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Extension of Bipolar Q-Intuitionistic Fuzzy Ideals to Lower Level Sets and Homomorphisms via Regular Ordered Semigroups

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Abstract. This paper investigates several (ς, ζ)-bipolar Q-intuitionistic fuzzy structures in an ordered semigroup $\mathfrak{S}$, including subsemigroups (GBQIFSSG), left ideals (GBQIFLId), right ideals (GBQIFRId), ideals (GBQIFId), and bi-ideals (GBQIFBId). We introduce a novel extension of GBQIFId, denoted (ς, ζ)-GBQIFI, and study its correspondence to lower level sets (FLId, FRId, FBId) within $\mathfrak{S}$. Several characterizations are established, showing conditions under which these fuzzy structures reduce to their crisp counterparts. Homomorphic and inverse images of GBQIFSSGs are analyzed, and the role of regularity in ordered semigroups is explored to derive equivalences between product and meet operations. Illustrative examples are provided to validate the proposed results and offer new insights into the interaction between bipolar fuzzy frameworks and algebraic structures.

1. Introduction

As the world developed, human ambiguity and uncertainty increased, and the theory of crisp sets proved insufficient for experts or decision analysts to address these complexities. Fuzzy set theory (FST), intuitionistic FST (IFST), Pythagorean fuzzy set theory (PFST), and spherical fuzzy set theory (SFST) are among the theories that have been developed as a result of the uncertainties. The importance of the concept and its applicability to a wide range of disciplines, including real analysis, measure theory, topology, group theory, logic, and groupoids, has been illustrated by the numerous publications on FSTs. Ordered semigroups have several applications in formal languages, computer arithmetic, error-correcting codes, and sequential machine theory. Rosenfeld [10] described fuzzy subgroups and their characteristics. Mordeson [9] created a particular fuzzy

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semigroup classification. Zhang [12] was the first to introduce bipolar valued FSTs (BFSTs), which he used for decision analysis and modeling. FSTs that have their MG range extended from $[0,1]$ to $[-1,1]$ are known as BFSTs. Bipolar intuitionistic fuzzy ideals (BIdFIs) are an expansion of both fuzzy ideals (FIs) and intuitionistic fuzzy ideals (IFIs) by fusing the concepts of bipolarity (positive and negative MGs) with IFSs (MG and NMG). The support of the FST is contained in $[0,1]$. Numerous academics from nearly every scientific discipline were drawn to the FST, conducting studies and using it in their work. The FST was initially used by Rosenfeld [10] to organize fuzzy groups in a group context. Kuroki [5–7] performed an interpretation of fuzzy semigroups (FSG), FId, and FBId in semigroups. Dib et al. [2] introduced FIds and FBIds in FSGs. Xie et al. [11] and Kehayopulu et al. [3] introduced the idea of regular and intra-regular ordered semigroups.

Khan et al. [4] investigated specific characterizations of intra-regular semigroups. But FST cannot account for people's negative opinions or negative supporting grades; one expansion of FST is the idea of a bipolar fuzzy (BF) set. Therefore, the BF set theory (BFST) expands the range of FST $([0,1])$ to the BFST $([0,1],[-1,0])$ to encompass both positive and negative attitudes of humans. Both the negative supporting grade (NSG) and the positive supportive grade (PSG) of the BFST are found in $[-1,0]$ and $[0,1]$, respectively. Recently, Palanikumar et al. discussed the generalization of FSs via ordered semigroups [1] and bisemirings [8]. In contemporary algebra, the study of semigroups is a thriving field. But a semigroup need not have elements with inverses; the name suggests it is a variation of the group notion. Many scholars studied semigroups in the early stages from the viewpoints of rings and groups. But ring theory offers some guidance on how to develop the notion of ideals in the semigroup; it is possible to view the semigroup concept as the successful child of ring theory. Furthermore, the idea of a semigroup is a powerful strategy that has been applied by many academics in a variety of fields, including stochastic processes, mathematical biology, control theory, and nonlinear dynamical systems.

2. Preliminaries

Definition 2.1. *An ordered semigroup is a structure $(\mathfrak{S},\cdot,\leq)$, where $(\mathfrak{S},\cdot)$ is a semigroup and $\leq$ is a partial order relation on $\mathfrak{S}$ satisfying the following condition:*

$$\hbar \leq \rho \text{ and } \nu \leq \ell \Rightarrow \hbar\nu \leq \rho\ell, \forall \hbar,\rho,\nu,\ell \in \mathfrak{S}.$$

Here, $\mathfrak{S}$ refers to the ordered semigroup.

Definition 2.2. *Let X be an FS, if $\Bbbk_X$ is the characteristic function of X, then*

$$(\Bbbk_X)^{\kappa}(\zeta) := \begin{cases} \kappa \text{ if } \zeta \in X, \\ 0 \text{ otherwise}. \end{cases}$$

Definition 2.3. *An FS ζ of $\mathfrak{S}$ is known as FRId (FLId) of $\mathfrak{S}$ if*

(1) $\hbar \leq \rho \Rightarrow \zeta(\hbar) \geq \zeta(\rho)$,
(2) $\zeta(\hbar\rho) \geq \zeta(\hbar)$ *(resp.* $\zeta(\hbar\rho) \geq \zeta(\rho)$*)*, $\forall\, \hbar,\rho \in \mathfrak{S}$.

Definition 2.4. *An FS $\propto$ of $\mathfrak{S}$ is known as FBId of $\mathfrak{S}$ if*

(1) $\hbar \leq \rho \Rightarrow \propto(\hbar) \geq \propto(\rho)$,

(2) $\propto(\hbar\rho\nu) \geq \min\{\propto(\hbar), \propto(\nu)\}, \forall\, \hbar, \rho, \nu \in \mathfrak{S}$.

Remark 2.1. *$\mathfrak{S}$ is regular if and only if $\forall$ RId η and $\forall$ LId λ of $\mathfrak{S}$, $(\eta \sqcap \lambda] = (\eta \circ \lambda]$.*

Remark 2.2. *Let η, λ be subsets of $\mathfrak{S}$. Then*

(1) $(\Bbbk_{(\eta]} \vartriangle^{\zeta}_{\varsigma} \Bbbk_{(\lambda]}) = (\Bbbk_{(\eta \sqcup \lambda]})^{\zeta}_{\varsigma}$,

(2) $(\Bbbk_{(\eta]} \triangledown^{\zeta}_{\varsigma} \Bbbk_{(\lambda]}) = (\Bbbk_{(\eta \sqcap \lambda]})^{\zeta}_{\varsigma}$.

Remark 2.3. *Let $\eta, \lambda \subseteq \mathfrak{S}$ and $\{\eta_i \mid i \in I\}$ be a collection of subsets of $\mathfrak{S}$. We define*

(1) $(\eta] \subseteq (\lambda]$ *if and only if* $(\Bbbk_{(\eta]})^{\zeta}_{\varsigma} \geq (\Bbbk_{(\lambda]})^{\zeta}_{\varsigma}$,

(2) $(\sqcup_{i \in I} \Bbbk_{(\eta_i]})^{\zeta}_{\varsigma} = (\Bbbk_{\sqcup_{i \in I}(\eta_i]})^{\zeta}_{\varsigma}$,

(3) $(\sqcap_{i \in I} \Bbbk_{(\eta_i]})^{\zeta}_{\varsigma} = (\Bbbk_{\sqcap_{i \in I}(\eta_i]})^{\zeta}_{\varsigma}$.

3. New Generalization of Bipolar Q-Intuitionistic Fuzzy Ideals

Here $\varsigma, \zeta \in [0,1]$, $0 \geq \overleftarrow{\varsigma} > \overleftarrow{\zeta} \geq -1$ and $0 \leq \overrightarrow{\varsigma} < \overrightarrow{\zeta} \leq 1$.

Definition 3.1. *The GBQIFS $\propto = [(\mho, \supset), (\Finv, \Im)]$ of $\mathfrak{S}$ is a known as (ς, ζ)-GBQIFSSG of $\mathfrak{S}$ if*

(1) $\nu \leq \ell \Rightarrow \mho(\nu) \geq \mho(\ell), \supset(\nu) \leq \supset(\ell)$,

(2) $\max\{\mho(\nu \cdot \ell, \iota), \overrightarrow{\varsigma}\} \geq \min\{\mho(\nu, \iota), \mho(\ell, \iota), \overrightarrow{\zeta}\}$,
$\min\{\Finv(\nu \cdot \ell, \iota), \overrightarrow{\varsigma}\} \leq \max\{\Finv(\nu, \iota), \Finv(\ell, \iota), \overrightarrow{\zeta}\}$,

(3) $\min\{\supset(\nu \cdot \ell, \iota), \overleftarrow{\varsigma}\} \leq \max\{\supset(\nu, \iota), \supset(\ell, \iota), \overleftarrow{\zeta}\}$,
$\max\{\Im(\nu \cdot \ell, \iota), \overleftarrow{\varsigma}\} \geq \min\{\Im(\nu, \iota), \Im(\ell, \iota), \overleftarrow{\zeta}\}$, $\forall\, \nu, \ell \in \mathfrak{S}$ *and* $\iota \in Q \subseteq \mathfrak{S}$.

Definition 3.2. *The GBQIFS $\propto = [(\mho, \supset), (\Finv, \Im)]$ of $\mathfrak{S}$ is a known as (ς, ζ)-GBQIFLId of $\mathfrak{S}$ if*

(1) $\nu \leq \ell \Rightarrow \mho(\nu) \geq \mho(\ell)$, $\Finv(\nu) \leq \Finv(\ell)$, $\supset(\nu) \leq \supset(\ell)$, $\Im(\nu) \geq \Im(\ell)$,

(2) $\max\{\mho(\nu \cdot \ell, \iota), \overrightarrow{\varsigma}\} \geq \min\{\mho(\ell, \iota), \overrightarrow{\zeta}\}$,
$\min\{\Finv(\nu \cdot \ell, \iota), \overrightarrow{\varsigma}\} \leq \max\{\Finv(\ell, \iota), \overrightarrow{\zeta}\}$,

(3) $\min\{\supset(\nu \cdot \ell, \iota), \overleftarrow{\varsigma}\} \leq \max\{\supset(\ell, \iota), \overleftarrow{\zeta}\}$,
$\max\{\Im(\nu \cdot \ell, \iota), \overleftarrow{\varsigma}\} \geq \min\{\Im(\ell, \iota), \overleftarrow{\zeta}\}$, $\forall\, \nu, \ell \in \mathfrak{S}$ *and* $\iota \in Q \subseteq \mathfrak{S}$.

Definition 3.3. *The GBQIFS $\propto = [(\mho, \supset), (\Finv, \Im)]$ of $\mathfrak{S}$ is a known as (ς, ζ)-GBQIFRId of $\mathfrak{S}$ if*

(1) $\nu \leq \ell \Rightarrow \mho(\nu) \geq \mho(\ell)$, $\Finv(\nu) \leq \Finv(\ell)$, $\supset(\nu) \leq \supset(\ell)$, $\Im(\nu) \geq \Im(\ell)$,

(2) $\max\{\mho(\nu \cdot \ell, \iota), \overrightarrow{\varsigma}\} \geq \min\{\mho(\nu, \iota), \overrightarrow{\zeta}\}$,
$\min\{\Finv(\nu \cdot \ell, \iota), \overrightarrow{\varsigma}\} \leq \max\{\Finv(\nu, \iota), \overrightarrow{\zeta}\}$,

(3) $\min\{\supset(\nu \cdot \ell, \iota), \overleftarrow{\varsigma}\} \leq \max\{\supset(\nu, \iota), \overleftarrow{\zeta}\}$,
$\max\{\Im(\nu \cdot \ell, \iota), \overleftarrow{\varsigma}\} \geq \min\{\Im(\nu, \iota), \overleftarrow{\zeta}\}$, $\forall\, \nu, \ell \in \mathfrak{S}$ *and* $\iota \in Q \subseteq \mathfrak{S}$.

