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Stability Analysis of Multi-Delay Differential Equations Using Aboodh Transform Method: Parkinson's Disease Model

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Abstract. In this study, we investigate the stability of multi - delay differential equations (MDDEs) associated with Parkinson's disease model using the Aboodh integral transform method, which is a powerful way to make delay systems easier to understand. The Aboodh integral transform does a good job of dealing with delay terms by turning them into algebraic statements in the transform domain. This makes it easy to write the characteristic equation. We find the conditions under which different MDDEs are stable by looking at the roots of the characteristic equation. The suggested method has been tested in Aboodh integral transform, all of which have situations where multiple delays have a big effect on how the system works. The Parkinson's disease model is asymptotic stability stable and will return to a steady state eventually after disturbances, and reverting to its initial condition. The Aboodh integral transform turns delay terms into exponential factors in the transform domain, which makes stability analysis easier by turning it into an algebraic root - finding problem that can be handled. The analytical models show that the method is more accurate and uses less computing power for systems with long delays or interactions between multiple delays.

1. Introduction

Delay Differential Equations (DDES), are a category of mathematical equations that are used to represent systems in which the development of a state variable is dependent not only on its current value but also on its values at prior times. These equations are especially helpful in expressing dynamics in situations where delays arise as a result of physical, biological, or communication

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constraints. Distinctive differential equations contain memory effects by explicitly incorporating terms that depend on previous states. This is in contrast to ordinary differential equations, which only take into account instantaneous interactions. Because of this characteristic, they are crucial for the study of phenomena in which timing plays a significant role, such as biological rhythms, control systems, and economic cycles. DDEs, in their most basic form, are responsible for the inherent lag that exists in the systems responsiveness to changes. In biology, for instance, the growth rate of a population may be dependent on the size of the population at some earlier time due to delayed effects such as gestation or maturation periods. This is because delayed effects can occur at a later time. In a similar to the performance of feedback control systems in engineering frequently incorporates a temporal lag as a result of delays in signal transmission or processing.

The presence of delays, which effectively enhance the dimensionality of the system and introduce a dependence on a full history rather than simply the current state, is the source of this richness in dynamics. Because DDEs are dependent on a history function, which defines the state of the system over a period of time prior to the initial time, analyzing them is more difficult than analyzing ordinary differential equations. Most of the time, analytical solutions are uncommon and are often restricted to particular situations, such as linear or basic nonlinear systems. As a consequence of this, researchers frequently engage in numerical approaches in order to solve and investigate DDEs. Utilizing techniques like as continuous Runge-Kutta, collocation methods, and spectral approaches are all included in these methods. When it comes to dealing with the complexity that are brought about by delays, dedicated solutions, such as MATLAB's dde23 or other specialized tools, are utilized extensively. These solvers take into account discontinuities and are able to handle systems that have numerous delays or delays that are depending on the state in an efficient manner.

The Aboodh Transform (AT) is a newly invented integral transform that was developed to overcome obstacles in the process of solving mathematical problems that involve differential equations, integral equations, and other complex functions. This method, which was developed by Khalid Aboodh and is named after him, offers an alternative to traditional transforms such as Laplace and Fourier. It was developed with the intention of managing functions that may demonstrate rapid growth or possess convoluted beginning conditions. The operation of the AT involves transforming these functions from their original domain into a transformed domain. In this transformed domain, the complexity of differentiation, integration, or convolution are simplified to algebraic forms that are easier to manage. The transformation in question makes it possible to manipulate equations in a straightforward manner and provides a means of resolving issues that would otherwise be difficult to handle analytically. The capacity of the AT to manage functions that exhibit odd behavior or quick growth is one of the most prominent advantages of this particular transformation.

