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Intrinsic Study of a Special Anisotropic Conformal Transformation

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Abstract. In this paper, we present a coordinate-free investigation of a special anisotropic conformal transformation of a Finsler function L. Specifically, for a Finsler metric (M, L) and a one-form $\mathfrak{B}$, we consider the transformation

$$\widetilde{L}(x,\dot{x}) = e^{\sigma(x,\dot{x})}L(x,\dot{x}) = e^{\frac{\mathfrak{B}(x,\dot{x})}{L(x,\dot{x})}}L(x,\dot{x}).$$

We examine various geometric objects associated with the transformed metric $\widetilde{L}(x,\dot{x})$ in terms of the corresponding objects of the original metric L. In particular, we derive the expressions for the metric tensor, Cartan tensor, and other related geometric quantities for $\widetilde{L}$, and we characterize the conditions under which the metric tensor of $\widetilde{L}$ is non-degenerate. To further explore geometric structures such as the geodesic spray, Barthel connection, and Berwald connection of $\widetilde{L}(x,\dot{x})$, we restrict the one-form $\mathfrak{B}$ to one induced by a concurrent π-vector field. Under this assumption, we compute the curvature of the Barthel connection associated with $\widetilde{L}$. An explicit example is also provided.

1. Introduction

Finsler geometry extends Riemannian geometry by allowing the norm of a tangent vector to vary with both position and direction. This direction dependence makes Finsler spaces especially suited for modeling anisotropic structures, where physical or geometric properties change based on direction. Such flexibility has attracted attention in both pure mathematics and theoretical physics.

One important class of transformations in Finsler geometry is the conformal change, which modifies the Finsler function $L(x,\dot{x})$ by a positive scalar factor. Unlike standard conformal transformations that depend solely on the base point x, anisotropic conformal changes involve scaling functions that also depend on the direction $\dot{x}$.

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Such direction-dependent rescalings influence key geometric entities including the Finsler metric tensor, Cartan tensor, and the connections that govern parallel transport and curvature. Studying these changes sheds light on the internal structure of Finsler manifolds and allows the construction of special types of metrics such as Landsberg and non-Berwaldian metrics.

From the viewpoint of physics, anisotropic conformal transformations offer a useful tool in modeling space-time geometries where physical laws may vary with direction. These ideas appear in extensions of general relativity, cosmological models with anisotropies, and approaches to quantum gravity. The flexibility of Finsler geometry, especially when equipped with anisotropic modifications, provides a natural setting for these theories. These transformations take the form

$$\widetilde{L}(x,\dot{x}) = e^{\sigma(x,\dot{x})}L(x,\dot{x}),$$

where $\sigma(x,\dot{x})$ is a smooth real-valued function on the slit tangent bundle $TM \setminus \{0\}$. Recently, in [16], the authors demonstrated the challenges that arise when considering an anisotropic conformal change with a general factor $\sigma(x,\dot{x})$. So, a particularly interesting case arises when $\sigma(x,\dot{x}) = \frac{\mathfrak{B}(x,\dot{x})}{L(x,\dot{x})}$, with $\mathfrak{B}$ being a one-form. This leads to

$$\widetilde{L}(x,\dot{x}) = e^{\sigma(x,\dot{x})}L(x,\dot{x}) = e^{\frac{\mathfrak{B}(x,y)}{L(x,\dot{x})}}L(x,\dot{x}) \tag{1.1}$$

where $\sigma(x,\dot{x}) := \frac{\mathfrak{B}}{L}$. A prominent class of special Finsler metrics is the family of (α,β)-metrics, defined on a Riemannian manifold (M,α) in the presence of a 1-form β. These metrics have been widely explored in the literature due to their rich geometric structure and diverse applications (see, for example, [1,2,7–11]). Many of these metrics play a fundamental role in Finsler geometry (see references therein). One notable example of an (α,β)-metric is the exponential (α,β)-metric. In the context of the above anisotropic transformation, a similar structure arises—however, it involves the Finsler metric L in place of the Riemannian metric α. Therefore, this can be regarded as a generalized exponential (α,β)-metric.

This paper focuses on a coordinate-free study of anisotropic conformal transformations of Finsler functions, particularly those defined using one-forms derived from concurrent π-vector fields. We derive expressions for the resulting geometric quantities associated with the transformed metric $\widetilde{L}$, examine the conditions for non-degeneracy, and compute connections and curvatures relevant to the new geometry. An illustrative example is also included.

Using the pullback formalism of Finsler geometry, we undertake a coordinate-free investigation of the special anisotropic conformal transformation (1.1) which is defined as a deformation of a Finsler metric L (not necessarily Riemannian) by a π-form $\mathfrak{B}$. We study various geometric objects associated with $\widetilde{L}$, including the metric tensor and Cartan tensor. Furthermore, we provide a condition for the non-degeneracy of the metric tensor $\widetilde{g}$: $\widetilde{g}$ is non-degenerate if and only if

$$\mathfrak{B}^2 + \mathfrak{B}L - L^2(p^2+1) \neq 0.$$

To derive the geodesic spray and Berwald curvature of $\widetilde{L}$, we focus on a Finsler manifold (M,L) that admits a concurrent π-vector field $\overline{p}$. We then compute the associated π-form $\mathbf{B} := i_{\overline{p}}g$, where

g is the metric tensor of L, and define the corresponding scalar π-form as $\mathfrak{B}(x,\dot{x}) := \mathbf{B}(\overline{\eta})$. Using this setup, we determine intrinsic expressions for the canonical spray $\widetilde{G}$ and Barthel connection $\widetilde{\Gamma}$ associated with $\widetilde{L}$. Additionally, we establish the transformation law of the curvature tensor of the Barthel connection and show that the canonical sprays G and $\widetilde{G}$ satisfy the relation

$$\widetilde{G} = G - \frac{L^2(L-2\mathfrak{B})}{\mathfrak{B}^2 + \mathfrak{B}L - L^2(p^2+1)}\, C - \frac{L^4}{\mathfrak{B}^2 + \mathfrak{B}L - L^2(p^2+1)}\, \gamma\overline{p},$$

where C denotes the Liouville vector field and $p^2 := \mathbf{B}(\overline{p}) = g(\overline{p},\overline{p})$.

As an illustrative example, consider the Finsler manifold $M = \{(r,\theta,\phi,t) \in U \subset \mathbb{R}^4 \mid r,\ \theta \neq 0\}$, and define the conic Finsler metric L by

$$L = \sqrt{r^2\left(\frac{\theta^2\dot{\phi}^2 + 2\dot{\theta}\dot{\phi}}{\dot{\theta}}\right)^2 + \dot{r}^2 + \dot{t}^2}$$

for $(r,\theta,\phi,t;\dot{r},\dot{\theta},\dot{\phi},\dot{t}) \in \mathcal{T}U \subset \mathbb{R}^4 \times \mathbb{R}^4$. In this case, the corresponding π-form has components $\mathbf{B}_1 = r$, $\mathbf{B}_2 = \mathbf{B}_3 = 0$, $\mathbf{B}_4 = t$, leading to the scalar π-form $\mathfrak{B}(x,\dot{x}) = r\dot{r} + t\dot{t}$. That is, we have

$$\widetilde{L}(x,\dot{x}) = L(x,\dot{x})\, e^{\frac{\mathfrak{B}(x,\dot{x})}{L(x,\dot{x})}} = \sqrt{r^2\left(\frac{\theta^2\dot{\phi}^2 + 2\dot{\theta}\dot{\phi}}{\dot{\theta}}\right)^2 + \dot{r}^2 + \dot{t}^2}\, e^{\frac{r\dot{r}+t\dot{t}}{\sqrt{r^2\left(\frac{\theta^2\dot{\phi}^2+2\dot{\theta}\dot{\phi}}{\dot{\theta}}\right)^2 + \dot{r}^2 + \dot{t}^2}}}.$$

2. Notations and Preliminaries

Let M be a smooth manifold of dimension n, and let (TM,π,M) denote its tangent bundle with the corresponding tangent map $(TTM,d\pi,TM)$. The vertical bundle $V(TM)$ is defined as the kernel of the differential map: $V(TM) := \ker(d\pi)$. The pullback of the tangent bundle is denoted by $\pi^{-1}(TM)$. These structures yield the following short exact sequence of vector bundle morphisms [4]:

$$0 \longrightarrow \pi^{-1}(TM) \xrightarrow{\gamma} TTM \xrightarrow{\rho} \pi^{-1}(TM) \longrightarrow 0,$$

where $\mathcal{T}M := TM \setminus \{0\}$ is the slit tangent bundle, γ is the canonical injection, and $\rho := (\pi_{TM}, d\pi)$.