Definition 3.4. *The GBQIFS $\propto = [(\mho, \supset), (\Finv, \Im)]$ of $\mathfrak{S}$ is a known as (ς, ζ)-GBQIFBId of $\mathfrak{S}$ if*

(1) $\nu \leq \ell \Rightarrow \mho(\nu) \geq \mho(\ell)$, $\Finv(\nu) \leq \Finv(\ell)$, $\supset(\nu) \leq \supset(\ell)$, $\Im(\nu) \geq \Im(\ell)$,

(2) $\max\{\mho(\nu\cdot\ell,\iota),\overrightarrow{\varsigma}\}\geq\min\{\mho(\nu,\iota),\mho(\ell,\iota),\overrightarrow{\zeta}\}$,
$\min\{\Finv(\nu\cdot\ell,\iota),\overrightarrow{\varsigma}\}\leq\max\{\Finv(\nu,\iota),\Finv(\ell,\iota),\overrightarrow{\zeta}\}$,
$\max\{\mho(\nu\cdot\ell\cdot\jmath,\iota),\overrightarrow{\varsigma}\}\geq\min\{\mho(\nu,\iota),\mho(\jmath,\iota),\overrightarrow{\zeta}\}$,
$\min\{\Finv(\nu\cdot\ell\cdot\jmath,\iota),\overrightarrow{\varsigma}\}\leq\max\{\Finv(\nu,\iota),\Finv(\jmath,\iota),\overrightarrow{\zeta}\}$,

(3) $\min\{\backepsilon(\nu\cdot\ell,\iota),\overleftarrow{\varsigma}\}\leq\max\{\backepsilon(\nu,\iota),\backepsilon(\ell,\iota),\overleftarrow{\zeta}\}$,
$\max\{\mathfrak{I}(\nu\cdot\ell,\iota),\overleftarrow{\varsigma}\}\geq\min\{\mathfrak{I}(\nu,\iota),\mathfrak{I}(\ell,\iota),\overleftarrow{\zeta}\}$,
$\min\{\backepsilon(\nu\cdot\ell\cdot\jmath,\iota),\overleftarrow{\varsigma}\}\leq\max\{\backepsilon(\nu,\iota),\backepsilon(\jmath,\iota),\overleftarrow{\zeta}\}$,
$\max\{\mathfrak{I}(\nu\cdot\ell\cdot\jmath,\iota),\overleftarrow{\varsigma}\}\geq\min\{\mathfrak{I}(\nu,\iota),\mathfrak{I}(\jmath,\iota),\overleftarrow{\zeta}\}$, $\forall\ \nu,\ell,\jmath\in\mathfrak{S}$ *and* $\iota\in Q\subseteq\mathfrak{S}$.

Example 3.1. *Let* $\mathfrak{S}=\{\mathfrak{b}_1,\mathfrak{b}_2,\mathfrak{b}_3,\mathfrak{b}_4\}$, *where* $\cdot$ *is defined on* $\mathfrak{S}$.

$\cdot$	$\mathfrak{b}_1$	$\mathfrak{b}_2$	$\mathfrak{b}_3$	$\mathfrak{b}_4$
$\mathfrak{b}_1$	$\mathfrak{b}_1$	$\mathfrak{b}_1$	$\mathfrak{b}_1$	$\mathfrak{b}_1$
$\mathfrak{b}_2$	$\mathfrak{b}_1$	$\mathfrak{b}_2$	$\mathfrak{b}_3$	$\mathfrak{b}_4$
$\mathfrak{b}_3$	$\mathfrak{b}_1$	$\mathfrak{b}_3$	$\mathfrak{b}_3$	$\mathfrak{b}_3$
$\mathfrak{b}_4$	$\mathfrak{b}_1$	$\mathfrak{b}_3$	$\mathfrak{b}_3$	$\mathfrak{b}_3$

The order relation: $\{(\mathfrak{b}_1,\mathfrak{b}_1),(\mathfrak{b}_1,\mathfrak{b}_2),(\mathfrak{b}_1,\mathfrak{b}_3),(\mathfrak{b}_1,\mathfrak{b}_4),(\mathfrak{b}_2,\mathfrak{b}_2),(\mathfrak{b}_2,\mathfrak{b}_3),(\mathfrak{b}_2,\mathfrak{b}_4),(\mathfrak{b}_3,\mathfrak{b}_3),(\mathfrak{b}_4,\mathfrak{b}_3),(\mathfrak{b}_4,\mathfrak{b}_4)\}$.
Define a GBQIFS $\lambda=[(\mho,\backepsilon),(\Finv,\mathfrak{I})]:\mathfrak{S}\to[0,1]\times[-1,0]$ *as follows:*

$$[(\mho,\backepsilon),(\Finv,\mathfrak{I})](\mathfrak{b}_1,\iota)=((0.28,-0.30),(0.54,-0.63),\iota)$$

$$[(\mho,\backepsilon),(\Finv,\mathfrak{I})](\mathfrak{b}_2,\iota)=((0.33,-0.36),(0.44,-0.59),\iota)$$

$$[(\mho,\backepsilon),(\Finv,\mathfrak{I})](\mathfrak{b}_3,\iota)=((0.48,-0.47),(0.28,-0.35),\iota)$$

$$[(\mho,\backepsilon),(\Finv,\mathfrak{I})](\mathfrak{b}_4,\iota)=((0.41,-0.42),(0.35,-0.45),\iota).$$

Thus, λ *is a* $(0.45,0.55)$*-GBQIFSSG of* $\mathfrak{S}$ *and* ι *is in any subset of* $\mathfrak{S}$.

Theorem 3.1. *Let a GBQIFS* $\propto_{\varsigma}$ *be a* (ς,ζ)*-GBQIFSSG (GBQIFLId, GBQIFRId, GBQIFBId) of* $\mathfrak{S}$*. Hence, the lower level set is an SSG (LId, RId, BId) of* $\mathfrak{S}$*, where* $\mho_{\overrightarrow{\varsigma}}=\{\nu\in\mathfrak{S}\mid\mho(\nu,\iota)>\overrightarrow{\varsigma}\}$, $\Finv_{\overrightarrow{\varsigma}}=\{\nu\in\mathfrak{S}\mid\Finv(\nu,\iota)<\overrightarrow{\varsigma}\}$, $\backepsilon_{\overleftarrow{\varsigma}}=\{\nu\in\mathfrak{S}\mid\backepsilon(\nu,\iota)<\overleftarrow{\varsigma}\}$, *and* $\mathfrak{I}_{\overleftarrow{\varsigma}}=\{\nu\in\mathfrak{S}\mid\mathfrak{I}(\nu,\iota)>\overleftarrow{\varsigma}\}$.

Proof. Suppose that $\propto_{\overrightarrow{\varsigma}}$ is a (ς,ζ)-GBQIFSSG of $\mathfrak{S}$. Let $\nu,\ell\in\mathfrak{S}$ be such that $\nu,\ell\in\mho_{\overrightarrow{\varsigma}}$. Then $\mho(\nu,\iota)>\overrightarrow{\varsigma},\mho(\ell,\iota)>\overrightarrow{\varsigma}$. Thus, $\max\{\mho(\nu\cdot\ell,\iota),\overrightarrow{\varsigma}\}\geq\min\{\mho(\nu,\iota),\mho(\ell,\iota),\overrightarrow{\zeta}\}>\min\{\overrightarrow{\varsigma},\overrightarrow{\varsigma},\overrightarrow{\zeta}\}=\overrightarrow{\varsigma}$. Thus, $\mho(\nu\cdot\ell,\iota)>\overrightarrow{\varsigma}$, so $\nu\cdot\ell\in\mho_{\overrightarrow{\varsigma}}$. Thus, $\mho_{\overrightarrow{\varsigma}}$ is an SSG of $\mathfrak{S}$. Let $\nu,\ell\in\mathfrak{S}$ be such that $\nu,\ell\in\Finv_{\overrightarrow{\varsigma}}$. Then $\Finv(\nu,\iota)<\overrightarrow{\varsigma},\Finv(\ell,\iota)<\overrightarrow{\varsigma}$. Thus, $\min\{\Finv(\nu\cdot\ell,\iota),\overrightarrow{\varsigma}\}\leq\max\{\Finv(\nu,\iota),\Finv(\ell,\iota),\overrightarrow{\zeta}\}<\max\{\overrightarrow{\varsigma},\overrightarrow{\varsigma},\overrightarrow{\zeta}\}=\overrightarrow{\zeta}$. Thus, $\Finv(\nu\cdot\ell,\iota)<\overrightarrow{\varsigma}$, so $\nu\cdot\ell\in\Finv_{\overrightarrow{\varsigma}}$. Thus, $\Finv_{\overrightarrow{\varsigma}}$ is an SSG of $\mathfrak{S}$. Suppose that $\propto_{\overleftarrow{\varsigma}}$ is a (ς,ζ)-GBQIFSSG of $\mathfrak{S}$. Let $\nu,\ell\in\mathfrak{S}$ be such that $\nu,\ell\in\backepsilon_{\overleftarrow{\varsigma}}$. Then $\backepsilon(\nu,\iota)<\overleftarrow{\varsigma},\backepsilon(\ell,\iota)<\overleftarrow{\varsigma}$. Thus, $\min\{\backepsilon(\nu\cdot\ell,\iota),\overleftarrow{\varsigma}\}\leq\max\{\backepsilon(\nu,\iota),\backepsilon(\ell,\iota),\overleftarrow{\zeta}\}<\max\{\overleftarrow{\varsigma},\overleftarrow{\varsigma},\overleftarrow{\zeta}\}=\overleftarrow{\varsigma}$. Thus, $\backepsilon(\nu\cdot\ell,\iota)<\overleftarrow{\varsigma}$, so $\nu\cdot\ell\in\backepsilon_{\overleftarrow{\varsigma}}$. Thus, $\backepsilon_{\overleftarrow{\varsigma}}$ is an SSG of $\mathfrak{S}$. Let $\nu,\ell\in\mathfrak{S}$ be such that $\nu,\ell\in\mathfrak{I}_{\overleftarrow{\varsigma}}$. Then $\mathfrak{I}(\nu,\iota)>\overleftarrow{\varsigma},\mathfrak{I}(\ell,\iota)>\overleftarrow{\varsigma}$. Thus, $\max\{\mathfrak{I}(\nu\cdot\ell,\iota),\overleftarrow{\varsigma}\}\geq\min\{\mathfrak{I}(\nu,\iota),\mathfrak{I}(\ell,\iota),\overleftarrow{\zeta}\}>\min\{\overleftarrow{\varsigma},\overleftarrow{\varsigma},\overleftarrow{\zeta}\}=\overleftarrow{\zeta}$. Thus, $\mathfrak{I}(\nu\cdot\ell,\iota)>\overleftarrow{\varsigma}$, so $\nu\cdot\ell\in\mathfrak{I}_{\overleftarrow{\varsigma}}$. Thus, $\mathfrak{I}_{\overleftarrow{\varsigma}}$ is an SSG of $\mathfrak{S}$. □

Theorem 3.2. *A subset η of $\mathfrak{S}$ is an SSG (LId, RId, BId) of $\mathfrak{S}$ if and only if the GBQIFS $\propto = [(\mho, \eth), (\dashv, \Im)]$ of $\mathfrak{S}$ is defined as follows:*

$$\mho(\nu,\iota) = \begin{cases} \geq \overrightarrow{\zeta} \text{ if } \nu \in (\eta] \\ \overrightarrow{\varsigma} \text{ otherwise} \end{cases} \qquad \dashv(\nu,\iota) = \begin{cases} \leq \overrightarrow{\zeta} \text{ if } \nu \in (\eta] \\ \overrightarrow{\varsigma} \text{ otherwise} \end{cases}$$

$$\eth(\nu,\iota) = \begin{cases} \leq \overleftarrow{\zeta} \text{ if } \nu \in (\eta] \\ \overleftarrow{\varsigma} \text{ otherwise} \end{cases} \qquad \Im(\nu,\iota) = \begin{cases} \geq \overleftarrow{\zeta} \text{ if } \nu \in (\eta] \\ \overleftarrow{\varsigma} \text{ otherwise} \end{cases}$$

is a (ς,ζ)-GBQIFSSG (GBQIFLId, GBQIFRId, GBQIFBId) of $\mathfrak{S}$.