Aboodh [1] discussed in comparison to traditional transforms such as the Laplace Transform (LT), the AT makes calculations more straightforward. It is possible that the ATis beneficial for functions that are difficult to handle using other approaches due to their complexity or difficulty. In light of the fact that the paper focused primarily on the mathematical formulation and applications to certain cases, it is possible that additional work is required in order to properly investigate its expanded utility or computational features.Several integral transforms, including the LT, the Fourier Transform, and others, were explored by Aboodh [2–5] to determine the relationships that exist between the ATand other well - established integral transforms. It highlights the usefulness of the AT well as the complementing nature of the procedure when it comes to addressing mathematical difficulties. Developed an understanding of the mathematical relationships that exist between the AT and other integral transforms.

In order to solve fractional - order partial differential equations, specifically those that model spatial diffusion in biological populations, Ojo [21] developed and applied the AT iterative Method. The methods primary focused on enhancing the accuracy and efficiency of solving fractional - order equations, which are increasingly being utilized to describe anomalous diffusion and memory effects in biological systems. Analyses of special functions such as Bessel functions were performed by Aggarwal [7,8], and Chaudhary [14].

Khan [17] worked on applying the AT to the study of integro - differential equations in a more methodical way. In many areas, like physics, engineering, and applied sciences, these equations are used to model complex systems mathematically.Tao [23] gave us a fast and accurate semi - analytical method for fixing fractional differential equations. He also introduced the Homotopy Perturbation Method to help fix problems that come up when fixing fractional differential equations. Fractional calculus takes the idea of regular differentiation and integration to orders that are not integers.Benattia [12,13] showed that the AT, which is a fairly new integral transform, can be used to solve first - order differential equations in the complex domain with constant coefficients.

For the purpose of solving linear Volterra integro - differential equations of the second kind, Aggarwal [6,9] shown that the AT can be utilized as an effective approach. Aboodh transform in three dimensions, which entails converting functions with respect to three variables, is extended by Alfaqeih [10], and Murali [20]. It is resolving partial differential equations involving three independent variables is examined, along with its operational characteristics, including linearity and shifting rules.

Alshikh [11] provided a comprehensive comparison of three integral transforms: the classical LT, the Elzaki and AT, and the Riemann transform. The research concentrated on the properties, computational efficiency, and applications of these methods in the solution of differential equations. Mohammed [19] adomian decomposition method is a semi - analytical approach that approximates solutions by decomposing the equation into a series of solvable components. Adomian polynomials are employed to manage the nonlinear and delayed terms, enabling systematic computation without the need for linearization or discretization. Cimen [15] concentrated on the LT as a method for resolving linear second - order DDEs. This method is significant in that it converts differential equations into algebraic equations, provides a theoretical framework for solving these equations using the LT, and provides examples that illustrate and validate their theoretical results.

Villasana [25], and Keane [16] investigated the effects of delays on the dynamics of climate systems, which frequently exhibit inherent time gaps as a result of a variety of processes. They investigate the circumstances in which the system maintains stability or may exhibit oscillatory or chaotic behavior. Shayak [22] demonstrated that the delayed model is more effective in fitting the observed infection curves than models that do not account for delays by applying the model to real - world data from the Covid - 19 pandemic. This underscores the significance of taking time delays into account when predicting and managing the virus's spread. Lucio Torelli [24] studied the stability of the backward Euler approach allows it to be used reliably in solving DDEs without the possibility of numerical instability. This makes it a useful tool for practitioners dealing with time - delay systems. The stability of numerical techniques when used to DDE is a topic of interest. This work connects the study of stable ODEs to the more complicated DDEs.

In this manuscript, we examine the solution of MDDEs of the form

$$\varphi'(\tau) - a_1\varphi(\tau) - a_2\varphi(\tau-\eta_1) - a_3\varphi(\tau-\eta_2) = h(\tau),\ \tau > 0 \tag{1.1}$$

$$\varphi(\tau) = \chi(\tau),\ \ \tau \leq 0. \tag{1.2}$$

using Aboodh integral transform.

