation theory developed in [19, Theorem 5.8.4]. Let $\mathcal{L}_r$ (resp. $\mathcal{L}_l$) be the closed subalgebra of $\mathcal{L}(X)$ (resp. $\mathcal{L}(Y)$) containing the set $\{R_{\lambda,r}(A,B) \ : \ \lambda \in \varrho(A,B)\}$ and I_X (resp. $\{R_{\lambda,l}(A,B) \ : \ \lambda \in \varrho(A,B)\}$ and I_Y) and $\mathcal{M}_r$ and $\mathcal{M}_l$ are their spaces of maximal ideals respectively. Then there exists a continuous $\widetilde{\mathbb{C}}$-valued function α_j on $\mathcal{M}_j$ such that for all $\lambda \in \varrho(A,B)$ and $m \in \mathcal{M}_j$,

$$R_{\lambda,j}(A,B)(m) = \frac{1}{(\lambda - \alpha_j(m))}$$

$\mathcal{M}_j$ and $\alpha_j(\mathcal{M}_j) = \{\alpha_j(m) \ : \ m \in \mathcal{M}_j\}$ are holomorphic, $j = r, l$. Particularly, as $\lambda_0 \in \varrho(A,B)$, then according to (4.4), $\alpha_j(\mathcal{M}_j) = \widetilde{\sigma}(A,B)$ and the space of maximal ideals of algebras $\mathcal{L}_j$ are homeomorphic, $j = r, l$. Thus, for all $m \in \mathcal{M}_j$, $j = r, l$,

$$\begin{aligned} f_j(A,B)(m) &= \frac{1}{2\pi i}\int_\gamma f(\lambda)R_{\lambda,j}(A,B)(m)d\lambda \\ &= \frac{1}{2\pi i}\int_\gamma f(\lambda)\frac{1}{(\lambda - \alpha_j(m))}d\lambda = f(\alpha_j(m)). \end{aligned}$$

Consequently, $\sigma(f_j(A,B))$ is the range of the function f on $\alpha_r(\mathcal{M}_r) = \alpha_l(\mathcal{M}_l) = \widetilde{\sigma}(A,B)$ and $\sigma(f_j(A,B)) = f(\widetilde{\sigma}(A,B))$, $j = r, l$. ∎

The following result is well-known, it concerns linear operator pencils having a discrete spectrum. It has been proved by Ditkin [14], we get here this result directly as a consequence of the previous theorem. Indeed if $\lambda_0 \in \varrho(A,B)$ and B is compact it follows from (3.6) that $\widetilde{\sigma}(A,B) = \Phi_0^{-1}(\sigma(R_{\lambda_0,j}(A,B)))$. Since $R_{\lambda_0,j}(A,B)$ is compact, $\sigma(R_{\lambda_0,j}(A,B))\setminus\{0\}$ consists of eigenvalues $(\mu_k)_k$ with finite-dimensional eigenspaces. The only possible point of accumulation of $\sigma(R_{\lambda_0,j}(A,B))$ is 0, if $(\mu_k)_k$ is infinite, $\lim\limits_{k\to+\infty} \mu_k = 0$. Thus,

$$\widetilde{\sigma}(A,B) = \{\Phi_0^{-1}(\mu_k)\}_k \cup \{\infty\}.$$

Theorem 4.3. *Let $A, B \in \mathcal{L}(X,Y)$. Suppose that $\varrho(A,B) \neq \emptyset$. Then,*
(1) If B is of finite rank, $\sigma(A,B)$ is of finite cardinal.
(2) If B is compact, then $\widetilde{\sigma}(A,B)$ is at most countable and consists only of eigenvalues of finite algebraic multiplicity which accumulate at most at infinity.

Proof. (1) Let $\lambda_0 \in \varrho(A,B)$. $\widetilde{\sigma}(A,B)$ is homeomorphic to $\sigma(R_{\lambda,l}(A,B))$. Or, $R_{\lambda l}(A,B)$ is of finite rank, thus $\widetilde{\sigma}(A,B) = \sigma(A,B)$ is a finite set.
(2) B is a limit of finite-rank operators B_n and $\widetilde{\sigma}(A,B) \subseteq \bigcup\limits_n \sigma(A,B_n)$ is at most countable as a countable union of finite sets. If $\widetilde{\sigma}(A,B) = (\lambda_n)_n$ is infinite and $\lim\limits_{n\to+\infty} \lambda_n = \lambda$, then there exists $n_0 \in \mathbb{N}$ such that $\lambda = \lambda_{n_0}$ since $\widetilde{\sigma}(A,B)$ is closed, which necessarily implies that $\lambda = \lambda_{n_0} = \infty$. ∎

5. Isolated points of linear operator pencils

Let λ_0 be an isolated point of $\widetilde{\sigma}(A,B)$, thus, $\lambda_0 \neq \infty$ and there exists $\delta_0 > 0$ such that $\{\lambda \in \mathbb{C} \ : \ |\lambda - \lambda_0| < \delta_0\} \cap \widetilde{\sigma}(A,B) = \{\lambda_0\}$ and $\gamma_0 \cap \sigma(A,B) = \emptyset$ if $\gamma_0 = \{\lambda \in \mathbb{C} \ : \ |\lambda - \lambda_0| = \delta_0\}$ with clockwise orientation. The left and right Riesz projectors corresponding to λ_0 and the pair (A,B) are respectively defined in $\mathcal{L}(Y)$ and $\mathcal{L}(X)$ by:

$$P_{\lambda_0,j}(A,B) = -\frac{1}{2\pi i}\int_{\gamma_0} R_{\lambda,j}(A,B)d\lambda \quad ; \quad j = l, r \tag{5.1}$$

which corresponds to $f(\lambda) = -1$ in the functional calculus formula 4.2.

In this section we investigate the isolated points of the spectrum of a pair of bounded linear operators $A, B \in \mathcal{L}(X,Y)$.