The *almost tangent structure* (also known as the vertical endomorphism) J on TM is defined by the composition $J := \gamma \circ \rho$. The ring of smooth functions on TM is denoted by $C^\infty(TM)$, and the module of smooth sections of $\pi^{-1}(TM)$ is denoted by $\mathfrak{X}(\pi(M))$, whose elements are called π-vector fields and are typically written as $\overline{X}$.

The *Liouville vector field* (or the fundamental π-vector field), denoted by C, is given by $C := \gamma\overline{\eta}$, where $\overline{\eta}(u) := (u,u)$ for all $u \in \mathcal{T}M$.

We now recall foundational aspects of the Klein–Grifone approach to Finsler geometry; see [4–6] for more comprehensive treatments.

A *nonlinear connection* on M is a vector-valued 1-form Γ on TM that is smooth on $\mathcal{T}M$ and continuous on TM, satisfying:

$$J\Gamma = J, \quad \Gamma J = -J.$$

The associated horizontal and vertical projectors are defined by:

$$h := \frac{1}{2}(I + \Gamma), \quad v := \frac{1}{2}(I - \Gamma).$$

The torsion and curvature tensors of Γ are given respectively by:

$$t := \frac{1}{2}[J, \Gamma], \quad \mathfrak{R} := -\frac{1}{2}[h, h].$$

Given a linear connection D on $\pi^{-1}(TM)$, the corresponding *connection map* K is defined by:

$$K : TTM \to \pi^{-1}(TM), \quad X \mapsto D_X\overline{\eta}.$$

The horizontal subspace at $u \in TM$ is:

$$H_u(TM) := \{X \in T_u(TM) : K(X) = 0\}.$$

The connection D is said to be *regular* if every tangent space decomposes as:

$$T_u(TM) = V_u(TM) \oplus H_u(TM), \quad \forall u \in TM.$$

In this case, the restrictions $\rho|_{H(TM)}$ and $K|_{V(TM)}$ are isomorphisms, and the inverse $\beta := (\rho|_{H(TM)})^{-1}$ is called the *horizontal map* associated with D.

Let D be a regular connection on $\pi^{-1}(TM)$, with horizontal map β and torsion and curvature tensors $\mathbf{T}$ and $\mathbf{K}$, respectively. Then the following hold:

(1) For a π-tensor field A of type $(0, p)$, the horizontal and vertical covariant derivatives are defined by:

$$(\overset{h}{D} A)(\overline{X}, \overline{X}_1, \ldots, \overline{X}_p) := (D_{\beta\overline{X}}A)(\overline{X}_1, \ldots, \overline{X}_p),$$
$$(\overset{v}{D} A)(\overline{X}, \overline{X}_1, \ldots, \overline{X}_p) := (D_{\gamma\overline{X}}A)(\overline{X}_1, \ldots, \overline{X}_p).$$

(2) The torsion components of D are:

$$Q(\overline{X}, \overline{Y}) := \mathbf{T}(\beta\overline{X}, \beta\overline{Y}),$$
$$T(\overline{X}, \overline{Y}) := \mathbf{T}(\gamma\overline{X}, \beta\overline{Y}),$$
$$V(\overline{X}, \overline{Y}) := \mathbf{T}(\gamma\overline{X}, \gamma\overline{Y}).$$

(3) The curvature tensors of D are given by:

$$R(\overline{X}, \overline{Y})\overline{Z} := \mathbf{K}(\beta\overline{X}, \beta\overline{Y})\overline{Z},$$
$$P(\overline{X}, \overline{Y})\overline{Z} := \mathbf{K}(\beta\overline{X}, \gamma\overline{Y})\overline{Z},$$
$$S(\overline{X}, \overline{Y})\overline{Z} := \mathbf{K}(\gamma\overline{X}, \gamma\overline{Y})\overline{Z}.$$

(4) The (v)h-, (v)hv-, and (v)v-torsion tensors are:

$$\widehat{R}(\overline{X}, \overline{Y}) := R(\overline{X}, \overline{Y})\overline{\eta}, \quad \widehat{P}(\overline{X}, \overline{Y}) := P(\overline{X}, \overline{Y})\overline{\eta}, \quad \widehat{S}(\overline{X}, \overline{Y}) := S(\overline{X}, \overline{Y})\overline{\eta}.$$

Finsler Structures and Cartan Connection.

Definition 2.1. *A* Finsler manifold *is a pair (M, L) where M is a smooth n-dimensional manifold and $L : TM \to \mathbb{R}$ is a function satisfying:*

- **(a):** $L(u) > 0$ *for all* $u \in \mathcal{T}M$, *and* $L(0) = 0$,
- **(b):** L *is* C^∞ *on* $\mathcal{T}M$ *and continuous on* TM,
- **(c):** L *is positively 1-homogeneous:* $\mathcal{L}_C L = L$,
- **(d):** *The 2-form* $\Omega := dd_J E$, *where* $E := \frac{1}{2}L^2$, *is non-degenerate.*

The induced Finsler metric g on $\pi^{-1}(TM)$ is defined by:

$$g(\rho X, \rho Y) := \Omega(JX, Y), \quad \forall X, Y \in \mathfrak{X}(TM).$$

If the above conditions hold only on a conic subset $U \subset \mathcal{T}M$, the pair (M, L) is called a conic Finsler manifold.

Remark 2.1. *Motivated by potential applications in both theoretical and applied settings such as general relativity, we can extend the classical notions of Finsler geometry by introducing pseudo-Finsler and conic pseudo-Finsler spaces. These generalizations allow for more flexibility in the geometric structure, particularly by relaxing the strong convexity and positivity conditions on the Finsler metric.*

A vector field G on TM is a *semi-spray* if $JG = C$; it is a *spray* if it is 2-homogeneous, i.e., $[C, G] = G$.

Proposition 2.1 ([5,6]). *Let (M, L) be a Finsler manifold. Then:*

- **(a):** *There exists a unique spray G (called the* canonical spray*) satisfying* $i_G dd_J E = -dE$.
- **(b):** *The* Barthel connection *is defined by* $\Gamma := [J, G]$.

Theorem 2.1 ([13]). *Given a Finsler manifold (M, L) with Finsler metric g, there exists a unique regular connection ∇ on $\pi^{-1}(TM)$ such that:*

- **(i):** ∇ *is metric:* $\nabla g = 0$,
- **(ii):** *The (h)h-torsion vanishes:* $Q = 0$,
- **(iii):** *The (h)hv-torsion satisfies symmetry:* $g(T(\overline{X}, \overline{Y}), \overline{Z}) = g(T(\overline{X}, \overline{Z}), \overline{Y})$.

This connection is called the Cartan connection.