2. Preliminaries

The purpose of this part is to provide fundamental definitions in order to provide a foundation for understanding various vital concepts to the DDEs using AT method.

Definition 2.1. *[1] Let*

$$A = \{\varphi(\tau) : \exists\, M, k_1, k_2 > 0, |\varphi(\tau)| < Me^{-v\tau}\}, \tag{2.1}$$

where M must be finite number, k_1, k_2 may be infinite or finite. The Aboodh integral transform is characterized by

$$A[\varphi(\tau)] = K(v) = \frac{1}{v}\int_0^\infty \varphi(\tau)e^{-v\tau}d\tau,\ \ \tau \geq 0,\ k_1 \leq v \leq k_2.$$

3. Analytical Approaches to DDEs

In this section, we illustrate the general solution of MDDEs using Aboodh integral transform.

Lemma 3.1. *Let $\varphi(\tau-\eta)$ be a real valued function. Then*

$$A[\varphi(\tau-\eta)] = \frac{1}{v}\int_0^\infty e^{-v\tau}\varphi(\tau-\eta)d\tau = \chi(v) + e^{-v\eta}K(v),$$

where $\chi(v) = \dfrac{e^{-v\eta}}{v}\displaystyle\int_{-\eta}^{0} e^{-v\tau}\varphi(\tau)dt.$

Theorem 3.1. *Suppose that* (1.1) *satisfies* (2.1). *Then, $A[\varphi(\tau)]$ is the exact solution of* (1.1) - (1.2).

Proof. Integrating (1.1) with respect to $(0,\tau)$, we have

$$\varphi(\tau)-\chi(0)-a\int_0^\tau \varphi(u)du-b\int_0^\tau \varphi(u-\eta_1)du$$
$$-c\int_0^\tau \varphi(u-\eta_2)du=\int_0^\tau g(u)du. \tag{3.1}$$

Using $u-\eta_i=\tau_i,\ i=1,2,...,n$, we have

$$\int_0^\tau \varphi(u-\eta_i)du=\int_{-\eta_i}^0 \chi(\tau_i)d\tau_i+\int_0^{\tau-\eta_i}\varphi(\tau_i)d\tau_i. \tag{3.2}$$

From (3.1), we obtain

$$\varphi(\tau)+l_1(\tau)-a\int_0^\tau \varphi(u)du-b\int_0^{\tau-\eta_1}\varphi(\tau_1)d\tau_1$$
$$-c\int_0^{\tau-\eta_2}\varphi(\tau_2)d\tau_2=\int_0^\tau g(u)du, \tag{3.3}$$

where

$$l_1(\tau)=-\chi(0)-b\int_{-\eta_1}^0 \chi(\tau_1)d\tau_1-c\int_{-\eta_2}^0 \chi(\tau_2)d\tau_2$$
$$|\varphi(\tau)|\leq |l_1(\tau)|+l_2(\tau)+|a|\int_0^\tau |\varphi(u)|du+\int_0^\tau |g(u)|du,$$

where

$$l_2(\tau)=|b|\int_0^\tau |\varphi(\tau_1)|d\tau_1+|c|\int_0^\tau |\varphi(\tau_2)|d\tau_2.$$

Since $\varphi(\tau)$ is piecewise continuous, then

$$|g(u)|<Me^{-vu},\ (u>0).$$

We have $\int_0^\tau |g(u)|du\leq \frac{Me^{-v\tau}}{-v}$. Hence

$$|\varphi(\tau)|\leq |l_1(\tau)|+l_2(\tau)+|a|\int_0^\tau |\varphi(u)|du-\frac{Me^{-v\tau}}{v}. \tag{3.4}$$