Theorem 5.1. *Let $A, B \in \mathcal{L}(X,Y)$ and λ_0 an isolated point of $\widetilde{\sigma}(A,B)$. Then the following hold:*
(1) $P_{\lambda_0,j}(A,B)$, $j = l$(respectively, $j = r$) are projections operators in $\mathcal{L}(Y)$ (respectively, $\mathcal{L}(X)$).
(2) The spaces X and Y can be written as a direct sum

$$X = R(P_{\lambda_0,r}(A,B)) \oplus N(P_{\lambda_0,r}(A,B)) \text{ and } Y = R(P_{\lambda_0,l}(A,B)) \oplus N(P_{\lambda_0,l}(A,B)).$$

(3) $AP_{\lambda_0,r}(A,B) = P_{\lambda_0,l}(A,B)A$.
(4) Let $X = Y$, then we have $N(A - \lambda_0 B) \subset R(P_{\lambda_0,r}(A,B))$ and if B commutes with $R_\lambda(A,B)$, then $N(A - \lambda_0 B) \subset R(P_{\lambda_0,l}(A,B))$.
(5) If $X = Y$ is a Hilbert space, $\lambda_0 \in \mathbb{R}$, A and B are self-adjoint, B is invertible

and $AB^{-1} = B^{-1}A$, then $P_{\lambda_0,j}(A,B)$ where $j = l,r$, are the orthogonal projections onto $N(A - \lambda_0 B)$. In particular, $R(P_{\lambda_0,j}(A,B)) = N(A - \lambda_0 B)$ where $j = l,r$.

Proof. (1) Let $\gamma_1 = \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| = \delta_1\}$ such that $\delta_0 < \delta_1$, $\{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < \delta_1\} \cap \widetilde{\sigma}(A,B) = \{\lambda_0\}$ and $\gamma_1 \subset \varrho(A,B)$. In view of the resolvent identity (3.2) we obtain:

$$\begin{aligned}
P^2_{\lambda_0,j}(A,B) &= -\frac{1}{4\pi^2}\int_{\gamma_0}\int_{\gamma_1}\frac{[R_{\lambda,j}(A,B) - R_{\mu,j}(A,B)]}{(\lambda-\mu)}d\lambda d\mu \\
&= \frac{1}{4\pi^2}\int_{\gamma_0} R_{\lambda,j}(A,B)[\int_{\gamma_1}\frac{d\mu}{(\mu-\lambda)}]d\lambda \\
&\quad -\frac{1}{4\pi^2}\int_{\gamma_1} R_{\mu,j}(A,B)[\int_{\gamma_0}\frac{d\lambda}{(\mu-\lambda)}]d\mu = P_{\lambda_0,j}(A,B), \quad j = l,r
\end{aligned}$$

because $\int_{\gamma_1}\frac{d\mu}{(\mu-\lambda)} = 2\pi i$ and $\int_{\gamma_0}\frac{d\lambda}{(\mu-\lambda)} = 0$.
(2) follows directly from (1).
(3)

$$\begin{aligned}
AP_{\lambda_0,r}(A,B) &= -\frac{1}{2\pi i}\int_{\gamma_0} AR_{\lambda,r}(A,B)d\lambda \\
&= -\frac{1}{2\pi i}\int_{\gamma_0} \lambda BR_{\lambda}(A,B)Bd\lambda \\
&= -\frac{1}{2\pi i}\int_{\gamma_0} BR_{\lambda}(A,B)((\lambda B - A) + A)d\lambda \\
&= -\frac{1}{2\pi i}\int_{\gamma_0} R_{\lambda,l}(A,B)Ad\lambda = P_{\lambda_0,l}(A,B)A.
\end{aligned}$$

(4) Let $x \in N(A - \lambda_0 B)$. Then for all $\lambda \in \gamma_0$, $(A - \lambda B)x = (\lambda_0 - \lambda)Bx$ or else $x = (\lambda_0 - \lambda)(A - \lambda B)^{-1}Bx = (\lambda_0 - \lambda)R_{\lambda,r}(A,B)x$. Thus,

$$P_{\lambda_0,r}(A,B)x = -\frac{1}{2\pi i}\int_{\gamma_0}\frac{d\lambda}{(\lambda_0 - \lambda)}x = x.$$

So $x \in R(P_{\lambda_0,r}(A,B))$. Now if B commutes with $R_\lambda(A,B)$, $P_{\lambda_0,r}(A,B) = P_{\lambda_0,l}(A,B)$.
(5) We use now the parametrization $\lambda = \lambda_0 + \delta_0 e^{it}$, $-\pi \le t \le \pi$, of all point λ of γ_0, then

$$\begin{aligned}
P_{\lambda_0,l}(A,B) &= -\frac{\delta_0}{2\pi}\int_{-\pi}^{\pi} B[A - (\lambda_0 + \delta_0 e^{it})B]^{-1}e^{it}dt, \\
P^*_{\lambda_0,l}(A,B) &= -\frac{\delta_0}{2\pi}\int_{-\pi}^{\pi} [A - (\overline{\lambda_0} + \delta_0 e^{-it})B]^{-1}Be^{-it}dt.
\end{aligned}$$

By the change $s = -t$, we obtain since λ_0 is real

$$P^*_{\lambda_0,l}(A,B) = -\frac{\delta_0}{2\pi}\int_{-\pi}^{\pi} B[A - (\lambda_0 + \delta_0 e^{is})B]^{-1}e^{is}ds = P_{\lambda_0,l}(A,B).$$

Similarly we obtain $P^*_{\lambda_0,r}(A,B) = P_{\lambda_0,r}(A,B)$. It remains now to show that $R(P_{\lambda_0,j}(A,B)) \subset N(A-\lambda_0 B),\ \ j = l, r$. Indeed,

$$\begin{aligned}(A-\lambda_0 B)P_{\lambda_0,j}(A,B) &= -\frac{1}{2\pi i}\int_{\gamma_0}(A-\lambda_0 B)(A-\lambda B)^{-1}Bd\lambda \\ &= -\frac{1}{2\pi i}\int_{\gamma_0}(\lambda-\lambda_0)B(A-\lambda B)^{-1}Bd\lambda \\ &= -\frac{1}{2\pi i}\int_{\gamma_0}(\lambda-\lambda_0)(AB^{-1}-\lambda I_X)^{-1}Bd\lambda.\end{aligned}$$

