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Statistical Inference of Stress-Strength Reliability for Burr Distributions Based on Ranked Set Sampling

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Abstract. A fundamental issue in several studies is the need for cost-effective sampling, particularly when measuring a significant characteristic is expensive, uncomfortable, or time-consuming. In terms of precision achieved per unit of sample, the ranked set sampling (RSS) approach offers a practical way to achieve observational economy. In the current work, ten frequentist estimation strategies are considered for the reliability of the stress strength parameter $\bar{\lambda} = P[T < Z]$, where T and Z are independent random variables following the Burr III and Burr XII distributions, respectively, that share the same shape parameter. Percentiles and weighted least squares, Anderson-Darling, maximum likelihood, minimum spacing absolute log distance, least squares, Cram'er-von Mises, maximum product of spacing, right-tailed Anderson-Darling, and minimum spacing absolute distance are some recommended estimation methods for the RSS and simple random sample methods. The effectiveness of the proposed RSS-based approximations is evaluated using simulation work employing certain accuracy standards. We conclude that the maximum product spacing and percentile approaches are the lowest in the mean squared error values for the reliability estimate when compared to those of the other alternatives. Two real data sets that trade share data and the prices of the 31 distinct children's wooden toys are used to provide further findings.

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1. Introduction

McIntyre [1] initially proposed the idea of ranked set sampling (RSS) as a method for enhancing the accuracy of the sample mean as an estimator of the population mean. Ranked Set Sampling (RSS) is a suitable method when judging a collection of sampling units is fast, accurate, and cost-free, as stated by McIntyre [1]. This approach involves choosing m sets of size m each at random from a population and visually rating the units in each set, or using some other low-cost method. The unit with the lowest ranking among the first m units is chosen for actual quantification. The unit ranked second-lowest among the second batch of m units is selected for actual quantification. The procedure is repeated until the unit with the highest ranking is picked from the qth set for actual quantification. The RSS with size $m_1^* = mq$ is then obtained by repeating this method q times. The characters "m" and "q" stand for the set size and cycle count, respectively. The set size m should be regarded as being essential to the RSS process. In order to get additional details about the important variable, we would want to take the optimal value of m. Wolfe [2] mentioned that set sizes larger than five would almost certainly result in too many ranking mistakes, and as a result, they would not likely considerably improve the RSS's effectiveness. The RSS is used in a variety of fields, including medicine, agriculture, veterinary science, and forestry. A number of authors have used RSS in a variety of scientific fields, including environment and ecology (Al-Saleh and Al-Shrafat [3]; Tiwari and Pandey [4]). Application of RSS design in environmental investigations for real data set (Zamanzade and Mahdizadeh [5]), agriculture (McIntyre [1] and Husby et al. [6]), quality control (Al-Omari and Haq [7]), fisheries research (Wang et al. [8]), forestry (Halls and Dell [9]), medicine (Samawi and Al-Sagheer [10]), and engineering (Hassan et al. [11]).

The stress-strength (S-S) model, which was developed in reliability theory, represents the lifetime of a component with random strength Z and random stress T. When a component is put under more stress than it can handle, it breaks, and it works when $T < Z$. Hence, $\lambdabar = P[T < Z]$ is a measure of component reliability. Numerous engineering and life testing issues have used it successfully. The S-S model theory was initially put forward in Reference [12]. The S-S model's estimate for several independent distributions has been researched by numerous academics. For instance, the exponential distribution (Beg [13]), the gamma distribution (Constantine et al. [14]), the normal distribution (Downton ([15]), the Burr distribution (Awad and Gharraf [16]), the Weibull distribution (Kundu and Gupta [17]), the Frechet distribution (Abbas and Tang [18]), the generalized exponential distribution (Kundu and Gupta [19]), and the Exponentiated Weibull distribution (Hassan and Al-Sulami [20]).

Recently, numerous scholars assessed the S-S model's inference for several independent distributions using RSS data and some of its modifications. In [21] and [22], the S-S reliability estimate for independent exponential populations was examined. The estimate of S-S reliability for the independent Weibull distributions was studied by [23]. The median RSS (MRSS) and RSS approaches were used in [24] to analyze the estimator of $\lambdabar$ for independent exponentiated Pareto distributions. Reference [25] investigated the Bayesian and maximum likelihood (ML) estimates of the S-S reliability for the generalized exponential distribution. When stress and strength are independently distributed as a generalized inverse exponential from the MRSS scheme, [26] examined the S-S reliability estimator. The reliability estimator of λ for a generalized Pareto distribution has been provided by [27]. Reference [28] investigated the estimation of λ based on extreme RSS data when both stress and strength random variables have inverse Lomax distributions. Reference [29] examined the estimation of λ based on extreme RSS data when both stress and strength random variables have Burr XII distributions. The S-S reliability of the inverted Topp-Leone distribution was estimated using Bayesian and non-Bayesian methods by [30]. In Ref. [31], the estimate of λ for the generalized exponential distribution using neotric and MRSS data was examined. When the stress and the strength are both independent Burr Type X random variables, Reference [32] evaluated the point and interval estimate of λ based on RSS. Reference [33] looked at the estimation of λ when the stress and strength random variables have the same or different distributions. Reference [34] determined the unit Gompertz distribution's S-S reliability using a variety of estimation techniques based on RSS. Reference [35] investigated the S-S reliability for the inverted Kumaraswamy distribution based on RSS. Reference [36] examined the S-S reliability for the exponentiated exponential distribution based on advanced RSS designs.

In the literature, few publications have compared the ML technique with various classical methods, in the context of RSS, as was the case for the simple random sample (SRS). This paper's primary objective is to examine and contrast the performance of the ML estimators in the setting of RSS and SRS with nine other estimation methods, namely least squares (LS), the maximum product of spacing (MXPS), the method of Cramer-von-Mises (CM), Anderson-Darling (AD), percentiles (PC), minimum spacing absolute-log distance (MNSALD), right tail AD (RAD), minimum spacing log distance (MNSLD), and weighted LS (WLS). When the stress and the strength are independent Burr Type III (BIII) and Burr XII (BXII) random variables, respectively, our primary goal in this work is to estimate $\lambda = P[T < Z]$ utilizing ten various methodologies. We carried out an in-depth simulation analysis to evaluate the effectiveness of suggested estimates for various sample sizes using some accuracy criteria. Based on our simulation research, we provide a set of useful recommendations for selecting estimators when using RSS in the context of faulty ranking and small sample sizes. The utility of the proposed estimators is explained through an analysis of actual data sets.

The study has the following organizational structure: The significance of the BIII and BXII distributions and their applications are explained in Section 2. In the same section, the reliability parameter λ formula is also introduced. The ML estimate (MLE) of λ under RSS and SRS is given in Section 3. Section 4 discusses the S-S reliability estimate based on maximum and minimum product spacing methods. In Section 5, the AD, RTD, and CM techniques are used to get the S-S reliability estimate. Section 6 provides the reliability estimate based on the LS, PC, and WLS approaches. In Sections 7 and 8, simulation research and its application to real-world circumstances are explored, along with comparisons of RSS and SRS projections. The paper's argument comes to an end in Section 9.

2. Burr Type XII and Type III Distributions

Reference [37] described the twelve different types of cumulative distribution functions (CDFs) that make up the Burr system of distributions, which provide a wide range of density shapes. The selection of the suitable Burr distribution parameters, as highlighted by [38], encompasses the majority of the Pearson family's curve-shape properties. Due to its numerous applications in areas like reliability, failure time modeling, and acceptability sampling plans, the BXII model attracted greater attention from researchers. It was used by [39] to simulate inpatient costs in English hospitals. To create an economical statistical design of the control chart for the non-normally distributed data, [40] employed the BXII distribution. The estimation techniques based on the BXII distribution were provided by [41–47]. The probability density function (PDF) of the BXII distribution with two shape parameters $\vartheta > 0$, and $\omega_1 > 0$, is given by:

$$f(z) = \vartheta\omega_1 z^{\vartheta-1}\left(1+z^{\vartheta}\right)^{-\omega_1-1}, \qquad z>0. \tag{2.1}$$

The CDF for the BXII distribution is given by

$$F(z) = 1-\left(1+z^{\vartheta}\right)^{-\omega_1}, \qquad z>0. \tag{2.2}$$

On the other side, the BIII distribution is the inverse of the BXII distribution. Its properties have undergone thorough analysis and have been employed in a variety of scientific domains. In studies of income, wage, and wealth distributions, it was referred to as the Dagum distribution [48]. In the actuarial literature, it is referred to as the inverse Burr distribution [49]. In the field of meteorology, it is known as the Kappa distribution [50]. Additionally, this distribution has been used in financial, environmental, survivability, and reliability theory research [51–53]. The benefits and characteristics of the BIII distribution have been highlighted by [38] and [54]. Several writers gave the statistical inference of the BIII distribution, for instance, [55–57]. The PDF of the BIII distribution with shape parameters $\vartheta > 0$, and $\omega_2 > 0$, is given by:

$$g(t) = \vartheta\omega_2 t^{-\vartheta-1}\left(1+t^{-\vartheta}\right)^{-\omega_2-1}, \qquad t>0. \tag{2.3}$$

The CDF for the BIII distribution is given by

$$G(t) = \left(1+t^{-\vartheta}\right)^{-\omega_2}, \qquad t>0. \tag{2.4}$$

Let $Z \sim$ BXII (ϑ, ω_1), and $T \sim$BIII (ϑ, ω_2), are independent random variables with CDF $F_Z(.)$ and $G_T(.)$. The S-S reliability expression of$\lambda\!\!\!^{-}$ is then provided by

$$\begin{aligned}\bar{\lambda} &= \int_0^{\infty} f(z)G_T(z)dz = \int_0^{\infty} \vartheta\omega_1 z^{\vartheta-1}(1+z^{\vartheta})^{-\omega_1-1}\left(1+z^{-\vartheta}\right)^{-\omega_2}dz \\ &= \omega_1 \mathrm{B}\left(\omega_2+1,\omega_1\right) = \left[\frac{\Gamma(\omega_1+1)\Gamma(\omega_2+1)}{\Gamma(\omega_1+\omega_2+1)}\right],\end{aligned} \tag{2.5}$$

where $\Gamma(.)$ is the gamma function. The S-S parameter $\lambdabar$ depends on the shape parameters ω_1 and ω_2.

3. Maximum Likelihood Estimator of $\lambdabar$

This section provides the MLE of $\lambdabar$ when the strength and stress are independent random variables with BXII and BIII distributions for common shape parameter ϑ. The MLE of $\lambdabar$ is established based on RSS and SRS methods.

