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ANALYSIS OF NONLINEAR FRACTIONAL NABLA DIFFERENCE EQUATIONS

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Abstract. In this paper, we establish sufficient conditions on global existence and uniqueness of solutions of nonlinear fractional nabla difference systems and investigate the dependence of solutions on initial conditions and parameters.

1. Introduction

Discrete fractional calculus deals with sums and differences of arbitrary orders. Looking into the literature of fractional difference calculus, two approaches are found: one using the Δ - point of view (called the fractional delta difference approach) and another using the ∇ - perspective (called the nabla fractional difference approach). The theory for fractional nabla difference calculus was initiated by Gray and Zhang [18], Atici and Eloe [9] and Anastassiou [17], where basic approaches, definitions and properties of fractional sums and differences were reported. Recently, a series of papers continuing research on fractional nabla difference equations has appeared [10, 11, 12, 14, 16, 19, 20, 21, 22, 23, 26]. But a very little progress has been made to develop fractional nabla difference systems [11, 24].

In the following example, we illustrate the advantage of fractional order nabla difference system over integer order nabla difference system.

Example 1. *Consider the following two systems.*

$$\nabla_0 \mathbf{u}(t) = \beta t^{\overline{\beta-1}}, \quad 0 < \beta < 1, \quad t \in \mathbb{N}_1, \tag{1.1}$$

$$\nabla^{\alpha}_{0*} \mathbf{u}(t) = \beta t^{\overline{\beta-1}}, \quad 0 < \alpha < \beta < 1, \quad t \in \mathbb{N}_1, \tag{1.2}$$

where ∇^{α}_{0*} is the Caputo type fractional nabla difference operator.

The solution of (1.1) is given by

$$\mathbf{u}(t) = \mathbf{u}(0) + t^{\overline{\beta}}, \quad t \in \mathbb{N}_0, \tag{1.3}$$

Clearly (1.3) tends to ∞ as $t \to \infty$ for $0 < \beta < 1$ and thus it is unstable. But the solution of (1.2) is given by

$$\mathbf{u}(t) = \mathbf{u}(0) + \frac{\Gamma(1-\beta)}{\Gamma(1-\beta+\alpha)} t^{\overline{\alpha-\beta}}, \quad t \in \mathbb{N}_0. \tag{1.4}$$

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Clearly (1.4) tends to 0 as $t \to \infty$ for $0 < \alpha < \beta < 1$ and therefore it is stable, which implies that the fractional order system may have additional attractive feature over the integer order system.

On the other hand, several authors [13, 15, 31, 32] used fixed point theorems to discuss existence, uniqueness and stability properties of fractional differential systems. Motivated by this fact, in this paper, we initiate the study on global existence and uniqueness of solutions of nonlinear fractional nabla difference systems.

The present paper is organized as follows: Section 2 contains preliminaries on nabla discrete fractional calculus and functional analysis. We consider a system of nonlinear fractional nabla difference equations and obtain sufficient conditions on global existence and uniqueness of solutions and the dependence of solutions on initial conditions and parameters in sections 3 and 4 respectively.

2. Preliminaries

We shall use the following notations, definitions and known results of discrete fractional calculus [8, 9, 24, 29] throughout this article. For any a, $b \in \mathbb{R}$, $\mathbb{N}_a = \{a, a+1, a+2, ...\}$, $\mathbb{N}_{a,b} = \{a, a+1, a+2, ..., b\}$ where $a < b$.

Definition 2.1. For any α, $t \in \mathbb{R}$, the α rising function is defined by

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \quad t \in \mathbb{R} \setminus \{......, -2, -1, 0\}, \quad 0^{\overline{\alpha}} = 0.$$

We observe the following properties of rising factorial function.

Lemma 2.1. *Assume the following factorial functions are well defined.*

1. $t^{\overline{\alpha}}(t+\alpha)^{\overline{\beta}} = t^{\overline{\alpha+\beta}}$.
2. *If* $t \le r$ *then* $t^{\overline{\alpha}} \le r^{\overline{\alpha}}$.
3. *If* $\alpha < t \le r$ *then* $r^{\overline{-\alpha}} \le t^{\overline{-\alpha}}$.

Definition 2.2. Let $u : \mathbb{N}_a \to \mathbb{R}$, $\alpha \in \mathbb{R}^+$ and choose $N \in \mathbb{N}_1$ such that $N - 1 < \alpha < N$.

(1) (Nabla Difference) The first order backward difference or nabla difference of u is defined by

$$\nabla u(t) = u(t) - u(t-1), \quad t \in \mathbb{N}_{a+1},$$

and the N^{th} - order nabla difference of u is defined recursively by

$$\nabla^N u(t) = \nabla(\nabla^{N-1} u(t)), \quad t \in \mathbb{N}_{a+N}.$$

In addition, we take ∇^0 as the identity operator.

(2) (Fractional Nabla Sum) The α^{th} - order fractional nabla sum of u is given by

$$\nabla_a^{-\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{\alpha-1}} u(s), \quad t \in \mathbb{N}_a \tag{2.1}$$

where $\rho(s) = s - 1$. Also, we define the trivial sum by $\nabla_a^{-0} u(t) = u(t)$ for $t \in \mathbb{N}_a$.

(3) (R - L Nabla Fractional Difference) The α^{th} - order Riemann - Liouville type nabla fractional difference of u is given by

$$\nabla_a^{\alpha} u(t) = \nabla^N \Big[\nabla_a^{-(N-\alpha)} u(t)\Big], \quad t \in \mathbb{N}_{a+N}. \tag{2.2}$$

For $\alpha = 0$, we set $\nabla_a^0 u(t) = u(t),\ t \in \mathbb{N}_a$.

(4) (Caputo Fractional Nabla Difference) The α^{th} - order Caputo type fractional nabla difference of u is given by

$$\nabla_{a*}^{\alpha} u(t) = \nabla_a^{-(N-\alpha)} \Big[\nabla^N u(t)\Big], \quad t \in \mathbb{N}_{a+N}. \tag{2.3}$$

For $\alpha = 0$, we set $\nabla_{a*}^0 u(t) = u(t),\ t \in \mathbb{N}_a$.

Theorem 2.2. *(Power Rule) Let $\alpha > 0$ and $\mu > -1$. Then,*

(1) $\nabla_a^{-\alpha}(t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(t-a)^{\overline{\mu+\alpha}}, \quad t \in \mathbb{N}_a.$

(2) $\nabla_a^{\alpha}(t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}(t-a)^{\overline{\mu-\alpha}},\ t \in \mathbb{N}_{a+N}.$

Let $f : \mathbb{N}_a \times \mathbb{R} \to \mathbb{R}$, $u : \mathbb{N}_a \to \mathbb{R}$ and $0 < \alpha < 1$. Consider a nonautonomous fractional nabla difference equation of Riemann - Liouville type together with an initial condition of the form

$$\nabla_{a-1}^{\alpha} u(t) = f(t, u(t)), \quad t \in \mathbb{N}_{a+1}, \tag{2.4}$$

$$\nabla_{a-1}^{-(1-\alpha)} u(t)\Big|_{t=a} = u(a) = u_0. \tag{2.5}$$

Then, from [30], u is a solution of the initial value problem (2.4) - (2.5) if and only if it has the following representation

$$u(t) = \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} u_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} f(s, u(s)), \quad t \in \mathbb{N}_a. \tag{2.6}$$

If we consider a nonautonomous fractional nabla difference equation of Caputo type together with an initial condition of the form

$$\nabla_{a*}^{\alpha} u(t) = f(t, u(t)), \quad t \in \mathbb{N}_{a+1}, \tag{2.7}$$

$$u(a) = u_0. \tag{2.8}$$

Then, u is a solution of the initial value problem (2.7) - (2.8) if and only if it has the following representation

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} f(s, u(s)), \quad t \in \mathbb{N}_a. \tag{2.9}$$

Now we present some important definitions and theorems of functional analysis [3, 7] which will be useful in establishing main results.