Now we can choose δ_0 such that $(\lambda-\lambda_0)(AB^{-1}-\lambda I_X)^{-1}$ extends to analytic function on $\{\lambda \in \mathbb{C} :\ \ |\lambda-\lambda_0| < \delta_0\}$. Hence by Cauchy's theorem, the last integral is identically zero which gives $(A-\lambda_0 B)P_{\lambda_0,j}(A,B) = 0$. ∎

Note also that if λ_0 be an isolated point of $\widetilde{\sigma}(A,B)$, the Laurent series for the resolvent $(\lambda B - A)^{-1}$ in a small neighborhood of the isolated singularity λ_0 is given by

$$(\lambda B - A)^{-1} = \sum_{n=-\infty}^{+\infty}(\lambda-\lambda_0)^n S_n, \tag{5.2}$$

where

$$S_n = -\frac{1}{2\pi i}\int_{\gamma_0}\frac{1}{(\lambda-\lambda_0)^{n+1}}R_\lambda(A,B)d\lambda\ ,\ n \in \mathbb{Z}.$$

The coefficients S_n are bounded operators and satisfies the following properties:
(i) $S_nBS_m = (1-\tau_n-\tau_m)S_{n+m}$, where $\tau_n = 1$ if $n \geq 0$ and $\tau_n = 0$ if $n < 0$. Indeed, assume that $\lambda_0 = 0$, since 0 is an isolated point of $\widetilde{\sigma}(A,B)$, then there exists $\delta > 0$ such that $\{\lambda \in \mathbb{C}\ :\ \ |\lambda| < \delta\} \cap \widetilde{\sigma}(A,B) = \{0\}$. Denote $\gamma_r = \{\lambda \in \mathbb{C}\ :\ \ |\lambda| = r\}$ for $0 < r < \delta$. Let $r < r_1$, we have by using the resolvent identities that

$$\begin{aligned}S_nBS_m &= \frac{1}{(2\pi i)^2}\int_{\gamma_r}\int_{\gamma_{r_1}}\lambda^{-n-1}\mu^{-m-1}R_\lambda(A,B)BR_\mu(A,B)d\lambda d\mu \\ &= \frac{1}{(2\pi i)^2}\int_{\gamma_r}\int_{\gamma_{r_1}}\frac{\lambda^{-n-1}\mu^{-m-1}}{(\mu-\lambda)}[(\lambda B-A)^{-1}-(\mu B-A)^{-1}]d\lambda d\mu.\end{aligned}$$

By computing the double integral on the right in any order and the fact that

$$\begin{aligned}\frac{1}{2\pi i}\int_{\gamma_r}\frac{\lambda^{-n-1}}{(\lambda-\mu)}d\lambda &= -\tau_n\mu^{-n-1} \\ \frac{1}{2\pi i}\int_{\gamma_{r_1}}\frac{\mu^{-m-1}}{(\lambda-\mu)}d\mu &= (\tau_m-1)\lambda^{-m-1}.\end{aligned}$$

We obtain

$$\begin{aligned}S_nBS_m &= \frac{(1-\tau_n-\tau_m)}{2\pi i}\int_{\gamma_r}\lambda^{-n-m-2}(\lambda B-A)^{-1}d\lambda \\ &= (1-\tau_n-\tau_m)S_{n+m+1}.\end{aligned} \tag{5.3}$$

(ii) Multiplying (5.3) on the left and the right by $(\lambda B - A)$, we obtain

$$(\lambda B - A)\sum_{n=-\infty}^{+\infty}(\lambda-\lambda_0)^n S_n = I_Y \text{ and } \sum_{n=-\infty}^{+\infty}(\lambda-\lambda_0)^n S_n(\lambda B - A) = I_X.$$

Thus,

$$\begin{aligned} I_Y &= \sum_{n=-\infty}^{+\infty} (\lambda-\lambda_0)^n [BS_{n-1} + (\lambda_0 B - A)S_n], \\ I_X &= \sum_{n=-\infty}^{+\infty} (\lambda-\lambda_0)^n [S_{n-1}B + S_n(\lambda_0 B - A)]. \end{aligned}$$

The uniqueness of the Laurent series expansion yields $I_X = S_{-1}B + S_0(\lambda_0 B - A)$, $I_Y = BS_{-1} + (\lambda_0 B - A)S_0$ and $S_{n-1}B + S_n(\lambda_0 B - A) = 0$, $BS_{n-1} + (\lambda_0 B - A)S_n$ for all $n \neq 0$. Then,

$$\begin{cases} S_{-1}B = I_X - S_0(\lambda_0 B - A) \\ BS_{-1} = I_Y - (\lambda_0 B - A)S_0 \\ S_{n-1}B = S_n(A - \lambda_0 B)\ ,\ n \neq 0 \\ BS_{n-1} = (A - \lambda_0 B)S_n\ ,\ n \neq 0. \end{cases}$$

From the standard terminology of the complex theory, we call the operator S_{-1} in the Laurent series (5.2) the residue operator at λ_0. By taking $n = m = -1$ in (5.3), BS_{-1} and $S_{-1}B$ are projections which coincide respectively with the left and right Riesz projectors $P_{\lambda_0,l}(A,B)$ and $P_{\lambda_0,r}(A,B)$ at λ_0.

Definition 5.2. Let $X = Y$ and $A, B \in \mathcal{L}(X)$ are with non empty resolvent set $\varrho(A,B)$. We say that A and B commute in the sense of resolvent if for all $\lambda \in \varrho(A,B)$,

$$R_{\lambda,l}(A,B) = R_{\lambda,r}(A,B).$$

Remark 5.3. If A and B commute in the sense of resolvent then for all $\lambda \in \varrho(A,B)$, we deduce that

$$\begin{aligned} AR_\lambda(A,B) &= R_\lambda(A,B)A \\ BR_\lambda(A,B)B &= B^2R_\lambda(A,B) = R_\lambda(A,B)B^2 \\ ABR_\lambda(A,B) &= R_\lambda(A,B)AB = R_\lambda(A,B)BA = BAR_\lambda(A,B). \end{aligned}$$