Let $Z_{h_1(h_1)c_1}$ be the order statistics (OS) of the h_1-th sample ($h_1 = 1, 2, ..., m_z$) of size m_z, in the c_1-th cycle ($c_1 = 1, 2, ..., q_z$) of size q_z from BXII (ω_1, ϑ). Also, let $T_{h_2(h_2)c_2}$ be the OS of the h_2-th sample ($h_2 = 1, 2, ..., m_t$) of m_t, in the c_2-th cycle ($c_2 = 1, 2, ..., q_t$) of size q_t from BIII (ω_2, ϑ). Here, $m_1^* = q_z m_z$, and $m_2^* = q_t m_t$, are the sample sizes of $Z_{h_1(h_1)c_1}$ and $T_{h_2(h_2)c_2}$, respectively, where m_z, m_t are the set sizes and q_z, q_t are the cycles number. In this study, we write $Z_{h_1c_1}$ and $T_{h_2c_2}$ instead of using $Z_{h_1(h_1)c_1}$ and $T_{h_2(h_2)c_2}$, for simplified form. If the ranking of the observations is perfect, the PDFs of $Z_{h_1c_1}$ and $T_{h_2c_2}$ are exactly the PDFs of h_1-th and h_2-th OS. Regarding this, the PDF of $Z_{h_1c_1}$ for $\omega_1, \vartheta, z_{h_1c_1} > 0$, is as bellow:

$$\begin{aligned} f_{Z_{h_1c_1}}(z_{h_1c_1}) &= \frac{m_z!}{(h_1-1)!(m_z-h_1)!} f(z_{h_1c_1}) \Big[F(z_{h_1c_1})\Big]^{h_1-1} \Big[1 - F(z_{h_1c_1})\Big]^{m_z-h_1} \\ &= \frac{m_z! \ \omega_1 \vartheta z_{h_1c_1}^{\vartheta-1}}{(h_1-1)!(m_z-h_1)!} \left(1 + z_{h_1c_1}^{\vartheta}\right)^{-\omega_1(m_z-h_1+1)-1} \left[1 - \left(1 + z_{h_1c_1}^{\vartheta}\right)^{-\omega_1}\right]^{h_1-1}. \end{aligned} \tag{3.1}$$

Similarly, the PDF of $T_{h_2c_2}$ for $\omega_2, \vartheta, t_{h_2c_2} > 0$, is given by

$$g_{T_{h_2c_2}}(z_{h_2c_2}) = \frac{m_t! \ \omega_2 \vartheta t_{h_2c_2}^{-\vartheta-1}}{(h_2-1)!(m_t-h_2)!} \left(1 + t_{h_2c_2}^{-\vartheta}\right)^{-(\omega_2 h_2+1)} \left[1 - \left(1 + t_{h_2c_2}^{-\vartheta}\right)^{-\omega_2}\right]^{m_t-h_2}. \tag{3.2}$$

In the case of the RSS approach with the perfect ranking assumption, the likelihood function (LF) based on (3.1) and (3.2) is determined as follows:

$$\begin{aligned} \ell \propto & \prod_{c_1=1}^{q_z} \prod_{h_1=1}^{m_z} \omega_1 \vartheta z_{h_1c_1}^{\vartheta-1} \left(1 + z_{h_1c_1}^{\vartheta}\right)^{-\omega_1(m_z-h_1+1)-1} \left[1 - \left(1 + z_{h_1c_1}^{\vartheta}\right)^{-\omega_1}\right]^{h_1-1} \\ & \times \prod_{c_2=1}^{q_t} \prod_{h_2=1}^{m_t} \omega_2 \vartheta t_{h_2c_2}^{-\vartheta-1} \left(1 + t_{h_2c_2}^{-\vartheta}\right)^{-(\omega_2 h_2+1)} \left[1 - \left(1 + t_{h_2c_2}^{-\vartheta}\right)^{-\omega_2}\right]^{m_t-h_2}. \end{aligned} \tag{3.3}$$

Following that, the log-LF of (3.3) is given by:

$$\begin{aligned} \ell_1 \propto & \ m_1^* \ln(\vartheta\omega_1) + \sum_{c_1=1}^{q_z} \sum_{h_1=1}^{m_z} \left\{(\vartheta-1) \ln z_{h_1c_1} - [\omega_1(m_z-h_1+1)-1] \ln\left(1 + z_{h_1c_1}^{\vartheta}\right)\right\} \\ & + \sum_{c_1=1}^{q_z} \sum_{h_1=1}^{m_z} (h_1-1) \ln\left[1 - \left(1 + z_{h_1c_1}^{\vartheta}\right)^{-\omega_1}\right] \ m_2^* \ln(\vartheta\omega_2) + \sum_{c_2=1}^{q_t} \sum_{h_2=1}^{m_t} (m_t-h_2) \ln\left[1 - \left(1 + t_{h_2c_2}^{-\vartheta}\right)^{-\omega_2}\right] \\ & - \sum_{c_2=1}^{q_t} \sum_{h_2=1}^{m_t} \left\{(\vartheta+1) \ln t_{h_2c_2} + (\omega_2 h_2+1) \ln\left(1 + t_{h_2c_2}^{-\vartheta}\right)\right\}. \end{aligned} \tag{3.4}$$

Assuming that the shape parameter ϑ is known, we can derive the likelihood equations below by taking the first derivative of (3.4) with respect to ω_1 and ω_2

$$\frac{\partial \ell_1}{\partial \omega_1} = \frac{m_1^*}{\omega_1} + \sum_{c_1=1}^{q_z}\sum_{h_1=1}^{m_z} \frac{(h_1-1)\log\left(1+z_{h_1c_1}^{\vartheta}\right)}{\left[\left(1+z_{h_1c_1}^{\vartheta}\right)^{\omega_1}-1\right]} - \sum_{c_1=1}^{q_z}\sum_{h_1=1}^{m_z}\left[(m_z-h_1+1)\right]\log\left(1+z_{h_1c_1}^{\vartheta}\right), \tag{3.5}$$

and

$$\frac{\partial \ell_1}{\partial \omega_2} = \frac{m_2^*}{\omega_2} - \sum_{c_2=1}^{q_t}\sum_{h_2=1}^{m_t}\left\{h_2\log\left(1+t_{h_2c_2}^{-\vartheta}\right)\right\} + \sum_{c_2=1}^{q_t}\sum_{h_2=1}^{m_t}\frac{(m_t-h_2)\log\left(1+t_{h_2c_2}^{-\vartheta}\right)}{\left[1-\left(1+t_{h_2c_2}^{-\vartheta}\right)^{-\omega_2}\right]}. \tag{3.6}$$

It is obvious that the solutions to (3.5) and (3.6) cannot be directly determined. As a result, we use iterative techniques to obtain the MLE $\hat{\omega}_1$, of ω_1, and $\hat{\omega}_2$, of ω_2. Therefore, the MLE$\hat{\lambda}$ ofλ according to the invariance property of the MLE is given by

$$\hat{\lambda} = \left[\frac{\Gamma(\hat{\omega}_1+1)\Gamma(\hat{\omega}_2+1)}{\Gamma(\hat{\omega}_1+\hat{\omega}_2+1)}\right].$$

After that, the MLE$\tilde{\lambda}$ ofλ is obtained in the case of the SRS. Suppose that $Z_{k_1}, k_1 = (1,2,...,m_1^*)$ and $k_2 = (1,2,...,m_2^*)$ be a two independent SRS from BXII$\sim(\omega_1,\vartheta)$ and BIII$\sim(\omega_2,\vartheta)$ respectively. The LF in this situation is given by:

$$\ell_2 = \prod_{k_1=1}^{m_1^*}\vartheta\omega_1 z_{k_1}^{\vartheta-1}\left(1+z_{k_1}^{\vartheta}\right)^{-\omega_1-1}\prod_{k_2=1}^{m_2^*}\vartheta\omega_2 t_{k_2}^{-\vartheta-1}\left(1+t_{k_2}^{-\vartheta}\right)^{-\omega_2-1}. \tag{3.7}$$

The log-LF based on SRS is then supplied for (3.7) by

$$\begin{aligned}\ell_3 = {} & m_1^*\log(\vartheta\omega_1) + (\vartheta-1)\sum_{k_1=1}^{m_1^*}\log z_{k_1} - (\omega_1+1)\sum_{k_1=1}^{m_1^*}\log\left(1+z_{k_1}^{\vartheta}\right) + m_2^*\log(\vartheta\omega_2) \\ & -(\vartheta+1)\sum_{k_2=1}^{m_2^*}\log t_{k_2} - (\omega_2+1)\sum_{k_2=1}^{m_2^*}\log\left(1+t_{k_2}^{-\vartheta}\right).\end{aligned} \tag{3.8}$$

By determining the first derivative of (3.8) with respect to ω_1, ω_2 and considering that the shape parameter ϑ is known, we can get the likelihood equations below:

$$\frac{\partial \ell_3}{\partial \omega_1} = \frac{m_1^*}{\omega_1} - \sum_{k_1=1}^{m_1^*}\log\left(1+z_{k_1}^{\vartheta}\right), \qquad \frac{\partial \ell_3}{\partial \omega_2} = \frac{m_2^*}{\omega_2} - \sum_{k_2=1}^{m_2^*}\log\left(1+t_{k_2}^{-\vartheta}\right). \tag{3.9}$$

Setting the non-linear equations (3.9) above with zero and solved them using numerical method, the MLE $\tilde{\omega}_1$, of ω_1, and $\tilde{\omega}_2$, of ω_2, are presented as the results. Consequently, the MLE$\tilde{\lambda}$ ofλ using SRS is obtained by putting $\tilde{\omega}_1, \tilde{\omega}_2$ in (2.5), according to the invariance property.

4. Product of Spacings Estimators ofλ

In this section, three estimators of λ, namely, MXPS estimate (MXPSE), MNSALD estimate (MNSALDE), and MNSLD estimate (MNSLDE) are produced when Z and T are independent random variables that follow the BXII and BIII distributions, respectively. These estimators are offered using RSS and SRS techniques under the assumption that the shape parameter is known.

A. The Maximum Product of Spacings

The MXPS approach was presented by [58, 59] as an alternative for the ML technique for estimating the parameters of continuous distributions. They demonstrated that regardless of whether ML estimation exists or not, the MXPE technique offers consistent and asymptotically efficient estimators.

Let $Z_{(1:m_1^*)}, Z_{(2:m_1^*)}, ..., Z_{(m_1^*:m_1^*)}$ be an OS of RSS drawn from BXII distribution with sample size $m_1^* = q_z m_z$, where m_z is set size and q_z is the cycle numbers. The uniform spacings of size $m_1^* = q_z m_z$, is defined as:

$\gamma_{i_1}(\omega_1) = F\left(z_{(i_1:m_1^*)}\right) - F\left(z_{(i_1-1:m_1^*)}\right), \quad i_1 = 1, 2, ..., m_1^*$, where $F\left(z_{(0:m_1^*)}\right) = 0, \quad F\left(z_{(m_1^*+1:m_1^*)}\right) = 1$, such that $\sum_{i_1=1}^{m_1^*+1} \gamma_{i_1}(\omega_1) = 1$.

Similarly, suppose that $T_{(1:m_2^*)}, T_{(2:m_2^*)}, ..., T_{(m_2^*:m_2^*)}$ be an OS of RSS drawn from BIII distribution with sample size $m_2^* = q_t m_t$, where m_t is set size and q_t is the cycle numbers. The uniform spacings of size $m_2^* = q_t m_t$, is defined as $\gamma_{i_2}(\omega_2) = G\left(t_{(i_2:m_2^*)}\right) - G\left(t_{(i_2-1:m_2^*)}\right), \quad i_2 = 1, 2, ..., m_2^*$, where $G\left(t_{(0:m_2^*)}\right) = 0, \; G\left(t_{(m_2^*+1:m_2^*)}\right) = 1$, such that $\sum_{i_2=1}^{m_2^*+1} \gamma_{i_2}(\omega_2) = 1$.