Lemma 2.1 ([13]). *Let (M, L) be a Finsler manifold and let β be the horizontal map of the Cartan connection ∇. Then:*

(a) $(D^\circ_{\gamma\overline{X}} g)(\overline{Y}, \overline{Z}) = 2\mathbf{T}(\overline{X}, \overline{Y}, \overline{Z}), \quad \nabla_{\gamma\overline{X}} g = 0,$

(b) $(D^\circ_{\beta\overline{X}} g)(\overline{Y}, \overline{Z}) = -2\widehat{\mathbf{P}}(\overline{X}, \overline{Y}, \overline{Z}), \quad \nabla_{\beta\overline{X}} g = 0,$

where $\widehat{\mathbf{P}}$ is defined by $\widehat{\mathbf{P}}(\overline{X}, \overline{Y}, \overline{Z}) := g(\widehat{P}(\overline{X}, \overline{Y}), \overline{Z})$.

Lemma 2.2. *For a Finsler manifold (M, L):*

- **(a):** $d_J L(\gamma\overline{X}) = 0, \quad D^\circ_{\gamma\overline{X}} L = dL(\gamma\overline{X}) = d_J L(\beta\overline{X}) = \ell(\overline{X}),$
- **(b):** $d_h L(\beta\overline{X}) = D^\circ_{\beta\overline{X}} L = dL(\beta\overline{X}) = 0,$

(c): $(D^{\circ}_{\gamma\overline{X}}\ell)(\overline{Y}) = (\nabla_{\gamma\overline{X}}\ell)(\overline{Y}) = L^{-1}\hbar(\overline{X}, \overline{Y})$,

(d): $dd_J E(\gamma\overline{X}, \beta\overline{Y}) = g(\overline{X}, \overline{Y})$,

where g is the Finsler metric and $\ell := L^{-1} i_{\overline{\eta}} g$ is the normalized supporting element.

3. Special Anisotropic Conformal Change

In this section, we present an intrinsic study of the special anisotropic conformal change. By deforming the underlying Finsler structure using a factor $\sigma(x, \dot{x})$, we introduce as special anisotropic conformal change, as defined below.

Definition 3.1 ([14]). *Let (M, L) be a Finsler manifold and let D° denote the Berwald connection on $\mathcal{P}$. A π-vector field $\overline{Y} \in C(\pi)$ is said to be independent of the directional argument y if and only if*

$$D^{\circ}_{\gamma\overline{X}}\overline{Y} = 0 \quad \text{for all } \overline{X} \in C(\pi).$$

Similarly, a scalar or vector π-form ω is independent of the directional argument $\dot{x}$ if and only if

$$D^{\circ}_{\gamma\overline{X}}\,\omega = 0 \quad \text{for all } \overline{X} \in C(\pi).$$

Definition 3.2. *Let (M, L) be a Finsler manifold. Consider the special anisotropic deformation of the Finsler structure L given by*

$$\widetilde{L}(x, \dot{x}) = e^{\sigma(x,\dot{x})} L(x, \dot{x}) = e^{\frac{\mathfrak{B}(x,\dot{x})}{L(x,\dot{x})}} L(x, \dot{x}), \tag{3.1}$$

where $\sigma(x, \dot{x}) := \frac{\mathfrak{B}(x,\dot{x})}{L(x,\dot{x})}$, and $\mathfrak{B}(x, \dot{x}) := \mathbf{B}(\overline{\eta})$ is defined by a scalar π-form $\mathbf{B}$ that is independent of the directional argument y. If $\widetilde{L}$ defines a Finsler structure on M, then it can be referred also as a generalized exponential (α, β)-metric.

To investigate the geometry associated with $\widetilde{L}$, we require the following lemmas.

Lemma 3.1. *Under the transformation $L \mapsto \widetilde{L}$, the vertical component of the Berwald connection remains invariant:*

$$\widetilde{D}^{\circ}_{\gamma\overline{X}}\,\overline{Y} = D^{\circ}_{\gamma\overline{X}}\,\overline{Y}.$$

Proof. This follows from the fact that the difference between the horizontal maps $\widetilde{\beta}$ and β is a vertical vector field, together with the identity

$$D^{\circ}_{\gamma\overline{X}}\overline{Y} = \rho[\gamma\overline{X}, \beta\overline{Y}],$$

the vanishing of $\rho \circ \gamma$, and the integrability of the vertical distribution (cf. [13]). □

Lemma 3.2. *Let (M, L) be a Finsler manifold equipped with a scalar π-form $\mathbf{B}$ independent of $\dot{x}$, and let $\overline{p}$ be the associated π-vector field defined via $i_{\overline{p}}\, g := \mathbf{B}$. Then the function $\mathfrak{B}(x, y) := \mathbf{B}(\overline{\eta})$ satisfies the following:*

(a): $d_J\mathfrak{B}(\gamma\overline{X}) = 0, \quad D^{\circ}_{\gamma\overline{X}}\mathfrak{B} = d\mathfrak{B}(\gamma\overline{X}) = d_J\mathfrak{B}(\beta\overline{X}) = \mathbf{B}(\overline{X})$.

(b): $d_h\,\mathfrak{B}(\beta\overline{X}) = D^{\circ}_{\beta\overline{X}}\,\mathfrak{B} = d\mathfrak{B}(\beta\overline{X}) = L\,\ell(D^{\circ}_{\beta\overline{X}}\,\overline{p})$, and $d\mathfrak{B}(G) = L\,\ell(D^{\circ}_{G}\,\overline{p})$.

(c): $D^\circ_{\gamma\overline{X}}\overline{p} = -2T(\overline{X},\overline{p})$.

Proof. Straightforward computations; the proof is omitted. □

We now derive the behavior of the normalized supporting element ℓ and the angular metric tensor $\hbar$ under the anisotropic conformal change (3.1).

Proposition 3.1. *Under the transformation* (3.1), *the following hold:*

(1) *The supporting form $\widetilde{\ell}$ is related to ℓ by*

$$\widetilde{\ell}(\overline{X}) = \frac{e^{\mathfrak{B}/L}(L-\mathfrak{B})}{L}\,\ell(\overline{X}) + e^{\mathfrak{B}/L}\,\mathbf{B}(\overline{X}).$$

(2) *The angular metric tensors $\widetilde{\hbar}$ and $\hbar$ are related via*

$$\begin{aligned}\widetilde{\hbar}(\overline{X},\overline{Y}) = {} & \frac{e^{2\mathfrak{B}/L}(L-\mathfrak{B})}{L}\,\hbar(\overline{X},\overline{Y}) + e^{2\mathfrak{B}/L}\,\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y}) \\ & + \frac{\mathfrak{B}^2 e^{2\mathfrak{B}/L}}{L^2}\,\ell(\overline{X})\,\ell(\overline{Y}) - \frac{\mathfrak{B}e^{2\mathfrak{B}/L}}{L}\Big[\mathbf{B}(\overline{X})\,\ell(\overline{Y}) + \mathbf{B}(\overline{Y})\,\ell(\overline{X})\Big].\end{aligned} \tag{3.2}$$

Proof. (1) Since $\rho\circ\gamma = 0$ and $\rho\circ\beta = \rho\circ\widetilde{\beta} = \mathrm{id}_{\mathfrak{X}(\pi(M))}$, we have:

$$\widetilde{\ell}(\overline{X}) = d_J\widetilde{L}(\widetilde{\beta}\overline{X}) = d_J\widetilde{L}(\beta\overline{X}).$$

Differentiating $\widetilde{L} = Le^{\mathfrak{B}/L}$ gives:

$$\widetilde{\ell}(\overline{X}) = \left(\frac{\partial\widetilde{L}}{\partial L}\right)\ell(\overline{X}) + \left(\frac{\partial\widetilde{L}}{\partial\mathfrak{B}}\right)\mathbf{B}(\overline{X}) = \frac{e^{\mathfrak{B}/L}(L-\mathfrak{B})}{L}\,\ell(\overline{X}) + e^{\mathfrak{B}/L}\mathbf{B}(\overline{X}).$$