Then, $P_{\lambda_0,l}(A,B) = P_{\lambda_0,r}(A,B) = P_{\lambda_0}(A,B)$, $AP_{\lambda_0}(A,B) = P_{\lambda_0}(A,B)A$, $BP_{\lambda_0}(A,B) = P_{\lambda_0}(A,B)B$ and $S_nB = BS_n$ for all $n \in \mathbb{Z}$, if A and B commute in the sense of resolvent. By setting $R_{\lambda,B}(A,B) = -R_{\lambda,l}(A,B) = -R_{\lambda,r}(A,B)$, $D = S_{-2}B = BS_{-2}$ and $E = -BS_0 = -S_0B$, the relation (S_nBS_m) gives

$$\begin{aligned} BS_{-k} &= D^{k-1} \text{ for } k \geq 2 \\ BS_k &= -E^{k+1} \text{ for } k \geq 0. \end{aligned}$$

The Laurent series (5.2) around λ_0 is equivalent to

$$R_{\lambda,B}(A,B) = \sum_{n=1}^{+\infty} \frac{D^n}{(\lambda-\lambda_0)^{n+1}} + \frac{P_{\lambda_0}(A,B)}{(\lambda-\lambda_0)} - \sum_{n=1}^{+\infty} (\lambda-\lambda_0)^n E^{n+1}. \tag{5.4}$$

Thus,

$$\begin{cases} R_{\lambda,B}(A,B)P_{\lambda_0}(A,B) = \sum\limits_{n=1}^{+\infty} \frac{D^n}{(\lambda-\lambda_0)^{n+1}} + \frac{P_{\lambda_0}(A,B)}{(\lambda-\lambda_0)} \\ R_{\lambda,B}(A,B)(I_X - P_{\lambda_0}(A,B)) = -\sum\limits_{n=1}^{+\infty} (\lambda-\lambda_0)^n E^{n+1}, \end{cases}$$

where

$$(A-\lambda_0 B)P_{\lambda_0}(A,B) = (AP_{\lambda_0}(A,B)-\lambda_0 B)P_{\lambda_0}(A,B) = D$$

and

$$(A-\lambda_0 B)E = I_X - P_{\lambda_0}(A,B).$$

Hence

$$(5.5)\qquad \begin{cases} E = (A-\lambda_0 B)^{-1}_{|R(I_X - P_{\lambda_0}(A,B))}(I_X - P_{\lambda_0}(A,B)) \\ EP_{\lambda_0}(A,B) = P_{\lambda_0}(A,B)E = 0 \\ DE = ED = 0 \\ D = DP_{\lambda_0}(A,B) = P_{\lambda_0}(A,B)D. \end{cases}$$

Now suppose that λ_0 is a pole of the resolvent $(A-\lambda B)$ of order m, then $S_{-m}\neq 0$ and $S_n = 0$ for all $n > m$. Since $BS_{-1} = S_{-1}B = P_{\lambda_0}(A,B)$, it follows that $(A-\lambda B)^{m-1}P_{\lambda_0}(A,B) = S_{-m}\neq 0$ and $(A-\lambda B)^m P_{\lambda_0}(A,B) = S_{-m-1} = 0$, then the operator $D = (A-\lambda_0 B)P_{\lambda_0}(A,B)$ is nilpotent of order m.

Now, we give the following fundamental results:

Theorem 5.4. *Let $A, B \in \mathcal{L}(X)$ such that A and B commute in the sense of resolvent. If λ_0 is an isolated point in the spectrum $\widetilde{\sigma}(A,B)$, then λ_0 is a pole of the resolvent of order $m\in\mathbb{N}^*$. Laurent series around λ_0 is given by* (5.2) *with the residue operator $BS_{-1} = S_{-1}B$ coincides with the Riesz projection $P_{\lambda_0}(A,B)$ associated to λ_0 and the relations* (5.5) *are satisfied. On the other hand, the operator $D = (A-\lambda_0 B)P_{\lambda_0}(A,B)$ is nilpotent of order m.*

The discrete spectrum of the pair (A,B) denoted $\sigma_d(A,B)$ is the set of isolated points $\lambda\in\mathbb{C}$ of the spectrum $\sigma(A,B)$ such that the corresponding Riesz projectors $P_{\lambda,j}(A,B)$ are finite dimensional. Thus, $\sigma_d(A,B)\subset\sigma_p(A,B)$. Define also the essential spectra of the pair (A,B) by:

$$(5.6)\qquad \sigma_{ess}(A,B) = \widetilde{\sigma}(A,B)\backslash\sigma_d(A,B).$$

The largest open set of $\widetilde{\mathbb{C}}$ on which the resolvent $R_\lambda(A,B)$ is finitely meromorphic is precisely $\varrho_{ess}(A,B) = \sigma_d(A,B)\cup\varrho(A,B) = \widetilde{\mathbb{C}}\backslash\sigma_{ess}(A,B)$. Let $X = Y$, $\lambda\in\varrho_{ess}(A,B)$ and let $P_{\lambda,j}(A,B)$ be the corresponding finite rank Riesz projector, $j = l, r$. Since $R(P_{\lambda,j}(A,B))$ and $N(P_{\lambda,j}(A,B))$ are $P_{\lambda,j}(A,B)$-invariant, $j = l, r$, we may define the operators:

$$(5.7)\qquad Q_{\lambda,j}(A,B) = (A-\lambda B)(I - P_{\lambda,j}(A,B)) + P_{\lambda,j}(A,B)\ ;\ j = l, r.$$

With respect to the decomposition $X = R(P_{\lambda,j}(A,B))\oplus N(P_{\lambda,j}(A,B))$, $j = l, r$, we can write:

$$(5.8)\qquad Q_{\lambda,j}(A,B) = (A-\lambda B)_{|N(P_{\lambda,j}(A,B))}\oplus I_X\ ;\ j = l, r.$$

Since $\sigma((A-\lambda B)_{|N(P_{\lambda,j}(A,B))}) = \widetilde{\sigma}(A,B)\backslash\{0\}$, $Q_{\lambda,j}(A,B)$ has bounded inverse denoted by $\mathcal{R}_{\lambda,j}(A,B)$, $j = l, r$. $\mathcal{R}_{\lambda,l}(A,B)$ and $\mathcal{R}_{\lambda,r}(A,B)$ are called respectively the left Browder and the right Browder resolvent operator of the pair (A,B), that is,

$$(5.9)\qquad \begin{aligned} \mathcal{R}_{\lambda,j}(A,B) &= ((A-\lambda B)_{|N(P_{\lambda,j}(A,B))})^{-1}(I - P_{\lambda,j}(A,B)) + P_{\lambda,j}(A,B) \\ j &= l, r,\ \lambda\in\varrho_{ess}(A,B). \end{aligned}$$