Suppose that the shape parameter ϑ is known, then the MXPSE $\hat{\omega}_1^{\text{MX}}$ of ω_1, and the MXPSE $\hat{\omega}_2^{\text{MX}}$ of ω_2, is provided by maximizing the following geometric mean of the spacing $\left\{\prod_{i_1=1}^{m_1^*+1} \gamma_{i_1}(\omega_1)\right\}^{1/(m_1^*+1)} \left\{\prod_{i_2=1}^{m_2^*+1} \gamma_{i_2}(\omega_2)\right\}^{1/(m_2^*+1)}$, or alternatively, by maximizing the function $\frac{1}{m_1^*+1}\left\{\sum_{i_1=0}^{m_1^*+1} \ln[\gamma_{i_1}(\omega_1)]\right\} + \frac{1}{m_2^*+1}\left\{\sum_{i_2=0}^{m_2^*+1} \ln[\gamma_{i_2}(\omega_2)]\right\}$. Thus, the MXPSE $\hat{\omega}_1^{\text{MX}}$ of ω_1, and $\hat{\omega}_2^{\text{MX}}$ of ω_2, can also be obtained by numerically resolving the following nonlinear equations:

$$\frac{1}{m_1^*+1}\left\{\sum_{i_1=0}^{m_1^*+1} \frac{1}{\gamma_{i_1}(\omega_1)}\left[\left(1+z_{(i_1:m_1^*)}^{\vartheta}\right)^{-\omega_1}\ln(1+z_{(i_1:m_1^*)}^{\vartheta}) - \left(1+z_{(i_1-1:m_1^*)}^{\vartheta}\right)^{-\omega_1}\ln(1+z_{(i_1-1:m_1^*)}^{\vartheta})\right]\right\} = 0, \tag{4.1}$$

and

$$\frac{1}{m_2^*+1}\left\{\sum_{i_2=0}^{m_2^*+1} \frac{1}{\gamma_{i_2}(\omega_2)}\left[\left(1+t_{(i_2-1:m_2^*)}^{-\vartheta}\right)^{-\omega_2}\ln(1+t_{(i_2-1:m_2^*)}^{-\vartheta}) - \left(1+t_{(i_2:m_2^*)}^{-\vartheta}\right)^{-\omega_2}\ln(1+t_{(i_2:m_2^*)}^{-\vartheta})\right]\right\} = 0. \tag{4.2}$$

Thus, it is possible to determine the MXPSE $\hat{\hbar}^{\text{MX}}$ of $\hbar$ inserting $\hat{\omega}_1^{\text{MX}}$ and $\hat{\omega}_2^{\text{MX}}$ produced from (4.1) and (4.2) in (2.5).

In addition to above, the MXPSE $\tilde{\hbar}^{\text{MX}}$ of $\hbar$ in case of SRS is obtained. Let $Z_{(1)}, Z_{(2)}, ..., Z_{(m_1^*)}$ be an OS of SRS drawn from BXII distribution with sample size m_1^*. The uniform spacings is defined as $\ddot{\gamma}_{i_1}(\omega_1) = F\left(z_{(i_1)}\right) - F\left(z_{(i_1-1)}\right), \quad i_1 = 1, 2, ..., m_1^*$, where $F\left(z_{(0)}\right) = 0, \quad F\left(z_{(m_1^*+1)}\right) = 1$, such that $\sum_{i_1=1}^{m_1^*+1} \ddot{\gamma}_{i_1}(\omega_1) = 1$. Similarly, suppose that $T_{(1)}, T_{(2)}, ..., T_{(m_2^*)}$ be an OS of SRS drawn from

BIII distribution with sample size m_2^*. The uniform spacings is defined as $\ddot{\gamma}_{i_2}(\omega_2) = G\left(t_{(i_2)}\right) - G\left(t_{(i_2-1)}\right)$, $i_2 = 1, 2, ..., m_2^*$, where $G\left(t_{(0)}\right) = 0$, $G\left(t_{(m_2^*+1)}\right) = 1$, such that $\sum_{i_2=1}^{m_2^*+1} \ddot{\gamma}_{i_2}(\omega_2) = 1$.

Suppose that the shape parameter ϑ is known, then the MXPSE $\ddot{\omega}_1^{\text{MX}}$ of ω_1, and $\ddot{\omega}_2^{\text{MX}}$ of ω_2, is provided by maximizing the following $\frac{1}{m_1^*+1}\left\{\sum_{i_1=0}^{m_1^*+1} \ln[\ddot{\gamma}_{i_1}(\omega_1)]\right\} + \frac{1}{m_2^*+1}\left\{\sum_{i_2=0}^{m_2^*+1} \ln[\ddot{\gamma}_{i_2}(\omega_2)]\right\}$. Thus, the MXPSE $\ddot{\omega}_1^{\text{MX}}$ and $\ddot{\omega}_2^{\text{MX}}$ can also be obtained by numerically resolving the following nonlinear equations:

$$\frac{1}{m_1^*+1}\left\{\sum_{i_1=0}^{m_1^*+1} \frac{1}{\ddot{\gamma}_{i_1}(\omega_1)}\left[(1+z_{(i_1)}^{\vartheta})^{-\omega_1}\ln(1+z_{(i_1)}^{\vartheta}) - (1+z_{(i_1-1)}^{\vartheta})^{-\omega_1}\ln(1+z_{(i_1-1)}^{\vartheta})\right]\right\} = 0, \tag{4.3}$$

and

$$\frac{1}{m_2^*+1}\left\{\sum_{i_2=0}^{m_2^*+1} \frac{1}{\ddot{\gamma}_{i_2}(\omega_2)}\left[(1+t_{(i_2-1:m_2^*)}^{-\vartheta})^{-\omega_2}\ln(1+t_{(i_2-1:m_2^*)}^{-\vartheta}) - (1+t_{(i_2:m_2^*)}^{-\vartheta})^{-\omega_2}\ln(1+t_{(i_2:m_2^*)}^{-\vartheta})\right]\right\} = 0. \tag{4.4}$$

Thus, it is possible to determine the MXPSE $\tilde{\lambda}^{\text{MX}}$ of λ by inserting $\tilde{\omega}_1^{\text{MX}}$ and $\tilde{\omega}_2^{\text{MX}}$ produced from (4.3) and (4.4) in (2.5).

B. The Minimum Product of Spacings

Here, the S-S reliability estimators of λ based on the MNSLD and MNSALD methods are provided. To do so, it must obtain the estimators of ω_1 and ω_2. The idea for the minimal spacing distance estimate method was first put out by [60]. These estimators are generated by minimizing the following functions,

$$\sum_{i_1=1}^{m_1^*+1} \Xi\left[\gamma_{i_1}(\omega_1), H_1(m_1^*)\right], \qquad \sum_{i_2=1}^{m_2^*+1} \Xi\left[\gamma_{i_2}(\omega_2), H_2(m_2^*)\right],$$

where $\Xi(a_1, a_2)$ is suitable distance, $H_j(m_j^*) = \frac{1}{m_j^*+1}$, $j = 1, 2$. Some choices are absolute distance $\Xi(a_1, a_2) = |a_1 - a_2|$ and log absolute distance $\Xi(a_1, a_2) = \left|\log a_1 - \log a_2\right|$. As a result, the MNSLDE $\hat{\omega}_1^{\text{MNA}}, \hat{\omega}_2^{\text{MNA}}$ can be obtained by minimizing the following functions with respect to ω_1, and ω_2

$$\sum_{i_1=1}^{m_1^*+1} \left|\gamma_{i_1}(\omega_1) - H_1(m_1^*)\right|, \qquad \sum_{i_2=1}^{m_2^*+1} \left|\gamma_{i_2}(\omega_2) - H_2(m_2^*)\right|. \tag{4.5}$$

Equivalently to (4.5), the MNSADE $\hat{\omega}_1^{\text{MNA}}, \hat{\omega}_2^{\text{MNA}}$ is provided by solving the nonlinear equations

$$\sum_{i_1=1}^{m_1^*+1} \frac{\gamma_{i_1}(\omega_1) - H_1(m_1^*)}{\left|\gamma_{i_1}(\omega_1) - H_1(m_1^*)\right|}\left[(1+z_{(i_1:m_1^*)}^{\vartheta})^{-\omega_1}\ln(1+z_{(i_1:m_1^*)}^{\vartheta}) - (1+z_{(i_1-1:m_1^*)}^{\vartheta})^{-\omega_1}\ln(1+z_{(i_1-1:m_1^*)}^{\vartheta})\right] = 0,$$

and

$$\sum_{i_2=1}^{m_2^*+1} \frac{\gamma_{i_2}(\omega_2) - H_2(m_2^*)}{\left|\gamma_{i_2}(\omega_2) - H_2(m_2^*)\right|}\left[(1+t_{(i_2-1:m_2^*)}^{-\vartheta})^{-\omega_2}\ln(1+t_{(i_2-1:m_2^*)}^{-\vartheta}) - (1+t_{(i_2:m_2^*)}^{-\vartheta})^{-\omega_2}\ln(1+t_{(i_2:m_2^*)}^{-\vartheta})\right] = 0.$$

Equivalently, the MNSLDE $\hat{\omega}_1^{\text{MNL}}, \hat{\omega}_2^{\text{MNL}}$ of ω_1, ω_2 can be obtained by minimizing the following functions

$$\sum_{i_1=1}^{m_1^*+1} \left|\log \gamma_{i_1}(\omega_1) - \log H_1(m_1^*)\right|, \qquad \sum_{i_2=1}^{m_2^*+1} \left|\log \gamma_{i_2}(\omega_2) - \log H_2(m_2^*)\right|. \tag{4.6}$$

Alternative to (4.6), $\hat{\omega}_1^{\text{MNL}}, \hat{\omega}_2^{\text{MNL}}$ can be obtained by minimizing the following equations:

$$\sum_{i_1=1}^{m_1^*+1} \frac{\log \gamma_{i_1}(\omega_1) - \log H_1(m_1^*)}{\left|\log \gamma_{i_1}(\omega_1) - \log H_1(m_1^*)\right|} \frac{\left[(1+z^{\vartheta}_{(i_1:m_1^*)})^{-\omega_1} \ln(1+z^{\vartheta}_{(i_1:m_1^*)}) - (1+z^{\vartheta}_{(i_1-1:m_1^*)})^{-\omega_1} \ln(1+z^{\vartheta}_{(i_1-1:m_1^*)})\right]}{\gamma_{i_1}(\omega_1)} = 0,$$

and

$$\sum_{i_2=1}^{m_2^*+1} \frac{\gamma_{i_2}(\omega_2) - H_2(m_2^*)}{\left|\log \gamma_{i_2}(\omega_2) - \log H_2(m_2^*)\right|} \frac{\left[(1+t^{-\vartheta}_{(i_2-1:m_2^*)})^{-\omega_2} \ln(1+t^{-\vartheta}_{(i_2-1:m_2^*)}) - (1+t^{-\vartheta}_{(i_2:m_2^*)})^{-\omega_2} \ln(1+t^{-\vartheta}_{(i_2:m_2^*)})\right]}{\gamma_{i_2}(\omega_2)} = 0.$$

Hence, the MNSADE$\hat{\lambda}^{\text{MNA}}$ and MSALDE$\hat{\lambda}^{\text{MNL}}$ are obtained as follows

$$\hat{\lambda}^{\text{MNA}} = \left[\frac{\Gamma(\hat{\omega}_1^{\text{MNA}}+1)\Gamma(\hat{\omega}_2^{\text{MNA}}+1)}{\Gamma(\hat{\omega}_1^{\text{MNA}}+\hat{\omega}_2^{\text{MNA}}+1)}\right], \quad \hat{\lambda}^{\text{MNL}} = \left[\frac{\Gamma(\hat{\omega}_1^{\text{MNL}}+1)\Gamma(\hat{\omega}_2^{\text{MNL}}+1)}{\Gamma(\hat{\omega}_1^{\text{MNL}}+\hat{\omega}_2^{\text{MNL}}+1)}\right].$$

To obtain the MNSADE$\tilde{\lambda}^{\text{MNA}}$ and MNSLDE$\tilde{\lambda}^{\text{MNL}}$ based on SRS, we apply the above procedure by using the following OS of SRS $Z_{(1)}, Z_{(2)}, ..., Z_{(m_1^*)}$ and $T_{(1)}, T_{(2)}, ..., T_{(m_2^*)}$.