(2) Using Lemmas 2.2, 3.1 and 3.2:

$$\begin{aligned}\widetilde{\hbar}(\overline{X},\overline{Y}) &= \widetilde{L}(\widetilde{D}^\circ_{\gamma\overline{X}}\widetilde{\ell})(\overline{Y}) = \widetilde{L}(D^\circ_{\gamma\overline{X}}\widetilde{\ell})(\overline{Y}) \\ &= \widetilde{L}\,D^\circ_{\gamma\overline{X}}\left\{\frac{e^{\mathfrak{B}/L}(L-\mathfrak{B})}{L}\,\ell(\overline{Y}) + e^{\mathfrak{B}/L}\,\mathbf{B}(\overline{Y})\right\} \\ &= \widetilde{L}\left\{\left(D^\circ_{\gamma\overline{X}}\left(\frac{e^{\mathfrak{B}/L}(L-\mathfrak{B})}{L}\right)\right)\ell(\overline{Y}) + \left(D^\circ_{\gamma\overline{X}}\left(e^{\mathfrak{B}/L}\right)\right)\mathbf{B}(\overline{Y})\right\} \\ &\quad + \widetilde{L}\left\{\left(\frac{e^{\mathfrak{B}/L}(L-\mathfrak{B})}{L}\right)(D^\circ_{\gamma\overline{X}}\ell)(\overline{Y}) + e^{\mathfrak{B}/L}\,(D^\circ_{\gamma\overline{X}}\mathbf{B})(\overline{Y})\right\} \\ &= Le^{\mathfrak{B}/L}\left\{\left(\frac{\mathfrak{B}^2 e^{\mathfrak{B}/L}}{L^3}\,\ell(\overline{X}) - \frac{\mathfrak{B}e^{\mathfrak{B}/L}}{L^2}\,\mathbf{B}(\overline{X})\right)\ell(\overline{Y})\right. \\ &\quad \left. + \left(-\frac{\mathfrak{B}e^{\mathfrak{B}/L}}{L^2}\,\ell(\overline{X}) + \frac{e^{\mathfrak{B}/L}}{L}\,\mathbf{B}(\overline{X})\right)\mathbf{B}(\overline{Y})\right\} \\ &\quad + Le^{\mathfrak{B}/L}\left\{\left(\frac{e^{\mathfrak{B}/L}(L-\mathfrak{B})}{L}\right)(L^{-1}\,\hbar(\overline{X},\overline{Y}) + 0)\right\}.\end{aligned}$$

which simplifies to the expression in (3.2). □

4. The Metric and Cartan Tensors

In this section, we calculate some geometric objects associated to $\widetilde{L}(x,\dot{x})$ in terms of the objects associated with L. The following proposition shows the relationship between g and $\widetilde{g}$.

Proposition 4.1. *The Finsler metric $\widetilde{g}$ associated with the special anisotropic change* (3.1) *is given by the following relation:*

$$\begin{aligned}\widetilde{g}(\overline{X},\overline{Y}) &= \frac{e^{\frac{2\mathfrak{B}}{L}}(L-\mathfrak{B})}{L}\, g(\overline{X},\overline{Y}) + 2e^{\frac{2\mathfrak{B}}{L}}\,\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y}) + \frac{\mathfrak{B}e^{\frac{2\mathfrak{B}}{L}}(2\mathfrak{B}-L)}{L^2}\,\ell(\overline{X})\,\ell(\overline{Y})\\ &\quad + \frac{e^{\frac{2\mathfrak{B}}{L}}(L-2\mathfrak{B})}{L}\left\{\mathbf{B}(\overline{X})\,\ell(\overline{Y}) + \mathbf{B}(\overline{Y})\,\ell(\overline{X})\right\}.\end{aligned}$$

Consequently, the Cartan torsion $\widetilde{\mathbf{T}}$, under the special anisotropic change, has the form

$$\begin{aligned}2\widetilde{\mathbf{T}}(\overline{X},\overline{Y},\overline{Z}) &= 2\frac{e^{\frac{2\mathfrak{B}}{L}}(L-\mathfrak{B})}{L}\;T(\overline{X},\overline{Y},\overline{Z}) + \frac{\mathfrak{B}e^{\frac{2\mathfrak{B}}{L}}(2\mathfrak{B}-L)}{L^3}\{\hbar(\overline{X},\overline{Z})\,\ell(\overline{Y}) + \hbar(\overline{Y},\overline{Z})\,\ell(\overline{X})\}\\ &\quad + \frac{e^{\frac{2\mathfrak{B}}{L}}(L-2\mathfrak{B})}{L^2}\left\{\mathbf{B}(\overline{X})\,\hbar(\overline{Y},\overline{Z}) + \mathbf{B}(\overline{Y})\,\hbar(\overline{X},\overline{Z})\right\} + \left(D^\circ_{\gamma\overline{Z}}2e^{\frac{2\mathfrak{B}}{L}}\right)\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y})\\ &\quad + \left(D^\circ_{\gamma\overline{Z}}\frac{e^{\frac{2\mathfrak{B}}{L}}(L-\mathfrak{B})}{L}\right) g(\overline{X},\overline{Y}) + \left(D^\circ_{\gamma\overline{Z}}\frac{\mathfrak{B}e^{\frac{2\mathfrak{B}}{L}}(2\mathfrak{B}-L)}{L^2}\right)\ell(\overline{X})\,\ell(\overline{Y})\\ &\quad + \left(D^\circ_{\gamma\overline{Z}}\frac{e^{\frac{2\mathfrak{B}}{L}}(L-2\mathfrak{B})}{L}\right)\left\{\mathbf{B}(\overline{X})\,\ell(\overline{Y}) + \mathbf{B}(\overline{Y})\,\ell(\overline{X})\right\}.\end{aligned}$$

Proof. In view of the change (3.1), using Proposition 3.1, we have the following:

$$\begin{aligned}\widetilde{\ell}(\overline{X}) &= \frac{e^{\mathfrak{B}/L}(L-\mathfrak{B})}{L}\,\ell(\overline{X}) + e^{\mathfrak{B}/L}\,\mathbf{B}(\overline{X}).\\ \widetilde{\hbar}(\overline{X},\overline{Y}) &= \frac{e^{\frac{2\mathfrak{B}}{L}}(L-\mathfrak{B})}{L}\,\hbar(\overline{X},\overline{Y}) + e^{\frac{2\mathfrak{B}}{L}}\,\mathbf{B}(\overline{X})\mathbf{B}(\overline{Y}) + \frac{\mathfrak{B}^2e^{\frac{2\mathfrak{B}}{L}}}{L^2}\,\ell(\overline{X})\,\ell(\overline{Y})\\ &\quad - \frac{\mathfrak{B}\,e^{\frac{2\mathfrak{B}}{L}}}{L}\left\{\mathbf{B}(\overline{X})\,\ell(\overline{Y}) + \mathbf{B}(\overline{Y})\,\ell(\overline{X})\right\}.\end{aligned}$$

Hence, using the definition of the angular metric tensor $\widetilde{\hbar} := \widetilde{g} - \widetilde{\ell}\otimes\widetilde{\ell}$, the expression of $\widetilde{g}$ is obtained. Moreover, using the expression of the metric $\widetilde{g}$, taking into account the fact that $(D^\circ_{\gamma\overline{Z}}g)(\overline{X},\overline{Y}) = 2\mathbf{T}(\overline{X},\overline{Y},\overline{Z})$ (Lemma 2.1), it follows the required expression of the Cartan torsion $\widetilde{\mathbf{T}}$. □