Definition 2.3. $\mathbb{R}^n$ is the space of all ordered n-tuples of real numbers. Clearly, $\mathbb{R}^n$ is a Banach space with respect to the supremum norm. A closed ball with radius r centered at the origin of $\mathbb{R}^n$ is defined by

$$B_{\mathbf{0}}^{\infty}(r) = \{\mathbf{u} = (u_1, u_2, ..., u_n) \in \mathbb{R}^n : \|\mathbf{u}\|_\infty \le r\}.$$

Definition 2.4. $l^\infty = l^\infty(\mathbb{R})$ is the space of all real sequences defined on the set of positive integers where any individual sequence is bounded with respect to the usual supremum norm. Clearly l^∞ is a Banach space under the supremum norm. A closed ball with radius r centered on the null sequence of l^∞ is defined by

$$B_0^\infty(r) = \{u = \{u(t)\}_{t=0}^\infty \in l^\infty : \|u\|_\infty \le r\}.$$

Definition 2.5. A subset S of l^∞ is uniformly Cauchy (or equi - Cauchy), if for every $\epsilon > 0$, there exists $k \in \mathbb{N}_1$ such that $|u(t_1) - u(t_2)| < \epsilon$ whenever $t_1, t_2 \in \mathbb{N}_{k+1}$, for any $u = \{u(t)\}_{t=0}^\infty$ in S.

Theorem 2.3. *(Discrete Arzela - Ascoli's Theorem) A bounded uniformly Cauchy subset S of l^∞ is relatively compact.*

Theorem 2.4. *(Krasnoselskii's Fixed Point Theorem) Let S be a nonempty, closed, convex and bounded subset of a Banach space X, and let $A : X \to X$ and $B : S \to X$ be two operators such that*

(1) *A is a contraction with constant $L < 1$,*
(2) *B is continuous, BS resides in a compact subset of X,*
(3) *$[x = Ax + By,\ y \in S] \Longrightarrow x \in S$.*

Then the operator equation $Ax + Bx = x$ has a solution in S.

Theorem 2.5. *(Generalized Banach Fixed Point Theorem) Let S be a nonempty, closed subset of a Banach space $(X, \|.\|)$, and let a $\gamma_n \ge 0$ for every $n \in \mathbb{N}_0$ and such $\sum_{n=0}^\infty \gamma_n$ converges. Moreover, let the mapping $T : S \to S$ satisfy the inequality*

$$\|T^n u - T^n v\| \le \gamma_n \|u - v\|$$

for every $n \in \mathbb{N}_1$ and any $u, v \in S$. Then, T has a uniquely defined fixed point u^*. Furthermore, for any $u_0 \in S$, the sequence $(T^n u_0)_{n=1}^\infty$ converges to this fixed point u^*.

Theorem 2.6. *(Schauder Fixed Point Theorem) Let S be a nonempty, closed and convex subset of a Banach space X. Let $T : S \to S$ be a continuous mapping such that TS is a relatively compact subset of X. Then T has at least one fixed point in S. That is, there exists an $x \in S$ such that $Tx = x$.*

Definition 2.6. Let X be a Banach space with respect to a norm $\|.\|$. Define the set

$$\mathbb{S} = \mathbb{S}(X) = \{u : u = \{u(t)\}_{t=0}^\infty,\ u(t) \in X\}.$$

Then, $\mathbb{S}$ is a linear space of sequences of elements of X under obvious definition of addition and scalar multiplication. Now we employ the notation

$$u = \{u(t)\}_{t=0}^\infty, \quad \|u\|_\infty = \sup_{t \in \mathbb{N}_0} |u(t)|,$$

and define the set

$$\mathbb{S}^\infty(X) = \{u : u \in \mathbb{S}(X) \text{ with } \|u\|_\infty < \infty\}.$$

Clearly $\mathbb{S}^\infty(X)$ is a Banach space consisting of elements of $\mathbb{S}(X)$, with respect to the supremum norm.

Definition 2.7. From Definitions 2.4 and 2.6, we observe that $l^\infty = l^\infty(\mathbb{R}) = \mathbb{S}^\infty(\mathbb{R})$. Now we choose $X = \mathbb{R}^n$ in Definition 2.6 to define

$$\mathbf{l}^\infty = \mathbf{l}^\infty(\mathbb{R}^n) = \mathbb{S}^\infty(\mathbb{R}^n) = \{\mathbf{u} : \mathbf{u} = \{\mathbf{u}(t)\}_{t=0}^\infty, \mathbf{u}(t) \in \mathbb{R}^n \text{ with } \|\mathbf{u}\|_\infty < \infty\}.$$

Thus, $\mathbf{l}^\infty$ denotes the Banach space comprising sequences of vectors with respect to the supremum norm $\|.\|_\infty$ defined by

$$\|\mathbf{u}\|_\infty = \sup_{t\in\mathbb{N}_0} \|\mathbf{u}(t)\|.$$

A closed ball with radius r centered on the null sequence in $\mathbf{l}^\infty$ is defined by

$$B_{\mathbf{0}}^\infty(r) = \{\mathbf{u} = \{\mathbf{u}(t)\}_{t=0}^\infty \in \mathbf{l}^\infty : \|\mathbf{u}\|_\infty \le r\}.$$

3. Existence & Uniqueness

In this section we prove existence and uniqueness theorems pertaining to the initial value problems associated with a system of fractional nabla difference equations of the form

$$(3.1)\quad \nabla_{-1}^\alpha \mathbf{u}(t) = \mathbf{f}(t,\mathbf{u}(t)), \quad \nabla_{-1}^{-(1-\alpha)}\mathbf{u}(t)\Big|_{t=0} = \mathbf{u}(0) = \mathbf{c}, \quad 0<\alpha<1, \quad t\in\mathbb{N}_1$$

and

$$(3.2)\qquad \nabla_{0*}^\alpha \mathbf{u}(t) = \mathbf{f}(t,\mathbf{u}(t)), \quad \mathbf{u}(0) = \mathbf{c}, \quad 0<\alpha<1, \quad t\in\mathbb{N}_1,$$

where ∇_{-1}^α and ∇_{0*}^α are the Riemann - Liouville and Caputo type fractional difference operators, $\mathbf{u}(t)$ is an n-vector whose components are functions of the variable t, $\mathbf{c}$ is a constant n-vector and $\mathbf{f}(t,\mathbf{u}(t))$ is an n-vector whose components are functions of the variable t and the n-vector $\mathbf{u}(t)$.