5. Minimum Distances Estimators of λ

Three estimation techniques, namely, CM, AD, and RAD for λ that are based on the minimization of the goodness-of-fit statistics with respect to λ are shown in this section. The difference between the estimated CDF and the empirical CDF serves as the foundation for this class of statistics. Let $Z_{(1:m_1^*)}, Z_{(2:m_1^*)}, ..., Z_{(m_1^*:m_1^*)}$ be an OS of RSS drawn from BXII distribution with sample size $m_1^* = q_z m_z$.

Also, suppose that $T_{(1:m_2^*)}, T_{(2:m_2^*)}, ..., T_{(m_2^*:m_2^*)}$ be an OS of RSS drawn from the BIII distribution with sample size $m_2^* = q_t m_t$. The CM estimate (CME) $\hat{\omega}_1^{\text{CM}}$, $\hat{\omega}_2^{\text{CM}}$ of ω_1, ω_2 are produced after minimizing the following functions:

$$\begin{aligned} \aleph &= \frac{-1}{12m_1^*} - \sum_{i_1=1}^{m_1^*} \left[1 - \left(1+z^{\vartheta}_{(i_1:m_1^*)}\right)^{-\omega_1} - H'_1(m_1, i_1)\right]^2, \\ \aleph_1 &= \frac{-1}{12m_2^*} + \sum_{i_2=1}^{m_2^*} \left[\left(1+t^{-\vartheta}_{(i_2:m_2^*)}\right)^{-\omega_2} - H'_2(m_2, i_2)\right]^2, \end{aligned} \tag{5.1}$$

where $H'_j(m_j, i_j) = \frac{2j-1}{2m_j^*}$, $j = 1, 2$. These estimates can also be achieved by simultaneously resolving the following equations.

$$\sum_{i_1=1}^{m_1^*} \left[1 - \left(1+z^{\vartheta}_{(i_1:m_1^*)}\right)^{-\omega_1} - H'_1(m_1, i_1)\right]\left(1+z^{\vartheta}_{(i_1:m_1^*)}\right)^{-\omega_1} \ln\left(1+z^{\vartheta}_{(i_1:m_1^*)}\right) = 0, \tag{5.2}$$

and

$$\sum_{i_2=1}^{m_2^*}\left[\left(1+t_{(i_2:m_2^*)}^{-\vartheta}\right)^{-\omega_2}-H'_2(m_2,i_2)\right]\left(1+t_{(i_2:m_2^*)}^{-\vartheta}\right)^{-\omega_2}\ln\left(1+t_{(i_2:m_2^*)}^{-\vartheta}\right)=0. \tag{5.3}$$

The estimate $\hat{\lambda}^{\mathrm{CM}}$ based on CM approach is subsequently given after putting the findings of $\hat{\omega}_1^{\mathrm{CM}}$ and $\hat{\omega}_2^{\mathrm{CM}}$ from (5.2) and (5.3) in (2.5).

Second, after minimizing the following functions: the AD estimate (ADE) $\hat{\omega}_1^{\mathrm{AD}}, \hat{\omega}_2^{\mathrm{AD}}$ of ω_1, ω_2 are produced after minimizing the following functions:

$$\mathbb{Q}=-m_1^*-\frac{1}{m_1^*}\sum_{i_1=1}^{m_1^*}(2i_1-1)\left[\log\left[1-\left(1+z_{(i_1:m_1^*)}^{\vartheta}\right)^{-\omega_1}\right]+\log\left[\left(1+z_{(1+m_1^*-i_1:m_1^*)}^{\vartheta}\right)^{-\omega_1}\right]\right],$$

and

$$\mathbb{Q}_1=-m_2^*-\frac{1}{m_2^*}\sum_{i_2=1}^{m_2^*}(2i_2-1)\left[\log\left(1+t_{(i_2:m_2^*)}^{-\vartheta}\right)^{-\omega_2}+\log\left[1-\left(1+t_{(m_2^*+i_2-1:m_2^*)}^{-\vartheta}\right)^{-\omega_2}\right]\right].$$

Inserting $\hat{\omega}_1^{\mathrm{AD}}, \hat{\omega}_2^{\mathrm{AD}}$ in (2.5), the reliability estimator $\hat{\lambda}^{\mathrm{AD}}$ based on AD method is generated.

Thirdly, the following functions are minimized to generate $\hat{\omega}_1^{\mathrm{RAD}}, \hat{\omega}_2^{\mathrm{RAD}}$ of ω_1, ω_2

$$\mathbb{Q}'=\frac{m_1^*}{2}-2\sum_{i_1=1}^{m_1^*}1-\left(1+z_{(i_1:m_1^*)}^{\vartheta}\right)^{-\omega_1}-\frac{1}{m_1^*}\sum_{i_1=1}^{m_1^*}(2i_1-1)\log\left[\left(1+z_{(m_1^*+1-i_1:m_1^*)}^{\vartheta}\right)^{-\omega_1}\right],$$

and

$$\mathbb{Q}_1'=\frac{m_2^*}{2}-2\sum_{i_2=1}^{m_2^*}\left(1+t_{(i_2:m_2^*)}^{-\vartheta}\right)^{-\omega_2}-\frac{1}{m_2^*}\sum_{i_2=1}^{m_2^*}(2i_2-1)\left[\log\left[1-\left(1+t_{(1+m_2^*-i_2:m_2^*)}^{-\vartheta}\right)^{-\omega_2}\right]\right].$$

Inserting $\hat{\omega}_1^{\mathrm{RAD}}, \hat{\omega}_2^{\mathrm{RAD}}$ in (2.5), the reliability estimator $\hat{\lambda}^{\mathrm{RAD}}$ based on RAD method is generated. Finally, the SRS procedure is used to create reliability estimators, $\tilde{\lambda}^{\mathrm{CM}}$, $\tilde{\lambda}^{\mathrm{AD}}$, and $\tilde{\lambda}^{\mathrm{RAD}}$ based on CM, AD, and RAD methods, which are produced in a manner similar to that described above.

6. Other Estimators

This section offers another three more estimators that are used to create the reliability estimator of λ utilizing the LS, WLS, and PC techniques.

Let $Z_{(1:m_1^*)}, Z_{(2:m_1^*)}, ..., Z_{(m_1^*:m_1^*)}$ be an OS of RSS drawn from BXII distribution with sample size $m_1^*=q_z m_z$. Also, suppose that $T_{(1:m_2^*)}, T_{(2:m_2^*)}, ..., T_{(m_2^*:m_2^*)}$ be an OS of RSS drawn from the BIII distribution with sample size $m_2^*=q_t m_t$. The LS estimate (LSE) $\hat{\omega}_1^{\mathrm{L}}$, $\hat{\omega}_2^{\mathrm{L}}$ of ω_1, ω_2 are produced after minimizing the following functions

$$\sum_{i_1=1}^{m_1^*}\left[1-\left(1+z_{(i_1:m_1^*)}^{\vartheta}\right)^{-\omega_1}-\frac{i_1}{m_1^*-1}\right]^2,\quad \sum_{i_2=1}^{m_2^*}\left[\left(1+t_{(i_2:m_2^*)}^{-\vartheta}\right)^{-\omega_2}-\frac{i_2}{m_2^*-1}\right]^2.$$

Hence, the LSE $\hat{\omega}_1^{\mathrm{L}}$ and $\hat{\omega}_2^{\mathrm{L}}$ is the solution of the following equations:

$$\sum_{i_1=1}^{m_1^*}\left[1-\left(1+z_{(i_1:m_1^*)}^{\vartheta}\right)^{-\omega_1}-\frac{i_1}{m_1^*-1}\right]\left(1+z_{(i_1:m_1^*)}^{\vartheta}\right)^{-\omega_1}\ln\left(1+z_{(i_1:m_1^*)}^{\vartheta}\right)=0, \tag{6.1}$$

and

$$\sum_{i_2=1}^{m_2^*}\left[\left(1+t_{(i_2:m_2^*)}^{-\vartheta}\right)^{-\omega_2}-\frac{i_2}{m_2^*-1}\right]\left(1+t_{(i_2:m_2^*)}^{-\vartheta}\right)^{-\omega_2}\ln\left(1+t_{(i_2:m_2^*)}^{-\vartheta}\right)=0. \tag{6.2}$$

Consequently, by inserting (6.1) and (6.2) in (2.5), one may determine the LSE$\hat{\lambda}^{\mathrm{L}}$ ofλ based on RSS.

By minimizing the following functions, the WLS estimator (WLSE) $\hat{\omega}_1^{\mathrm{W}}$ of ω_1 and $\hat{\omega}_2^{\mathrm{W}}$ of ω_2 are generated

$$\begin{aligned}&\sum_{i_1=1}^{m_1^*}\dddot{H}(m_1^*,i_1)\left[1-\left(1+z_{(i_1:m_1^*)}^{\vartheta}\right)^{-\omega_1}-\frac{i_1}{m_1^*-1}\right]^2,\\&\sum_{i_2=1}^{m_2^*}\dddot{H}(m_2^*,i_2)\left[\left(1+t_{(i_2:m_2^*)}^{-\vartheta}\right)^{-\omega_2}-\frac{i_2}{m_2^*-1}\right]^2,\end{aligned} \tag{6.3}$$

where $\dddot{H}(m_j^*,i_j)=\frac{(m_j^*+1)^2(m_j^*+2)}{i_j(m_j^*-i_j+1)}$, $j=1,2$.

Instead of (6.3), $\hat{\omega}_1^{\mathrm{W}}$ and $\hat{\omega}_2^{\mathrm{W}}$ are created by solving non-linear equations

$$\sum_{i_1=1}^{m_1^*}\dddot{H}(m_1^*,i_1)\left[1-\left(1+z_{(i_1:m_1^*)}^{\vartheta}\right)^{-\omega_1}-\frac{i_1}{m_1^*-1}\right]\left(1+z_{(i_1:m_1^*)}^{\vartheta}\right)^{-\omega_1}\ln\left(1+z_{(i_1:m_1^*)}^{\vartheta}\right)=0, \tag{6.4}$$

and

$$\sum_{i_2=1}^{m_2^*}\dddot{H}(m_2^*,i_2)\left[\left(1+t_{(i_2:m_2^*)}^{-\vartheta}\right)^{-\omega_2}-\frac{i_2}{m_2^*-1}\right]\left(1+t_{(i_2:m_2^*)}^{-\vartheta}\right)^{-\omega_2}\ln\left(1+t_{(i_2:m_2^*)}^{-\vartheta}\right)=0. \tag{6.5}$$

As a result, by including (6.4) and (6.5) in (2.5), it is possible to calculate the WLSE$\hat{\lambda}^{\mathrm{W}}$ ofλ using RSS.