Theorem 4.1. *The metric tensor $\widetilde{g}$ of $\widetilde{L}$ is non-degenerate if and only if*

$$\mathfrak{B}^2 + \mathfrak{B}L - L^2\left(p^2+1\right) \neq 0. \tag{4.1}$$

That is, the special anisotropic change provide a Finsler structure (or, conic Finsler structure) if and only if the condition (4.1) *is satisfied.*

Proof. Assume that $\widetilde{g}$ be the Finsler metric associated with the special anisotropic change (3.1). To prove the non-degenerate property of $\widetilde{g}$. Suppose that $\widetilde{g}(\overline{X},\overline{Y})=0$ for all $\overline{X}\in\mathfrak{X}(\pi(M))$. By using Proposition 4.1, we obtain

$$\begin{aligned} 0 &= \frac{e^{\frac{2\mathfrak{B}}{L}}(L-\mathfrak{B})}{L}\, g(\overline{X},\overline{Y}) + 2e^{\frac{2\mathfrak{B}}{L}}\,\mathbf{B}(\overline{X})\,\mathbf{B}(\overline{Y}) + \frac{\mathfrak{B}e^{\frac{2\mathfrak{B}}{L}}(2\mathfrak{B}-L)}{L^2}\,\ell(\overline{X})\,\ell(\overline{Y}) \\ &\quad + \frac{e^{\frac{2\mathfrak{B}}{L}}(L-2\mathfrak{B})}{L}\left\{\mathbf{B}(\overline{X})\,\ell(\overline{Y}) + \mathbf{B}(\overline{Y})\,\ell(\overline{X})\right\}. \end{aligned}$$

From which, by substituting $\overline{X}=\overline{p}$, noting that $\ell(\overline{p})=\frac{\mathfrak{B}}{L}$ and $\mathbf{B}(\overline{p})=g(\overline{p},\overline{p})=: p^2$, one can show that

$$\zeta_1\,\ell(\overline{Y}) + \zeta_2\,\mathbf{B}(\overline{Y}) = 0, \tag{4.2}$$

where

$$\zeta_1 := \frac{e^{\frac{2\mathfrak{B}}{L}}(2\mathfrak{B}-L)(\mathfrak{B}-Lp)(\mathfrak{B}+Lp)}{L^3}, \qquad \zeta_2 := \frac{e^{\frac{2\mathfrak{B}}{L}}\left(-2\mathfrak{B}^2+2L^2p^2+L^2\right)}{L^2}.$$

Similarly, by substituting $\overline{X}=\overline{\eta}$, taking into account the facts that $\ell(\overline{\eta})=L$ and $\mathbf{B}(\overline{\eta})=\mathfrak{B}$, we obtain

$$\zeta_3\,\ell(\overline{Y}) + \zeta_4\,\mathbf{B}(\overline{Y}) = 0, \tag{4.3}$$

with

$$\zeta_3 := -e^{\frac{2\mathfrak{B}}{L}}(\mathfrak{B}-L), \qquad \zeta_4 := Le^{\frac{2\mathfrak{B}}{L}}.$$

Now, the system of the algebraic equations (4.2) and (4.3) has non-trivial solution (i.e. $\ell(\overline{Y})\neq 0$ and $\mathbf{B}(\overline{Y})\neq 0$) if and only if

$$\frac{e^{\frac{4\mathfrak{B}}{L}}\left(\mathfrak{B}^2+\mathfrak{B}L-L^2\left(p^2+1\right)\right)}{L} = 0.$$

Hence, as $\widetilde{L}\neq 0$ over $\mathcal{T}M$, then we conclude that

$$\mathfrak{B}^2+\mathfrak{B}L-L^2\left(p^2+1\right)=0.$$

Therefore, $\ell(\overline{Y})=\mathbf{B}(\overline{Y})=0$ if and only if the Finsler structure L and the π-form $\mathfrak{B}$ satisfy the condition

$$\mathfrak{B}^2+\mathfrak{B}L-L^2\left(p^2+1\right)\neq 0.$$

From which, one can show that $\overline{Y}$ vanishes. This means that the metric $\widetilde{g}$ is non-degenerate if and only if the condition (4.1) is satisfied. Hence, the proof is completed. □

Form now on, we consider that the $\widetilde{L}$ satisfies the condition (4.1).

5. Geodesic Spray and Berwald Connection

To simplify the calculations involved in determining the geodesic spray under the special anisotropic change, we focus on a particular type of 1-form. This leads to the following definition.

Definition 5.1 ([14]). *Let (M, L) be a Finsler manifold. A π-vector field $\overline{p} \in \mathfrak{X}(\pi(M))$ is said to be a* concurrent *π-vector field if it satisfies:*

$$\nabla_{\beta\overline{X}}\overline{p} = -\overline{X} = D^{\circ}_{\beta\overline{X}}\overline{p}, \quad \nabla_{\gamma\overline{X}}\overline{p} = 0 = D^{\circ}_{\gamma\overline{X}}\overline{p}. \tag{5.1}$$

Furthermore, if $\mathbf{B}$ is the π-form associated with $\overline{p}$ via the Finsler metric g, that is, $\mathbf{B} = i_{\overline{p}}\, g$, then $\mathbf{B}$ satisfies:

$$(\nabla_{\beta\overline{X}}\mathbf{B})(\overline{Y}) = -g(\overline{X}, \overline{Y}) = (D^{\circ}_{\beta\overline{X}}\mathbf{B})(\overline{Y}), \quad (\nabla_{\gamma\overline{X}}\mathbf{B})(\overline{Y}) = 0 = (D^{\circ}_{\gamma\overline{X}}\mathbf{B})(\overline{Y}).$$

An intrinsic treatment of concurrent π-vector fields within Finsler geometry was presented in [14], where their properties and characterizations were systematically examined. One such characterization is given below.

Lemma 5.1. *Let (M, L) be a Finsler manifold endowed with a scalar π-form $\mathbf{B}$ that does not depend on the directional argument $\dot{x}$, and let $\overline{p}$ be its associated concurrent π-vector field. Then the function $\mathfrak{B} := \mathbf{B}(\overline{\eta})$ satisfies:*

$$d\mathfrak{B}(G) = -L^2.$$

Proof. The result is obtained by applying part (b) of Lemma 3.2 in conjunction with Definition 5.1. □

Theorem 5.1 ([14]). *If $\overline{p}$ is a concurrent π-vector field, then both $\overline{p}$ and its associated π-form $\mathbf{B}$ are independent of the directional argument $\dot{x}$.*

We now derive the expression for the canonical spray $\widetilde{G}$ associated with $\widetilde{L}$, relating it to the geodesic spray G of the original metric L.

Theorem 5.2. *Under the special anisotropic change* (3.1)*, the corresponding canonical spray $\widetilde{G}$ is given by:*

$$\widetilde{G} = G - \frac{L^2(L - 2\mathfrak{B})}{\mathfrak{B}^2 + \mathfrak{B}L - L^2(p^2 + 1)}\, C - \frac{L^4}{\mathfrak{B}^2 + \mathfrak{B}L - L^2(p^2 + 1)}\, \gamma\overline{p},$$

where $C := \gamma\,\overline{\eta}$ denotes the Liouville vector field, and $p^2 := \mathbf{B}(\overline{p}) = g(\overline{p}, \overline{p})$.