Proof. Let $\nu,\ell \in \mathfrak{S}$ be such that $\nu,\ell \in (\eta]$. Then $\nu\cdot\ell \in (\eta]$. Thus, $\mho(\nu\ell,\iota) \geq \overrightarrow{\zeta}$ and $\dashv(\nu\ell,\iota) \leq \overrightarrow{\zeta}$. Thus, $\max\{\mho(\nu\cdot\ell,\iota),\overrightarrow{\varsigma}\} \geq \overrightarrow{\zeta} = \min\{\mho(\nu,\iota),\mho(\ell,\iota),\overrightarrow{\zeta}\}$ and $\min\{\dashv(\nu\cdot\ell,\iota),\overrightarrow{\varsigma}\} \leq \overrightarrow{\zeta} = \max\{\dashv(\nu,\iota),\dashv(\ell,\iota),\overrightarrow{\zeta}\}$. If $\nu \notin (\eta]$ or $\ell \notin (\eta]$, then $\min\{\mho(\nu,\iota),\mho(\ell,\iota),\overrightarrow{\zeta}\} = \overrightarrow{\varsigma}$ and $\max\{\dashv(\nu,\iota),\dashv(\ell,\iota),\overrightarrow{\zeta}\} = \overrightarrow{\zeta}$. That is, $\max\{\mho(\nu\cdot\ell,\iota),\overrightarrow{\varsigma}\} \geq \min\{\mho(\nu,\iota),\mho(\ell,\iota),\overrightarrow{\zeta}\}$ and $\min\{\dashv(\nu\cdot\ell,\iota),\overrightarrow{\varsigma}\} \leq \max\{\dashv(\nu,\iota),\dashv(\ell,\iota),\overrightarrow{\zeta}\}$. Let $\nu,\ell \in \mathfrak{S}$ be such that $\nu,\ell \in (\eta]$. Then $\nu\cdot\ell \in (\eta]$. Thus, $\eth(\nu\ell,\iota) \leq \overleftarrow{\zeta}$ and $\Im(\nu\ell,\iota) \geq \overleftarrow{\zeta}$. Thus, $\min\{\eth(\nu\cdot\ell,\iota),\overleftarrow{\varsigma}\} \leq \overleftarrow{\zeta} = \max\{\eth(\nu,\iota),\eth(\ell,\iota),\overleftarrow{\zeta}\}$ and $\max\{\Im(\nu\cdot\ell,\iota),\overleftarrow{\varsigma}\} \geq \overleftarrow{\zeta} = \min\{\Im(\nu,\iota),\Im(\ell,\iota),\overleftarrow{\zeta}\}$. If $\nu \notin (\eta]$ or $\ell \notin (\eta]$, then $\max\{\eth(\nu,\iota),\eth(\ell,\iota),\overleftarrow{\zeta}\} = \overleftarrow{\varsigma}$ and $\min\{\Im(\nu,\iota),\Im(\ell,\iota),\overleftarrow{\zeta}\} = \overleftarrow{\zeta}$. That is, $\min\{\eth(\nu\cdot\ell,\iota),\overleftarrow{\varsigma}\} \leq \max\{\eth(\nu,\iota),\eth(\ell,\iota),\overleftarrow{\zeta}\}$ and $\max\{\Im(\nu\cdot\ell,\iota),\overleftarrow{\varsigma}\} \geq \min\{\Im(\nu,\iota),\Im(\ell,\iota),\overleftarrow{\zeta}\}$. Thus, $\propto$ is a (ς,ζ)-GBQIFSSG of $\mathfrak{S}$.

Conversely, assume that $\propto = [\mho,\dashv]$ is a (ς,ζ)-GBQIFSSG of $\mathfrak{S}$. Let $\nu,\ell \in (\eta]$. Then $\mho(\nu,\iota) \geq \overrightarrow{\zeta}, \mho(\ell,\iota) \geq \overrightarrow{\zeta}$ and $\dashv(\nu,\iota) \leq \overrightarrow{\zeta}, \dashv(\ell,\iota) \leq \overrightarrow{\zeta}$. Now, $\propto = [\mho,\dashv]$ is a (ς,ζ)-GBQIFSSG of $\mathfrak{S}$. Thus, $\max\{\mho(\nu\cdot\ell,\iota),\overrightarrow{\varsigma}\} \geq \min\{\mho(\nu,\iota),\mho(\ell,\iota),\overrightarrow{\zeta}\} \geq \min\{\overrightarrow{\zeta},\overrightarrow{\zeta},\overrightarrow{\zeta}\} = \overrightarrow{\zeta}$ and $\min\{\dashv(\nu\cdot\ell,\iota),\overrightarrow{\varsigma}\} \leq \max\{\dashv(\nu,\iota),\dashv(\ell,\iota),\overrightarrow{\zeta}\} \leq \max\{\overrightarrow{\zeta},\overrightarrow{\zeta},\overrightarrow{\zeta}\} = \overrightarrow{\zeta}$. It follows that $\nu\ell \in (\eta]$. Let $\nu,\ell \in (\eta]$. Then $\eth(\nu,\iota) \leq \overleftarrow{\zeta}, \eth(\ell,\iota) \leq \overleftarrow{\zeta}$, and $\Im(\nu,\iota) \geq \overleftarrow{\zeta}, \Im(\ell,\iota) \geq \overleftarrow{\zeta}$. Now, $\propto = [\eth,\Im]$ is a (ς,ζ)-GBQIFSSG of $\mathfrak{S}$. Thus, $\min\{\eth(\nu\cdot\ell,\iota),\overleftarrow{\varsigma}\} \leq \max\{\eth(\nu,\iota),\eth(\ell,\iota),\overleftarrow{\zeta}\} \leq \max\{\overleftarrow{\zeta},\overleftarrow{\zeta},\overleftarrow{\zeta}\} = \overleftarrow{\zeta}$ and $\max\{\Im(\nu\cdot\ell,\iota),\overleftarrow{\varsigma}\} \geq \min\{\Im(\nu,\iota),\Im(\ell,\iota),\overleftarrow{\zeta}\} \geq \overleftarrow{\zeta}$, so $\nu\ell \in (\eta]$. Hence, η is an SSG of $\mathfrak{S}$. □

Definition 3.5. *Let $\propto = [(\mho,\eth),(\dashv,\Im)]$ be a (ς,ζ)-GBQIFSSG of $\mathfrak{S}$, and let $t,s \in (\varsigma,\zeta]$. Hence, the* level subset $\propto^{(t,s)}$ *of $\propto$ is defined as*

$$\propto^{(t,s)} = \left\{ x \in \mathfrak{S} \;\middle|\; \begin{array}{l} \mho(x) \geq t \text{ and } \dashv(x) \leq t, \\ \eth(x) \leq s \text{ and } \Im(x) \geq s \end{array} \right\}.$$

Theorem 3.3. *The GBQIFS $\propto = [(\mho,\eth),(\dashv,\Im)]$ is a (ς,ζ)-GBQIFSSG (GBQIFLId, GBQIFRId, GBQIFBId) of $\mathfrak{S}$ if and only if each level subset $\propto^{(t,s)}$ is an SSG (LId, RId, BId) of $\mathfrak{S}$, $\forall\, t \in (\overrightarrow{\varsigma},\overrightarrow{\zeta}]$.*

Proof. Assume that $\propto^{(t,s)}$ is an SSG of $\mathfrak{S}$. Let $\nu_1,\nu_2 \in \mathfrak{S}$. Let $t = \min\{\mho(\nu_1,\iota),\mho(\nu_2,\iota)\}$. Hence, $\nu_1,\nu_2 \in \mho_t$. Thus, $\max\{\mho(\nu_1\cdot\nu_2,\iota),\overrightarrow{\varsigma}\} \geq t = \min\{\mho(\nu_1,\iota),\mho(\nu_2,\iota),\overrightarrow{\zeta}\}$. Let $t = \max\{\dashv(\nu_1,\iota),\dashv(\nu_2,\iota)\}$. Hence, $\nu_1,\nu_2 \in \dashv_t$. Thus, $\min\{\dashv(\nu_1\cdot\nu_2,\iota),\overrightarrow{\varsigma}\} \leq t = \max\{\dashv(\nu_1,\iota),\dashv(\nu_2,\iota),\overrightarrow{\zeta}\}$. Let $s = \max\{\eth(\nu_1,\iota),\eth(\nu_2,\iota)\}$. Hence, $\nu_1,\nu_2 \in \eth_s$. Thus, $\min\{\eth(\nu_1\cdot\nu_2,\iota),\overleftarrow{\varsigma}\} \leq s = \max\{\eth(\nu_1,\iota),\eth(\nu_2,\iota),\overleftarrow{\zeta}\}$. Let $s = \min\{\Im(\nu_1,\iota),\Im(\nu_2,\iota)\}$. Hence, $\nu_1,\nu_2 \in \Im_s$. Thus, $\max\{\Im(\nu_1\cdot\nu_2,\iota),\overleftarrow{\varsigma}\} \geq s = \min\{\Im(\nu_1,\iota),\Im(\nu_2,\iota),\overleftarrow{\zeta}\}$. Hence, $\propto$ is a (ς,ζ)-GBQIFSSG of $\mathfrak{S}$.

Conversely, assume that $\propto$ is a (ς,ζ)-GBQIFSSG of $\mathfrak{S}$ and $\nu_1,\nu_2 \in\propto^{(t,s)}$. Hence, $\mho(\nu_1,\iota) \geq t, \mho(\nu_2,\iota) \geq t$. Thus, $\max\{\mho(\nu_1\cdot\nu_2,\iota),\overrightarrow{\varsigma}\} \geq \min\{\mho(\nu_1,\iota),\mho(\nu_2,\iota),\overrightarrow{\zeta}\} \geq t$, so $\nu_1\cdot\nu_2 \in\propto^{(t,s)}$. Now, $\dashv(\nu_1,\iota) \leq t, \dashv(\nu_2,\iota) \leq t$. But $\propto$ is a (ς,ζ)-GBQIFSSG of $\mathfrak{S}$, $\min\{\dashv(\nu_1\cdot\nu_2,\iota),\overrightarrow{\varsigma}\} \leq \max\{\dashv(\nu_1,\iota),\dashv(\nu_2,\iota),\overrightarrow{\zeta}\} \leq t$, so $\nu_1\cdot\nu_2 \in \dashv_t$. Hence, $Ↄ(\nu_1,\iota) \leq t, Ↄ(\nu_2,\iota) \leq s$. Thus, $\min\{Ↄ(\nu_1\cdot\nu_2,\iota),\overleftarrow{\varsigma}\} \leq \max\{Ↄ(\nu_1,\iota),Ↄ(\nu_2,\iota),\overleftarrow{\zeta}\} \leq s$, so $\nu_1\cdot\nu_2 \in Ↄ_t$. Hence, $\mathfrak{I}(\nu_1,\iota) \geq s, \mathfrak{I}(\nu_2,\iota) \geq s$. But $\mathfrak{I}$ is an SSG of $\mathfrak{S}$, $\max\{\mathfrak{I}(\nu_1\cdot\nu_2,\iota),\overleftarrow{\varsigma}\} \geq s$. Hence, $\propto_{(t,s)}$ is an SSG of $\mathfrak{S}$. □

Remark 3.1. *The GBQIFSSG $\propto$ of $\mathfrak{S}$ is a (ς,ζ)-GBQIFSSG of $\mathfrak{S}$, but reverse implication is need not be true. By Example 3.1, we get $\forall\iota \in \mathfrak{S}$,*

$$[(\mho,Ↄ),(\dashv,\mathfrak{I})](\flat_1,\iota) = ((0.05,-0.04),(0.22,-0.19),\iota)$$
$$[(\mho,Ↄ),(\dashv,\mathfrak{I})](\flat_2,\iota) = ((0.12,-0.09),(0.15,-0.12),\iota)$$
$$[(\mho,Ↄ),(\dashv,\mathfrak{I})](\flat_3,\iota) = ((0.22,-0.19),(0.05,-0.02),\iota)$$
$$[(\mho,Ↄ),(\dashv,\mathfrak{I})](\flat_4,\iota) = ((0.16,-0.14),(0.10,-0.07),\iota).$$

Thus, $\propto$ is a $(0.13,0.27)$-GBQIFSSG of $\mathfrak{S}$ but not a GBQIFSSG.