Furthermore, the PC estimate (PCE) $\hat{\omega}_1^{\mathrm{P}}$ of ω_1 and $\hat{\omega}_2^{\mathrm{P}}$ of ω_2 are generated by minimizing the following functions and assuming that $p_{i_j}=\frac{i_j}{m_j^*+1}$, $j=1,2$ is the estimate of $F\left(z_{(i_1:m_1^*)}\right)$ and $G\left(t_{(i_2:m_2^*)}\right)$, respectively,

$$\mathrm{P}_1=\sum_{i_1=1}^{m_1^*}\left[z_{(i_1:m_1^*)}-\left([1-p_{(i_1:m_1^*)}]^{-\omega_1}-1\right)^{1/\vartheta}\right],\quad \mathrm{P}_2=\sum_{i_2=1}^{m_2^*}\left[t_{(i_2:m_2^*)}-\left((p_{(i_1:m_1^*)})^{-\omega_2}-1\right)^{-1/\vartheta}\right].$$

The PCE$\hat{\lambda}^{\mathrm{W}}$ ofλ is produced after inserting $\hat{\omega}_1^{\mathrm{P}}$ and $\hat{\omega}_2^{\mathrm{P}}$ in (2.5).

Finally, suppose that $Z_{(1)},Z_{(2)},...,Z_{(m_1^*)}$ be an ordered SRS of size m_1^* drawn from BXII distribution, and $T_{(1)},T_{(2)},...,T_{(m_2^*)}$ be an ordered SRS of size m_2^* drawn from BIII distribution. The LSE$\tilde{\lambda}^{\mathrm{L}}$, WLSE $\tilde{\lambda}^{\mathrm{W}}$ and PCE$\tilde{\lambda}^{\mathrm{P}}$ are obtained using the above similar procedures.

7. Simulation-Based Research

This section performs a comprehensive simulation analysis to investigate the performance of various reliability estimators employed in the proposed methodologies. The analysis utilizes a suite of accuracy metrics, including mean, absolute bias (AB), mean squared error (MSE), and relative absolute bias (RAB). In addition, the efficiency (Eff) of each estimator is calculated. The simulation process itself follows a defined sequence of stages, which will be outlined in detail hereafter.

- The true parameter values of $(\omega_1, \omega_2, \vartheta)$ are selected as (1, 1, 3), (0.6,0.4,3), (0.7, 1, 3), (0.4, 0.5, 3), and (0.3, 0.3, 3). The corresponding true values of λ are as follows: 0.5, 0.72, 0.731, 0.844, and 0.901. For all trials, the number of cycles is selected to be $q_z = q_t = 10$.
- The RSS of Z and T, represented by $Z_{1(1)c_1}, Z_{2(2)c_1}, ..., Z_{m_z(m_z)c_1}$ in the c_1-th cycle ($c_1 = 1, 2, \ldots, q_z$) of size q_z and $T_{1(1)c_2}, T_{2(2)c_2}, ..., T_{m_t(m_t)c_2}$ in the c_2-th cycle ($c_2 = 1, 2, \ldots, q_t$) of size q_t, where the set sizes are $(m_z, m_t) =$ (3,3), (4,4) and (5,5). Hence, the sample sizes are $(m_1^*, m_2^*) =$ (30,30), (40,40), (50, 50).
- Under the selected estimation procedures, the estimates of parameters as well as their reliability estimates $\hat{\lambda}^{\text{ML}}$,$\hat{\lambda}^{\text{CM}}$,$\hat{\lambda}^{\text{L}}$,$\hat{\lambda}^{\text{W}}$,$\hat{\lambda}^{\text{MX}}$,$\hat{\lambda}^{\text{MNA}}$,$\hat{\lambda}^{\text{MNL}}$,$\hat{\lambda}^{\text{RAD}}$,$\hat{\lambda}^{\text{AD}}$ and $\hat{\lambda}^{P}$ are calculated.
- The mean, AB, RAB, MSE, and Eff of different S-S reliability estimates are calculated using the following relations:

$$mean = \frac{1}{1000}\sum_{i=1}^{1000}\hat{\lambda}_i^*,\ \text{AB} = \frac{1}{1000}\sum_{i=1}^{1000}\left|\hat{\lambda}_i^* - \lambda\right|,\ RAB = \frac{1}{1000}\sum_{i=1}^{1000}\left|\frac{\hat{\lambda}_i^* - \lambda}{\lambda}\right|,$$

$$\text{MSE} = \frac{1}{1000}\sum_{i=1}^{1000}\left(\hat{\lambda}_i^* - \lambda\right)^2, Eff(\lambda^*) = \frac{MSE(\hat{\lambda}^*)}{MSE(\tilde{\lambda}^*)}.$$

The estimation procedures' results are presented in two sets of tables. Tables 1, 2, 3, 4, and 5 summarize the mean, AB, RAB, and MSE , and Eff for various estimation methods under RSS, while Tables 6, 7, 8, 9, and 10 provide the same statistics for the estimation methods under SRS.

The efficiency of MSE estimates under RSS relative to the MSE estimates under SRS for different S-S reliability estimates is listed in Table 11.

Based on the numerical results shown in Tables 1-11 and Figures 1, 2, 3 and 4, the following conclusions can be drawn:

- The MSE of $\hat{\lambda}^{\text{ML}}$ at $\lambda = 0.5, 0.731, 0.844,$ and 0.901 for different set sizes have the highest values as seen in Figures 1-4.
- Figure 2 and 4 show that the MSE of $\hat{\lambda}^{\text{MX}}$ has the least values while the MSE of $\hat{\lambda}^{\text{ML}}$ has the highest values at different set sizes, for $\lambda = 0.731$ and 0.844.
- In most of the cases, as seen in Tables 1-5, the MSEs of $\hat{\lambda}^{\text{ML}}$,$\hat{\lambda}^{\text{CM}}$,$\hat{\lambda}^{\text{L}}$,$\hat{\lambda}^{\text{W}}$,$\hat{\lambda}^{\text{MX}}$,$\hat{\lambda}^{\text{MNA}}$,$\hat{\lambda}^{\text{MNL}}$,$\lambda^{\hat{\text{RAD}}}$,$\hat{\lambda}^{\text{AD}}$ and $\hat{\lambda}^{P}$ decrease with increases value of λ.

- Tables 1-5 show that $\hat{\lambda}^{CM}$ have the greatest RAB values among all other methods of estimation for all values of λ except $\lambda = 0.731$ while $\hat{\lambda}^{ML}$ have the least mean values among all other methods of estimation for $\lambda = 0.5,\ 072,\ 0.731,\ 0.844,\ 0.901$.
- From Table 11 it can be deduced that S-S reliability estimates via RSS are more efficient than the corresponding estimates under SRS based on their MSEs using different estimation methods.

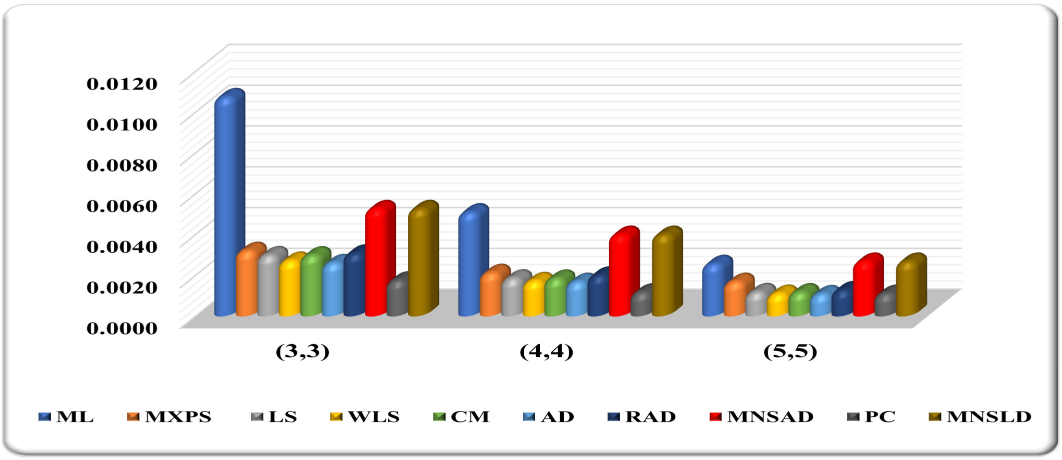

Figure 1. MSE of the reliability estimates based on RSS at different set sizes using various methods of estimation at $\lambda = 0.5$

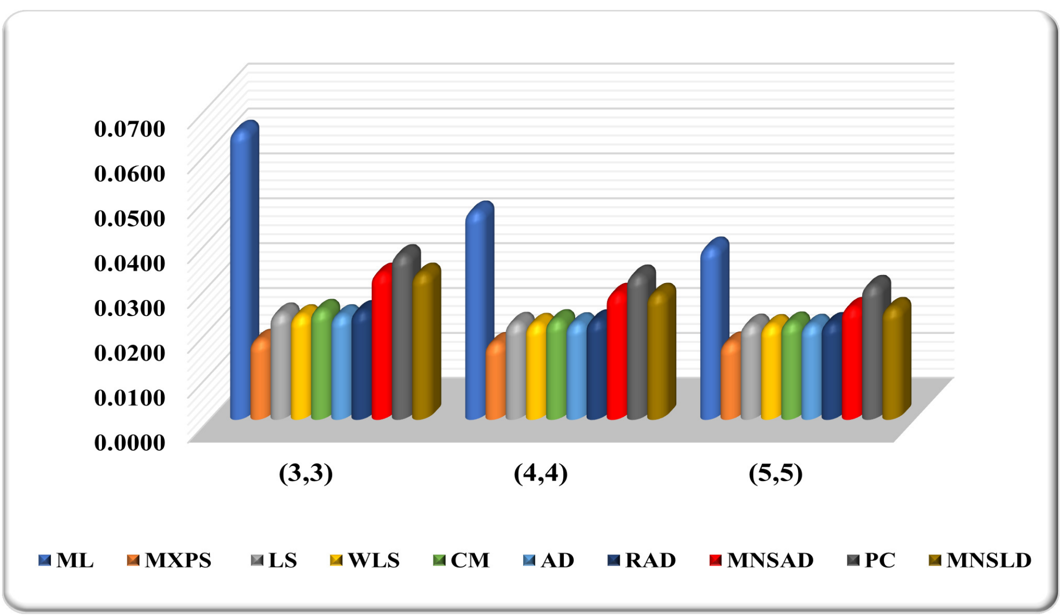

Figure 2. MSE of the reliability estimates based on RSS at different set sizes using various methods of estimation at $\lambda = 0.731$