Proof. Due to the Let $\widetilde{L}$ be *(3.1)*, taking into account the expression of the exterior π-form $\widetilde{\Omega} := \frac{1}{2} dd_J\,\widetilde{L}^2$, the fact that the difference between two sprays is a vertical vector field (i.e. $\widetilde{G} = G + \gamma\overline{\mu}$, for some π-vector field $\overline{\mu}$) and using Proposition 2.1, one can show that

$$\begin{aligned} -d\widetilde{E}(X) &= i_{\widetilde{G}}\,\widetilde{\Omega}(X) = i_{G+\gamma\overline{\mu}}\,(\tfrac{1}{2}\, dd_J\widetilde{L}^2)(X) \\ &= \tfrac{1}{2} i_G\, dd_J\widetilde{L}^2(X) + \tfrac{1}{2} i_{\gamma\overline{\mu}}\, dd_J\widetilde{L}^2(X). \end{aligned} \tag{5.2}$$

Therefore, after some computation together the fact thats $\beta\overline{\eta} = G$ and $X = hX + vX = \beta\rho X + \gamma KX$, together with Lemmas 2.2, 3.2 and 5.1, we have

$$\begin{aligned} d\widetilde{E}(X) &= \frac{1}{2} d\widetilde{L}^2(X) = \widetilde{L}\, d\widetilde{L}(X) \\ &= \left(Le^{\mathfrak{B}/L}\right)\left\{\frac{e^{\mathfrak{B}/L}(L-\mathfrak{B})}{L}\, dL(X) + e^{\mathfrak{B}/L}\, d\mathfrak{B}(X)\right\} \\ &= -e^{\frac{2\mathfrak{B}}{L}}(\mathfrak{B}-L)\, dL(X) + Le^{\frac{2\mathfrak{B}}{L}}\, d\mathfrak{B}(X). \end{aligned}$$

Also, we have

$$\begin{aligned} \frac{1}{2}\, i_G\, dd_J\widetilde{L}^2(X) &= \frac{1}{2}\{dd_J\widetilde{L}^2(\beta\overline{\eta}, X)\} \\ &= \frac{1}{2}\left\{G\cdot d_J\widetilde{L}^2(X) - X\cdot d_J\widetilde{L}^2(G) - d_J\widetilde{L}^2[G,X]\right\} \\ &= \frac{1}{2}\left\{G\cdot(2\widetilde{L}\widetilde{\ell}(\rho X)) - X\cdot(2\widetilde{L}\widetilde{\ell}(\overline{\eta})) - 2\widetilde{L}\widetilde{\ell}(\rho[G,X])\right\} \\ &= ((G\cdot\widetilde{L})\,\widetilde{\ell}(\rho X) + \widetilde{L}\, G\cdot\widetilde{\ell}(\rho X)) - (X\cdot\widetilde{L}^2) - \widetilde{L}\,\widetilde{\ell}(\rho[G,X]). \end{aligned}$$

From which taking into account Lemmas 3.2, 5.1 and the following facts

$$\begin{aligned} G\cdot\widetilde{L} &= d\widetilde{L}(G) = -L^2\, e^{\mathfrak{B}/L} \\ X\cdot\widetilde{L} &= d\widetilde{L}(X) = \frac{e^{\mathfrak{B}/L}(L-\mathfrak{B})}{L}\, dL(X) + e^{\mathfrak{B}/L}\, d\mathfrak{B}(X), \\ \widetilde{\ell}(\overline{X}) &= \frac{e^{\mathfrak{B}/L}(L-\mathfrak{B})}{L}\, \ell(\overline{X}) + e^{\mathfrak{B}/L}\, \mathbf{B}(\overline{X}), \\ \rho[G,X] &= \rho[G, hX + vX] = D^\circ_G\rho X - KX, \\ (D^\circ_G\, \mathbf{B})(\overline{X}) &= -g(\overline{X}, \overline{\eta}) = -L\,\ell(\overline{X}), \\ (D^\circ_G\, \ell)(\overline{X}) &= (\nabla_G\, \ell)(\overline{X}) = 0, \\ d\mathfrak{B}(X) &= \mathbf{B}(KX) - L\ell(\rho X), \\ dL(X) &= dL(\gamma KX) = \ell(KX). \end{aligned}$$

Using the above identities, then straightforward calculations imply the above relation reduces to

$$\begin{aligned} \frac{1}{2}\, i_G\, dd_J\widetilde{L}^2(X) &= -L^2\, e^{\mathfrak{B}/L}\left(\frac{e^{\mathfrak{B}/L}(L-\mathfrak{B})}{L}\, \ell(\rho X) + e^{\mathfrak{B}/L}\, \mathbf{B}(\rho X)\right) \\ &\quad + L\, e^{\mathfrak{B}/L}\, G\cdot\left(\frac{e^{\mathfrak{B}/L}(L-\mathfrak{B})}{L}\, \ell(\rho X) + e^{\mathfrak{B}/L}\, \mathbf{B}(\rho X)\right) \\ &\quad - 2L\, e^{\mathfrak{B}/L}\left(\frac{e^{\mathfrak{B}/L}(L-\mathfrak{B})}{L}\, dL(X) + e^{\mathfrak{B}/L}\, d\mathfrak{B}(X)\right) \\ &\quad - L\, e^{\mathfrak{B}/L}\left(\frac{e^{\mathfrak{B}/L}(L-\mathfrak{B})}{L}\, \ell(\rho[G,X]) + e^{\mathfrak{B}/L}\, \mathbf{B}(\rho[G,X])\right) \\ &= -L\, e^{\frac{2\mathfrak{B}}{L}}(L-2\mathfrak{B})\ell(\rho X) - 2L^2\, e^{\frac{2\mathfrak{B}}{L}}\, \mathbf{B}(\rho X) + e^{\frac{2\mathfrak{B}}{L}}(\mathfrak{B}-L)\, dL(X) \\ &\quad - Le^{\frac{2\mathfrak{B}}{L}}\, d\mathfrak{B}(X). \end{aligned}$$

On the other hand, using Proposition 4.1, we have

$$
\begin{aligned}
\frac{1}{2}\, i_{\gamma\overline{\mu}}\, dd_J\widetilde{L}^2(X) \quad &= \widetilde{g}(\overline{\mu}, \rho X)\\
&= \frac{e^{\frac{2\mathfrak{B}}{L}}(L-\mathfrak{B})}{L}\, g(\overline{\mu}, \rho X) + 2e^{\frac{2\mathfrak{B}}{L}}\, \mathbf{B}(\overline{\mu})\, \mathbf{B}(\rho X) + \frac{\mathfrak{B}e^{\frac{2\mathfrak{B}}{L}}(2\mathfrak{B}-L)}{L^2}\, \ell(\overline{\mu})\, \ell(\rho X)\\
&+ \frac{e^{\frac{2\mathfrak{B}}{L}}(L-2\mathfrak{B})}{L}\, \{\mathbf{B}(\overline{\mu})\, \ell(\rho X) + \mathbf{B}(\rho X)\, \ell(\overline{\mu})\}.
\end{aligned}
$$

Plugging the last two relations into Equation (5.2), after some calculation, it follows that

$$
\begin{aligned}
& e^{\frac{2\mathfrak{B}}{L}}(\mathfrak{B}-L)\, dL(X) - Le^{\frac{2\mathfrak{B}}{L}}\, d\mathfrak{B}(X)\\
= \quad & -L\, e^{\frac{2\mathfrak{B}}{L}}(L-2\mathfrak{B})\ell(\rho X) - 2L^2\, e^{\frac{2\mathfrak{B}}{L}}\, \mathbf{B}(\rho X) + e^{\frac{2\mathfrak{B}}{L}}(\mathfrak{B}-L)\, dL(X)\\
& -Le^{\frac{2\mathfrak{B}}{L}}\, d\mathfrak{B}(X) + \frac{e^{\frac{2\mathfrak{B}}{L}}(L-2\mathfrak{B})}{L}\, \{\mathbf{B}(\overline{\mu})\, \ell(\rho X) + \mathbf{B}(\rho X)\, \ell(\overline{\mu})\}\\
& + \frac{e^{\frac{2\mathfrak{B}}{L}}(L-\mathfrak{B})}{L}\, g(\overline{\mu}, \rho X) + 2e^{\frac{2\mathfrak{B}}{L}}\, \mathbf{B}(\overline{\mu})\, \mathbf{B}(\rho X) + \frac{\mathfrak{B}e^{\frac{2\mathfrak{B}}{L}}(2\mathfrak{B}-L)}{L^2}\, \ell(\overline{\mu})\, \ell(\rho X).
\end{aligned}
$$