Let $\mathbf{u} : \mathbb{N}_0 \to \mathbf{l}^\infty$ and $\mathbf{f} : \mathbb{N}_0 \times \mathbf{l}^\infty \to \mathbf{l}^\infty$. Analogous to (2.6), $\mathbf{u} = \{\mathbf{u}(t)\}_{t=0}^\infty \in \mathbf{l}^\infty$ is any solution of the initial value problem (3.1) if and only if

$$(3.3)\qquad \mathbf{u}(t) = \frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\mathbf{c} + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{u}(s)), \quad t\in\mathbb{N}_0.$$

Analogous to (2.9), $\mathbf{u} = \{\mathbf{u}(t)\}_{t=0}^\infty \in \mathbf{l}^\infty$ is any solution of the initial value problem (3.2) if and only if

$$(3.4)\qquad \mathbf{u}(t) = \mathbf{c} + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{u}(s)), \quad t\in\mathbb{N}_0.$$

Define the operators

$$(3.5)\quad T\mathbf{u}(t) = \frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\mathbf{c} + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{u}(s)), \quad t\in\mathbb{N}_0,$$

$$(3.6)\quad T'\mathbf{u}(t) = \mathbf{c} + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{u}(s)), \quad t\in\mathbb{N}_0,$$

$$(3.7)\quad A\mathbf{u}(t) = \frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\mathbf{c}, \quad t\in\mathbb{N}_0,$$

$$(3.8)\quad B\mathbf{u}(t) = \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{u}(s)), \quad t\in\mathbb{N}_0.$$

It is evident from (3.3) - (3.6) that $\mathbf{u}$ is a fixed point of T if and only if $\mathbf{u}$ is a solution of (3.1) and $\mathbf{u}$ is a fixed point of T' if and only if $\mathbf{u}$ is a solution of (3.2).

First we use Krasnoselskii's fixed point theorem (Theorem 2.4) to establish global existence of solutions of (3.1). Clearly A is a contraction mapping with constant 0, implies condition (1) of Theorem 2.4 holds.

Theorem 3.1. *(Global Existence) If* $\mathbf{f}$ *is continuous with respect to the second variable and there exist constants* $\beta_1 \in [\alpha, 1)$ *and* $L_1 \geq 0$ *such that*

$$\|\mathbf{f}(t, \mathbf{u}(t))\| \leq L_1 t^{\overline{-\beta_1}}, \quad t \in \mathbb{N}_1, \tag{3.9}$$

then the nonautonomous initial value problem (3.1) has at least one bounded solution in $\mathbf{l}^\infty$.

Proof. To prove condition (2) of Theorem 2.4, we define a set

$$S_1 = \{\mathbf{u} : \|\mathbf{u}(t)\| \leq \|\mathbf{c}\| + L_1\Gamma(1-\beta_1),\ t \in \mathbb{N}_1\}.$$

Clearly S_1 is a nonempty, closed, bounded and convex subset of $\mathbf{l}^\infty$. First, we show that B maps S_1 into S_1. Using Lemma 2.1, Theorem 2.2 and (3.9), we have

$$\begin{aligned}
\|B\mathbf{u}(t)\| &\leq \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\,\|\mathbf{f}(s,\mathbf{u}(s))\| \\
&\leq \frac{L_1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}} s^{\overline{-\beta_1}} \\
&= L_1\nabla_0^{-\alpha} t^{\overline{-\beta_1}} \\
&= \frac{L_1\Gamma(1-\beta_1)}{\Gamma(1-\beta_1+\alpha)} t^{\overline{-(\beta_1-\alpha)}} \\
&\leq \frac{L_1\Gamma(1-\beta_1)}{\Gamma(1-\beta_1+\alpha)} (1)^{\overline{-(\beta_1-\alpha)}} \\
&= L_1\Gamma(1-\beta_1) \\
&\leq \|\mathbf{c}\| + L_1\Gamma(1-\beta_1), \quad t \in \mathbb{N}_1,
\end{aligned}$$

implies $BS_1 \subset S_1$. Next, we show that B is continuous on S_1. Let $\epsilon > 0$ be given. Then there exists $m \in \mathbb{N}_1$ such that, for $t \in \mathbb{N}_{m+1}$,

$$\frac{L_1\Gamma(1-\beta_1)}{\Gamma(1-\beta_1+\alpha)} t^{\overline{-(\beta_1-\alpha)}} < \frac{\epsilon}{2}.$$

Let $\{\mathbf{u}_k\}$, $(k = 1, 2, ...)$ be a sequence in S_1 such that $\mathbf{u}_k \to \mathbf{u}$ in S_1. Then, we have $\|\mathbf{u}_k - \mathbf{u}\|_\infty \to 0$ as $k \to \infty$. Since $\mathbf{f}$ is continuous with respect to the second variable, we get $\|\mathbf{f}(t,\mathbf{u}_k) - \mathbf{f}(t,\mathbf{u})\|_\infty \to 0$ as $k \to \infty$. For $t \leq m$,

$$\begin{aligned}
\|B\mathbf{u}_k(t) - B\mathbf{u}(t)\| &\leq \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\|\mathbf{f}(s,\mathbf{u}_k(s)) - \mathbf{f}(s,\mathbf{u}(s))\| \\
&\leq \left[\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\right]\left[\sup_{s\in\{1,2,......,m\}}\|\mathbf{f}(s,\mathbf{u}_k(s)) - \mathbf{f}(s,\mathbf{u}(s))\|\right] \\
&= \frac{t^{\overline{\alpha}}}{\Gamma(\alpha+1)}\|\mathbf{f}(s,\mathbf{u}_k) - \mathbf{f}(s,\mathbf{u})\|_\infty \\
&\to 0 \quad \text{as} \quad k \to \infty.
\end{aligned}$$

For $t \in \mathbb{N}_{m+1}$,

$$\begin{aligned}\|B\mathbf{u}_k(t) - B\mathbf{u}(t)\| &\leq \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{\alpha-1}} [\|\mathbf{f}(s, \mathbf{u}_k(s))\| + \|\mathbf{f}(s, \mathbf{u}(s))\|] \\ &\leq \frac{2L_1\Gamma(1-\beta_1)}{\Gamma(1-\beta_1+\alpha)} t^{\overline{-(\beta_1-\alpha)}} < \epsilon.\end{aligned}$$

Thus we have, $\|B\mathbf{u}_k - B\mathbf{u}\|_\infty \to 0$ as $k \to \infty$, implies B is continuous. Now, we show that BS_1 is relatively compact. Let t_1, $t_2 \in \mathbb{N}_{m+1}$ such that $t_2 > t_1$. Then, we have

$$\begin{aligned}\|B\mathbf{u}(t_1) - B\mathbf{u}(t_2)\| &\leq \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t_1} (t_1-\rho(s))^{\overline{\alpha-1}} \|\mathbf{f}(s, \mathbf{u}(s))\| + \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t_2} (t_2-\rho(s))^{\overline{\alpha-1}} \|\mathbf{f}(s, \mathbf{u}(s))\| \\ &\leq \frac{L_1\Gamma(1-\beta_1)}{\Gamma(1-\beta_1+\alpha)} t_1^{\overline{-(\beta_1-\alpha)}} + \frac{L_1\Gamma(1-\beta_1)}{\Gamma(1-\beta_1+\alpha)} t_2^{\overline{-(\beta_1-\alpha)}} < \epsilon.\end{aligned}$$

Thus $\{B\mathbf{u} : \mathbf{u} \in S_1\}$ is a bounded and uniformly Cauchy subset of $\mathbf{l}^\infty$. Hence, by Theorem 2.3, BS_1 is relatively compact.