Definition 3.6. *The GBQIFS $\Bbbk_\eta$ is defined as*

$$\Bbbk_\eta^{\overleftarrow{T}}(\nu,\iota) = \begin{cases} \overrightarrow{\zeta} & \text{if } \nu \in (\eta] \\ \overrightarrow{\varsigma} & \text{otherwise} \end{cases} \qquad \Bbbk_\eta^{\overleftarrow{F}}(\nu,\iota) = \begin{cases} \overrightarrow{\varsigma} & \text{if } \nu \in (\eta] \\ \overrightarrow{\zeta} & \text{otherwise} \end{cases}$$

$$\Bbbk_\eta^{\overrightarrow{T}}(\nu,\iota) = \begin{cases} \overleftarrow{\zeta} & \text{if } \nu \in (\eta] \\ \overleftarrow{\varsigma} & \text{otherwise} \end{cases} \qquad \Bbbk_\eta^{\overrightarrow{F}}(\nu,\iota) = \begin{cases} \overleftarrow{\varsigma} & \text{if } \nu \in (\eta] \\ \overleftarrow{\zeta} & \text{otherwise} \end{cases}$$

Theorem 3.4. *The subset η of $\mathfrak{S}$ is an SSG (LId, RId, BId) of $\mathfrak{S}$ if and only if the GBQIFS $\Bbbk_{(\eta]}$ is a (ς,ζ)-GBQIFSSG (GBQIFLId, GBQIFRId, GBQIFBId) of $\mathfrak{S}$.*

Proof. Suppose that η is an SSG of $\mathfrak{S}$. Hence, $\Bbbk_{(\eta]}$ is a GBQIFSSG of $\mathfrak{S}$, so $\Bbbk_{(\eta]}$ is a (ς,ζ)-GBQIFSSG of $\mathfrak{S}$.

Conversely, assume that $\Bbbk_{(\eta]}$ is a (ς,ζ)-GBQIFSSG of $\mathfrak{S}$. Let $\nu,\ell \in \mathfrak{S}$ be such that $\nu,\ell \in (\eta]$. Hence, $\Bbbk_{(\eta]}^{\overleftarrow{T}}(\nu,\iota) = \overrightarrow{\zeta} = \Bbbk_{(\eta]}^{\overleftarrow{T}}(\ell,\iota) = \overrightarrow{\zeta}$. But $\Bbbk_{(\eta]}^{\overleftarrow{T}}$ is a (ς,ζ)-GBQIFSSG, we have

$$\begin{aligned} \max\{\Bbbk_{(\eta]}^{\overleftarrow{T}}(\nu\cdot\ell,\iota),\overrightarrow{\varsigma}\} &\geq \min\{\Bbbk_{(\eta]}^{\overleftarrow{T}}(\nu,\iota),\Bbbk_{(\eta]}^{\overleftarrow{T}}(\ell,\iota),\overrightarrow{\zeta}\} \\ &= \min\{\overrightarrow{\zeta},\overrightarrow{\zeta},\overrightarrow{\zeta}\} \\ &= \overrightarrow{\zeta} \end{aligned}$$

as $\overrightarrow{\varsigma} < \overrightarrow{\zeta} \Rightarrow \Bbbk_{(\eta]}^{\overleftarrow{T}}(\nu\cdot\ell,\iota) \geq \overrightarrow{\zeta}$. Thus, $\nu\cdot\ell \in (\eta]$.

Let $\nu, \ell \in \mathfrak{S}$ be such that $\nu, \ell \in (\eta]$. Hence, $\Bbbk^{\overleftarrow{F}}_{(\eta]}(\nu, \iota) = \overrightarrow{\varsigma} = \Bbbk^{\overleftarrow{F}}_{(\eta]}(\ell, \iota) = \overrightarrow{\varsigma}$. But $\Bbbk^{\overleftarrow{F}}_{(\eta]}$ is a (ς, ζ)-GBQIFSSG, we have

$$\begin{aligned}\min\{\Bbbk^{\overleftarrow{F}}_{(\eta]}(\nu \cdot \ell, \iota), \overrightarrow{\varsigma}\} &\leq \max\{\Bbbk^{\overleftarrow{F}}_{(\eta]}(\nu, \iota), \Bbbk^{\overleftarrow{F}}_{(\eta]}(\ell, \iota), \overrightarrow{\zeta}\} \\ &= \max\{\overrightarrow{\varsigma}, \overrightarrow{\varsigma}, \overrightarrow{\zeta}\} \\ &= \overrightarrow{\zeta}\end{aligned}$$

as $\overrightarrow{\varsigma} < \overrightarrow{\zeta} \Rightarrow \Bbbk^{\overleftarrow{F}}_{(\eta]}(\nu \cdot \ell, \iota) \leq \overrightarrow{\varsigma}$. Thus, $\nu \cdot \ell \in (\eta]$.

Thus, η is an SSG of $\mathfrak{S}$.

Let $\nu, \ell \in \mathfrak{S}$ be such that $\nu, \ell \notin (\eta]$. Hence, $\Bbbk^{\overleftarrow{T}}_{(\eta]}(\nu, \iota) = \overrightarrow{\varsigma} = \Bbbk^{\overleftarrow{T}}_{(\eta]}(\ell, \iota) = \overrightarrow{\varsigma}$. But $\Bbbk^{\overleftarrow{T}}_{(\eta]}$ is a (ς, ζ)-GBQIFSSG, we have

$$\begin{aligned}\max\{\Bbbk^{\overleftarrow{T}}_{(\eta]}(\nu \cdot \ell, \iota), \overrightarrow{\varsigma}\} &\geq \min\{\Bbbk^{\overleftarrow{T}}_{(\eta]}(\nu, \iota), \Bbbk^{\overleftarrow{T}}_{(\eta]}(\ell, \iota), \overrightarrow{\zeta}\} \\ &= \min\{\overrightarrow{\varsigma}, \overrightarrow{\varsigma}, \overrightarrow{\zeta}\} \\ &= \overrightarrow{\varsigma}\end{aligned}$$

as $\overrightarrow{\varsigma} < \overrightarrow{\zeta} \Rightarrow \Bbbk^{\overleftarrow{T}}_{(\eta]}(\nu \cdot \ell, \iota) \geq \overrightarrow{\varsigma}$. Thus, $\nu \cdot \ell \notin (\eta]$.

Let $\nu, \ell \in \mathfrak{S}$ be such that $\nu, \ell \notin (\eta]$. Hence, $\Bbbk^{\overleftarrow{F}}_{(\eta]}(\nu, \iota) = \overrightarrow{\zeta} = \Bbbk^{\overleftarrow{F}}_{(\eta]}(\ell, \iota) = \overrightarrow{\zeta}$. But $\Bbbk^{\overleftarrow{F}}{}_{(\eta]}$ is a (ς, ζ)-GBQIFSSG, we have

$$\begin{aligned}\min\{\Bbbk^{\overleftarrow{F}}_{(\eta]}(\nu \cdot \ell, \iota), \overrightarrow{\varsigma}\} &\leq \max\{\Bbbk^{\overleftarrow{F}}_{(\eta]}(\nu, \iota), \Bbbk^{\overleftarrow{F}}_{(\eta]}(\ell, \iota), \overrightarrow{\zeta}\} \\ &= \max\{\overrightarrow{\zeta}, \overrightarrow{\zeta}, \overrightarrow{\zeta}\} \\ &= \overrightarrow{\zeta}\end{aligned}$$

as $\overrightarrow{\varsigma} < \overrightarrow{\zeta} \Rightarrow \Bbbk^{\overleftarrow{F}}_{(\eta]}(\nu \cdot \ell, \iota) \leq \overrightarrow{\zeta}$. Thus, $\nu \cdot \ell \notin (\eta]$.

Let $\Bbbk_{(\eta]}$ be a (ς, ζ)-GBQIFSSG of $\mathfrak{S}$. Let $\nu, \ell \in \mathfrak{S}$ be such that $\nu, \ell \in (\eta]$. Hence, $\Bbbk^{\overrightarrow{T}}_{(\eta]}(\nu, \iota) = \overleftarrow{\zeta} = \Bbbk^{\overrightarrow{T}}_{(\eta]}(\ell, \iota) = \overleftarrow{\zeta}$. But $\Bbbk^{\overrightarrow{T}}_{(\eta]}$ is a (ς, ζ)-GBQIFSSG, we have

$$\begin{aligned}\min\{\Bbbk^{\overrightarrow{T}}_{(\eta]}(\nu \cdot \ell, \iota), \overleftarrow{\varsigma}\} &\leq \max\{\Bbbk^{\overrightarrow{T}}_{(\eta]}(\nu, \iota), \Bbbk^{\overrightarrow{T}}_{(\eta]}(\ell, \iota), \overleftarrow{\zeta}\} \\ &= \max\{\overleftarrow{\zeta}, \overleftarrow{\zeta}, \overleftarrow{\zeta}\} \\ &= \overleftarrow{\zeta}\end{aligned}$$

as $\overleftarrow{\varsigma} > \overleftarrow{\zeta} \Rightarrow \Bbbk^{\overrightarrow{T}}_{(\eta]}(\nu \cdot \ell, \iota) \leq \overleftarrow{\zeta}$. Thus, $\nu \cdot \ell \in (\eta]$.

Let $\nu, \ell \in \mathfrak{S}$ be such that $\nu, \ell \in (\eta]$. Hence, $\Bbbk^{\overrightarrow{F}}_{(\eta]}(\nu, \iota) = \overleftarrow{\varsigma} = \Bbbk^{\overrightarrow{F}}_{(\eta]}(\ell, \iota) = \overleftarrow{\varsigma}$. But $\Bbbk^{\overrightarrow{F}}_{(\eta]}$ is a (ς, ζ)-GBQIFSSG, we have

$$\begin{aligned}\max\{\Bbbk^{\overrightarrow{F}}_{(\eta]}(\nu \cdot \ell, \iota), \overleftarrow{\varsigma}\} &\geq \min\{\Bbbk^{\overrightarrow{F}}_{(\eta]}(\nu, \iota), \Bbbk^{\overrightarrow{F}}_{(\eta]}(\ell, \iota), \overleftarrow{\zeta}\} \\ &= \min\{\overleftarrow{\varsigma}, \overleftarrow{\varsigma}, \overleftarrow{\zeta}\} \\ &= \overleftarrow{\zeta}\end{aligned}$$

as $\overleftarrow{\varsigma} > \overleftarrow{\zeta} \Rightarrow \Bbbk^{\overrightarrow{F}}_{(\eta]}(\nu\cdot\ell,\iota) \geq \overleftarrow{\varsigma}$. Thus, $\nu\cdot\ell\in(\eta]$.

Thus, η is an SSG of $\mathfrak{S}$.

Let $\nu,\ell\in\mathfrak{S}$ be such that $\nu,\ell\notin(\eta]$. Hence, $\Bbbk^{\overrightarrow{T}}_{(\eta]}(\nu,\iota)=\overleftarrow{\varsigma}=\Bbbk^{\overrightarrow{T}}_{(\eta]}(\ell,\iota)=\overleftarrow{\varsigma}$. But $\Bbbk^{\overrightarrow{T}}_{(\eta]}$ is a (ς,ζ)-GBQIFSSG, we have

$$\begin{aligned}\min\{\Bbbk^{\overrightarrow{T}}_{(\eta]}(\nu\cdot\ell,\iota),\overleftarrow{\zeta}\} &\leq \max\{\Bbbk^{\overrightarrow{T}}_{(\eta]}(\nu,\iota),\Bbbk^{\overrightarrow{T}}_{(\eta]}(\ell,\iota),\overleftarrow{\zeta}\}\\ &= \max\{\overleftarrow{\varsigma},\overleftarrow{\varsigma},\overleftarrow{\zeta}\}\\ &= \overleftarrow{\varsigma}\end{aligned}$$

as $\overleftarrow{\varsigma} > \overleftarrow{\zeta} \Rightarrow \Bbbk^{\overrightarrow{T}}_{(\eta]}(\nu\cdot\ell,\iota) \leq \overleftarrow{\varsigma}$. Thus, $\nu\cdot\ell\notin(\eta]$.