Adopting to the non-degenerate property of the Finsler metric g, the above relation reduces to

$$
\begin{aligned}
\frac{e^{\frac{2\mathfrak{B}}{L}}(L-\mathfrak{B})}{L}\, \overline{\mu} \quad &= \left\{e^{\frac{2\mathfrak{B}}{L}}(L-2\mathfrak{B}) - \frac{e^{\frac{2\mathfrak{B}}{L}}(L-2\mathfrak{B})}{L^2}\, \mathbf{B}(\overline{\mu}) - \frac{\mathfrak{B}e^{\frac{2\mathfrak{B}}{L}}(2\mathfrak{B}-L)}{L^3}\, \ell(\overline{\mu})\right\}\overline{\eta}\\
&+ \left\{2L^2\, e^{\frac{2\mathfrak{B}}{L}} - 2e^{\frac{2\mathfrak{B}}{L}}\, \mathbf{B}(\overline{\mu}) - \frac{e^{\frac{2\mathfrak{B}}{L}}(L-2\mathfrak{B})}{L}\, \ell(\overline{\mu})\right\}\overline{p}, \qquad (5.3)
\end{aligned}
$$

where $\ell(\overline{\mu})$ and $\mathbf{B}(\overline{\mu})$ are geometric quantities given by the following system

$$
\begin{aligned}
A_1\, \ell(\overline{\mu}) + B_1\, \mathbf{B}(\overline{\mu}) \quad &= \quad C_1,\\
A_2\, \ell(\overline{\mu}) + B_2\, \mathbf{B}(\overline{\mu}) \quad &= \quad C_2, \qquad (5.4)
\end{aligned}
$$

with coefficients determined by

$$
\begin{aligned}
A_1 \quad &:= \quad \frac{e^{\frac{2\mathfrak{B}}{L}}(L-\mathfrak{B})}{L}, \quad B_1 := e^{\frac{2\mathfrak{B}}{L}}, \quad C_1 := L^2 e^{\frac{2\mathfrak{B}}{L}},\\
A_2 \quad &:= \quad \frac{e^{\frac{2\mathfrak{B}}{L}}(2\mathfrak{B}-L)\left(\mathfrak{B}^2 - L^2p^2\right)}{L^3},\\
B_2 \quad &:= \quad \frac{e^{\frac{2\mathfrak{B}}{L}}\left(L^2\left(2p^2+1\right) - 2\mathfrak{B}^2\right)}{L^2},\\
C_2 \quad &:= \quad -e^{\frac{2\mathfrak{B}}{L}}\left(2\mathfrak{B}^2 - \mathfrak{B}L - 2L^2p^2\right),\\
p^2 \quad &:= \quad \mathbf{B}(\overline{p}).
\end{aligned}
$$

Making use of the condition (4.1), the system (5.4) has the following solution

$$
\ell(\overline{\mu}) \quad = \quad \frac{L^3(\mathfrak{B}-L)}{\mathfrak{B}^2 + \mathfrak{B}L - L^2\left(p^2+1\right)}, \quad \mathbf{B}(\overline{\mu}) = -\frac{L^2\left(-2\mathfrak{B}^2 + \mathfrak{B}L + L^2p^2\right)}{\mathfrak{B}^2 + \mathfrak{B}L - L^2\left(p^2+1\right)}.
$$

Consequently, in view of Equation (5.3) taking into account the assumption $\widetilde{G} = G + \gamma\overline{\mu}$, it follows that the canonical sprays G and $\widetilde{G}$, are related by

$$\widetilde{G} = G - \frac{L^2(L-2\mathfrak{B})}{\mathfrak{B}^2 + \mathfrak{B}L - L^2\,(p^2+1)}\, C - \frac{L^4}{\mathfrak{B}^2 + \mathfrak{B}L - L^2\,(p^2+1)}\, \gamma\overline{p}.$$

This completes the proof. □

Theorem 5.3. *The Barthel connection $\widetilde{\Gamma}$ associated with the special anisotropic conformal change* (3.1), *is given by*

$$\widetilde{\Gamma} = \Gamma - \lambda_1\, J - d_J\lambda_1 \otimes \gamma\overline{\eta} + d_J\lambda_2 \otimes \gamma\overline{p},$$

where λ_1 and λ_2 are scalar functions determined by

$$\lambda_1 \;:=\; \frac{L^2(L-2\mathfrak{B})}{\mathfrak{B}^2 + \mathfrak{B}L - L^2\,(p^2+1)}, \quad \lambda_2 := \frac{-L^4}{\mathfrak{B}^2 + \mathfrak{B}L - L^2\,(p^2+1)}.$$

Consequently, the horizontal map $\widetilde{\beta}$ associated with the anisotropic conformal change (3.1) *has the form*

$$\widetilde{\beta}\overline{X} = \beta\overline{X} - \frac{1}{2}\left\{\lambda_1\,\gamma\overline{X} + d_J\lambda_1(\beta\overline{X})\,\gamma\overline{\eta} - d_J\lambda_2(\beta\overline{X})\,\gamma\overline{p}\right\}.$$

Proof. The proof follows from $\widetilde{\Gamma} = [J, \widetilde{G}]$, Theorem 5.2, the formula [3]:

$$[fX, J] = f[X, J] + df \wedge i_X J - d_J f \otimes X,$$

taking into account the given expressions of λ_1 and λ_2, and the facts that

$$d_J p^2 \;=\; 0, \;\; i_{\gamma\overline{\eta}}\, J = 0 = i_{\gamma\overline{p}}\, J, \;\; [\gamma\overline{p}, J]X = 0.$$

□

Theorem 5.4. *The Barthel curvature tensor $\widetilde{\mathfrak{R}}$ associated with the special anisotropic conformal change* (3.1) *is determined by*

$$\widetilde{\mathfrak{R}} = \mathfrak{R} - [h, \mathbb{L}] - N_{\mathbb{L}},$$

where $N_{\mathbb{L}} := \frac{1}{2}[\mathbb{L}, \mathbb{L}]$ is the Nijenhuis torsion of a vector 1-form $\mathbb{L}$ defined by

$$\mathbb{L} := -\frac{1}{2}\left\{\lambda_1\, J + d_J\lambda_1 \otimes \gamma\overline{\eta} - d_J\lambda_2 \otimes \gamma\overline{p}\right\}. \tag{5.5}$$

Proof. The proof follows from Theorem 5.3, together with the fact that $\widetilde{\mathfrak{R}} = -\frac{1}{2}[\widetilde{h}, \widetilde{h}]$, and taking into account the proprties of the Frölicher-Nijenhuis bracket. □

The Berwald vertical counterpart is given by Lemma 3.1 and the Berwald horizontal counterpart is given by the following result.