Now we prove condition (3) of Theorem 2.4. Let us suppose, for a fixed $\mathbf{v} \in S_1$, $\mathbf{u} = A\mathbf{u} + B\mathbf{v}$. Using Lemma 2.1, Theorem 2.2 and (3.9), we have

$$\begin{aligned}\|\mathbf{u}(t)\| &\leq \|A\mathbf{u}(t)\| + \|B\mathbf{v}(t)\| \\ &\leq \frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} \|\mathbf{c}\| + \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{\alpha-1}} \|\mathbf{f}(s, \mathbf{v}(s))\| \\ &\leq \frac{(1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} \|\mathbf{c}\| + \frac{L_1\Gamma(1-\beta_1)}{\Gamma(1-\beta_1+\alpha)} t^{\overline{-(\beta_1-\alpha)}} \\ &\leq \|\mathbf{c}\| + \frac{L_1\Gamma(1-\beta_1)}{\Gamma(1-\beta_1+\alpha)} (1)^{\overline{-(\beta_1-\alpha)}} \\ &= \|\mathbf{c}\| + L_1\Gamma(1-\beta_1), \ t \in \mathbb{N}_1.\end{aligned}$$

Thus $\mathbf{u} \in S_1$. According to Theorem 2.4, T has a fixed point in S_1 which is a solution of (3.1). Hence the proof. □

Theorem 3.2. *(Global Existence) If* $\mathbf{f}$ *is continuous with respect to the second variable and there exist constants* $\beta_2 \in [\alpha, 1)$ *and* $L_2 \geq 0$ *such that*

$$\|\mathbf{f}(t, \mathbf{u}(t))\| \leq L_2 t^{\overline{-\beta_2}} \|\mathbf{u}(t)\|, \quad t \in \mathbb{N}_1, \tag{3.10}$$

then the nonautonomous initial value problem (3.1) has at least one bounded solution in $\mathbf{l}^\infty$ *provided that*

$$L_2\Gamma(1-\beta_2) < 1. \tag{3.11}$$

Proof. Define

$$S_2 = \Big\{\mathbf{u} : \|\mathbf{u}(t)\| \leq \frac{\|\mathbf{c}\|}{[1 - L_2\Gamma(1-\beta_2)]}, \ t \in \mathbb{N}_1\Big\}.$$

Clearly S_2 is a nonempty, closed, bounded and convex subset of $\mathbf{l}^\infty$. First, we show that B maps S_2 into S_2. Using Lemma 2.1, Theorem 2.2 and (3.10), we have

$$\begin{aligned}
\|B\mathbf{u}(t)\| &\leq \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\,\|\mathbf{f}(s,\mathbf{u}(s))\| \\
&\leq \frac{L_2}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}s^{\overline{-\beta_2}}\,\|\mathbf{u}(s)\| \\
&\leq \frac{L_2\|\mathbf{c}\|}{[1-L_2\Gamma(1-\beta_2)]}\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}s^{\overline{-\beta_2}} \\
&= \frac{L_2\|\mathbf{c}\|}{[1-L_2\Gamma(1-\beta_2)]}\nabla_0^{-\alpha}t^{\overline{-\beta_2}} \\
&= \frac{L_2\|\mathbf{c}\|}{[1-L_2\Gamma(1-\beta_2)]}\frac{\Gamma(1-\beta_2)}{\Gamma(1-\beta_2+\alpha)}t^{\overline{-(\beta_2-\alpha)}} \\
&\leq \frac{L_2\|\mathbf{c}\|}{[1-L_2\Gamma(1-\beta_2)]}\frac{\Gamma(1-\beta_2)}{\Gamma(1-\beta_2+\alpha)}(1)^{\overline{-(\beta_2-\alpha)}} \\
&= \frac{L_2\|\mathbf{c}\|\Gamma(1-\beta_2)}{[1-L_2\Gamma(1-\beta_2)]} \\
&= \frac{\|\mathbf{c}\|}{[1-L_2\Gamma(1-\beta_2)]}-\|\mathbf{c}\| \\
&\leq \frac{\|\mathbf{c}\|}{[1-L_2\Gamma(1-\beta_2)]}, \quad t\in\mathbb{N}_1,
\end{aligned}$$

implies $BS_2 \subset S_2$. The remaining proof of condition (2) is similar to that of Theorem 3.1 and we omit it.

Now we prove condition (3) of Theorem 2.4. Let us suppose, for a fixed $\mathbf{v}\in S_2$, $\mathbf{u}=A\mathbf{u}+B\mathbf{v}$. Using Lemma 2.1, Theorem 2.2 and (3.10), we have

$$\begin{aligned}
\|\mathbf{u}(t)\| &\leq \|A\mathbf{u}(t)\|+\|B\mathbf{v}(t)\| \\
&\leq \frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\|\mathbf{c}\|+\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\|\mathbf{f}(s,\mathbf{v}(s))\| \\
&\leq \frac{(1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\|\mathbf{c}\|+\frac{L_2\|\mathbf{c}\|\Gamma(1-\beta_2)}{[1-L_2\Gamma(1-\beta_2)]} \\
&\leq \|\mathbf{c}\|+\frac{L_2\|\mathbf{c}\|\Gamma(1-\beta_2)}{[1-L_2\Gamma(1-\beta_2)]} \\
&= \frac{\|\mathbf{c}\|}{[1-L_2\Gamma(1-\beta_2)]},\ t\in\mathbb{N}_1.
\end{aligned}$$

Thus $\mathbf{u}\in S_2$. According to Theorem 2.4, T has a fixed point in S_2 which is a solution of (3.1) - (3.2). Hence the proof. □

Now we apply Schauder fixed point theorem (Theorem 2.6) to establish global existence of solutions of (3.2).

Theorem 3.3. *(Global Existence) If* $\mathbf{f}$ *satisfies the hypothesis of Theorem 3.1, then the nonautonomous initial value problem (3.2) has at least one bounded solution in* $\mathbf{l}^\infty$.

