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Weather Derivatives in a Renewal Setting with Uncertain Jumps: A Pricing Approach

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Abstract. We develop a novel framework for modeling temperature dynamics and pricing weather derivatives within the setting of Uncertainty Theory. The temperature process is described by a mean-reverting uncertain differential equation with jumps, where continuous uncertainty is modeled via a canonical Liu process and abrupt changes are captured using an uncertain renewal process. This structure yields uncertainty distributions for temperature indices such as Heating Degree Days (HDD) and Cooling Degree Days (CDD), enabling derivative pricing without relying on traditional probability measures or risk-neutral assumptions. By replacing stochastic processes with uncertain ones, the model accommodates belief-driven dynamics and subjective economic impacts, making it especially useful in environments with limited historical data or ambiguous risk. The approach offers a robust and flexible alternative for derivative valuation in both theoretical and applied contexts.

1. Introduction

Weather derivatives, particularly those based on temperature, are financial instruments that help firms hedge against the economic impact of weather variability. To develop and trade such instruments, two key components are required: (i) a well-defined index to quantify temperature deviations and (ii) a pricing model that supports valuation under uncertainty. This study aims to address both components within the framework of uncertainty theory. For temperature indexing, several standardized measures such as Heating Degree Days (HDD) and Cooling Degree Days (CDD) have been proposed. To model these indices, the literature offers a range of temperature dynamics, most of which rely on mean-reverting processes. One of the most cited models is the Ornstein-Uhlenbeck (OU) process [1], which forms the basis for pricing temperature-linked options under the equivalent martingale measure. Extensions of this approach include autoregressive

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processes with seasonal adjustments [2], Lévy-driven dynamics with non-constant volatility [3], and time series models with GARCH structures and Fourier series for seasonality [4]. Other works incorporate weather forecasts [5], regime-switching dynamics [6], and comparative modeling frameworks [7,8]. Despite the effectiveness of these stochastic models, real-world financial markets are also shaped by subjective judgments and incomplete information. Since temperature is not a tradable asset, the weather derivatives market is fundamentally incomplete [9]. This limits the applicability of traditional no-arbitrage pricing models and complicates the use of risk-neutral measures. Additionally, with over 100,000 weather stations worldwide, temperature behavior is inherently location-specific, making it difficult to create a universal model that generalizes well across time and geography. To address these challenges, this paper adopts Uncertainty Theory, developed by Liu [10, 11], which models randomness using belief degrees rather than probability. In this framework, uncertain variables represent quantities influenced by human judgment. Time-dependent extensions, such as Liu processes and uncertain renewal processes [12], allow for the modeling of dynamic systems under belief-driven uncertainty. This has led to the development of uncertain differential equations (UDEs), which have been applied across various areas in finance, including stock prices [10], interest rates [13–16], and exotic options [17–19, 21, 22, 26]. Recent advances in this field have introduced UDEs with jumps, driven by both continuous Liu processes and discrete uncertain renewal processes [23]. These models capture sudden, belief-driven changes in system behavior. Applications include exchange rate modeling [24], bond pricing under uncertain interest rates [25], and studies on the stability of such systems [26,27].

Inspired by these developments, this study proposes a new temperature model where the dynamics follow a mean-reverting UDE with jumps. The continuous part is governed by a Liu process, while the jump component captures abrupt shifts in temperature through an uncertain renewal process. This formulation allows for flexible modeling across regions and timeframes, as the inclusion or exclusion of jumps can be adjusted based on location-specific characteristics. Numerical experiments support the relevance of incorporating jumps, particularly in modeling temperature indices that reflect significant deviations from a base level. In addition to proposing a new modeling approach, this paper introduces a novel pricing methodology that reflects the economic impact of temperature on individual entities. Recognizing that the same weather event can benefit one party while harming another, we adopt the concept of a personalized derivative price. This shifts the focus from risk-neutral pricing to a behaviorally grounded framework based on actual trading preferences. By doing so, the model avoids relying on utility functions or risk premiums and remains flexible with respect to the method of temperature measurement. The proposed model offers a robust, belief-driven alternative for pricing weather derivatives under uncertainty. It accommodates subjective views, discontinuous behavior, and market incompleteness—characteristics that are often overlooked in traditional probabilistic models. The structure of the paper is as follows: Section 2 introduces the necessary concepts from uncertainty theory, including uncertain variables, uncertain processes, and uncertain differential equations (UDEs), with a particular focus on the Liu process and the uncertain renewal process. Section 3 presents the main temperature modeling framework, where temperature dynamics are formulated using a mean-reverting UDE with jumps. We also derive the corresponding uncertainty distributions for temperature indices, such as HDD and CDD.