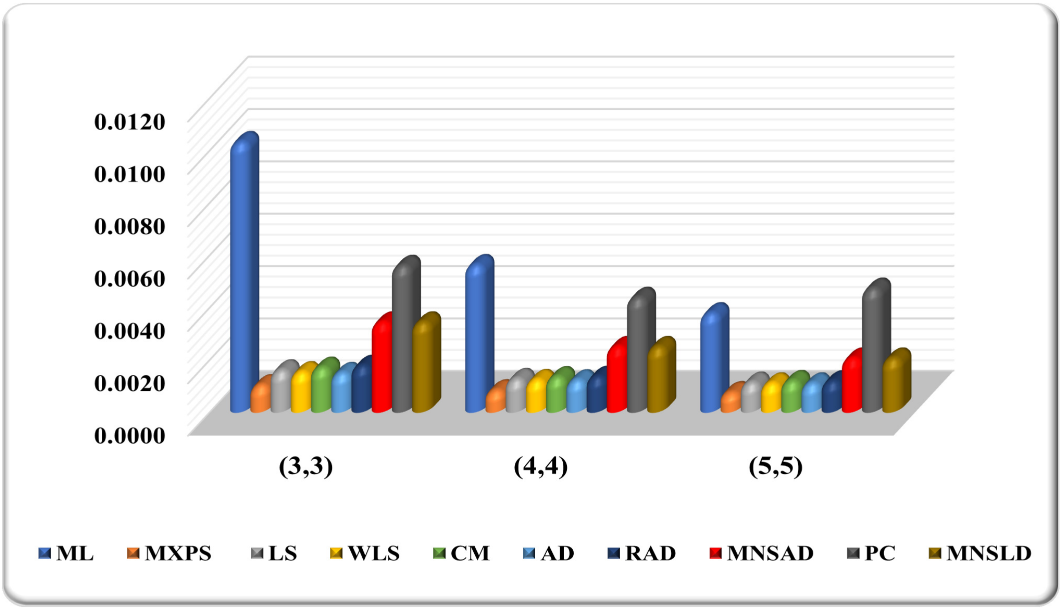

Figure 3. MSE of the reliability estimates based on RSS at different set sizes using various methods of estimation at $\bar{\lambda} = 0.844$

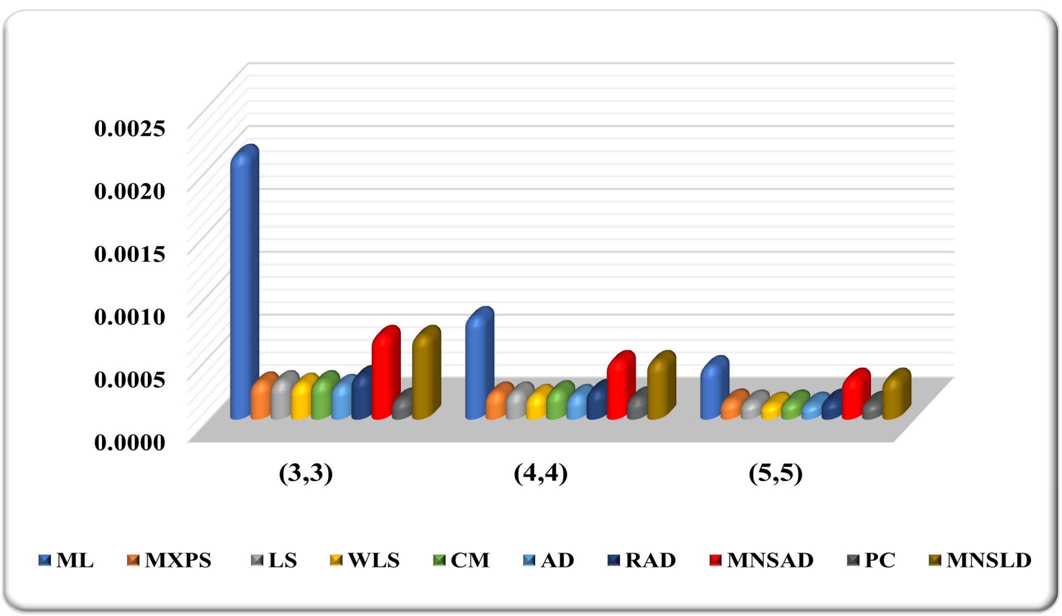

Figure 4. MSE of the reliability estimates based on RSS at different set sizes using various methods of estimation at $\bar{\lambda} = 0.901$

Table 1. Statistical indicators of the estimate's reliability for various techniques of estimation based on RSS at $\lambdabar = 0.5$

(m_1^*, m_2^*)		ML	MXPS	LS	WLS	CM	AD	RAD	MNSAD	PC	MNSLD
(3,3)	AB	0.0897	0.02356	0.00232	0.0025	0.0007	0.00204	0.00011	0.01682	0.00994	0.02358
	mean	0.4103	0.52356	0.50232	0.5025	0.4993	0.50204	0.49989	0.48318	0.49006	0.52358
	MSE	0.01064	0.0031	0.00289	0.00262	0.00288	0.0025	0.00297	0.00515	0.00165	0.00515
	RAB	0.1794	0.04712	0.00464	0.005	1.02509	0.00407	0.00022	0.03364	0.01988	0.04716
(4,4)	AB	0.05779	0.01847	0.00173	0.00135	0.00058	0.0011	0.00017	0.01554	0.01244	0.0185
	mean	0.44221	0.51847	0.50173	0.50135	0.49942	0.5011	0.49983	0.48446	0.51244	0.5185
	MSE	0.00499	0.00202	0.00175	0.00162	0.00174	0.00158	0.00189	0.00389	0.00106	0.00389
	RAB	0.11559	0.03694	0.00346	0.0027	1.01446	0.00221	0.00034	0.03108	0.02488	0.037
(5,5)	AB	0.0379	0.01965	0.00338	0.00348	0.00154	0.00307	0.00152	0.01082	0.00403	0.01968
	mean	0.4621	0.51965	0.50338	0.50348	0.50154	0.50307	0.50152	0.48918	0.50403	0.51968
	MSE	0.00246	0.00158	0.00108	0.001	0.00107	0.00098	0.00117	0.00257	0.001	0.00257
	RAB	0.0758	0.0393	0.00675	0.00696	1.0023	0.00614	0.00304	0.02163	0.00806	0.03935

Table 2. Statistical indicators of the estimate's reliability for various techniques of estimation based on RSS at $\lambdabar = 0.72$

(m_1^*, m_2^*)		ML	MXPS	LS	WLS	CM	AD	RAD	MNSAD	PC	MNSLD
(3,3)	AB	0.02648	0.08526	0.07312	0.07364	0.07134	0.07315	0.07087	0.05786	0.05828	0.08526
	mean	0.74602	0.8048	0.79265	0.79318	0.79087	0.79269	0.79041	0.77739	0.77782	0.8048
	MSE	0.00208	0.0081	0.00632	0.00632	0.00607	0.00622	0.00614	0.00524	0.00852	0.00524
	RAB	0.03681	0.1185	0.10162	0.10235	0.16554	0.10167	0.0985	0.08041	0.081	0.1185
(4,4)	AB	0.04997	0.08405	0.07354	0.07364	0.07218	0.07339	0.07218	0.06205	0.06149	0.08405
	mean	0.76951	0.80358	0.79308	0.79317	0.79172	0.79293	0.79172	0.78159	0.78102	0.80358
	MSE	0.0032	0.00761	0.00602	0.00599	0.00583	0.00594	0.00589	0.00521	0.0076	0.00521
	RAB	0.06945	0.11681	0.10221	0.10234	0.16049	0.102	0.10032	0.08624	0.08546	0.11681
(5,5)	AB	0.06072	0.08212	0.07346	0.07332	0.07236	0.07324	0.07251	0.06564	0.06323	0.08211
	mean	0.78026	0.80165	0.79299	0.79285	0.7919	0.79277	0.79205	0.78517	0.78277	0.80165
	MSE	0.00411	0.00716	0.0058	0.00574	0.00564	0.00573	0.00569	0.00523	0.00731	0.00523
	RAB	0.08439	0.11412	0.10209	0.1019	0.1603	0.10178	0.10078	0.09122	0.08788	0.11412

Table 3. Statistical indicators of the estimate's reliability for various techniques of estimation based on RSS at $ƛ = 0.731$

(m_1^*, m_2^*)		**ML**	**MXPS**	**LS**	**WLS**	**CM**	**AD**	**RAD**	**MNSAD**	**PC**	**MNSLD**
(3,3)	AB	0.24752	0.12309	0.14341	0.14286	0.14619	0.14329	0.14492	0.16516	0.15888	0.12309
	mean	0.48398	0.6084	0.58808	0.58864	0.5853	0.58821	0.58657	0.56634	0.57261	0.6084
	MSE	0.06367	0.01709	0.02295	0.02256	0.02376	0.02256	0.02348	0.03184	0.03594	0.03184
	RAB	0.33837	0.16827	0.19606	0.1953	0.02959	0.19588	0.19811	0.22578	0.2172	0.16828
(4,4)	AB	0.21057	0.12281	0.13974	0.13937	0.14186	0.14001	0.14158	0.15644	0.15283	0.12281
	mean	0.52092	0.60869	0.59175	0.59213	0.58963	0.59148	0.58992	0.57505	0.57866	0.60869
	MSE	0.04567	0.01638	0.02086	0.02064	0.02146	0.02078	0.02144	0.02735	0.03141	0.02735
	RAB	0.28787	0.16788	0.19104	0.19052	0.04001	0.19141	0.19354	0.21386	0.20893	0.16788
(5,5)	AB	0.1917	0.12462	0.13979	0.13974	0.14149	0.14002	0.14072	0.14835	0.14952	0.12462
	mean	0.53979	0.60687	0.59171	0.59175	0.59	0.59148	0.59078	0.58315	0.58197	0.60688
	MSE	0.03765	0.01651	0.02047	0.02037	0.02095	0.02044	0.02081	0.02418	0.02911	0.02418
	RAB	0.26207	0.17036	0.1911	0.19103	0.04264	0.19141	0.19237	0.2028	0.2044	0.17036

Table 4. Statistical indicators of the estimate's reliability for various techniques of estimation based on RSS at $ƛ = 0.844$

(m_1^*, m_2^*)		**ML**	**MXPS**	**LS**	**WLS**	**CM**	**AD**	**RAD**	**MNSAD**	**PC**	**MNSLD**
(3,3)	AB	0.0953	0.01621	0.02716	0.02704	0.02879	0.02716	0.02883	0.04126	0.03803	0.01621
	mean	0.74894	0.82804	0.81709	0.81721	0.81546	0.81708	0.81542	0.80299	0.80621	0.82804
	MSE	0.01022	0.00093	0.0015	0.00143	0.0016	0.0014	0.00167	0.00333	0.00549	0.00333
	RAB	0.11289	0.0192	0.03217	0.03203	0.51719	0.03217	0.03414	0.04887	0.04505	0.0192
(4,4)	AB	0.07029	0.01745	0.02644	0.02648	0.02767	0.02661	0.02749	0.03669	0.03197	0.01745
	mean	0.77395	0.8268	0.8178	0.81776	0.81658	0.81763	0.81676	0.80756	0.81227	0.8268
	MSE	0.00552	0.00073	0.00117	0.00113	0.00124	0.00112	0.00125	0.00239	0.00427	0.00239
	RAB	0.08326	0.02067	0.03132	0.03137	0.52306	0.03152	0.03256	0.04345	0.03787	0.02067
(5,5)	AB	0.05737	0.01826	0.0262	0.02622	0.02718	0.02637	0.02675	0.03393	0.03289	0.01825
	mean	0.78687	0.82599	0.81805	0.81802	0.81706	0.81788	0.81749	0.81032	0.81135	0.82599
	MSE	0.00368	0.00065	0.00103	0.001	0.00109	0.001	0.00108	0.00195	0.00462	0.00195
	RAB	0.06796	0.02162	0.03103	0.03106	0.52306	0.03123	0.03169	0.04019	0.03896	0.02162