This clearly extends the resolvent $R_{\lambda,j}(A,B)$ from $\varrho(A,B)$ to $\varrho_{ess}(A,B)$ and admits the following properties for $j = l, r$:

$$\begin{array}{rcl} P_{\lambda,j}(A,B)\mathcal{R}_{\lambda,j}(A,B) & = & \mathcal{R}_{\lambda,j}(A,B)P_{\lambda,j}(A,B), \\ P_{\lambda,j}(A,B)Q_{\lambda,j}(A,B) & = & Q_{\lambda,j}(A,B)P_{\lambda,j}(A,B) = P_{\lambda,j}(A,B). \end{array} \tag{5.10}$$

Proposition 5.5. *Let $A, B \in \mathcal{L}(X)$ and $\lambda, \mu \in \varrho_{ess}(A,B)$, then for $j = l, r$,*

$$\begin{array}{rcl} \mathcal{R}_{\lambda,j}(A,B) - \mathcal{R}_{\mu,j}(A,B) & = & (\lambda-\mu)\mathcal{R}_{\lambda,j}(A,B)B\mathcal{R}_{\mu,j}(A,B) \\ & & +\mathcal{R}_{\lambda,j}(A,B)\mathcal{M}_j(\lambda,\mu)\mathcal{R}_{\mu,j}(A,B) \end{array}$$

where $\mathcal{M}_j(\lambda,\mu)$ is a finite rank operator defined on X by:

$$\mathcal{M}_j(\lambda,\mu) = [(A-(\lambda+1)B)P_{\lambda,j}(A,B) \ -(A-(\mu+1)B)P_{\mu,j}(A,B)]$$

Proof. By computing $\mathcal{R}_{\lambda,j}(A,B) - \mathcal{R}_{\mu,j}(A,B)$, we directly obtain:

$$\mathcal{M}(\lambda,\mu) = [(A-\lambda B - I_X)P_{\lambda,j}(A,B) - (A-\mu B - I_X)P_{\mu,j}(A,B)]\,.$$

∎

The Browder resolvent, through its properties mentioned above, can be used to study the question of existence of solutions of boundary value problems with singularities defined by a given boundary condition:

$$\begin{cases} Ax = \lambda Bx + f \\ \Gamma x = \varphi. \end{cases}$$

Where $f \in X$, Γ is a boundary operator and λ is a spectral parameter such that $\lambda^{-1} \in \varrho_{ess}(A,B)$. For more details one can consult [23].

Remark 5.6. As the matter of fact, this decomposition is not "the simplest"; there are many different definitions of $\sigma_{ess}(A,B)$ for $A, B \in \mathcal{L}(X,Y)$:
1) $\lambda \in \sigma_{ess,1}(A,B)$ if $(A-\lambda B)$ is not semi-Fredholm ($T \in \mathcal{L}(X,Y)$ is semi-Fredolm if $R(T)$ is closed in Y and $N(T)$ or the quotient space $Y/R(T)$ are finite-dimensional);
2) $\lambda \in \sigma_{ess,2}(A,B)$ if $R(A-\lambda B)$ is not closed in Y or $N(A-\lambda B)$ is infinite-dimensional in X;
3) $\lambda \in \sigma_{ess,3}(A,B)$ if $(A-\lambda B)$ is not Fredholm ($T \in \mathcal{L}(X,Y)$ is Fredolm if $R(T)$ is closed in Y and $N(T)$ and $Y/R(T)$ are finite-dimensional);
4) $\lambda \in \sigma_{ess,4}(A,B)$ if $(A-\lambda B)$ is not Fredholm with index zero (recall that $index(T) = \dim N(T) - \dim Y/R(T) = \dim N(T) - co\dim R(T)$);
5) $\sigma_{ess,5}(A,B)$ is the union of $\sigma_{ess,1}(A,B)$ with all components of $\mathbb{C}\backslash\sigma_{ess,1}(A,B)$ that do not intersect with the resolvent set $\varrho(A,B)$.

Note that,

$$\sigma_{ess,1}(A,B) \subset \sigma_{ess,2}(A,B) \subset \sigma_{ess,3}(A,B) \subset \sigma_{ess,4}(A,B) \subset \sigma_{ess,5}(A,B) \subset \widetilde{\sigma}(A,B) \tag{5.11}$$

and that the essential spectrum $\sigma_{ess,i}(A,B)$ is invariant under compact perturbations for $i = 1,2,3,4$, but not for $i = 5$. The case $i = 4$ gives the part of the spectrum that is independent of compact perturbations, that is,

$$\sigma_{ess,4}(A,B) = \bigcap \lim_{K \in \in \mathcal{K}(X,Y)} \sigma(A+K,B)$$

where $\mathcal{K}(X,Y)$ denotes the set of compact operators from X to Y. As a generalization of the usual notion of Wolf essential spectrum, the essential spectrum of linear operator pencils was introduced by Faierman, Mennicken and Moller in [15].

Note that if $B = I_X$, we recover the usual definition of the essential spectra of a bounded linear operator, that is, $\sigma_{ess,i}(A, I_X) = \sigma_{ess,i}(A)$, $i = 1, ..., 4$. We denote the dimension of the null space or nullity of an operator $T \in \mathcal{L}(X, Y)$ by $n(T)$ and the codimension of the range or defect of T by $d(T)$. The ascent of T, $\alpha(T)$, is the smallest integer p such that $N(T^p) = N(T^{p+1})$, and the descent of T, $\beta(T)$, is the smallest integer q such that $R(T^q) = R(T^{q+1})$. (It may happen that $\alpha(T) = \infty$ or $\beta(T) = \infty$). One of the central questions in the study of the essential spectra of bounded linear operators consists in showing when different notions of essential spectrum coincide and studying the invariance of $\sigma_{ess,i}(A, B)$ by some class of perturbations. For a detailed study, see [1]. The following result is given in [20].