By Gronwall's inequality, we have

$$|\varphi(\tau)|\leq c_1+D_1-\frac{Me^{-v\tau}}{v}+c_2+D_2$$
$$+|a|e^{c_1\tau+D_1\tau}\left(c_1\tau+D_1\tau+\frac{M}{\alpha^2}\left(e^{-v\tau}-1\right)\right),$$

where

$$c_1=|\chi(0)|+|b|\int_{-\eta_1}^0 |\chi(\tau_1)|d\tau_1,$$
$$c_2=|b|\int_0^\tau |\varphi(\tau_1)|d\tau_1,$$
$$D_1=|c|\int_{-\eta_2}^0 |\chi(\tau_2)|d\tau_2,$$

$$D_2 = |c| \int_0^{\tau} |\varphi(\tau_2)| d\tau_2.$$

□

Theorem 3.2. *Let $K(v)$ is the Abooth transformation of $\varphi(\tau)$, and $\chi(\tau),\ \ \tau \in [-\infty, 0]$ be continuous. Then, the exact solution of* (1.1) - (1.2) *is*

$$K(v) = \frac{\varphi(0) + vb\chi_1(v) + vc\chi_2(v)}{v^2 - av - be^{-v\eta_1} - ce^{-v\eta_2}},$$

when $h(\tau) = 0$.

Proof. Since

$$A[\varphi'(\tau)] - aA[\varphi(\tau)] - bA[\varphi(\tau - \eta_1)] - cA[\varphi(\tau - \eta_2)] = 0 \tag{3.5}$$

$$A[\varphi'(\tau)] = vK(v) - \frac{\varphi(0)}{v},$$

$$A[\varphi(\tau)] = K(v),$$

$$A[\varphi(\tau - \eta_i)] = \frac{1}{v} \int_0^{\infty} e^{-v\tau} \varphi(\tau - \eta_i) d\tau. \tag{3.6}$$

Using $t - \eta_i = \tau_i$, for $i = 1, 2, ..., n$, we obtain

$$\begin{aligned} A[\varphi(\tau - \eta_i)] &= \frac{e^{-v\eta_i}}{v} \int_{-\eta_i}^{0} e^{-v\tau_i} \varphi(\tau_i) d\tau_i \\ &\quad + \frac{e^{-v\eta_i}}{v} \int_0^{\infty} e^{-v\tau_i} \varphi(\tau_i) d\tau_i \\ A[\varphi(\tau - \eta_i)] &= \chi_i(v) + e^{-v\eta_i} K(v), \end{aligned}$$

where

$$\chi_i(v) = \frac{e^{-v\eta_i}}{v} \int_{-\eta_i}^{0} e^{-v\tau_i} \varphi(\tau_i) d\tau_i.$$

We can write equation (3.5)

$$[v - a - be^{-v\eta_1} - ce^{-v\eta_2}]K(v) = \frac{\varphi(0)}{v} + b\chi_1(v) + c\chi_2(v)$$

$$K(v) = \frac{\varphi(0) + vb\chi_1(v) + vc\chi_2(v)}{v^2 - av - be^{-v\eta_1} - ce^{-v\eta_2}}.$$

Taking Aboodh inverse transform, we have

$$k(\tau) = A^{-1}\left[\frac{\varphi(0) + vb\chi_1(v) + vc\chi_2(v)}{v^2 - av - be^{-v\eta_1} - ce^{-v\eta_2}}\right].$$

Which is the exact solution of (1.1) - (1.2). □

Theorem 3.3. *Let $h(\tau) = 0$ in* (1.1)*, $K(v)$ is the Aboodh transformation of $\varphi(\tau)$, and $\chi(\tau),\ \ \tau \in [-\infty, 0]$ be continuous. Then, the exact solution of* (1.1) - (1.2) *is*

$$K(v) = \frac{\varphi(0)(kv - v^2) + bv\lambda_1 + cv\lambda_2}{v(kv - v^2)[v - a - be^{-v\eta_1} - ce^{-v\eta_2}]},$$

where $\lambda_1 = (e^{-v\eta_1} - e^{-k\eta_1})$*, and* $\lambda_2 = (e^{-v\eta_2} - e^{-k\eta_2})$*.*