Let $\nu,\ell\in\mathfrak{S}$ be such that $\nu,\ell\notin(\eta]$. Hence, $\Bbbk^{\overrightarrow{F}}_{(\eta]}(\nu,\iota)=\overleftarrow{\zeta}=\Bbbk^{\overrightarrow{F}}_{(\eta]}(\ell,\iota)=\overleftarrow{\zeta}$. But $\Bbbk^{\overrightarrow{F}}_{(\eta]}$ is a (ς,ζ)-GBQIFSSG, we have

$$\begin{aligned}\max\{\Bbbk^{\overrightarrow{F}}_{(\eta]}(\nu\cdot\ell,\iota),\overleftarrow{\varsigma}\} &\geq \min\{\Bbbk^{\overrightarrow{F}}_{(\eta]}(\nu,\iota),\Bbbk^{\overrightarrow{F}}_{(\eta]}(\ell,\iota),\overleftarrow{\zeta}\}\\ &= \min\{\overleftarrow{\zeta},\overleftarrow{\zeta},\overleftarrow{\zeta}\}\\ &= \overleftarrow{\zeta}\end{aligned}$$

as $\overleftarrow{\varsigma} > \overleftarrow{\zeta} \Rightarrow \Bbbk^{\overrightarrow{F}}_{(\eta]}(\nu\cdot\ell,\iota) \geq \overleftarrow{\zeta}$. Thus, $\nu\cdot\ell\notin(\eta]$.

Thus, η is an SSG of $\mathfrak{S}$. ◻

Definition 3.7. *The subsets and their product $\propto\circ\varsigma$ is defined as follows:*

$$(\propto^{\overleftarrow{T}}\circ\varsigma^{\overleftarrow{T}})(\nu,\iota)=\begin{cases}\sup\limits_{(s,t)\in\eta_\nu}\{\propto^{\overleftarrow{T}}(s)\vartriangle\varsigma^{\overleftarrow{T}}(t)\} & \text{if } \eta_\nu\neq\emptyset\\ 0 & \text{otherwise}\end{cases}$$

$$(\propto^{\overleftarrow{F}}\circ\varsigma^{\overleftarrow{F}})(\nu,\iota)=\begin{cases}\inf\limits_{(s,t)\in\eta_\nu}\{\propto^{\overleftarrow{F}}(s)\triangledown\varsigma^{\overleftarrow{F}}(t)\} & \text{if } \eta_\nu\neq\emptyset\\ 1 & \text{otherwise}\end{cases}$$

$$(\propto^{\overrightarrow{T}}\circ\varsigma^{\overrightarrow{T}})(\nu,\iota)=\begin{cases}\inf\limits_{(s,t)\in\eta_\nu}\{\propto^{\overrightarrow{T}}(s)\triangledown\varsigma^{\overrightarrow{T}}(t)\} & \text{if } \eta_\nu\neq\emptyset\\ 0 & \text{otherwise}\end{cases}$$

$$(\propto^{\overrightarrow{F}}\circ\varsigma^{\overrightarrow{F}})(\nu,\iota)=\begin{cases}\sup\limits_{(s,t)\in\eta_\nu}\{\propto^{\overrightarrow{F}}(s)\vartriangle\varsigma^{\overrightarrow{F}}(t)\} & \text{if } \eta_\nu\neq\emptyset\\ -1 & \text{otherwise}\end{cases}$$

Definition 3.8. *We define*

(1) $(\mho)^{\overrightarrow{\zeta}}_{\overrightarrow{\varsigma}}(\nu,\iota)=(\mho(\nu,\iota)\vartriangle\overrightarrow{\zeta})\triangledown\overrightarrow{\varsigma}$,

(2) $(\Finv)^{\overrightarrow{\zeta}}_{\overrightarrow{\varsigma}}(\nu,\iota)=(\Finv(\nu,\iota)\triangledown\overrightarrow{\zeta})\vartriangle\overrightarrow{\varsigma}$,

(3) $(\Game)^{\overleftarrow{\zeta}}_{\overleftarrow{\varsigma}}(\nu,\iota)=(\Game(\nu,\iota)\triangledown\overleftarrow{\zeta})\vartriangle\overleftarrow{\varsigma}$,

(4) $(\mathfrak{I})^{\overleftarrow{\zeta}}_{\overleftarrow{\varsigma}}(\nu,\iota) = (\mathfrak{I}(\nu,\iota) \vartriangle \overleftarrow{\zeta}) \triangledown \overleftarrow{\varsigma}, \forall\, \nu \in \mathfrak{S}$.

Lemma 3.1. *Let η, λ be subsets of $\mathfrak{S}$. Then $(\Bbbk_{(\eta]} \circ^{\zeta}_{\varsigma} \Bbbk_{(\lambda]}) = (\Bbbk_{(\eta\Lambda\lambda]})^{\zeta}_{\varsigma}$.*

Proof. Let $\nu \in \mathfrak{S}$. If $\nu \in (\eta\Lambda\lambda]$, then $(\Bbbk_{(\eta\Lambda\lambda]})(\nu,\iota) = \overrightarrow{\zeta}$. But $\nu \le ab$ for certain $a \in (\eta], b \in (\lambda]$, we have $(a,b) \in \eta_\nu$, so $\eta_\nu \neq \emptyset$. Thus,

$$
\begin{aligned}
(\Bbbk^{\overleftarrow{T}}_{(\eta]} \circ \Bbbk^{\overleftarrow{T}}_{(\lambda]})(\nu,\iota) &= \sup_{\nu=yz} \min\{\Bbbk^{\overleftarrow{T}}_{(\eta]}(y,\iota), \Bbbk^{\overleftarrow{T}}_{(\lambda]}(z,\iota)\} \\
&\ge \min\{\Bbbk^{\overleftarrow{T}}_{(\eta]}(a,\iota), \Bbbk^{\overleftarrow{T}}_{(\lambda]}(b,\iota)\} \\
&= \overrightarrow{\zeta}, \\
(\Bbbk^{\overleftarrow{F}}_{(\eta]} \circ \Bbbk^{\overleftarrow{F}}_{(\lambda]})(\nu,\iota) &= \inf_{\nu=yz} \max\{\Bbbk^{\overleftarrow{F}}_{(\eta]}(y,\iota), \Bbbk^{\overleftarrow{F}}_{(\lambda]}(z,\iota)\} \\
&\le \max\{\Bbbk^{\overleftarrow{F}}_{(\eta]}(a,\iota), \Bbbk^{\overleftarrow{F}}_{(\lambda]}(b,\iota)\} \\
&= \overrightarrow{\varsigma}.
\end{aligned}
$$

Thus, $(\Bbbk_{(\eta]} \circ \Bbbk_{(\lambda]})(\nu,\iota) = (\Bbbk_{(\eta\Lambda\lambda]})(\nu,\iota)$.

If $\nu \in (\eta\Lambda\lambda]$, then $(\Bbbk_{(\eta\Lambda\lambda]})(\nu,\iota) = \overleftarrow{\zeta}$. But $\nu \le ab$ for certain $a \in (\eta], b \in (\lambda]$, we have $(a,b) \in \eta_\nu$, so $\eta_\nu \neq \emptyset$. Thus,

$$
\begin{aligned}
(\Bbbk^{\overrightarrow{T}}_{(\eta]} \circ \Bbbk^{\overrightarrow{T}}_{(\lambda]})(\nu,\iota) &= \inf_{\nu=yz} \max\{\Bbbk^{\overrightarrow{T}}_{(\eta]}(y,\iota), \Bbbk^{\overrightarrow{T}}_{(\lambda]}(z,\iota)\} \\
&\le \max\{\Bbbk^{\overrightarrow{T}}_{(\eta]}(a,\iota), \Bbbk^{\overrightarrow{T}}_{(\lambda]}(b,\iota)\} \\
&= \overleftarrow{\zeta}, \\
(\Bbbk^{\overrightarrow{F}}_{(\eta]} \circ \Bbbk^{\overrightarrow{F}}_{(\lambda]})(\nu,\iota) &= \sup_{\nu=yz} \min\{\Bbbk^{\overrightarrow{F}}_{(\eta]}(y,\iota), \Bbbk^{\overrightarrow{F}}_{(\lambda]}(z,\iota)\} \\
&\ge \min\{\Bbbk^{\overrightarrow{F}}_{(\eta]}(a,\iota), \Bbbk^{\overrightarrow{F}}_{(\lambda]}(b,\iota)\} \\
&= \overleftarrow{\varsigma}.
\end{aligned}
$$

Thus, $(\Bbbk_{(\eta]} \circ \Bbbk_{(\lambda]})(\nu,\iota) = (\Bbbk_{(\eta\Lambda\lambda]})(\nu,\iota)$.

If $\nu \notin (\eta\Lambda\lambda]$, then $(\Bbbk^{\overleftarrow{T}}_{(\eta\Lambda\lambda]})(\nu,\iota) = \overrightarrow{\varsigma}$ and $(\Bbbk^{\overleftarrow{F}}_{(\eta\Lambda\lambda]})(\nu,\iota) = \overrightarrow{\zeta}$. But $\nu \le ab$ for certain $a \notin (\eta], b \notin (\lambda]$. Now,

$$
\begin{aligned}
(\Bbbk^{\overleftarrow{T}}_{(\eta]} \circ \Bbbk^{\overleftarrow{T}}_{(\lambda]})(\nu,\iota) &= \sup_{\nu=yz} \min\{\Bbbk^{\overleftarrow{T}}_{(\eta]}(y,\iota), \Bbbk^{\overleftarrow{T}}_{(\lambda]}(z,\iota)\} \\
&\ge \min\{\Bbbk^{\overleftarrow{T}}_{(\eta]}(a,\iota), \Bbbk^{\overleftarrow{T}}_{(\lambda]}(b,\iota)\} \\
&= \overrightarrow{\varsigma}, \\
(\Bbbk^{\overleftarrow{F}}_{(\eta]} \circ \Bbbk^{\overleftarrow{F}}_{(\lambda]})(\nu,\iota) &= \inf_{\nu=yz} \max\{\Bbbk^{\overleftarrow{F}}_{(\eta]}(y,\iota), \Bbbk^{\overleftarrow{F}}_{(\lambda]}(z,\iota)\} \\
&\le \max\{\Bbbk^{\overleftarrow{F}}_{(\eta]}(a,\iota), \Bbbk^{\overleftarrow{F}}_{(\lambda]}(b,\iota)\} \\
&= \overrightarrow{\zeta}.
\end{aligned}
$$

If $\nu \notin (\eta\Lambda\lambda]$, then $(\Bbbk^{\overrightarrow{T}}_{(\eta\Lambda\lambda]})(\nu,\iota) = \overleftarrow{\varsigma}$ and $(\Bbbk^{\overrightarrow{F}}_{(\eta\Lambda\lambda]})(\nu,\iota) = \overleftarrow{\zeta}$. But $\nu \leq ab$ for certain $a \notin (\eta], b \notin (\lambda]$. Now,

$$\begin{aligned}(\Bbbk^{\overrightarrow{T}}_{(\eta]} \circ \Bbbk^{\overrightarrow{T}}_{(\lambda]})(\nu,\iota) &= \inf_{\nu=yz} \max\{\Bbbk^{\overrightarrow{T}}_{(\eta]}(y,\iota), \Bbbk^{\overrightarrow{T}}_{(\lambda]}(z,\iota)\}\\ &\leq \max\{\Bbbk^{\overrightarrow{T}}_{(\eta]}(a,\iota), \Bbbk^{\overrightarrow{T}}_{(\lambda]}(b,\iota)\}\\ &= \overleftarrow{\varsigma},\end{aligned}$$

$$\begin{aligned}(\Bbbk^{\overrightarrow{F}}_{(\eta]} \circ \Bbbk^{\overrightarrow{F}}_{(\lambda]})(\nu,\iota) &= \sup_{\nu=yz} \min\{\Bbbk^{\overrightarrow{F}}_{(\eta]}(y,\iota), \Bbbk^{\overrightarrow{F}}_{(\lambda]}(z,\iota)\}\\ &\geq \min\{\Bbbk^{\overrightarrow{F}}_{(\eta]}(a,\iota), \Bbbk^{\overrightarrow{F}}_{(\lambda]}(b,\iota)\}\\ &= \overleftarrow{\zeta}.\end{aligned}$$

Thus, $(\Bbbk_{(\eta]} \circ \Bbbk_{(\lambda]})(\nu,\iota) = (\Bbbk_{(\eta\Lambda\lambda]})(\nu,\iota)$. □

Theorem 3.5. *If η is a (ς,ζ)-GBQIFLId (GBQIFSSG, GBQIFRId) of $\mathfrak{S}$, then $(\eta)^{\zeta}_{\varsigma}$ is a GBQIFLId (GBQIF-SSG, GBQIFRId) of $\mathfrak{S}$.*