Proposition 5.7. *Let $A, B \in \mathcal{L}(X, Y)$. Then the following hold:*
(1) $\sigma_{ess,3}(A, B)$ is closed subset of $\mathbb{C}$.
(2) index$(A - \lambda B)$ is constant on any component of $\mathbb{C}\backslash\sigma_{ess,3}(A, B)$.
(3) $n(A - \lambda B)$ and $d(A - \lambda B)$ are constant on any component of $\mathbb{C}\backslash\sigma_{ess,3}(A, B)$ except on a discreet set of points at which they have larger values.
(4) If $\mathbb{C}\backslash\sigma_{ess,3}(A, B)$ is connected and $\varrho(A, B)$ is not empty, then

$$\sigma_{ess,3}(A, B) = \sigma_{ess,4}(A, B).$$

The following result is a generalization of [24, Theorem 1].

Theorem 5.8. *Let $A, B \in \mathcal{L}(X)$ such that A and B commute in the sense of resolvent and $\lambda \in \sigma(A, B)$. The following statements are equivalent:*
(1) $\lambda \in \sigma_{ess,2}(A, B)$;
(2) λ is a pole of the resolvent $R_\lambda(A, B)$ of finite rank;
(3) $\alpha(A - \lambda B) = \beta(A - \lambda B) < \infty$ and $n(A - \lambda B) < \infty$.

Proof. The equivalence of (1) and (2) can be obtained in the same manner as in the proof of [11, Lemma 17]. (2) $\Longrightarrow$ (3). If λ is a pole of $(A - \lambda B)^{-1}$ of order m, then $N((A - \lambda B)^m) = N((A - \lambda B)^{m+1})$. Indeed, as $N((A - \lambda B)^m) \subset N((A - \lambda B)^{m+1})$ it suffices to prove the inverse inclusion. We proceed by contradiction. Let $x \in N((A - \lambda B)^{m+1})$ and $x \notin N((A - \lambda B)^m)$, that is, the vector $y = (A - \lambda B)^m x \neq 0$, it follows that $(A - \lambda B)y = 0$. This implies, by (4) of Theorem 5.1, that $P_{\lambda,l}(A, B)y = P_{\lambda,r}(A, B)y = P_\lambda(A, B)y = y$. Consequently,

$$0 = (A - \lambda B)^m P_\lambda(A, B)x = P_\lambda(A, B)(A - \lambda B)^m x = P_\lambda(A, B)y = y,$$

which is a contradiction. Hence, $N((A - \lambda B)^m) = N((A - \lambda B)^{m+1})$ and $\alpha(A - \lambda B) \leq m$. Now, notice that $(A - \lambda B)^{m-1} P_\lambda(A, B) \neq 0$ which guarantees the existence of some vector $x \in R(P_\lambda(A, B))$ such that $(A - \lambda B)^{m-1}x = (A - \lambda B)^{m-1} P_\lambda(A, B)x \neq 0$. From $(A - \lambda B)^m x = (A - \lambda B)^m P_\lambda(A, B)x = 0$, it follows that

$$N((A - \lambda B)^m) \neq N((A - \lambda B)^{m-1}). \tag{5.12}$$

This shows $\alpha(A - \lambda B) \geq m$. Thus, $\alpha(A - \lambda B) = m$. Now if we consider the decomposition $\widetilde{\sigma}(A, B) = \{\lambda\} \cup (\widetilde{\sigma}(A, B)\backslash \{\lambda\})$, then $(A - \lambda B)^n$ is invertible on $N(P_\lambda(A, B))$ for all $n \in \mathbb{N}$ and $(A - \lambda B)^m P_\lambda(A, B) = 0$ implies that $(A - \lambda B)^m = 0$ on $R(P_\lambda(A, B))$. Consequently, $R((A - \lambda B)^m) = N(P_\lambda(A, B)) = R((A - \lambda B)^{m+1})$. Thus, $(A - \lambda B)$ has finite descent $\beta(A - \lambda B) = m$. (3) $\Longrightarrow$ (1) Assume that $\alpha(A - \lambda B) = \beta(A - \lambda B) = m < \infty$. Then $N(P_\lambda(A, B)) = R((A - \lambda B)^m) = R((A - \lambda B)^n)$ and $R(P_\lambda(A, B)) = N((A - \lambda B)^m) = N((A - \lambda B)^n)$ for all $n \geq m$.

It follows that $D^n = (A-\lambda B)^n P_\lambda(A,B) = 0$ for all $n \geq m$, and so λ is a pole of the resolvent of order k with $k \leq m$. But from (5.12), necessarily $k = m$. ∎

Remark 5.9. If $A, B \in \mathcal{L}(X)$ commute in the sense of resolvent and λ is a pole of order m of the resolvent, then $\lambda \in \sigma_p(A,B)$ and

$$X = N((A-\lambda B)^n) \oplus R((A-\lambda B)^n) \text{ for all } n \geq m.$$

Now, we introduce an important class of bounded operators which involves the concept of semi-regularity see e.g. Muller [26] and Rakocevic [27] , Mbekhta and Ouahab [25].

Definition 5.10. Let $A \in \mathcal{L}(X)$. The algebraic core $C(A)$ of A is defined to be the greatest subspace M of X for which $A(M) = M$. The reduced minimum modulus of A is defined by

$$\gamma(A) = \begin{cases} \inf\limits_{x \notin N(A)} \frac{\|Ax\|}{dist(x,N(A))} & \text{if } A \neq 0 \\ \infty \text{ if } A = 0. \end{cases}$$

A is said to be semi-regular if $R(A)$ is closed and $N(A^n) \subseteq R(A)$ for all $n \in \mathbb{N}$. A is said to admit a generalized Kato decomposition or A is of generalized Kato type, if there exists a pair of closed subspaces (M,N) of X such that:
(i) $X = M \oplus N$.
(ii) $A(M) \subset M$ and $A_{|M}$ is semi-regular.
(iii) $A(N) \subset N$ and $A_{|N}$ is quasi-nilpotent.