Proof. Since $\varphi(\tau) = e^{kt},\ \tau \leq 0$ be the solution of the equation

$$\varphi'(\tau) = a\varphi(\tau) + b\varphi(\tau) + c\varphi(\tau),$$

where

$$k = a + b + c,\ \ \log\varphi = k\tau + k_1,\ \ \varphi = e^{k\tau + k_1}.$$

Using Abooth integral transform, we obtain

$$A[\varphi'(\tau)] - aA[\varphi(\tau)] - bA[\varphi(\tau - \eta_1)] - cA[\varphi(\tau - \eta_2)] = 0 \tag{3.7}$$

$$A[\varphi'(\tau)] = vK(v) - \frac{\varphi(0)}{v},$$

$$A[\varphi(\tau)] = K(v),$$

$$A[\varphi(\tau - \eta_i)] = \frac{1}{v}\int_0^\infty e^{-v\tau}\varphi(\tau - \eta_i)dt.$$

Utilizing $t - \eta_i = \tau_i,\ i = 1, 2, ..., n$, we obtain

$$\begin{aligned} A[\varphi(\tau - \eta_i)] &= \frac{e^{-v\eta_i}}{v}\int_{-\eta_i}^0 e^{-v\tau_i}\varphi(\tau_i)d\tau_i \\ &\quad + \frac{e^{-v\eta_i}}{v}\int_0^\infty e^{-v\tau_i}\varphi(\tau_i)d\tau_i \\ A[\varphi(\tau - \eta_1)] &= \chi_1(v) + e^{-v\eta_1}K(v), \end{aligned}$$

where

$$\chi_1(v) = \frac{e^{-v\eta_1}}{v}\int_{-\eta_1}^0 e^{-v\tau_1}\varphi(\tau_1)d\tau_1$$

$$\chi_1(v) = \frac{1}{(kv - v^2)[e^{-v\eta_1} - e^{-k\eta_1}]}.$$

Similarly

$$A[\varphi(\tau - \eta_2)] = \chi_2(v) + e^{-v\eta_2}K(v),$$

where

$$\chi_2(v) = \frac{e^{-v\eta_2}}{v}\int_{-\eta_2}^0 e^{-v\tau_2}\varphi(\tau_2)d\tau_2$$

$$\chi_2(v) = \frac{1}{(kv - v^2)[e^{-v\eta_2} - e^{-k\eta_2}]}.$$

We can rewrite equation (3.7), we obtain

$$K(v) = \frac{\varphi(0)(kv - v^2) + bv\lambda_1 + cv\lambda_2}{v(kv - v^2)[v - a - be^{-v\eta_1} - ce^{-v\eta_2}]},$$

where $\lambda_1 = (e^{-v\eta_1} - e^{-k\eta_1})$, and $\lambda_2 = (e^{-v\eta_2} - e^{-k\eta_2})$. Taking Aboodh inverse on both sides

$$k(\tau) = A^{-1}\left[\frac{\varphi(0)(kv - v^2) + bv\lambda_1 + cv\lambda_2}{v(kv - v^2)[v - a - be^{-v\eta_1} - ce^{-v\eta_2}]}\right].$$

Which is the exact solution of (1.1) - (1.2). □

4. Applications

The purpose of analyzing the stability of delay differential equations using the Aboodh transform is to understand how delays influence system behavior and to determine whether the system will remain bounded and predictable over time. The impact of analyzing the stability using the Aboodh transform is profound in both theoretical and practical domains. Stable systems result in predictable and bounded behavior, ensuring reliability in applications. This stability reduces operational risks, lowers maintenance costs, and enhances system efficiency. Conversely, instability can lead to unbounded growth, erratic oscillations, or even system failure.