Proof. Suppose that η is a (ς,ζ)-GBQIFLId of $\mathfrak{S}$. Let $\nu,\ell \in \mathfrak{S}$. Then

$$\begin{aligned}\max\{(\mho)^{\overrightarrow{\zeta}}_{\overrightarrow{\varsigma}}(\nu\cdot\ell,\iota), \overrightarrow{\varsigma}\} &= \max\{(\{\mho(\nu\cdot\ell,\iota) \vartriangle \overrightarrow{\zeta}\} \triangledown \overrightarrow{\varsigma}), \overrightarrow{\varsigma}\}\\ &= \{\mho(\nu\cdot\ell,\iota) \vartriangle \overrightarrow{\zeta}\} \triangledown \overrightarrow{\varsigma}\\ &= \{\mho(\nu\cdot\ell,\iota) \triangledown \overrightarrow{\varsigma}\} \vartriangle \{\overrightarrow{\zeta} \triangledown \overrightarrow{\varsigma}\}\\ &= \{(\mho(\nu\cdot\ell,\iota) \triangledown \overrightarrow{\varsigma}) \triangledown \overrightarrow{\varsigma}\} \vartriangle \overrightarrow{\zeta}\\ &\geq \{(\mho(\ell,\iota) \vartriangle \overrightarrow{\zeta}) \triangledown \overrightarrow{\varsigma}\} \vartriangle \overrightarrow{\zeta}\\ &\geq (\mho)^{\overrightarrow{\zeta}}_{\overrightarrow{\varsigma}}(\ell,\iota) \vartriangle \overrightarrow{\zeta},\end{aligned}$$

$$\begin{aligned}\min\{(\dashv)^{\overrightarrow{\zeta}}_{\overrightarrow{\varsigma}}(\nu\cdot\ell,\iota), \overrightarrow{\varsigma}\} &= \min\{(\{\dashv(\nu\cdot\ell,\iota) \triangledown \overrightarrow{\zeta}\} \vartriangle \overrightarrow{\varsigma}), \overrightarrow{\varsigma}\}\\ &= \{\dashv(\nu\cdot\ell,\iota) \triangledown \overrightarrow{\zeta}\} \vartriangle \overrightarrow{\varsigma}\\ &= \{(\dashv(\nu\cdot\ell,\iota) \vartriangle \overrightarrow{\varsigma}) \vartriangle \overrightarrow{\varsigma}\} \triangledown \overrightarrow{\varsigma}\\ &\leq \{(\dashv(\ell,\iota) \triangledown \overrightarrow{\zeta}) \vartriangle \overrightarrow{\varsigma}\} \triangledown \overrightarrow{\varsigma}\\ &= \{(\dashv(\ell,\iota) \triangledown \overrightarrow{\zeta}) \triangledown \overrightarrow{\zeta}\} \vartriangle \overrightarrow{\varsigma}\\ &\leq (\dashv)^{\overrightarrow{\zeta}}_{\overrightarrow{\varsigma}}(\ell,\iota) \triangledown \overrightarrow{\zeta},\end{aligned}$$

$$
\begin{aligned}
\min\{(\backepsilon)_{\overleftarrow{\varsigma}}^{\overleftarrow{\zeta}}(\nu\cdot\ell,\iota),\overleftarrow{\varsigma}\} &= \min\{(\{\backepsilon(\nu\cdot\ell,\iota)\triangledown\overleftarrow{\zeta}\}\vartriangle\overleftarrow{\varsigma}),\overleftarrow{\varsigma}\}\\
&= \{\backepsilon(\nu\cdot\ell,\iota)\triangledown\overleftarrow{\zeta}\}\vartriangle\overleftarrow{\varsigma}\\
&= \{\backepsilon(\nu\cdot\ell,\iota)\vartriangle\overleftarrow{\varsigma}\}\triangledown\{\overleftarrow{\zeta}\vartriangle\overleftarrow{\varsigma}\}\\
&= \{(\backepsilon(\nu\cdot\ell,\iota)\vartriangle\overleftarrow{\varsigma})\vartriangle\overleftarrow{\varsigma}\}\triangledown\overleftarrow{\zeta}\\
&\le \{(\backepsilon(\ell,\iota)\triangledown\overleftarrow{\zeta})\vartriangle\overleftarrow{\varsigma}\}\triangledown\overleftarrow{\zeta}\\
&= \{(\backepsilon(\ell,\iota)\triangledown\overleftarrow{\zeta})\triangledown\overleftarrow{\zeta}\}\vartriangle\overleftarrow{\varsigma}\\
&\le (\backepsilon)_{\overleftarrow{\varsigma}}^{\overleftarrow{\zeta}}(\ell,\iota)\triangledown\overleftarrow{\zeta},\\
\max\{(\Im)_{\overleftarrow{\varsigma}}^{\overleftarrow{\zeta}}(\nu\cdot\ell,\iota),\overleftarrow{\varsigma}\} &= \max\{(\{\Im(\nu\cdot\ell,\iota)\vartriangle\overleftarrow{\zeta}\}\triangledown\overleftarrow{\varsigma}),\overleftarrow{\varsigma}\}\\
&= \{\Im(\nu\cdot\ell,\iota)\vartriangle\overleftarrow{\zeta}\}\triangledown\overleftarrow{\varsigma}\\
&= \{\Im(\nu\cdot\ell,\iota)\triangledown\overleftarrow{\varsigma}\}\vartriangle\{\overleftarrow{\zeta}\triangledown\overleftarrow{\varsigma}\}\\
&\ge \{(\Im(\ell,\iota)\vartriangle\overleftarrow{\zeta})\triangledown\overleftarrow{\varsigma}\}\vartriangle\overleftarrow{\zeta}\\
&= \{(\Im(\ell,\iota)\vartriangle\overleftarrow{\zeta})\vartriangle\overleftarrow{\zeta}\}\triangledown\overleftarrow{\varsigma}\\
&\ge (\Im)_{\overleftarrow{\varsigma}}^{\overleftarrow{\zeta}}(\ell,\iota)\vartriangle\overleftarrow{\zeta}.
\end{aligned}
$$

Thus, $(\eta)_{\varsigma}^{\zeta}=[(\mho)_{\varsigma}^{\zeta},(\Finv)_{\varsigma}^{\zeta}]$ is a GBQIFLId of $\mathfrak{S}$. □

Theorem 3.6. *Let η be a (ς,ζ)-GBQIFRId and λ be a (ς,ζ)-GBQIFLId of $\mathfrak{S}$. Then $((\eta\circ\lambda])_{\varsigma}^{\zeta}\subseteq(\eta\sqcap_{\varsigma}^{\zeta}\lambda]$.*

Proof. Let $\eta=[(\mho_\eta,\backepsilon_\eta),(\Finv_\eta,\Im_\eta)]$ be a (ς,ζ)-GBQIFRId and $\lambda=[(\mho_\lambda,\backepsilon_\lambda),(\Finv_\lambda,\Im_\lambda)]$ be a (ς,ζ)-GBQIFLId of $\mathfrak{S}$. Let $(\nu,\ell,\iota)\in I_\jmath$. If $I_\jmath\neq\emptyset$, then $\jmath\le\nu\ell$. Thus, $\mho_\eta(\jmath,\iota)\ge\mho_\eta(\nu\ell)\ge\mho_\eta(\nu,\iota)$ and $\Finv_\eta(\jmath,\iota)\le\Finv_\eta(\nu\ell)\le\Finv_\eta(\nu,\iota)$. Similarly, $\mho_\lambda(\jmath,\iota)\ge\mho_\lambda(\nu\ell)\ge\mho_\lambda(\nu,\iota)$ and $\Finv_\lambda(\jmath,\iota)\le\Finv_\lambda(\nu\ell)\le\Finv_\lambda(\nu,\iota)$. Let $(\nu,\ell,\iota)\in I_\jmath$. If $I_\jmath\neq\emptyset$, then $\jmath\ge\nu\ell$. Thus, $\backepsilon_\eta(\jmath,\iota)\le\backepsilon_\eta(\nu\ell)\le\backepsilon_\eta(\nu,\iota)$ and $\Im_\eta(\jmath,\iota)\ge\Im_\eta(\nu\ell)\ge\Im_\eta(\nu,\iota)$. Similarly, $\backepsilon_\lambda(\jmath,\iota)\le\backepsilon_\lambda(\nu\ell)\le\backepsilon_\lambda(\nu,\iota)$ and $\Im_\lambda(\jmath,\iota)\ge\Im_\lambda(\nu\ell)\ge\Im_\lambda(\nu,\iota)$. Now,

$$
\begin{aligned}
(\mho_{(\eta\circ\lambda]})_{\overrightarrow{\varsigma}}^{\overrightarrow{\zeta}}(\jmath,\iota) &= (\mho_{(\eta\circ\lambda]}(\jmath,\iota)\vartriangle\overrightarrow{\zeta})\triangledown\overrightarrow{\varsigma}\\
&= [\sup_{\jmath\le\nu\ell}\{\mho_\eta(\nu,\iota)\vartriangle\mho_\lambda(\ell,\iota)\}\vartriangle\overrightarrow{\zeta}]\triangledown\overrightarrow{\varsigma}\\
&= [\sup_{\jmath\le\nu\ell}\{\mho_\eta(\nu,\iota)\vartriangle\mho_\lambda(\ell,\iota)\}\vartriangle\overrightarrow{\zeta}\vartriangle\overrightarrow{\zeta}]\triangledown\overrightarrow{\varsigma}\\
&= [\sup_{\jmath\le\nu\ell}\{(\mho_\eta(\nu,\iota)\vartriangle\overrightarrow{\zeta})\vartriangle(\mho_\lambda(\ell,\iota)\vartriangle\overrightarrow{\zeta})\}\vartriangle\overrightarrow{\zeta}]\triangledown\overrightarrow{\varsigma}\\
&\le (\{(\mho_\eta(\jmath,\iota)\triangledown\overrightarrow{\varsigma})\vartriangle(\mho_\lambda(\jmath,\iota)\triangledown\overrightarrow{\varsigma})\}\vartriangle\overrightarrow{\zeta})\triangledown\overrightarrow{\varsigma}\\
&= \{((\mho_\eta(\jmath,\iota)\vartriangle\mho_\lambda(\jmath,\iota))\triangledown\overrightarrow{\varsigma})\vartriangle\overrightarrow{\zeta}\}\triangledown\overrightarrow{\varsigma}\\
&= \{(\mho_\eta\vartriangle\mho_\lambda)(\jmath,\iota)\vartriangle\overrightarrow{\zeta}\}\triangledown\overrightarrow{\varsigma}\\
&= (\mho_{\eta\sqcup_{\overrightarrow{\varsigma}}^{\overrightarrow{\zeta}}\lambda})(\jmath,\iota),
\end{aligned}
$$

$$
\begin{aligned}
(\Finv_{(\eta\circ\lambda]})^{\overrightarrow{\zeta}}_{\overrightarrow{\varsigma}}(\jmath,\iota) &= (\Finv_{(\eta\circ\lambda]}(\jmath,\iota) \triangledown \overrightarrow{\zeta}\,) \vartriangle \overrightarrow{\varsigma} \\
&= [\inf_{\jmath\leq \nu\ell}\{\Finv_{\eta}(\nu,\iota) \triangledown \Finv_{\lambda}(\ell,\iota)\} \triangledown \overrightarrow{\zeta}\,] \vartriangle \overrightarrow{\varsigma} \\
&= [\inf_{\jmath\leq \nu\ell}\{\Finv_{\eta}(\nu,\iota) \triangledown \Finv_{\lambda}(\ell,\iota)\} \triangledown \overrightarrow{\zeta} \triangledown \overrightarrow{\zeta}\,] \vartriangle \overrightarrow{\varsigma} \\
&= [\inf_{\jmath\leq \nu\ell}\{(\Finv_{\eta}(\nu,\iota) \triangledown \overrightarrow{\zeta}\,) \triangledown (\Finv_{\lambda}(\ell,\iota) \triangledown \overrightarrow{\zeta}\,)\} \triangledown \overrightarrow{\zeta}\,] \vartriangle \overrightarrow{\varsigma} \\
&\geq (\{(\Finv_{\eta}(\jmath,\iota) \vartriangle \overrightarrow{\varsigma}\,) \triangledown (\Finv_{\lambda}(\jmath,\iota) \vartriangle \overrightarrow{\varsigma}\,)\} \triangledown \overrightarrow{\zeta}\,) \vartriangle \overrightarrow{\varsigma} \\
&= \{((\Finv_{\eta}(\jmath,\iota) \triangledown \Finv_{\lambda}(\jmath,\iota)) \vartriangle \overrightarrow{\varsigma}\,) \triangledown \overrightarrow{\zeta}\,\} \vartriangle \overrightarrow{\varsigma} \\
&= \{((\Finv_{\eta} \triangledown \Finv_{\lambda})(\jmath,\iota) \triangledown \overrightarrow{\zeta}\,\} \vartriangle \overrightarrow{\varsigma} \\
&= (\Finv_{\eta\sqcap^{\overrightarrow{\zeta}}_{\overrightarrow{\varsigma}}\lambda})(\jmath,\iota),
\end{aligned}
$$