Note that if $A \in \mathcal{L}(X)$, $R(A)$ is closed in X if and only if $\gamma(A) > 0$ and $\gamma(A) = \gamma(A^*)$. If A is semi-regular, then $\gamma(A^n) \geq (\gamma(A))^n$ and A^n is semi-regular for all $n \in \mathbb{N}$, $C(A)$ is closed and $C(A) = \bigcap\limits_{n\in\mathbb{N}} R(A^n)$. A is semi-regular if and only if A^* is semi-regular. On the other hand if a pair of closed subspaces (M,N) of X reduces A ($X = M \oplus N$, $A(M) \subset M$ and $A(N) \subset N$), then A is semi-regular if and only if $A_{|M}$ and $A_{|N}$ are semi-regular. If $A_{|N}$ is nilpotent, A is said to be of Kato type [22]. Semi-regular operators are of Kato type with $M = X$ and $N = \{0\}$. If 0 is an isolated point in $\sigma(A)$, or equivalently 0 is a pole of the resolvent of A, then A is of generalized Kato type [8]. Using rather direct technique different from [3], we extend the results to semi-regular operators and those who admit a generalized Kato decomposition. Indeed, an immediate and direct generalization of [3, Theorem 1.31] we provided the following result:

Theorem 5.11. *Let $A, B \in \mathcal{L}(X)$, A be semi-regular and $BC(A) = C(A)$. Then $(A-\lambda B)$ is semi-regular for all $|\lambda| < \frac{\gamma(A)}{\|B\|}$.*

For $A, B \in \mathcal{L}(X)$, let us define the generalized Kato spectrum for the pair (A,B) as follows:

$$\sigma_{gk}(A,B) = \{\lambda \in \mathbb{C} \ : \ (A-\lambda B) \text{ is not of generalized Kato type}\} \tag{5.13}$$

$\sigma_{gk}(A,B)$ is not necessarily non-empty. For example, each pair of quasi-nilpotent (resp. nilpotent) operator A and $B = I_X$ has empty generalized Kato spectrum. The next theorem is a generalization of Theorem 2.2 of [21].

Theorem 5.12. *Let $A, B \in \mathcal{L}(X)$, A be of generalized Kato type and $BC(A) = C(A)$. Then there exists an open disc $D(0,\varepsilon)$ for which $(A-\lambda B)$ is semi-regular for all $\lambda \in D(0,\varepsilon) \backslash \{0\}$.*

Proof. $B = I_X$ corresponds to the Theorem 2.2 of [21]. If $B \neq I_X$, note that $X = M \oplus N$, $A(M) \subset M$, $A_{|M}$ is semi-regular, $A(N) \subset N$ and $A_{|N}$ is quasi-nilpotent. If $M = \{0\}$, A is quasi-nilpotent and thus

$$(A - \lambda B) = (A - \lambda I_X)\left[I_X - \lambda(A - \lambda I_X)^{-1}(B - I_X)\right]$$

is invertible if $|\lambda| < \frac{1}{\|(A-\lambda I_X)^{-1}\|\|B-I_X\|} = \eta$. This shows that $(A - \lambda B)$ is semi-regular in $D(0, \eta)\setminus\{0\}$. If $M \neq \{0\}$, $A = \begin{pmatrix} A_{|M} & 0 \\ 0 & A_{|N} \end{pmatrix}$ and

$$(A - \lambda B) = \begin{pmatrix} (A - \lambda B)_{|M} & 0 \\ 0 & (A - \lambda B)_{|N} \end{pmatrix}.$$

Since $A_{|N}$ is quasi-nilpotent, $(A - \lambda I_X)$ is invertible in $\mathcal{L}(X)$ for all λ non-zero complex number. Then, $(A - \lambda B)_{|N}$ is invertible and semi-regular for $|\lambda| < \eta$. As $A_{|M}$ is semi-regular operator, then $\gamma(A_{|M}) > 0$ and by Theorem 29, $(A - \lambda B)_{|M}$ is semi-regular for all $|\lambda| < \frac{\gamma(A_{|M})}{\|B\|}$. Consequently, $(A - \lambda B)$ is semi-regualar for all $|\lambda| < \varepsilon$, where $\varepsilon = \min(\eta, \frac{\gamma(A_{|M})}{\|B\|})$. ∎

We deduce in particular from this theorem that the generalized Kato spectrum of a pair of bounded operators is a closed subset. The following result gives the relation between the closed range spectrum $\sigma_{ess,2}(A, B)$ and the generalized Kato spectrum $\sigma_{gk}(A, B)$ of a pair (A, B) of bounded operators which extend some results of [6, 7, 21].

Theorem 5.13. *Let $A, B \in \mathcal{L}(X)$ such that $BC(A) = C(A)$.*
(1) If $\lambda \in \sigma_{ess,2}(A, B)$ is non-isolated point then $\lambda \in \sigma_{gk}(A, B)$.
(2) The symmetric difference $\sigma_{gk}(A, B)\Delta\sigma_{ess,2}(A, B)$ is at most countable.

Proof. (1) Let $\lambda \in \sigma_{ess,2}(A, B)$ be a non-isolated point and assume that $(A - \lambda B)$ is of generalized Kato type. Then by Theorem 29 there exists an open disc $D(\lambda, \epsilon)$ such that $(A - \mu B)$ is semi-regular in $D(\lambda, \epsilon)\setminus\{\lambda\}$, so that $R(A - \mu B)$ is closed in X for all $\mu \in D(\lambda, \epsilon)\setminus\{\lambda\}$. This contradicts our assumption that λ is a non-isolated point.
(2) We have

$$\sigma_{gk}(A, B)\Delta\sigma_{ess,2}(A, B)$$

is equal to

$$(\sigma_{gk}(A, B) \cap (\mathbb{C}\setminus\sigma_{ess,2}(A, B))) \cup (\sigma_{ess,2}(A, B) \cap (\mathbb{C}\setminus\sigma_{gk}(A, B))).$$