Proof. Define a set

$$S_3 = \{\mathbf{u} : \mathbf{u}(0) = \mathbf{c},\ \|\mathbf{u}(t) - \mathbf{c}\| \leq L_1\Gamma(1-\beta_1),\ t \in \mathbb{N}_1\}.$$

Clearly S_3 is a nonempty, closed, bounded and convex subset of $\mathbf{l}^\infty$. First, we show that T' maps S_3 into S_3. Using Lemma 2.1, Theorem 2.2 and (3.9), we have

$$\begin{aligned}
\|T'\mathbf{u}(t) - \mathbf{c}\| &\leq \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\,\|\mathbf{f}(s,\mathbf{u}(s))\| \\
&\leq \frac{L_1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}s^{\overline{-\beta_1}} \\
&= L_1\nabla_0^{-\alpha}t^{\overline{-\beta_1}} \\
&= \frac{L_1\Gamma(1-\beta_1)}{\Gamma(1-\beta_1+\alpha)}t^{\overline{-(\beta_1-\alpha)}} \\
&\leq \frac{L_1\Gamma(1-\beta_1)}{\Gamma(1-\beta_1+\alpha)}(1)^{\overline{-(\beta_1-\alpha)}} \\
&= L_1\Gamma(1-\beta_1), \quad t \in \mathbb{N}_1,
\end{aligned}$$

and $T'\mathbf{u}(0) = \mathbf{c}$, implies $T'S_3 \subset S_3$. Next, we show that T' is continuous on S_3. Let $\epsilon > 0$ be given. Then there exists $m \in \mathbb{N}_1$ such that, for $t \in \mathbb{N}_{m+1}$,

$$\frac{L_1\Gamma(1-\beta_1)}{\Gamma(1-\beta_1+\alpha)}t^{\overline{-(\beta_1-\alpha)}} < \frac{\epsilon}{2}. \tag{3.12}$$

Let $\{\mathbf{u}_k\}$, $(k = 1, 2, ...)$ be a sequence in S_3 such that $\mathbf{u}_k \to \mathbf{u}$ in S_3. Then, we have $\|\mathbf{u}_k - \mathbf{u}\|_\infty \to 0$ as $k \to \infty$. Since $\mathbf{f}$ is continuous with respect to the second variable, we get $\|\mathbf{f}(t,\mathbf{u}_k) - \mathbf{f}(t,\mathbf{u})\|_\infty \to 0$ as $k \to \infty$. For $t \leq m$,

$$\begin{aligned}
\|T'\mathbf{u}_k(t) - T'\mathbf{u}(t)\| &\leq \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\|\mathbf{f}(s,\mathbf{u}_k(s)) - \mathbf{f}(s,\mathbf{u}(s))\| \\
&\leq \left[\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\right]\left[\sup_{s\in\{1,2,......,m\}}\|\mathbf{f}(s,\mathbf{u}_k(s)) - \mathbf{f}(s,\mathbf{u}(s))\|\right] \\
&= \frac{t^{\overline{\alpha}}}{\Gamma(\alpha+1)}\|\mathbf{f}(s,\mathbf{u}_k) - \mathbf{f}(s,\mathbf{u})\|_\infty \\
&\to 0 \quad \text{as} \quad k \to \infty.
\end{aligned}$$

For $t \in \mathbb{N}_{m+1}$,

$$\begin{aligned}
\|T'\mathbf{u}_k(t) - T'\mathbf{u}(t)\| &\leq \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}[\|\mathbf{f}(s,\mathbf{u}_k(s))\| + \|\mathbf{f}(s,\mathbf{u}(s))\|] \\
&\leq \frac{2L_1\Gamma(1-\beta_1)}{\Gamma(1-\beta_1+\alpha)}t^{\overline{-(\beta_1-\alpha)}} < \epsilon.
\end{aligned}$$

Thus we have, $\|T'\mathbf{u}_k - T'\mathbf{u}\|_\infty \to 0$ as $k \to \infty$, implies T' is continuous. Now, we show that $T'S_3$ is relatively compact. Let $t_1,\ t_2 \in \mathbb{N}_{m+1}$ such that $t_2 > t_1$. Then, we have

$$\begin{aligned}\|T'\mathbf{u}(t_1) - T'\mathbf{u}(t_2)\| &\leq \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t_1}(t_1-\rho(s))^{\overline{\alpha-1}}\|\mathbf{f}(s,\mathbf{u}(s))\| + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t_2}(t_2-\rho(s))^{\overline{\alpha-1}}\|\mathbf{f}(s,\mathbf{u}(s))\| \\ &\leq \frac{L_1\Gamma(1-\beta_1)}{\Gamma(1-\beta_1+\alpha)}t_1^{\overline{-(\beta_1-\alpha)}} + \frac{L_1\Gamma(1-\beta_1)}{\Gamma(1-\beta_1+\alpha)}t_2^{\overline{-(\beta_1-\alpha)}} < \epsilon.\end{aligned}$$

Thus $\{T'\mathbf{u} : \mathbf{u} \in S_3\}$ is a bounded and uniformly Cauchy subset of $\mathbf{l}^\infty$. Hence, by Theorem 2.3, $T'S_3$ is relatively compact. According to Theorem 2.6, T' has a fixed point in S_3 which is a solution of (3.2). Hence the proof. □

We use generalized Banach fixed point theorem (Theorem 2.5) to prove the uniqueness of solutions of (3.1) and (3.2).

Theorem 3.4. *(Global Uniqueness) If* $\mathbf{f}$ *is continuous with respect to the second variable and there exist constants* $\gamma \in [\alpha, 1)$ *and* $M \geq 0$ *such that*

$$\|\mathbf{f}(t,\mathbf{u}) - \mathbf{f}(t,\mathbf{v})\|_\infty \leq Mt^{\overline{-\gamma}}\|\mathbf{u}-\mathbf{v}\|_\infty, \quad t \in \mathbb{N}_1, \tag{3.13}$$

for any pair of elements $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbf{l}^\infty$*. Then the initial value problems (3.1) and (3.2) have unique bounded solution in* $\mathbf{l}^\infty$ *provided that*

$$c = M\Gamma(1-\gamma) < 1. \tag{3.14}$$

Proof. Let us define the iterates of operator T as follows:

$$T^1 = T, \quad T^n = ToT^{n-1}, \quad n \in \mathbb{N}_1.$$

It is sufficient to prove that T^n is a contraction operator for sufficiently large n. Actually, we have

$$\|T^n\mathbf{u} - T^n\mathbf{v}\|_\infty \leq c^n\|\mathbf{u}-\mathbf{v}\|_\infty \tag{3.15}$$

where the constant c depends only on M and γ. In fact, using Lemma 2.1, Theorem 2.2 and (3.13), we get

$$\begin{aligned}\|T\mathbf{u}(t) - T\mathbf{v}(t)\| &\leq \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\|\mathbf{f}(s,\mathbf{u}(s)) - \mathbf{f}(s,\mathbf{v}(s))\| \\ &\leq \frac{M}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}s^{\overline{-\gamma}}\|\mathbf{u}-\mathbf{v}\|_\infty \\ &= M\nabla_0^{-\alpha}t^{\overline{-\gamma}}\|\mathbf{u}-\mathbf{v}\|_\infty \\ &= \frac{M\Gamma(1-\gamma)}{\Gamma(1-\gamma+\alpha)}t^{\overline{-(\gamma-\alpha)}}\|\mathbf{u}-\mathbf{v}\|_\infty \\ &\leq \frac{M\Gamma(1-\gamma)}{\Gamma(1-\gamma+\alpha)}(1)^{\overline{-(\gamma-\alpha)}}\|\mathbf{u}-\mathbf{v}\|_\infty \\ &= c\|\mathbf{u}-\mathbf{v}\|_\infty,\end{aligned}$$

implies

$$\|T\mathbf{u} - T\mathbf{v}\|_\infty \leq c\|\mathbf{u}-\mathbf{v}\|_\infty. \tag{3.16}$$