Table 5. Statistical indicators of the estimate's reliability for various techniques of estimation based on RSS at ƛ $= 0.901$

(m_1^*, m_2^*)		**ML**	**MXPS**	**LS**	**WLS**	**CM**	**AD**	**RAD**	**MNSAD**	**PC**	**MNSLD**
(3,3)	AB	0.03988	0.00671	0.00053	0.00033	0.00152	0.00042	0.00201	0.00756	0.00005	0.00671
	mean	0.86157	0.90815	0.90091	0.90112	0.89992	0.90102	0.89944	0.89389	0.90149	0.90815
	MSE	0.00208	0.00027	0.00028	0.00026	0.00028	0.00025	0.00032	0.00064	0.00014	0.00064
	RAB	0.04424	0.00744	0.00059	0.00036	0.66457	0.00046	0.00222	0.00838	0.00005	0.00744
(4,4)	AB	0.02345	0.00592	0.00002	0.00013	0.00072	0.00005	0.00074	0.00606	0.00114	0.00592
	mean	0.87799	0.90737	0.90146	0.90157	0.90072	0.9015	0.9007	0.89538	0.90031	0.90737
	MSE	0.00079	0.00019	0.00019	0.00017	0.0002	0.00017	0.00021	0.00044	0.00015	0.00044
	RAB	0.02601	0.00657	0.00002	0.00014	0.66555	0.00006	0.00082	0.00672	0.00126	0.00657
(5,5)	AB	0.01622	0.00492	0.00017	0.00006	0.00076	0.00019	0.00074	0.00521	0.00212	0.00492
	mean	0.88523	0.90637	0.90128	0.90138	0.90069	0.90125	0.9007	0.89623	0.89933	0.90637
	MSE	0.0004	0.00013	0.00012	0.0001	0.00012	0.0001	0.00013	0.0003	0.00011	0.0003
	RAB	0.01799	0.00546	0.00019	0.00007	0.66581	0.00021	0.00082	0.00578	0.00235	0.00546

Table 6. Statistical indicators of the estimate's reliability for various techniques of estimation based on SRS at ƛ $= 0.5$

(m_1^*, m_2^*)		**ML**	**MXPS**	**LS**	**WLS**	**CM**	**AD**	**RAD**	**MNSAD**	**PC**	**MNSLD**
(30,30)	AB	0.0912	0.01792	0.0028	0.00269	0.00574	0.00265	0.00396	0.03204	0.00931	0.01792
	mean	0.4088	0.51792	0.4972	0.49731	0.49426	0.49735	0.49604	0.46796	0.50931	0.51792
	MSE	0.01119	0.00427	0.00527	0.00484	0.00529	0.00461	0.00553	0.00814	0.00403	0.00814
	RAB	0.18241	0.03584	0.0056	0.00538	1.07837	0.00529	0.00792	0.06408	0.01862	0.03583
(40,40)	AB	0.07014	0.0161	0.0005	0.00063	0.00278	0.0007	0.00246	0.01725	0.0018	0.01609
	mean	0.42986	0.5161	0.4995	0.49937	0.49722	0.4993	0.49754	0.48275	0.4982	0.51609
	MSE	0.00635	0.00318	0.00399	0.00362	0.004	0.00348	0.00409	0.00543	0.00384	0.00543
	RAB	0.14028	0.0322	0.001	0.00127	1.03455	0.0014	0.00491	0.03451	0.0036	0.03219
(50,50)	AB	0.0053	0.01415	0.00089	0.00099	0.00271	0.00074	0.00287	0.01521	0.00464	0.01415
	mean	0.4947	0.51415	0.49911	0.49901	0.49729	0.49926	0.49713	0.48479	0.49536	0.51415
	MSE	0.00249	0.00262	0.0032	0.00291	0.00321	0.00283	0.00329	0.00437	0.00275	0.00437
	RAB	0.01061	0.02831	0.00178	0.00199	1.03069	0.00147	0.00575	0.03041	0.00928	0.0283

Table 7. Statistical indicators of the estimate's reliability for various techniques of estimation based on SRS at $\hbar = 0.72$

(m_1^*, m_2^*)		ML	MXPS	LS	WLS	CM	AD	RAD	MNSAD	PC	MNSLD
(30,30)	AB	0.06717	0.08374	0.07094	0.07128	0.06918	0.07133	0.06965	0.05451	0.06155	0.08374
	mean	0.7867	0.80327	0.79047	0.79081	0.78871	0.79086	0.78918	0.77405	0.78108	0.80327
	MSE	0.00611	0.00843	0.00706	0.00693	0.00683	0.00688	0.00696	0.00588	0.00949	0.00588
	RAB	0.09335	0.11638	0.09859	0.09906	0.14385	0.09913	0.09679	0.07576	0.08554	0.11638
(40,40)	AB	0.06863	0.08216	0.07182	0.07181	0.07047	0.0719	0.07003	0.06067	0.06134	0.08216
	mean	0.78816	0.80169	0.79136	0.79134	0.79001	0.79143	0.78957	0.78021	0.78088	0.80169
	MSE	0.00584	0.00777	0.00669	0.00655	0.00651	0.00649	0.00652	0.00559	0.00859	0.00559
	RAB	0.09538	0.11418	0.09982	0.0998	0.14706	0.09992	0.09733	0.08432	0.08525	0.11418
(50,50)	AB	0.06964	0.08117	0.07154	0.07171	0.07046	0.07182	0.07035	0.06313	0.06284	0.08117
	mean	0.78917	0.80071	0.79107	0.79125	0.79	0.79135	0.78988	0.78266	0.78237	0.8007
	MSE	0.00568	0.00736	0.00623	0.00614	0.00608	0.00612	0.00612	0.00544	0.00801	0.00544
	RAB	0.09678	0.11281	0.09943	0.09966	0.15801	0.09981	0.09777	0.08774	0.08733	0.11281

Table 8. Statistical indicators of the estimate's reliability for various techniques of estimation based on SRS at $\hbar = 0.731$

(m_1^*, m_2^*)		ML	MXPS	LS	WLS	CM	AD	RAD	MNSAD	PC	MNSLD
(30,30)	AB	0.24852	0.12836	0.1484	0.14817	0.15115	0.14808	0.14973	0.17032	0.16085	0.12836
	mean	0.48297	0.60313	0.58309	0.58332	0.58034	0.58342	0.58177	0.56117	0.57064	0.60313
	MSE	0.06404	0.01993	0.02686	0.02643	0.0277	0.02619	0.02717	0.03543	0.03781	0.03543
	RAB	0.33974	0.17548	0.20287	0.20256	0.01093	0.20243	0.20469	0.23284	0.21989	0.17548
(40,40)	AB	0.21542	0.12798	0.14378	0.14381	0.14584	0.14347	0.14605	0.1623	0.15594	0.12798
	mean	0.51607	0.60351	0.58771	0.58769	0.58565	0.58803	0.58544	0.56919	0.57555	0.60351
	MSE	0.04789	0.019	0.024	0.02375	0.0246	0.02355	0.02474	0.03108	0.03417	0.03108
	RAB	0.2945	0.17496	0.19656	0.19659	0.02922	0.19613	0.19967	0.22188	0.21319	0.17496
(50,50)	AB	0.1932	0.12915	0.14352	0.14344	0.1452	0.14319	0.14469	0.15688	0.1525	0.12916
	mean	0.5383	0.60234	0.58797	0.58805	0.58629	0.5883	0.5868	0.57461	0.579	0.60234
	MSE	0.03828	0.01878	0.02339	0.02313	0.02388	0.02299	0.02375	0.02827	0.03211	0.02827
	RAB	0.26411	0.17656	0.19621	0.1961	0.02675	0.19575	0.1978	0.21447	0.20847	0.17657

Table 9. Statistical indicators of the estimate's reliability for various techniques of estimation based on SRS at $\lambda = 0.844$

(m_1^*, m_2^*)		ML	MXPS	LS	WLS	CM	AD	RAD	MNSAD	PC	MNSLD
(30,30)	AB	0.0947	0.01895	0.03002	0.02992	0.03164	0.02992	0.03151	0.04521	0.04145	0.01895
	mean	0.74955	0.8253	0.81423	0.81433	0.81261	0.81433	0.81273	0.79903	0.80279	0.8253
	MSE	0.01036	0.00147	0.00257	0.00242	0.00269	0.00234	0.00265	0.00443	0.00732	0.00443
	RAB	0.11217	0.02244	0.03555	0.03544	0.51087	0.03544	0.03733	0.05355	0.0491	0.02244
(40,40)	AB	0.07463	0.02057	0.03033	0.03008	0.03155	0.03006	0.03083	0.04331	0.0366	0.02058
	mean	0.76962	0.82367	0.81392	0.81417	0.81269	0.81418	0.81341	0.80093	0.80765	0.82367
	MSE	0.00626	0.00131	0.00218	0.00205	0.00227	0.00202	0.0022	0.00357	0.00508	0.00357
	RAB	0.08839	0.02437	0.03592	0.03562	0.5108	0.03561	0.03652	0.0513	0.04335	0.02437
(50,50)	AB	0.05979	0.01927	0.02785	0.02788	0.02883	0.02751	0.02859	0.03665	0.03674	0.01927
	mean	0.78446	0.82498	0.8164	0.81637	0.81542	0.81673	0.81566	0.80759	0.80751	0.82498
	MSE	0.0039	0.00105	0.00173	0.00165	0.00179	0.00159	0.00182	0.00258	0.00554	0.00258
	RAB	0.07082	0.02282	0.03299	0.03302	0.51894	0.03259	0.03386	0.04341	0.04352	0.02282

Table 10. Statistical indicators of the estimate's reliability for various techniques of estimation based on SRS at $\lambda = 0.901$

(m_1^*, m_2^*)		ML	MXPS	LS	WLS	CM	AD	RAD	MNSAD	PC	MNSLD
(30,30)	AB	0.03989	0.00332	0.00383	0.00373	0.00481	0.00355	0.00471	0.01179	0.00419	0.00332
	mean	0.86156	0.90476	0.89762	0.89772	0.89663	0.89789	0.89674	0.88966	0.89725	0.90476
	MSE	0.00217	0.00045	0.0007	0.00064	0.00072	0.0006	0.0007	0.00101	0.00041	0.00101
	RAB	0.04425	0.00368	0.00425	0.00414	0.65627	0.00394	0.00522	0.01308	0.00465	0.00368
(40,40)	AB	0.02427	0.00465	0.00094	0.00091	0.00167	0.00081	0.00138	0.00809	0.00014	0.00465
	mean	0.87717	0.9061	0.9005	0.90054	0.89977	0.90063	0.90007	0.89336	0.90131	0.9061
	MSE	0.00084	0.00032	0.00046	0.00042	0.00047	0.0004	0.00049	0.00069	0.00029	0.00069
	RAB	0.02693	0.00516	0.00105	0.00101	0.66173	0.0009	0.00153	0.00897	0.00015	0.00516
(50,50)	AB	0.00378	0.00372	0.00129	0.0013	0.00188	0.0012	0.00225	0.00697	0.00159	0.00372
	mean	0.90522	0.90517	0.90015	0.90015	0.89956	0.90025	0.89919	0.89448	0.90303	0.90517
	MSE	0.00043	0.00024	0.00034	0.00031	0.00035	0.0003	0.00037	0.0005	0.00025	0.0005
	RAB	0.00419	0.00413	0.00143	0.00144	0.66227	0.00133	0.0025	0.00773	0.00176	0.00413