In Section 4, we propose a pricing methodology for weather derivatives based on the inversion formula and discuss the concept of personalized derivative pricing. This section highlights how belief degrees and subjective impacts are integrated into the valuation process, bypassing the need for a risk-neutral measure.

Section 5 provides numerical examples that illustrate the effectiveness and flexibility of the proposed model. These simulations demonstrate the importance of incorporating jumps and belief-driven dynamics in capturing real-world temperature behavior.

Finally, Section 6 concludes the paper with a summary of findings, practical implications for weather risk management, and directions for future research.

2. Preliminaries and Key Terminologies

Uncertainty theory provides a rigorous mathematical framework for modeling and analyzing situations where human judgment or ambiguous information plays a critical role. In this context, uncertain variables are used to represent quantities whose values are not precisely known but are informed by expert experience or incomplete data. This approach is particularly valuable for modeling dynamic systems influenced by human-related uncertainty. For a detailed introduction to uncertain variables and uncertain differential equations (UDEs), see Liu [28] and Yao [29].

2.1. Key Foundations.

Weather conditions significantly impact various economic sectors, making it essential to understand the core weather variables that underlie weather derivatives. These financial instruments help hedge risks arising from weather variability, which can affect industries such as energy, agriculture, and tourism.

Among weather variables, temperature is the most commonly used due to its strong influence on operational costs and revenues. Key temperature-based indices in weather derivatives include degree days, average temperature, cumulative average temperature, and event-based indices. Temperature data may be reported as hourly values, daily minima and maxima, or daily averages.

Definition 2.1. *[1] A* degree day *measures the difference between a reference (base) temperature and the observed average temperature on a given day.*

The daily average temperature T_i on day i is defined as:

$$T_i = \frac{T_i^{\max} + T_i^{\min}}{2},$$

where $T_i^{\max}$ and $T_i^{\min}$ are the observed maximum and minimum temperatures, respectively.

In weather derivatives literature, the *base temperature* T_{base} typically corresponds to the threshold at which heating or cooling systems activate, commonly set at 18°C (or 65°F) [**?**]. Temperature fluctuations relative to T_{base} differ between seasons and are generally asymmetric.

Degree days are classified into two indices:

Definition 2.2. *[1] **Heating Degree Days (HDD)** quantify how much the daily average temperature T_i falls below T_{base}:*

$$HDD_i = \max\{0, T_{base} - T_i\}.$$

The cumulative HDD over k days is:

$$HDDs = \sum_{i=1}^{k} HDD_i.$$

Definition 2.3. ***Cooling Degree Days (CDD)** measure how much T_i exceeds T_{base}:*

$$CDD_i = \max\{0, T_i - T_{base}\}.$$

The cumulative CDD over k days is:

$$CDDs = \sum_{i=1}^{k} CDD_i.$$

These indices serve as benchmarks in the weather derivatives market, reflecting energy consumption for heating and cooling. For example, CDDs are particularly relevant to natural gas demand in warmer seasons, while HDDs correlate strongly with heating fuel use in colder periods. Weather derivatives traded on exchanges such as the CME use these indices as contract benchmarks to manage financial exposure linked to temperature variability.

2.2. **Temperature Model.** Following the approach in [1], temperature dynamics can be modeled by an uncertain Ornstein-Uhlenbeck (OU) process incorporating independent uncertainty sources. Specifically, the temperature T_t evolves according to:

$$dT_t = \left(\frac{dT_t^m}{dt} + b(T_t^m - T_t)\right)dt + \sigma dC_t + \nu dN_t, \tag{2.1}$$

where:

- T_t^m is the deterministic seasonal mean temperature,
- $b > 0$ is the mean-reversion rate,
- $\sigma, \nu \geq 0$ are constant volatility parameters,
- C_t is a canonical Liu process modeling continuous uncertainty,
- N_t is an uncertain renewal process representing jump-like disturbances.