Therefore (3.15) is true for $n = 1$. Assuming (3.15) is valid for n, we obtain similarly

$$\begin{aligned}
\|T^{n+1}\mathbf{u}(t) - T^{n+1}\mathbf{v}(t)\| &= \|(ToT^n)\mathbf{u}(t) - (ToT^n)\mathbf{v}(t)\| \\
&\le \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\|\mathbf{f}(s,T^n\mathbf{u}(s)) - \mathbf{f}(s,T^n\mathbf{v}(s))\| \\
&\le \frac{M}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}s^{\overline{-\gamma}}\|T^n\mathbf{u} - T^n\mathbf{v}\|_\infty \\
&\le Mc^n\nabla_0^{-\alpha}t^{\overline{-\gamma}}\|\mathbf{u}-\mathbf{v}\|_\infty \\
&= \frac{Mc^n\Gamma(1-\gamma)}{\Gamma(1-\gamma+\alpha)}t^{\overline{-(\gamma-\alpha)}}\|\mathbf{u}-\mathbf{v}\|_\infty \\
&\le \frac{Mc^n\Gamma(1-\gamma)}{\Gamma(1-\gamma+\alpha)}(1)^{\overline{-(\gamma-\alpha)}}\|\mathbf{u}-\mathbf{v}\|_\infty \\
&= c^{n+1}\|\mathbf{u}-\mathbf{v}\|_\infty.
\end{aligned}$$

Thus, by the principle of mathematical induction on n, the statement (3.15) is true for each $n \in \mathbb{N}_1$. Since $c < 1$, the geometric series $\sum_{n=0}^{\infty} c^n$ converges. Hence T has a uniquely defined point $\mathbf{u}^*$ in S_1 (or S_2). This completes the proof. Similarly we can prove that T' has a uniquely defined point $\mathbf{u}^*$ in S_3. □

4. Dependence of Solutions on Initial Conditions and Parameters

The initial value problems (3.1) and (3.2) describes a model of a physical problem in which often some parameters such as lengths, masses, temperature, etc. are involved. The values of these parameters can be measured only up to a certain degree of accuracy. Thus, in (3.1) and (3.2), the initial value $\mathbf{c}$, the order of the difference operator α and the function $\mathbf{f}$, may be subject to some errors either by necessity or for convenience. Hence, it is important to know how the solution changes when these parameters are slightly altered. We shall discuss this question quantitatively in the following theorems.

Theorem 4.1. *Assume that* $\mathbf{f}$ *is continuous and satisfies (3.13) with respect to the second variable. Suppose* $\mathbf{u}$ *and* $\mathbf{v}$ *are the solutions of the initial value problems*

$$\nabla_{-1}^{\alpha+\epsilon}\mathbf{u}(t) = \mathbf{f}(t,\mathbf{u}(t)), \quad \nabla_{-1}^{-(1-\alpha-\epsilon)}\mathbf{u}(t)\Big|_{t=0} = \mathbf{u}(0) = \mathbf{c}, \quad t \in \mathbb{N}_1, \tag{4.1}$$

$$\nabla_{-1}^{\alpha}\mathbf{v}(t) = \mathbf{f}(t,\mathbf{v}(t)), \quad \nabla_{-1}^{-(1-\alpha)}\mathbf{v}(t)\Big|_{t=0} = \mathbf{v}(0) = \mathbf{c}, \quad t \in \mathbb{N}_1, \tag{4.2}$$

respectively, where $\epsilon > 0$ *and* $0 < \alpha < \alpha + \epsilon < 1$. *Then*

$$\|\mathbf{u}-\mathbf{v}\|_\infty = O(\epsilon) \tag{4.3}$$

provided that (3.14) holds.

Proof. We have

$$\mathbf{u}(t) = \frac{(t+1)^{\overline{\alpha+\epsilon-1}}}{\Gamma(\alpha+\epsilon)}\mathbf{c} + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha+\epsilon-1}}\mathbf{f}(s,\mathbf{u}(s)), \quad t \in \mathbb{N}_0,$$

$$\mathbf{v}(t) = \frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\mathbf{c} + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{v}(s)), \quad t \in \mathbb{N}_0.$$

Consider

$$\begin{aligned}
\|\mathbf{u}(t)-\mathbf{v}(t)\| &\leq \Big|\frac{(t+1)^{\overline{\alpha+\epsilon-1}}}{\Gamma(\alpha+\epsilon)}-\frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\Big|\|\mathbf{c}\| \\
&\quad +\Big\|\frac{1}{\Gamma(\alpha+\epsilon)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha+\epsilon-1}}\mathbf{f}(s,\mathbf{u}(s))-\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{v}(s))\Big\| \\
&\leq \Big|\frac{\Gamma(\alpha)}{\Gamma(\alpha+\epsilon)}(t+\alpha)^{\overline{\epsilon}}-1\Big|\frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\|\mathbf{c}\|_{\infty} \\
&\quad +\Big\|\frac{1}{\Gamma(\alpha+\epsilon)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha+\epsilon-1}}[\mathbf{f}(s,\mathbf{u}(s))-\mathbf{f}(s,\mathbf{v}(s))]\Big\| \\
&\quad +\Big\|\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{v}(s))\Big[1-\frac{\Gamma(\alpha)}{\Gamma(\alpha+\epsilon)}(t-s+\alpha)^{\overline{\epsilon}}\Big]\Big\|
\end{aligned}$$

$$\begin{aligned}
&\leq \Big|\frac{\Gamma(\alpha)}{\Gamma(t+\alpha)}\frac{\Gamma(\epsilon+t+\alpha)}{\Gamma(\epsilon+\alpha)}-1\Big|\frac{(2)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\|\mathbf{c}\|_{\infty} \\
&\quad +\frac{1}{\Gamma(\alpha+\epsilon)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha+\epsilon-1}}\|\mathbf{f}(s,\mathbf{u}(s))-\mathbf{f}(s,\mathbf{v}(s))\| \\
&\quad +\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\|\mathbf{f}(s,\mathbf{v}(s))\|\Big|1-\frac{\Gamma(\alpha)}{\Gamma(t-s+\alpha)}\frac{\Gamma(\epsilon+t-s+\alpha)}{\Gamma(\epsilon+\alpha)}\Big|, \quad t\in\mathbb{N}_1.
\end{aligned} \tag{4.4}$$