$$
\begin{aligned}
(\Game_{(\eta\circ\lambda]})^{\overleftarrow{\zeta}}_{\overleftarrow{\varsigma}}(\jmath,\iota) &= (\Game_{(\eta\circ\lambda]}(\jmath,\iota) \triangledown \overleftarrow{\zeta}\,) \vartriangle \overleftarrow{\varsigma} \\
&= [\inf_{\jmath\leq \nu\cdot\ell}\{\Game_{\eta}(\nu,\iota) \triangledown \Game_{\lambda}(\ell,\iota)\} \triangledown \overleftarrow{\zeta}\,] \vartriangle \overleftarrow{\varsigma} \\
&= [\inf_{\jmath\leq \nu\cdot\ell}\{\Game_{\eta}(\nu,\iota) \triangledown \Game_{\lambda}(\ell,\iota)\} \triangledown \overleftarrow{\zeta} \triangledown \overleftarrow{\zeta}\,] \vartriangle \overleftarrow{\varsigma} \\
&= [\inf_{\jmath\leq \nu\cdot\ell}\{(\Game_{\eta}(\nu,\iota) \triangledown \overleftarrow{\zeta}\,) \triangledown (\Game_{\lambda}(\ell,\iota) \triangledown \overleftarrow{\zeta}\,)\} \triangledown \overleftarrow{\zeta}\,] \vartriangle \overleftarrow{\varsigma} \\
&\geq (\{(\Game_{\eta}(\jmath,\iota) \vartriangle \overleftarrow{\varsigma}\,) \triangledown (\Game_{\lambda}(\jmath,\iota) \vartriangle \overleftarrow{\varsigma}\,)\} \triangledown \overleftarrow{\zeta}\,) \vartriangle \overleftarrow{\varsigma} \\
&= \{((\Game_{\eta}(\jmath,\iota) \triangledown \Game_{\lambda}(\jmath,\iota)) \vartriangle \overleftarrow{\varsigma}\,) \triangledown \overleftarrow{\zeta}\,\} \vartriangle \overleftarrow{\varsigma} \\
&= \{(\Game_{\eta} \triangledown \Game_{\lambda})(\jmath,\iota) \triangledown \overleftarrow{\zeta}\,\} \vartriangle \overleftarrow{\varsigma} \\
&= (\Game_{\eta\sqcup^{\overleftarrow{\zeta}}_{\overleftarrow{\varsigma}}\lambda})(\jmath,\iota),
\end{aligned}
$$

$$
\begin{aligned}
(\mathfrak{I}_{(\eta\circ\lambda]})^{\overleftarrow{\zeta}}_{\overleftarrow{\varsigma}}(\jmath,\iota) &= (\mathfrak{I}_{(\eta\circ\lambda]}(\jmath,\iota) \vartriangle \overleftarrow{\zeta}\,) \triangledown \overleftarrow{\varsigma} \\
&= [\sup_{\jmath\leq \nu\cdot\ell}\{\mathfrak{I}_{\eta}(\nu,\iota) \vartriangle \mathfrak{I}_{\lambda}(\ell,\iota)\} \vartriangle \overleftarrow{\zeta}\,] \triangledown \overleftarrow{\varsigma} \\
&= [\sup_{\jmath\leq \nu\cdot\ell}\{\mathfrak{I}_{\eta}(\nu,\iota) \vartriangle \mathfrak{I}_{\lambda}(\ell,\iota)\} \vartriangle \overleftarrow{\zeta} \vartriangle \overleftarrow{\zeta}\,] \triangledown \overleftarrow{\varsigma} \\
&= [\sup_{\jmath\leq \nu\cdot\ell}\{(\mathfrak{I}_{\eta}(\nu,\iota) \vartriangle \overleftarrow{\zeta}\,) \vartriangle (\mathfrak{I}_{\lambda}(\ell,\iota) \vartriangle \overleftarrow{\zeta}\,)\} \vartriangle \overleftarrow{\zeta}\,] \triangledown \overleftarrow{\varsigma} \\
&\leq (\{(\mathfrak{I}_{\eta}(\jmath,\iota) \triangledown \overleftarrow{\varsigma}\,) \vartriangle (\mathfrak{I}_{\lambda}(\jmath,\iota) \triangledown \overleftarrow{\varsigma}\,)\} \vartriangle \overleftarrow{\zeta}\,) \triangledown \overleftarrow{\varsigma} \\
&= \{((\mathfrak{I}_{\eta}(\jmath,\iota) \vartriangle \mathfrak{I}_{\lambda}(\jmath,\iota)) \triangledown \overleftarrow{\varsigma}\,) \vartriangle \overleftarrow{\zeta}\,\} \triangledown \overleftarrow{\varsigma} \\
&= \{(\mathfrak{I}_{\eta} \vartriangle \mathfrak{I}_{\lambda})(\jmath,\iota) \vartriangle \overleftarrow{\zeta}\,\} \triangledown \overleftarrow{\varsigma} \\
&= (\mathfrak{I}_{\eta\sqcap^{\overleftarrow{\zeta}}_{\overleftarrow{\varsigma}}\lambda})(\jmath,\iota).
\end{aligned}
$$

Let $\nu, \ell \notin I_\jmath$. If $I_\jmath = \emptyset$, then $(\mho_\eta \circ \mho_\lambda)(\jmath, \iota) = -1$ and $(\Finv_\eta \circ \Finv_\lambda)(\jmath, \iota) = 0$, so $\jmath \leq \nu \cdot \ell$. Thus,

$$\begin{aligned}
(\mho_{(\eta\circ\lambda]})^{\overrightarrow{\zeta}}_{\overrightarrow{\varsigma}}(\jmath,\iota) &= (\mho_{(\eta\circ\lambda]}(\jmath,\iota) \vartriangle \overrightarrow{\zeta}\,) \triangledown \overrightarrow{\varsigma} \\
&= -1 \triangledown \overrightarrow{\varsigma} \\
&\leq (\mho_{\eta\sqcup\lambda}(\jmath,\iota) \vartriangle \overrightarrow{\zeta}\,) \triangledown \overrightarrow{\varsigma} \\
&= (\mho_{\eta\sqcup\lambda}(\jmath,\iota) \vartriangle \overrightarrow{\zeta}\,), \\
(\Finv_{(\eta\circ\lambda]})^{\overrightarrow{\zeta}}_{\overrightarrow{\varsigma}}(\jmath,\iota) &= (\Finv_{(\eta\circ\lambda]}(\jmath,\iota) \triangledown \overrightarrow{\zeta}\,) \vartriangle \overrightarrow{\varsigma} \\
&= 0 \vartriangle \overrightarrow{\varsigma} \\
&= \overrightarrow{\varsigma} \\
&\geq (\Finv_{\eta\sqcap\lambda}(\jmath,\iota) \triangledown \overrightarrow{\zeta}\,) \vartriangle \overrightarrow{\varsigma} \\
&= (\Finv_{\eta\sqcap\lambda}(\jmath,\iota) \triangledown \overrightarrow{\zeta}\,).
\end{aligned}$$

Let $\nu, \ell \notin I_\jmath$. If $I_\jmath = \emptyset$, then $(\eth_\eta \circ \eth_\lambda)(\jmath, \iota) = 1$ and $(\mathfrak{I}_\eta \circ \mathfrak{I}_\lambda)(\jmath, \iota) = 0$, so $\jmath \leq \nu \cdot \ell$. Thus,

$$\begin{aligned}
(\eth_{(\eta\circ\lambda]})^{\overleftarrow{\zeta}}_{\overleftarrow{\varsigma}}(\jmath,\iota) &= (\eth_{(\eta\circ\lambda]}(\jmath,\iota) \triangledown \overleftarrow{\zeta}\,) \vartriangle \overleftarrow{\varsigma} \\
&= 1 \vartriangle \overleftarrow{\varsigma} \\
&\geq (\eth_{\eta\sqcup\lambda}(\jmath,\iota) \triangledown \overleftarrow{\zeta}\,) \vartriangle \overleftarrow{\varsigma} \\
&= (\eth_{\eta\sqcup\lambda}(\jmath,\iota) \triangledown \overleftarrow{\zeta}\,), \\
(\mathfrak{I}_{(\eta\circ\lambda]})^{\overleftarrow{\zeta}}_{\overleftarrow{\varsigma}}(\jmath,\iota) &= (\mathfrak{I}_{(\eta\circ\lambda]}(\jmath,\iota) \vartriangle \overleftarrow{\zeta}\,) \triangledown \overleftarrow{\varsigma} \\
&= 0 \triangledown \overleftarrow{\varsigma} \\
&= \overleftarrow{\varsigma} \\
&\leq (\mathfrak{I}_{\eta\sqcap\lambda}(\jmath,\iota) \vartriangle \overleftarrow{\zeta}\,) \triangledown \overleftarrow{\varsigma} \\
&= (\mathfrak{I}_{\eta\sqcap\lambda}(\jmath,\iota) \vartriangle \overleftarrow{\zeta}\,).
\end{aligned}$$

Thus, $((\eta \circ \lambda])^{\zeta}_{\varsigma} \subseteq (\eta \sqcap^{\zeta}_{\varsigma} \lambda]$. □

4. Regular Concept Applied to Various Q-Bipolar Fuzzy Ideals

In this section, we investigate the influence of the regularity property of ordered semigroups on various (ς, ζ)-bipolar Q-intuitionistic fuzzy ideals. Building on the definitions and results from earlier sections, we show how regularity establishes direct equivalences between certain fuzzy ideal operations and their crisp counterparts, particularly for products and meet operations. This connection not only unifies the fuzzy framework with the underlying algebraic structure but also provides simplified and elegant characterizations of these ideals.