The Parkinson's disease with delay differential equation is characterized by [18]

$$\dot{\varphi} = c_1\varphi(\tau) + c_2\varphi_{\eta_1} + c_3\varphi_{\eta_2}, \tag{4.1}$$

where $\varphi(\tau)$ is the state variable of Parkinson's disease interpreted with the measure of symptoms, $c_i,\ i = 1,2,3$ are coefficients reflect the pharmacological or physiological factors of $\varphi(\tau)$, and $\eta_i,\ i = 1,2$ represent the different time lags of the effects of medication ("on" or "off" meditation) or disease progression..

To solve this Parkinson's disease model applying Aboodh transform derived from Theorem 3.2 for equation (4.1), we have

$$K(v) = \frac{\varphi(0) + vc_2\chi_1(v) + vc_3\chi_2(v)}{v^2 - c_1 v - c_2 e^{-v\eta_1} - c_3 e^{-v\eta_2}}. \tag{4.2}$$

Taking Aboodh inverse transform for (4.2) on both sides, we obtain

$$k(\tau) = A^{-1}\left[\frac{\varphi(0)(kv - v^2) + c_2 v\lambda_1 + c_3 v\lambda_2}{v(kv - v^2)[v - c_1 - c_2 e^{-v\eta_1} - c_3 e^{-v\eta_2}]}\right].$$

Which is the exact solution of (4.1).

Case 1:: Let $c_i > 0,\ i = 1,2,3,\ \varphi(0) = 0.5, \eta_1 = 0.1, \eta_2 = 0.2$. Then $\varphi(\tau)$ diverges as $t \to \infty$. Hence, (4.1) is asymptotically stable.

Case 2:: Let $c_1 < 0, c_1 > 0,\ i = 2,3,\ \varphi(0) = 0.5, \eta_1 = 0.1, \eta_2 = 0.2$. Then $\varphi(\tau)$ converges to 0 as $tt \to \infty$. Hence, (4.1) is asymptotically stable.

Case 3:: Let $c_i > 0,\ i = 1,3,\ c_2 < 0,\ \varphi(0) = 0.5, \eta_1 = 0.1, \eta_2 = 0.2$. Then $\varphi(\tau)$ diverges as $tt \to \infty$. Hence, (4.1) is unstable.

Case 4:: Let $c_i > 0,\ i = 1,2,\ c_3 < 0,\ \varphi(0) = 0.5, \eta_1 = 0.1, \eta_2 = 0.2$. Then $\varphi(\tau)$ converges to 0 as $tt \to \infty$. Hence, (4.1) is asymptotically stable.

The asymptotic stability, as can be seen in figures 2, and 4, suggests that not only is the system stable, but it will also eventually return to a steady state after being disturbed due to the fact that it will eventually return to its initial condition. In addition to the fact that the system is completely stable, this is also the case. Over the course of time, the effects of relatively minor disturbances become less evident, and the system finally achieves a degree of stability with the passage of time.