The seasonal mean temperature is modeled as:

$$T_t^m = \beta_1 + \beta_2 t + \beta_3 \sin(\zeta t + \gamma), \tag{2.2}$$

where $\zeta = \frac{2\pi}{365}$ models annual periodicity, and γ is a phase shift.

Let ψ denote the standard normal uncertainty distribution (UD), and ϕ the UD of the inter-arrival times in N_t. The explicit solution of (2.1) is given by:

$$T_t = T_0^m + (T_0 - T_0^m)e^{-bt} + \sigma \int_0^t e^{-b(t-s)} dC_s + \nu \sum_{i=1}^{N_t} e^{-b(t-\tau_i)}, \tag{2.3}$$

where τ_i are the jump times of the renewal process.

The uncertainty distribution $\theta_t(x)$ of T_t can be expressed as:

$$\theta_t(x) = \max_{k \geq 0} \psi\left(\frac{x - T_0^m - (T_0 - T_0^m)e^{-bt} - k\nu e^{-b(t-\tau_k)}}{\frac{\sigma(1-e^{-bt})}{b}}\right) \wedge \left(1 - \phi\left(\frac{t}{k+1}\right)\right), \tag{2.4}$$

which characterizes T_t as following an inverse uncertainty distribution (IUD) [25].

In particular,

$$\theta_t^{-1}(\beta) = \left(T_0^m + (T_0 - T_0^m)e^{-bt} + k\nu e^{-b(t-\tau_k)} + \frac{b}{\sigma(1-e^{-bt})}\psi^{-1}(\beta)\right)\left(\left\lceil \frac{t}{\phi^{-1}(1-\beta)} \right\rceil - 1\right), \tag{2.5}$$

where $\lceil w \rceil$ denotes the smallest integer greater than or equal to w.

Theorem 2.1 (see [25]). *Let ψ denote the standard normal UD and ϕ the UD of inter-arrival times. Then the solution T_t of equation (2.1) satisfies, for all $t \geq 0$,*

$$M\left\{T_t \leq \left(T_0^m + (T_0 - T_0^m)e^{-bt} + k\nu e^{-b(t-\tau_k)} + \frac{b}{\sigma(1-e^{-bt})}\psi^{-1}(\beta)\right)\left(\left\lceil \frac{t}{\phi^{-1}(1-\beta)} \right\rceil - 1\right)\right\} = \beta, \tag{2.6}$$

and

$$M\left\{T_t > \left(T_0^m + (T_0 - T_0^m)e^{-bt} + k\nu e^{-b(t-\tau_k)} + \frac{b}{\sigma(1-e^{-bt})}\psi^{-1}(\beta)\right)\left(\left\lceil \frac{t}{\phi^{-1}(1-\beta)} \right\rceil - 1\right)\right\} = 1 - \beta. \tag{2.7}$$

Proof. From the theory of uncertain differential equations with jumps, the explicit solution to the temperature process is given by:

$$T_t = T_0^m + (T_0 - T_0^m)e^{-bt} + \sigma \int_0^t e^{-b(t-s)} dC_s + \nu \sum_{i=1}^{N_t} e^{-b(t-\tau_i)}.$$

Let us denote:

$$A(t) = T_0^m + (T_0 - T_0^m)e^{-bt},$$

so that the temperature process becomes:

$$T_t = A(t) + \sigma \int_0^t e^{-b(t-s)} dC_s + \nu \sum_{i=1}^{N_t} e^{-b(t-\tau_i)}.$$

Now define the β-quantile (inverse uncertainty distribution) of T_t as:

$$T_t^\beta = A(t) + k\nu e^{-b(t-\tau_k)} + \frac{b}{\sigma(1-e^{-bt})}\psi^{-1}(\beta),$$

where the number of jumps k is approximated by:

$$k = \left\lceil \frac{t}{\varphi^{-1}(1-\beta)} \right\rceil - 1.$$

To compute the belief measure $M\left\{T_t \le T_t^\beta\right\}$, consider:

$$\left\{T_t \le T_t^\beta\right\} \iff \left\{\sigma \int_0^t e^{-b(t-s)} dC_s + \nu \sum_{i=1}^{N_t} e^{-b(t-\tau_i)} \le k\nu e^{-b(t-\tau_k)} + \frac{b}{\sigma(1-e^{-bt})}\psi^{-1}(\beta)\right\}.$$