Theorem 4.1. *Let η be a (ς, ζ)-GBQIFBId and λ be a (ς, ζ)-GBQIFLId of $\mathfrak{S}$. Then $\mathfrak{S}$ is regular if and only if $((\eta \circ \lambda])^{\overrightarrow{\zeta}}_{\overrightarrow{\varsigma}} = (\eta \sqcap^{\overrightarrow{\zeta}}_{\overrightarrow{\varsigma}} \lambda]$.*

Proof. Let $(\nu,\ell,\iota)\in I_\jmath$. If $I_\jmath\neq\emptyset$, then $\jmath\leq\nu\ell$. Thus, $\mho_\eta(\jmath,\iota)\geq\mho_\eta(\nu\ell)\geq\mho_\eta(\nu,\iota)$ and $\Finv_\eta(\jmath,\iota)\leq\Finv_\eta(\nu\ell)\leq\Finv_\eta(\nu,\iota)$. Similarly, $\mho_\lambda(\jmath,\iota)\geq\mho_\lambda(\nu\ell)\geq\mho_\lambda(\nu,\iota)$ and $\Finv_\lambda(\jmath,\iota)\leq\Finv_\lambda(\nu\ell)\leq\Finv_\lambda(\nu,\iota)$. Let $(\nu,\ell,\iota)\in I_\jmath$. If $I_\jmath\neq\emptyset$, then $\jmath\geq\nu\ell$. Thus, $\eth_\eta(\jmath,\iota)\leq\eth_\eta(\nu\ell)\leq\eth_\eta(\nu,\iota)$ and $\mathfrak{I}_\eta(\jmath,\iota)\geq\mathfrak{I}_\eta(\nu\ell)\geq\mathfrak{I}_\eta(\nu,\iota)$. Similarly, $\eth_\lambda(\jmath,\iota)\leq\eth_\lambda(\nu\ell)\leq\eth_\lambda(\nu,\iota)$ and $\mathfrak{I}_\lambda(\jmath,\iota)\geq\mathfrak{I}_\lambda(\nu\ell)\geq\mathfrak{I}_\lambda(\nu,\iota)$. For $\jmath\in\mathfrak{S}$, there exists $x\in\mathfrak{S}$ such that $\jmath\leq(\jmath x\jmath)$. Hence, $\jmath, x\jmath\in I_\jmath$. Now,

$$\begin{aligned}
(\mho_{(\eta\circ\lambda]})^{\overrightarrow{\zeta}}_{\overrightarrow{\varsigma}}(\jmath,\iota) &= (\mho_{(\eta\circ\lambda]}(\jmath,\iota)\vartriangle\overrightarrow{\zeta})\triangledown\overrightarrow{\varsigma}\\
&= [\sup_{\jmath\leq\nu\cdot\ell}\{\mho_\eta(\nu,\iota)\vartriangle\mho_\lambda(\ell,\iota)\}\vartriangle\overrightarrow{\zeta}]\triangledown\overrightarrow{\varsigma}\\
&= [\sup_{\jmath\leq\nu\cdot\ell}\{\mho_\eta(\nu,\iota)\vartriangle\mho_\lambda(\ell,\iota)\}\vartriangle\overrightarrow{\zeta}\vartriangle\overrightarrow{\zeta}]\triangledown\overrightarrow{\varsigma}\\
&= [\sup_{\jmath\leq\nu\cdot\ell}\{(\mho_\eta(\nu,\iota)\vartriangle\overrightarrow{\zeta})\vartriangle(\mho_\lambda(\ell,\iota)\vartriangle\overrightarrow{\zeta})\}\vartriangle\overrightarrow{\zeta}]\triangledown\overrightarrow{\varsigma}\\
&\geq (\{(\mho_\eta(\jmath,\iota)\triangledown\overrightarrow{\varsigma})\vartriangle(\mho_\lambda(x\jmath,\iota)\triangledown\overrightarrow{\varsigma})\}\vartriangle\overrightarrow{\zeta})\triangledown\overrightarrow{\varsigma}\\
&= \{((\mho_\eta(\jmath,\iota)\vartriangle\mho_\lambda(\jmath,\iota))\triangledown\overrightarrow{\varsigma})\vartriangle\overrightarrow{\zeta}\}\triangledown\overrightarrow{\varsigma}\\
&= \{(\mho_\eta\vartriangle\mho_\lambda)(\jmath,\iota)\vartriangle\overrightarrow{\zeta}\}\triangledown\overrightarrow{\varsigma}\\
&= (\mho_{\eta\sqcap^{\overrightarrow{\zeta}}_{\overrightarrow{\varsigma}}\lambda})(\jmath,\iota),
\end{aligned}$$

$$\begin{aligned}
(\Finv_{(\eta\circ\lambda]})^{\overrightarrow{\zeta}}_{\overrightarrow{\varsigma}}(\jmath,\iota) &= (\Finv_{(\eta\circ\lambda]}(\jmath,\iota)\triangledown\overrightarrow{\zeta})\vartriangle\overrightarrow{\varsigma}\\
&= [\inf_{\jmath\leq\nu\cdot\ell}\{\Finv_\eta(\nu,\iota)\triangledown\Finv_\lambda(\ell,\iota)\}\triangledown\overrightarrow{\zeta}]\vartriangle\overrightarrow{\varsigma}\\
&= [\inf_{\jmath\leq\nu\cdot\ell}\{\Finv_\eta(\nu,\iota)\triangledown\Finv_\lambda(\ell,\iota)\}\triangledown\overrightarrow{\zeta}\triangledown\overrightarrow{\zeta}]\vartriangle\overrightarrow{\varsigma}\\
&= [\inf_{\jmath\leq\nu\cdot\ell}\{(\Finv_\eta(\nu,\iota)\triangledown\overrightarrow{\zeta})\triangledown(\Finv_\lambda(\ell,\iota)\triangledown\overrightarrow{\zeta})\}\triangledown\overrightarrow{\zeta}]\vartriangle\overrightarrow{\varsigma}\\
&\leq (\{(\Finv_\eta(\jmath)\vartriangle\overrightarrow{\varsigma})\triangledown(\Finv_\lambda(x\jmath,\iota)\vartriangle\overrightarrow{\varsigma})\}\triangledown\overrightarrow{\zeta})\vartriangle\overrightarrow{\varsigma}\\
&= \{((\Finv_\eta(\jmath,\iota)\triangledown\Finv_\lambda(\jmath,\iota))\vartriangle\overrightarrow{\varsigma})\triangledown\overrightarrow{\zeta}\}\vartriangle\overrightarrow{\varsigma}\\
&= \{(\Finv_\eta\triangledown\Finv_\lambda)(\jmath,\iota)\triangledown\overrightarrow{\zeta}\}\vartriangle\overrightarrow{\varsigma}\\
&= (\Finv_{\eta\sqcap^{\overrightarrow{\zeta}}_{\overrightarrow{\varsigma}}\lambda})(\jmath,\iota),
\end{aligned}$$

$$\begin{aligned}
(\eth_{(\eta\circ\lambda]})^{\overleftarrow{\zeta}}_{\overleftarrow{\varsigma}}(\jmath,\iota) &= (\eth_{(\eta\circ\lambda]}(\jmath,\iota)\triangledown\overleftarrow{\zeta})\vartriangle\overleftarrow{\varsigma}\\
&= [\inf_{\jmath\leq\nu\cdot\ell}\{\eth_\eta(\nu,\iota)\triangledown\eth_\lambda(\ell,\iota)\}\triangledown\overleftarrow{\zeta}]\vartriangle\overleftarrow{\varsigma}\\
&= [\inf_{\jmath\leq\nu\cdot\ell}\{\eth_\eta(\nu,\iota)\triangledown\eth_\lambda(\ell,\iota)\}\triangledown\overleftarrow{\zeta}\triangledown\overleftarrow{\zeta}]\vartriangle\overleftarrow{\varsigma}\\
&= [\inf_{\jmath\leq\nu\cdot\ell}\{(\eth_\eta(\nu,\iota)\triangledown\overleftarrow{\zeta})\triangledown(\eth_\lambda(\ell,\iota)\triangledown\overleftarrow{\zeta})\}\triangledown\overleftarrow{\zeta}]\vartriangle\overleftarrow{\varsigma}\\
&\geq ((\eth_\eta(\jmath,\iota)\vartriangle\overleftarrow{\varsigma})\triangledown(\eth_\lambda(x\jmath,\iota)\vartriangle\overleftarrow{\varsigma})\triangledown\overleftarrow{\zeta})\vartriangle\overleftarrow{\varsigma}
\end{aligned}$$

$$= \{((\eth_\eta(\jmath,\iota) \triangledown \eth_\lambda(\jmath,\iota)) \vartriangle \overleftarrow{\varsigma}) \triangledown \overleftarrow{\zeta}\} \vartriangle \overleftarrow{\varsigma}$$
$$= \{(\eth_\eta \triangledown \eth_\lambda)(\jmath,\iota) \triangledown \overleftarrow{\zeta}\} \vartriangle \overleftarrow{\varsigma}$$
$$= (\eth_{\eta \sqcup^{\overleftarrow{\zeta}}_{\overleftarrow{\varsigma}} \lambda})(\jmath,\iota),$$

$$(\Im_{(\eta\circ\lambda]})^{\overleftarrow{\zeta}}_{\overleftarrow{\varsigma}}(\jmath,\iota) = (\Im_{(\eta\circ\lambda]}(\jmath,\iota) \vartriangle \overleftarrow{\zeta}) \triangledown \overleftarrow{\varsigma}$$
$$= [\sup_{\jmath\le\nu\cdot\ell}\{\Im_\eta(\nu,\iota) \vartriangle \Im_\lambda(\ell,\iota)\} \vartriangle \overleftarrow{\zeta}] \triangledown \overleftarrow{\varsigma}$$
$$= [\sup_{\jmath\le\nu\cdot\ell}\{\Im_\eta(\nu,\iota) \vartriangle \Im_\lambda(\ell,\iota)\} \vartriangle \overleftarrow{\zeta} \vartriangle \overleftarrow{\zeta}] \triangledown \overleftarrow{\varsigma}$$
$$= [\sup_{\jmath\le\nu\cdot\ell}\{(\Im_\eta(\nu,\iota) \vartriangle \overleftarrow{\zeta}) \vartriangle (\Im_\lambda(\ell,\iota) \vartriangle \overleftarrow{\zeta})\} \vartriangle \overleftarrow{\zeta}] \triangledown \overleftarrow{\varsigma}$$
$$\le ((\Im_\eta(\jmath,\iota) \triangledown \overleftarrow{\varsigma}) \vartriangle (\Im_\lambda(x\jmath,\iota) \triangledown \overleftarrow{\varsigma}) \vartriangle \overleftarrow{\zeta}) \triangledown \overleftarrow{\varsigma}$$
$$= \{((\Im_\eta(\jmath,\iota) \vartriangle \Im_\lambda(\jmath,\iota)) \triangledown \overleftarrow{\varsigma}) \vartriangle \overleftarrow{\zeta}\} \triangledown \overleftarrow{\varsigma}$$
$$= \{(\Im_\eta \vartriangle \Im_\lambda)(\jmath,\iota) \vartriangle \overleftarrow{\zeta}\} \triangledown \overleftarrow{\varsigma}$$
$$= (\Im_{\eta \sqcup^{\overleftarrow{\zeta}}_{\overleftarrow{\varsigma}} \lambda})(\jmath,\iota).$$

Thus, $((\eta \circ \lambda])^{\zeta}_{\varsigma} \subseteq (\eta \sqcap^{\zeta}_{\varsigma} \lambda]$, by Theorem 3.6 and hence, $((\eta \circ \lambda])^{\zeta}_{\varsigma} = (\eta \sqcap^{\zeta}_{\varsigma} \lambda]$.

Conversely, assume that $((\eta \circ \lambda])^{\zeta}_{\varsigma} = \eta \sqcap^{\zeta}_{\varsigma} \lambda$. By Theorem 3.4, $\Bbbk_\eta$ is a (ς,ζ)-GBQIFBId and $\Bbbk_\lambda$ is a (ς,ζ)-GBQIFLId of $\mathfrak{S}$. By Remark 2.2, Remark 2.3 and Lemma 3.1, we have $(\Bbbk_{(\eta\sqcap\lambda]})^{\zeta}_{\varsigma} = (\Bbbk_\eta \sqcap^{\zeta}_{\varsigma} \Bbbk_\lambda) = (\Bbbk_\eta \circ \Bbbk_\lambda)^{\zeta}_{\varsigma} = (\Bbbk_{(\eta\circ\lambda]})^{\zeta}_{\varsigma}$, so $(\eta \sqcap^{\zeta}_{\varsigma} \lambda] = ((\eta \circ \lambda])^{\zeta}_{\varsigma}$. By Remark 2.1, we have $\mathfrak{S}$ is regular. □

5. Conclusion and future direction

Fuzzy set theory (FST) and its extensions, such as intuitionistic fuzzy sets (IFS) and bipolar fuzzy sets (BFS), offer enriched perspectives on membership and non-membership grades. By integrating these frameworks, we developed the concept of bipolar intuitionistic fuzzy structures (BIFS), which capture both positive and negative evaluations within ordered semigroups. In this work, we introduced and analyzed various (ς,ζ)-GBQIF structures, including subsemigroups (GBQIFSSG), left ideals (GBQIFLId), right ideals (GBQIFRId), ideals (GBQIFId), and bi-ideals (GBQIFBId), as well as their lower level set counterparts. We established characterizations relating these fuzzy structures to their crisp forms, examined their behavior under homomorphisms and inverse mappings, and explored equivalences arising in regular ordered semigroups. The results not only extend the theoretical foundation of bipolar fuzzy algebra but also open avenues for further research, including the incorporation of cubic and interval-valued fuzzy structures and the potential application of soft set theory to capture more complex uncertainty models.

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