Since

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big[\frac{\Gamma(\alpha)}{\Gamma(t+\alpha)}\frac{\Gamma(\epsilon+t+\alpha)}{\Gamma(\epsilon+\alpha)}-1\Big]=C_1 \text{ (a constant independent of } \epsilon)$$

and

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big[1-\frac{\Gamma(\alpha)}{\Gamma(t-s+\alpha)}\frac{\Gamma(\epsilon+t-s+\alpha)}{\Gamma(\epsilon+\alpha)}\Big]=C_2 \text{ (a constant independent of } \epsilon),$$

we have

$$\Big[\frac{\Gamma(\alpha)}{\Gamma(t+\alpha)}\frac{\Gamma(\epsilon+t+\alpha)}{\Gamma(\epsilon+\alpha)}-1\Big]=O(\epsilon), \tag{4.5}$$

$$\Big[1-\frac{\Gamma(\alpha)}{\Gamma(t-s+\alpha)}\frac{\Gamma(\epsilon+t-s+\alpha)}{\Gamma(\epsilon+\alpha)}\Big]=O(\epsilon). \tag{4.6}$$

Using (4.5) and (4.6) in (4.4), we get

$$\begin{aligned}
\|\mathbf{u}(t)-\mathbf{v}(t)\| &\leq \mathrm{O}(\epsilon)\alpha\|\mathbf{c}\|_\infty + M\|\mathbf{u}-\mathbf{v}\|_\infty \frac{1}{\Gamma(\alpha+\epsilon)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha+\epsilon-1}}s^{\overline{-\gamma}} \\
&\quad + \mathrm{O}(\epsilon)\|\mathbf{f}\|_\infty \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}s^{\overline{-\gamma}} \\
&= \mathrm{O}(\epsilon)\alpha\|\mathbf{c}\|_\infty + M\|\mathbf{u}-\mathbf{v}\|_\infty \nabla_0^{-(\alpha+\epsilon)}t^{\overline{-\gamma}} + \mathrm{O}(\epsilon)\|\mathbf{f}\|_\infty \nabla_0^{-\alpha}t^{\overline{-\gamma}} \\
&= \mathrm{O}(\epsilon)\alpha\|\mathbf{c}\|_\infty + M\|\mathbf{u}-\mathbf{v}\|_\infty \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha+\epsilon-\gamma)}t^{\overline{\alpha+\epsilon-\gamma}} + \mathrm{O}(\epsilon)\|\mathbf{f}\|_\infty \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}t^{\overline{\alpha-\gamma}} \\
&\leq \mathrm{O}(\epsilon)\alpha\|\mathbf{c}\|_\infty + M\|\mathbf{u}-\mathbf{v}\|_\infty \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha+\epsilon-\gamma)}(1)^{\overline{\alpha+\epsilon-\gamma}} + \mathrm{O}(\epsilon)\|\mathbf{f}\|_\infty \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}(1)^{\overline{\alpha-\gamma}} \\
&= \mathrm{O}(\epsilon)\alpha\|\mathbf{c}\|_\infty + M\|\mathbf{u}-\mathbf{v}\|_\infty \Gamma(1-\gamma) + \mathrm{O}(\epsilon)\|\mathbf{f}\|_\infty \Gamma(1-\gamma), \quad t \in \mathbb{N}_1.
\end{aligned}$$

Then, we have the relation

$$\|\mathbf{u}-\mathbf{v}\|_\infty \leq \frac{[\alpha\|\mathbf{c}\|_\infty + \|\mathbf{f}\|_\infty\Gamma(1-\gamma)]}{[1-M\Gamma(1-\gamma)]}\mathrm{O}(\epsilon)$$

implies

$$\|\mathbf{u}-\mathbf{v}\|_\infty = \mathrm{O}(\epsilon).$$

□

Corollary 1. *Assume that* **f** *is continuous and satisfies (3.13) with respect to the second variable. Suppose* **u** *and* **v** *are the solutions of the initial value problems*

$$\nabla_{0*}^{\alpha+\epsilon}\mathbf{u}(t) = \mathbf{f}(t,\mathbf{u}(t)), \quad \mathbf{u}(0)=\mathbf{c}, \quad t \in \mathbb{N}_1, \tag{4.7}$$

$$\nabla_{0*}^{\alpha}\mathbf{v}(t) = \mathbf{f}(t,\mathbf{v}(t)), \quad \mathbf{v}(0)=\mathbf{c}, \quad t \in \mathbb{N}_1, \tag{4.8}$$

respectively, where $\epsilon > 0$ *and* $0<\alpha<\alpha+\epsilon<1$. *Then*

$$\|\mathbf{u}-\mathbf{v}\|_\infty = O(\epsilon) \tag{4.9}$$

provided that (3.14) holds.

Theorem 4.2. *Assume that* **f** *is continuous and satisfies (3.13) with respect to the second variable. Suppose* **u** *and* **v** *are the solutions of the initial value problems*

$$\nabla_{-1}^{\alpha}\mathbf{u}(t) = \mathbf{f}(t,\mathbf{u}(t)), \quad \nabla_{-1}^{-(1-\alpha)}\mathbf{u}(t)\Big|_{t=0} = \mathbf{u}(0)=\mathbf{c}, \quad t \in \mathbb{N}_1, \tag{4.10}$$

$$\nabla_{-1}^{\alpha}\mathbf{v}(t) = \mathbf{f}(t,\mathbf{v}(t)), \quad \nabla_{-1}^{-(1-\alpha)}\mathbf{v}(t)\Big|_{t=0} = \mathbf{v}(0)=\mathbf{d}, \quad t \in \mathbb{N}_1, \tag{4.11}$$

respectively, where $0<\alpha<1$. *Then*

$$\|\mathbf{u}-\mathbf{v}\|_\infty = O(\|\mathbf{c}-\mathbf{d}\|_\infty) \tag{4.12}$$

provided that (3.14) holds.

Proof. We have

$$\mathbf{u}(t) = \frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\mathbf{c} + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{u}(s)), \quad t \in \mathbb{N}_0,$$

$$\mathbf{v}(t) = \frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\mathbf{d} + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{v}(s)), \quad t \in \mathbb{N}_0.$$

Consider

$$\begin{aligned}
\|\mathbf{u}(t)-\mathbf{v}(t)\| &\le \|\mathbf{c}-\mathbf{d}\|\frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\|\mathbf{f}(s,\mathbf{u}(s))-\mathbf{f}(s,\mathbf{v}(s))\| \\
&\le \|\mathbf{c}-\mathbf{d}\|_\infty\frac{(2)^{\overline{\alpha-1}}}{\Gamma(\alpha)}+M\|\mathbf{u}-\mathbf{v}\|_\infty\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}s^{\overline{-\gamma}} \\
&= \alpha\|\mathbf{c}-\mathbf{d}\|_\infty+M\|\mathbf{u}-\mathbf{v}\|_\infty\nabla_0^{-\alpha}t^{\overline{-\gamma}} \\
&= \alpha\|\mathbf{c}-\mathbf{d}\|_\infty+M\|\mathbf{u}-\mathbf{v}\|_\infty\frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}t^{\overline{\alpha-\gamma}} \\
&\le \alpha\|\mathbf{c}-\mathbf{d}\|_\infty+M\|\mathbf{u}-\mathbf{v}\|_\infty\frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}(1)^{\overline{\alpha-\gamma}} \\
&= \alpha\|\mathbf{c}-\mathbf{d}\|_\infty+M\|\mathbf{u}-\mathbf{v}\|_\infty\Gamma(1-\gamma), \quad t\in\mathbb{N}_1.
\end{aligned}$$