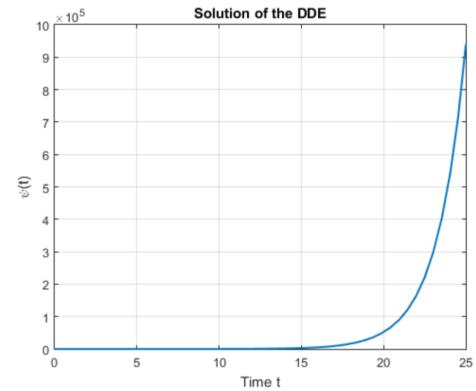

FIGURE 1. Divergence, unstable when $c_i > 0, \ i = 1, 2, 3$

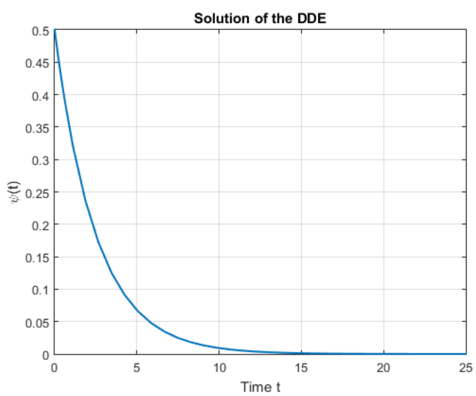

FIGURE 2. Convergence: Asymptotically stable when $c_1 < 0, c_1 > 0, \ i = 2, 3$

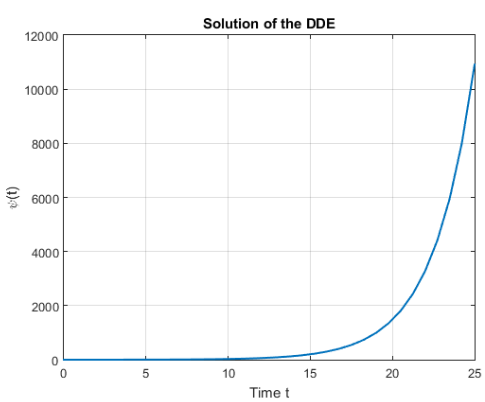

FIGURE 3. Divergence, unstable when $c_i > 0, \ i = 1, 3, c_2 < 0$

This might be regarded a more desired scenario in the treatment of sickness, which is when the patient's symptoms gradually settle over time, perhaps as a consequence of effective medicine that is taken over a longer length of time. This could be considered a more ideal position. In spite of

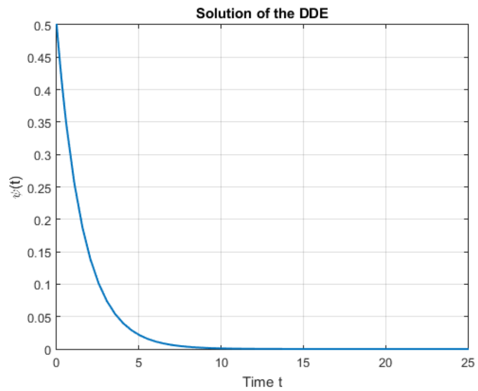

FIGURE 4. Convergence: Asymptotically stable when $c_i > 0,\ i = 1, 2,\ c_3 < 0$

the fact that symptoms can become more severe for a short period of time, it is very probable that they will eventually return to a state that is more manageable.

From figures 3 and 4, it is evident that when the solution is unstable, even little perturbations may result in significant changes in the behaviour of the system. This is particularly evident when the solution is unstable. In light of this, it is clear that the dynamics of the sickness are very sensitive to changes, which leads to implications that are difficult to forecast. Parkinson's disease in which the disease state can significantly worsen. It is probable that the patient need more careful supervision as well as maybe adjustments to the medicine that they are taking.

5. Conclusion

The assessment of the behavior of systems that are governed by such equations can be accomplished in a manner that is both efficient and reliable through the utilization of the AT Method for the stability study of MDDEs. The Aboodh transform is a powerful integral transform that provides a methodical approach to solving delay differential equations, even when there are many time delays present that need to be taken into consideration. As a result of the application of Parkinson's disease model using Aboodh transform, the stability criteria of the system are obtained in terms of the transformed variables. This offers the opportunity for a more comprehensive comprehension of the characteristics of the Parkinson's disease stability. The method offers a manner of dealing with the complexity that are brought about by temporal delays, which frequently result in instability in the dynamics of Parkinson's disease, when they are experienced. In most cases, the analysis entails locating the roots of characteristic equations that have been derived through the application of Parkinson's disease using Aboodh transform. This, leads to the determination of the stability of Parkinson's disease with multi delay differential equations. One can determine the conditions under which the system remains stable or becomes unstable owing to delays by evaluating these regions and determining the conditions. Finally, the Parkinson's disease model is asymptotic stability stable and will return to a steady state eventually after disturbances, and reverting to its initial condition. In case of illness, Parkinson's patient symptoms over time is effective and highly sensitive for long-term medication.

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