Proposition 5.1. *Under the special anisotropic conformal change* (3.1), *the Berwald horizontal counterpart is given by*

$$\begin{aligned}\widetilde{D^{\circ}}_{\widetilde{\beta\overline{X}}}\overline{Y} &= D^{\circ}{}_{\beta\overline{X}}\overline{Y} - \frac{1}{2}\{\lambda_1 D^{\circ}_{\gamma\overline{X}}\overline{Y} + d_J\lambda_1(\beta\overline{X})\, D^{\circ}_{\gamma\overline{\eta}}\overline{Y} \\ &\quad - d_J\lambda_1(\beta\overline{X})\,\overline{Y} - d_J\lambda_1(\beta\overline{Y})\,\overline{X} - d_J\lambda_2(\beta\overline{X})\, D^{\circ}_{\gamma\overline{p}}\overline{Y}\} \\ &\quad + \frac{1}{2}\left\{dd_J\lambda_1(\gamma\overline{Y},\beta\overline{X})\,\overline{\eta} - dd_J\lambda_2(\gamma\overline{Y},\beta\overline{X}))\,\overline{p}\right\}.\end{aligned}$$

Proof. The proof follows from the fact that $v := \gamma \circ K$, $h := \beta \circ \rho$, $\gamma D^{\circ}{}_{hX}\overline{Y} := v[hX, JY]$ and $D^{\circ}{}_{\gamma\overline{X}}\rho Y := \rho[\gamma\overline{X}, \beta\overline{Y}]$ ([13, Proposition 4.4]), taking into account Theorem 5.4, and the facts that the map $\gamma : \pi^{-1}(TM) \to VTM$ is an isomorphism, the Berwlad (v)v-curvature $\widetilde{S}^{\circ} = 0$, $[JX, JY] = J[X, JY] + J[JX, Y]$, $vJ = J$ and $Jv = 0$. □

We now present an example of a conic Finsler metric that admits a concurrent π-vector field. Subsequently, we compute the corresponding π-form associated with this structure. Detailed calculations for this example are available in the accompanying PDF and Maple files, which can be accessed at: https://github.com/salahelgendi/Example1_Intrinsic_anisotropic_change.

Example 5.1. Let $M = \{(r, \theta, \phi, t) \in U \subset \mathbb{R}^4 \mid r,\ \theta \neq 0\}$ and L be a conic Finsler metric given by

$$L = \sqrt{r^2\left(\frac{\theta^2\dot{\phi}^2 + 2\dot{\theta}\dot{\phi}}{\dot{\theta}}\right)^2 + \dot{r}^2 + \dot{t}^2},$$

where $(r, \theta, \phi, t; \dot{r}, \dot{\theta}, \dot{\phi}, \dot{t}) \in \mathcal{T}U \subset \mathbb{R}^4 \times \mathbb{R}^4$.

The metric tensor has the following non-vanishing components g_{ij}:

$$g_{11} = 1, \qquad g_{22} = \frac{r^2\,\theta^2\,\dot{\phi}^3(3\theta^2\dot{\phi} + 4\dot{\theta})}{\dot{\theta}^4}, \quad g_{23} = -\frac{2\,r^2\,\theta^2\dot{\phi}^2(2\theta^2\dot{\phi} + 3\dot{\theta})}{\dot{\theta}^3},$$

$$g_{33} = \frac{2\,r^2\,(3\theta^4\dot{\phi}^2 + 6\theta^2\dot{\theta}\dot{\phi} + 2\dot{\theta}^2)}{\dot{\theta}^2}, \quad g_{44} = 1.$$

The inverse metric tensor has the following non-vanishing components g^{ij}:

$$g^{11} = 1, \quad g^{44} = 1, \quad g^{22} = \frac{\dot{\theta}^4\,(3\theta^4\dot{\phi}^2 + 6\theta^2\dot{\theta}\dot{\phi} + 2\dot{\theta}^2)}{r^2\theta^3\dot{\phi}^3},$$

$$g^{23} = \frac{\left(2\theta^2\dot{\phi} + 3\dot{\theta}\right)\dot{\theta}^3}{r^2\dot{\phi}\left(\theta^6\dot{\phi}^3 + 6\theta^4\dot{\theta}\dot{\phi}^2 + 12\theta^2\dot{\theta}^2\dot{\phi} + 8\dot{\theta}^3\right)},$$

$$g^{33} = \frac{1}{2}\frac{\left(3\theta^2\dot{\phi} + 4\dot{\theta}\right)\dot{\theta}^2}{r^2\left(\theta^6\dot{\phi}^3 + 6\theta^4\dot{\theta}\dot{\phi}^2 + 12\theta^2\dot{\theta}^2\dot{\phi} + 8\dot{\theta}^3\right)}.$$

The the Cartan tensor has the following non-vanishing components C_{ijk}:

$$C_{222} = -\frac{6\,r^2\,\theta^2\,\dot{\phi}^3(\theta^2\dot{\phi} + \dot{\theta})}{\dot{\theta}^5}, \; C_{223} = \frac{6\,r^2\,\theta^2\,\dot{\phi}^2(\theta^2\dot{\phi} + \dot{\theta})}{\dot{\theta}^4},$$

$$C_{233} = -\frac{6\,r^2\,\theta^2\,\dot{\phi}(\theta^2\dot{\phi}+\dot{\theta})}{\dot{\theta}^3},\ C_{333} = \frac{6\,r^2\,\theta^2\,(\theta^2\dot{\phi}+\dot{\theta})}{\dot{\theta}^2}.$$

The coefficients G^i of the geodesic spray are given by

$$G^1 = -\frac{r\,\dot{\phi}^2\left(\theta^4\dot{\phi}^2 + 4\theta^2\dot{\theta}\dot{\phi} + 4\dot{\theta}^2\right)}{2\dot{\theta}^2}, \quad G^2 = \frac{(\theta\,\dot{r} - r\,\dot{\theta})\,\dot{\theta}}{r\,\theta}, \quad G^3 = \frac{\dot{r}\,\dot{\phi}}{r}, \quad G^4 = 0.$$

By performing direct computations, or alternatively using the Finsler package [15], one can obtain the coefficients of the Cartan connection. For instance,

$$\Gamma^2_{12} = \frac{1}{r}, \quad \Gamma^3_{13} = \frac{1}{r}, \quad \Gamma^1_{11} = \Gamma^3_{33} = 0, \quad \Gamma^i_{4j} = 0.$$

It is evident that this metric supports a concurrent π-vector field of the form $\overline{p} = p^i\,\overline{\partial_i}$, where $\overline{\partial_i}$ are the local basis vectors of the fibers of $\pi^{-1}(TM)$, with components defined by $p^2(x) = p^3(x) = 0$ and $p^1(x) = r, p^4(x) = t$. In this case, it follows that $p^iC_{ijk} = 0$, and for instance, we compute

$$p^i_{|j} = \delta_j p^i + p^h\Gamma^i_{hj} = \delta_j p^i + p^1\Gamma^i_{1j} + p^4\Gamma^i_{4j}.$$

$$p^1_{|1} = \delta_1 p^1 + p^1\Gamma^1_{11} = 1, \quad p^2_{|2} = \delta_2 p^2 + p^1\Gamma^2_{12} = 1, \quad p^3_{|3} = \delta_3 p^3 + p^1\Gamma^3_{13} = 1.$$

While all other components of $p^i_{|j}$ vanish. In addition, the corresponding π-form $\mathbf{B}$ has components $\mathbf{B}^2 = \mathbf{B}^3 = 0$, $\mathbf{B}^1 = r$, $\mathbf{B}^4 = t$, which implies the associated 1-form is given by $\mathfrak{B} = r\dot{r} + t\dot{t}$.

Therefore, we have

$$\widetilde{L}(x,\dot{x}) = L(x,\dot{x})\,e^{\frac{\mathfrak{B}(x,\dot{x})}{L(x,\dot{x})}} = \sqrt{r^2\left(\frac{\theta^2\dot{\phi}^2 + 2\dot{\theta}\dot{\phi}}{\dot{\theta}}\right)^2 + \dot{r}^2 + \dot{t}^2}\;e^{\frac{r\dot{r}+t\dot{t}}{\sqrt{r^2\left(\frac{\theta^2\dot{\phi}^2+2\dot{\theta}\dot{\phi}}{\dot{\theta}}\right)^2+\dot{r}^2+\dot{t}^2}}}.$$

Hence, the given Finsler structure $\widetilde{L}$ defines a conic Finsler structure over M, under the condition of non-degeneracy defined by (4.1).

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