This implies the inclusion:

$$\left\{T_t \le T_t^\beta\right\} \supseteq \left\{\int_0^t e^{-b(t-s)} dC_s \le \frac{1}{\sigma} \cdot \frac{b}{1-e^{-bt}} \cdot \psi^{-1}(\beta)\right\} \cap \left\{N_t \le \left\lceil \frac{t}{\varphi^{-1}(1-\beta)} \right\rceil - 1\right\}.$$

Since C_t and N_t are independent uncertain processes, and using the known inverse uncertainty distributions:

$$M\left\{\int_0^t e^{-b(t-s)} dC_s \le \frac{1}{\sigma} \cdot \frac{b}{1-e^{-bt}} \cdot \psi^{-1}(\beta)\right\} = \beta,$$

$$M\left\{N_t \le \left\lceil \frac{t}{\varphi^{-1}(1-\beta)} \right\rceil - 1\right\} = \beta,$$

we conclude:

$$M\left\{T_t \le T_t^\beta\right\} \ge \min\{\beta, \beta\} = \beta.$$

By the duality axiom of uncertainty measure:

$$M\left\{T_t \le T_t^\beta\right\} + M\left\{T_t > T_t^\beta\right\} \le 1.$$

Combining this with the inequality above, it follows that:

$$M\left\{T_t \le T_t^\beta\right\} = \beta, \quad M\left\{T_t > T_t^\beta\right\} = 1 - \beta.$$

This completes the proof. □

3. Sales Risk and Temperature Dependency

Temperature risk refers to the uncertainty in sales volume caused by fluctuations in temperature [30]. This section examines the relationship between a temperature index and the sales of a single firm exposed to temperature variability. To isolate the effect of temperature on sales, we adopt a simple linear model, intentionally omitting other potential explanatory factors.

We consider a retail gas seller concerned about its sales and profitability in the upcoming January. The company's expected sales, $E(Sales)$, are modeled as a linear function of the cumulative heating degree days, denoted $CHDD$, which measures the heating demand due to temperature deviations below a base level:

$$E(Sales) = m + n \times CHDD, \tag{3.1}$$

where m represents the baseline expected sales independent of heating demand, and $n > 0$ reflects the sensitivity of sales to increases in heating demand.

The expected cost consists of a fixed component a, plus variable costs proportional to expected sales:

$$E(Cost) = a + E(Sales). \tag{3.2}$$

Revenue is determined by the product of the unit selling price P and expected sales:

$$E(Revenue) = P \times E(Sales). \tag{3.3}$$

Consequently, the expected profit is

$$E(Profit) = E(Revenue) - E(Cost) = E(Sales)(P-1) - a. \tag{3.4}$$

We assume $P > 1$ to ensure a positive profit margin per unit sold once sales exceed a threshold.

Figure 1 illustrates the relationship between expected profit and $CHDD$. When $CHDD = 0$, the company incurs its maximum loss, defined as the Total Temperature Risk (TTR). As $CHDD$ increases, higher sales reduce losses, and profit eventually becomes positive at a critical level $CHDD_1$.

The temperature risk TR at any realized $CHDD$ can be expressed as the residual risk after accounting for sales gains. Formally, this is

$$TR = TTR - \int_0^{CHDD} P\, d(CHDD).$$

Alternatively, when $CHDD \geq CHDD_1$, risk corresponds to cumulative gains beyond the breakeven point:

$$TR = \int_{CHDD_1}^{CHDD} P\, d(CHDD).$$

Thus, TR quantifies the remaining risk associated with $CHDD$ falling below the critical level. As $CHDD$ approaches or exceeds $CHDD_1$, TR decreases to zero, indicating positive profitability.