Then, we have the relation

$$\|\mathbf{u}-\mathbf{v}\|_\infty \le \frac{\alpha\|\mathbf{c}-\mathbf{d}\|_\infty}{[1-M\Gamma(1-\gamma)]}$$

implies

$$\|\mathbf{u}-\mathbf{v}\|_\infty = \mathrm{O}(\|\mathbf{c}-\mathbf{d}\|_\infty).$$

□

Corollary 2. *Assume that* $\mathbf{f}$ *is continuous and satisfies (3.13) with respect to the second variable. Suppose* $\mathbf{u}$ *and* $\mathbf{v}$ *are the solutions of the initial value problems*

$$\nabla_{0*}^{\alpha}\mathbf{u}(t)=\mathbf{f}(t,\mathbf{u}(t)), \quad \mathbf{u}(0)=\mathbf{c}, \quad t\in\mathbb{N}_1, \tag{4.13}$$

$$\nabla_{0*}^{\alpha}\mathbf{v}(t)=\mathbf{f}(t,\mathbf{v}(t)), \quad \mathbf{v}(0)=\mathbf{d}, \quad t\in\mathbb{N}_1, \tag{4.14}$$

respectively, where $0<\alpha<1$*. Then*

$$\|\mathbf{u}-\mathbf{v}\|_\infty = O(\|\mathbf{c}-\mathbf{d}\|_\infty) \tag{4.15}$$

provided that (3.14) holds.

Theorem 4.3. *Assume that* $\mathbf{f}$ *and* $\mathbf{g}$ *are continuous and satisfies (3.13) with respect to the second variable. Suppose* $\mathbf{u}$ *and* $\mathbf{v}$ *are the solutions of the initial value problems*

$$\nabla_{-1}^{\alpha}\mathbf{u}(t)=\mathbf{f}(t,\mathbf{u}(t)), \quad \nabla_{-1}^{-(1-\alpha)}\mathbf{u}(t)\Big|_{t=0}=\mathbf{u}(0)=\mathbf{c}, \quad t\in\mathbb{N}_1, \tag{4.16}$$

$$\nabla_{-1}^{\alpha}\mathbf{v}(t)=\mathbf{g}(t,\mathbf{v}(t)), \quad \nabla_{-1}^{-(1-\alpha)}\mathbf{v}(t)\Big|_{t=0}=\mathbf{v}(0)=\mathbf{c}, \quad t\in\mathbb{N}_1, \tag{4.17}$$

respectively, where $0<\alpha<1$*. Then*

$$\|\mathbf{u}-\mathbf{v}\|_\infty = O(\|\mathbf{f}-\mathbf{g}\|_\infty) \tag{4.18}$$

provided that (3.14) holds.

Proof. We have

$$\mathbf{u}(t)=\frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\mathbf{c}+\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{u}(s)), \quad t\in\mathbb{N}_0,$$

$$\mathbf{v}(t)=\frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\mathbf{c}+\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\mathbf{g}(s,\mathbf{v}(s)), \quad t\in\mathbb{N}_0.$$

Consider

$$\begin{aligned}
\|\mathbf{u}(t)-\mathbf{v}(t)\| &\leq \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\|\mathbf{f}(s,\mathbf{u}(s))-\mathbf{g}(s,\mathbf{v}(s))\| \\
&= \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\|\mathbf{f}(s,\mathbf{u}(s))-\mathbf{f}(s,\mathbf{v}(s))+\mathbf{f}(s,\mathbf{v}(s))-\mathbf{g}(s,\mathbf{v}(s))\| \\
&\leq \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\|\mathbf{f}(s,\mathbf{u}(s))-\mathbf{f}(s,\mathbf{v}(s))\| \\
&\quad +\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\|\mathbf{f}(s,\mathbf{v}(s))-\mathbf{g}(s,\mathbf{v}(s))\| \\
&\leq [M\|\mathbf{u}-\mathbf{v}\|_\infty+\|\mathbf{f}-\mathbf{g}\|_\infty]\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}s^{\overline{-\gamma}} \\
&= [M\|\mathbf{u}-\mathbf{v}\|_\infty+\|\mathbf{f}-\mathbf{g}\|_\infty]\nabla_0^{-\alpha}t^{\overline{-\gamma}} \\
&= [M\|\mathbf{u}-\mathbf{v}\|_\infty+\|\mathbf{f}-\mathbf{g}\|_\infty]\frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}t^{\overline{\alpha-\gamma}} \\
&\leq [M\|\mathbf{u}-\mathbf{v}\|_\infty+\|\mathbf{f}-\mathbf{g}\|_\infty]\frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}(1)^{\overline{\alpha-\gamma}} \\
&= [M\|\mathbf{u}-\mathbf{v}\|_\infty+\|\mathbf{f}-\mathbf{g}\|_\infty]\Gamma(1-\gamma), \quad t\in\mathbb{N}_1.
\end{aligned}$$

Then, we have the relation

$$\|\mathbf{u}-\mathbf{v}\|_\infty \leq \frac{\Gamma(1-\gamma)}{[1-M\Gamma(1-\gamma)]}\|\mathbf{f}-\mathbf{g}\|_\infty$$

implies

$$\|\mathbf{u}-\mathbf{v}\|_\infty = \mathrm{O}(\|\mathbf{f}-\mathbf{g}\|_\infty).$$

□

Corollary 3. *Assume that* $\mathbf{f}$ *and* $\mathbf{g}$ *are continuous and satisfies (3.13) with respect to the second variable. Suppose* $\mathbf{u}$ *and* $\mathbf{v}$ *are the solutions of the initial value problems*

$$\nabla_{0*}^{\alpha}\mathbf{u}(t)=\mathbf{f}(t,\mathbf{u}(t)), \quad \mathbf{u}(0)=\mathbf{c}, \quad t\in\mathbb{N}_1, \tag{4.19}$$

$$\nabla_{0*}^{\alpha}\mathbf{v}(t)=\mathbf{g}(t,\mathbf{v}(t)), \quad \mathbf{v}(0)=\mathbf{c}, \quad t\in\mathbb{N}_1, \tag{4.20}$$

respectively, where $0<\alpha<1$. *Then*

$$\|\mathbf{u}-\mathbf{v}\|_\infty = O(\|\mathbf{f}-\mathbf{g}\|_\infty) \tag{4.21}$$

provided that (3.14) holds.

Definition 4.1. A solution $\tilde{\mathbf{u}}\in\mathbf{l}^\infty$ is said to be stable, if given $\epsilon>0$ and $t_0\geq 0$, there exists $\delta=\delta(\epsilon,t_0)$ such that $\|\mathbf{u}(t_0)-\tilde{\mathbf{u}}(t_0)\|_\infty<\delta \Rightarrow \|\mathbf{u}-\tilde{\mathbf{u}}\|_\infty<\epsilon$ for all $t\geq t_0$.

Theorem 4.4. *Assume that* $\mathbf{f}$ *is continuous and satisfies (3.13) with respect to the second variable. Then the solutions of (3.1) and (3.2) are stable provided that (3.14) holds.*

Proof. The proof is a direct consequence of Theorem 5.2 and Corollary 2. □

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