Table 11. Efficiency comparison of RSS vs. SRS for reliability estimation

ƛ	set size	ML	MPS	LS	WLS	CM	AD	RAD	MNSAD	PC	MNSLD
0.5	(3,3)	0.951	0.726	0.548	0.541	0.544	0.542	0.537	0.633	0.409	0.633
	(4,4)	0.786	0.635	0.439	0.448	0.435	0.454	0.462	0.716	0.276	0.716
	(5,5)	0.988	0.603	0.338	0.344	0.333	0.346	0.356	0.588	0.364	0.588
0.72	(3,3)	0.34	0.961	0.895	0.912	0.889	0.904	0.882	0.891	0.898	0.891
	(4,4)	0.548	0.979	0.9	0.915	0.896	0.915	0.903	0.932	0.885	0.932
	(5,5)	0.724	0.973	0.931	0.935	0.928	0.936	0.93	0.961	0.913	0.961
0.731	(3,3)	0.994	0.858	0.854	0.854	0.858	0.861	0.864	0.899	0.951	0.899
	(4,4)	0.954	0.862	0.869	0.869	0.872	0.882	0.867	0.88	0.919	0.88
	(5,5)	0.984	0.879	0.875	0.881	0.877	0.889	0.876	0.855	0.907	0.855
0.844	(3,3)	0.986	0.633	0.584	0.591	0.595	0.598	0.63	0.752	0.75	0.752
	(4,4)	0.882	0.557	0.537	0.551	0.546	0.554	0.568	0.669	0.841	0.669
	(5,5)	0.944	0.619	0.595	0.606	0.609	0.629	0.593	0.756	0.834	0.756
0.901	(3,3)	0.959	0.6	0.4	0.406	0.389	0.417	0.457	0.634	0.341	0.634
	(4,4)	0.94	0.594	0.413	0.405	0.426	0.425	0.429	0.638	0.517	0.638
	(5,5)	0.93	0.542	0.353	0.323	0.343	0.333	0.351	0.6	0.44	0.6

8. Application

This section provides an application using two data sets to show how the stress-strength estimation using BXII and BIII distributions can be applied effectively in practice. In this application, the model parameters are estimated using the RSS approach using the following methods: ML, MXPS, LS, WLS, CM, AD, RAD, MNSAD, PC, and MNSLD. For visual comparison, the plots of the fitted PDFs, CDFs, Probability–Probability (P-P), and Quantile–Quantile (Q-Q) of the BXII and BIII distributions are shown, respectively. The required computations are carried out using the R software.

The first data set refers to trade share data from [61]. The prices of the 31 distinct children's wooden toys that were offered for sale in an April 1991 craft store in Suffolk comprised the second data set studied by [62]. The data sets considered are detailed as follows:

Data set I: (n = 61) 0.1405, 0.15662, 0.1577, 0.16041, 0.16082, 0.22146, 0.29941, 0.31307, 0.32461, 0.32475 0.32948, 0.33002, 0.33788, 0.33971, 0.35232, 0.35886, 0.39325, 0.4176, 0.42584, 0.43558 0.44214, 0.44438, 0.45055, 0.45577, 0.46835, 0.47326, 0.4846, 0.48895, 0.50959, 0.51767 0.52777, 0.53469, 0.54334, 0.54424, 0.55081, 0.55272, 0.56064, 0.56075, 0.56713, 0.57528 0.58281, 0.60304, 0.60503, 0.61362, 0.62608, 0.63948, 0.64691, 0.6512, 0.68156, 0.69943 0.70482, 0.72923, 0.74297, 0.7455, 0.77985, 0.79838, 0.81471, 0.82296, 0.83024, 0.8342, 0.97936.

Data set II: (n = 31) 4.2, 1.12, 1.39, 2, 3.99, 2.15, 1.74, 5.81, 1.7, 2.85, 0.5, 0.99, 11.5, 5.12, 0.9, 1.99, 6.24, 2.6, 3, 12.2, 7.36, 4.75, 11.59, 8.69, 9.8, 1.85, 1.99, 1.35, 10, 0.65, 1.45.

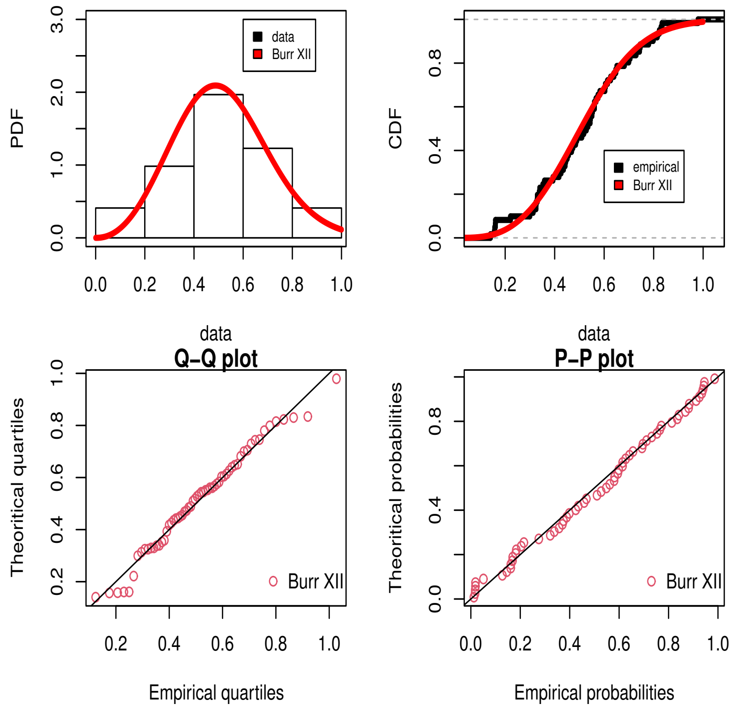

FIGURE 5. Estimated PDF, CDF, QQ-plot, and PP-plot for Burr XII according to data set I

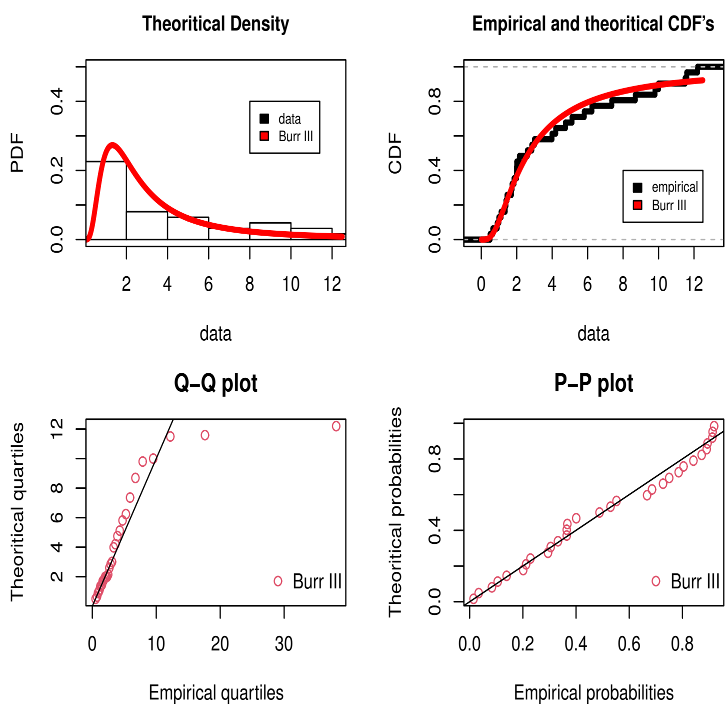

FIGURE 6. Estimated PDF, CDF, QQ-plot, and PP-plot for BIII according to data set II

Before commencing analysis of data sets I and II, goodness-of-fit tests are conducted to ensure they align well with the BXII and BIII distributions, respectively. The Q-Q plots are depicted in Figures 5 and 6 to visually assess this fit for each dataset. Additionally, the Kolmogorov-Smirnov test is computed for each dataset to obtain a statistical measure of the fit. The p-value for the first data set is 0.9546, indicating a good fit to the BXII distribution, while the second data set yields a p-value of 0.975, signifying a good fit to the BIII distribution. Using the previously described estimation techniques, the mean for S-S reliability is calculated for various set sizes, as shown in Table 12.

Table 12. Mean of $\hat{\lambda}^{ML}$, $\hat{\lambda}^{CM}$, $\hat{\lambda}^{L}$, $\hat{\lambda}^{W}$, $\hat{\lambda}^{MX}$, $\hat{\lambda}^{MNA}$, $\hat{\lambda}^{MNL}$, $\hat{\lambda}^{RAD}$, $\hat{\lambda}^{AD}$ and $\hat{\lambda}^{P}$ based on RSS at $q_z = q_t = 5$

(m_z, m_t)	**ML**	**MXPS**	**LS**	**WLS**	**CM**	**AD**	**RAD**	**MNSAD**	**PC**	**MNSLD**
(3,3)	0.0936	0.0481	0.0535	0.0304	0.0305	0.0517	0.0004	0.025	0.0021	0.0222
(4,4)	0.077	0.1099	0.0147	0.0048	0.0075	0.0165	0.004	0.0049	0.0017	0.0273
(5,5)	0.0845	0.0188	0.017	0.0022	0.0098	0.0253	0.0035	0.009	0.00003	0.009

Building upon the theoretical findings, the application of the RSS approach to real-world datasets where, the R package "RSSampling" is employed to generate the RSS design, which is then applied to datasets I and II. The analysis assumes the strength $Z \sim \text{BXII}(\vartheta, \omega_1)$, and the stress $T \sim \text{BIII}(\vartheta, \omega_2)$, where Z and T are independent. For each dataset and set size combination, the S-S estimates are computed using ten different methods (ML, MXPS, LS, WLS, CM, AD, RAD, MNSAD, PC, and MNSLD) across five analysis cycles.

9. Conclusion

This paper investigates S-S reliability $\lambda = P[T < Z]$ when stress (T) and strength (Z) are independent random variables following Burr III and Burr XII distributions, respectively, and both variables are drawn from an RSS design as well as SRS. Ten frequentist estimation methods are compared to assess their performance in estimating reliability. A numerical analysis is conducted to evaluate the behavior of these estimators based on metrics including the mean, AB, RAB, MSE, and Eff. The results suggest that estimators based on maximum product spacing and percentiles techniques provide the most reliable estimates under the RSS and SRS designs compared to other methods. Furthermore, the analysis indicates that estimators based on RSS generally outperform those based on SRS in terms of efficiency. Real-world data applications are presented to further support these findings.

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