Hence, from (1), the set $(\sigma_{ess,2}(A, B)\setminus\sigma_{gk}(A, B))$ is at most countable, we have

$$\mathbb{C}\setminus\sigma_{ess,2}(A, B) = \bigcup_{m=1}^{\infty}\left\{\lambda \in \mathbb{C} \ : \ \gamma(A - \lambda B) \geq \frac{1}{m}\right\}$$

and

$$\sigma_{gk}(A, B) \cap (\mathbb{C}\setminus\sigma_{ess,2}(A, B)) = \bigcup_{m=1}^{\infty}\sigma_{gk}(A, B) \cap \left\{\lambda \in \mathbb{C} \ : \ \gamma(A - \lambda B) \geq \frac{1}{m}\right\}.$$

The set $\mathcal{A}_m = \sigma_{gk}(A, B) \cap \left\{\lambda \in \mathbb{C} \ : \ \gamma(A - \lambda B) \geq \frac{1}{m}\right\}$is necessarily at most countable for all $m \geq 1$. Indeed, let ζ be a non-isolated point of $\mathcal{A}_m$, then there exists a sequence $(\lambda_k)_{k\in\mathbb{N}}$ in $\mathcal{A}_m$ such that $\lim\limits_{k\to+\infty}\lambda_k = \zeta$. Thus, $\gamma(A - \zeta B) \geq \frac{1}{m}$, since

$\left\{\lambda \in \mathbb{C} \ : \ \gamma(A-\lambda B) \geq \frac{1}{m}\right\}$ is closed in $\mathbb{C}$ (see e.g. [6]), and $\zeta \notin \sigma_{gk}(A,B)$ which contradicts the closedness of $\sigma_{gk}(A,B)$. ∎

6. Spectrum of the quotient of two bounded operators

Let here $X=Y$ be an infinite dimensional complex Hilbert space equipped with the inner product $\langle .;. \rangle$ and the associated norm $\|.\|$. The quotient A/B of bounded operators A and B on X, $B \neq 0$, is defined by the mapping $Bx \longrightarrow Ax$, $x \in X$ when $N(B) \subset N(A)$ and $A \neq B$. If $A=B$, take $A/B = I_X$. We note that the quotient of two bounded operators is not necessarily bounded whose domain is $R(B)$ and its rang is $R(A)$. The question of boundedness, compactness and invertibility of quotient operators is very important and for the reader's convenience, let us summarize all what has been obtained in [17].

Theorem 6.1. [17] *Let $A,B \in \mathcal{L}(X)$ such that $N(B) \subset N(A)$. Then the following hold:*
(1) A/B is bounded if and only if $R(A^*) \subset R(B^*)$.
(2) If $R(B)$ is closed in X then A/B is bounded.
(3) If $R(B)$ is closed in X and B is invertible, then $A/B = AB^{-1}$.
(4) If A/B is compact then A is compact. Conversely, if $R(B)$ is closed in X and A is compact then A/B is also compact.
(5) If $N(A) = N(B)$, then A/B is invertible and $(A/B)^{-1} = B/A$.
(6) If $N(A) = N(B)$ and $R(A)$ is closed in X, then A/B has a bounded inverse B/A.
(7) A/B has an everywhere defined and bounded inverse if and only if the operator A is invertible in $\mathcal{L}(X)$ and $(A/B)^{-1} = B/A = BA^{-1}$.

The aim of this section is to give some fundamental characterizations of the spectrum of quotient operators using the basic spectral properties of linear operator pencils. Note here that this is the first time where the notion of the spectrum of a quotient of two operators is studied by using the theory of linear operator pencils.

Remark 6.2. If $N(B) \subset N(A)$ then $N(B) \subset N(A-\lambda B)$ and $[(A/B) - \lambda I_X]$ is well defined by $(A-\lambda B)/B$ for all $\lambda \in \mathbb{C}$, then by property (7) of Theorem 6.1, we can write

$$\begin{aligned} \varrho\left(A/B\right) &= \{\lambda \in \mathbb{C} \ : \ (A-\lambda B)/B \text{ is invertible in } \mathcal{L}(X)\} \\ &= \{\lambda \in \mathbb{C} \ : \ (A-\lambda B) \text{ is invertible in } \mathcal{L}(X)\} = \varrho\left(A,B\right) \end{aligned} \tag{6.1}$$

and

$$\sigma\left(A/B\right) = \widetilde{\sigma}\left(A,B\right) \tag{6.2}$$

Thus, if $\lambda \in \varrho\left(A/B\right)$, then

$$\left[(A/B) - \lambda I_X\right]^{-1} = B(A-\lambda B)^{-1} = R_{\lambda,l}(A,B). \tag{6.3}$$

Using the results of the previous sections, we obtain the following properties on the spectra of quotient operators through those previously established on a pair of bounded linear operators.

Theorem 6.3. *Let $A,B \in \mathcal{L}(X)$ such that $N(B) \subset N(A)$. Then*
(1) If $0 \in \varrho\left(A\right)$ then $0 \in \varrho\left(A,B\right)$.
(2) If $N(A) = N(B)$ and $R(A)$ is closed in X, then $0 \in \varrho\left(A,B\right)$.
(3) If $R(B)$ is closed in X and A is compact, then

a) $\widetilde{\sigma}(A/B) = \{0\} \cup \{\lambda_j \ : \ j \in J\}$, *where either* $J = \emptyset$, *or* $J = \mathbb{N}$, *or* $J = \{1, ..., n\}$ *for some* $n \in \mathbb{N}$.
b) $\widetilde{\sigma}(A/B) \setminus \{0\} = \sigma_p(A, B)$. *Each* λ_j *is an eigenvalue having a finite multiplicity.*
c) If $J = \mathbb{N}$, *then* $(\lambda_j) \longrightarrow 0$ *as* $j \to \infty$. *This means that for all* $\varepsilon > 0$, *the set* $\sigma(A/B) \setminus D(0, \varepsilon)$ *is finite where* $D(0, \varepsilon) = \{\lambda \in \mathbb{C} \ : \ |\lambda| < \varepsilon\}$.
(4) If B *is compact and* $\varrho(A/B) \neq \emptyset$, *then* $\widetilde{\sigma}(A/B)$ *is at most countable and consists only of eigenvalues of finite algebraic multiplicity which accumulate at most at infinity.*

Remark 6.4. This is a first attempt to establish the link between the spectral theory of quotient operators and linear operator pencils. Our results give rise to other interesting perspectives on the study of quotients operators.

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