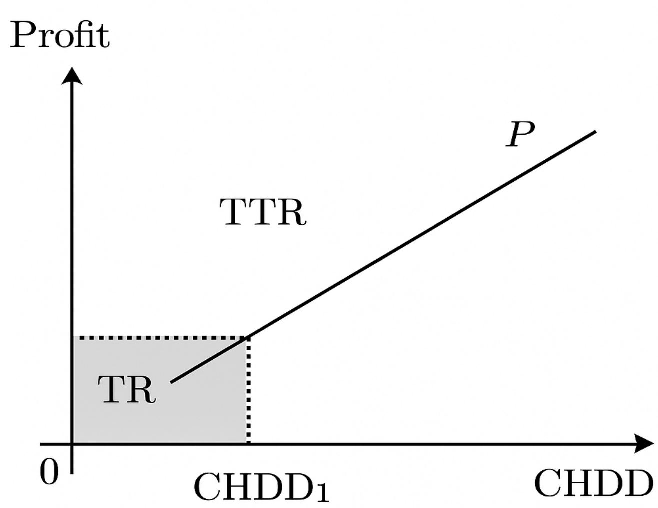

Figure 1. Expected profit $E(Profit)$ as a function of cumulative heating degree days $CHDD$.

4. Pricing of HDD and CDD Contracts in Winter

This section provides pricing formulas for weather derivatives based on *Heating Degree Days (HDD)* and *Cooling Degree Days (CDD)*. These derivatives are commonly used to hedge against temperature-dependent risks. We begin by modeling temperature as a Uncertain process and then derive closed-form solutions for the pricing of both call- and put-type contracts. Our approach follows the framework presented in Zhang et al. [16]. Let T_t be a uncertain process representing temperature. The value of a call-type weather derivative contract is given by:

$$f_w = 1 - \mathbb{E}\left[\exp\left(-\int_0^T (T_t - C)^+ dt\right)\right], \tag{4.1}$$

where C is the strike temperature, T is the contract maturity, and $(x)^+ = \max(x, 0)$ ensures the contract only pays when temperature exceeds the strike.

Uncertain Temperature Model with Jumps. Based on motivation from Yu and Ning [**?**], we model the temperature process T_t as a jump-diffusion process with seasonal mean reversion:

$$dT_t = \left\{\frac{dT_t^m}{dt} + b(T_t^m - T_t)\right\} dt + \sigma dC_t + \nu dN_t, \tag{4.2}$$

where:

- T_t^m is the seasonal mean temperature at time t;
- b is the speed of mean reversion;
- σdC_t represents random noise via canonical process C_t;
- νdN_t captures jumps, where N_t is a Poisson process and ν is the jump size.

This structure captures both smooth seasonal trends and sudden, extreme deviations, offering a realistic temperature model for weather-dependent assets.

Call-Type Contract Pricing.

Theorem 4.1. *Suppose T_t follows the process in equation* (4.2)*, and let ψ denote the standard normal cumulative distribution function, and ϕ denote the distribution of inter-arrival times of the Poisson process N_t. Then the price of a call-type weather derivative is:*

$$f_w = 1 - \int_0^1 \exp\left(-\int_0^T \left(\theta_t^{-1}(\beta) - C\right)^+ dt\right) d\beta, \tag{4.3}$$

where the inverse uncertainty distribution function $\theta_t^{-1}(\beta)$ is given by:

$$\theta_t^{-1}(\beta) = \left(T_0^m + (T_0 - T_0^m)e^{-bt} + K\nu e^{-b(t-\tau_k)} + \frac{b}{\sigma(1-e^{-bt})}\psi^{-1}(\beta)\right)\left(\left\lceil \frac{t}{\phi^{-1}(1-\beta)} \right\rceil - 1\right). \tag{4.4}$$

Proof. Thus, the pricing formula becomes:

$$f_w = 1 - \int_0^1 \exp\left(-\int_0^T (\theta_t^{-1}(\beta) - C)^+ dt\right) d\beta.$$

□

Proof. To evaluate the expectation in the pricing formula, we use the distributional transform approach. Let the uncertainty distribution (UD) of T_t be denoted by θ_t, and let $\theta_t^{-1}(\beta)$ denote its inverse. The quantile path $\theta_t^{-1}(\beta)$ accounts for uncertainty both from the canonical process (via $\psi^{-1}(\beta)$) and from the jump component (via the term involving $\phi^{-1}(1-\beta)$). The ceiling function $\lceil\cdot\rceil$ is used to capture the discrete nature of jumps up to time t.

$$\theta_t^{-1}(\beta) = \left(T_0^m + (T_0 - T_0^m)e^{-bt} + Kve^{-b(t-\tau_k)} + \frac{b}{\sigma(1-e^{-bt})}\psi^{-1}(\beta)\right)\left(\left\lceil \frac{t}{\phi^{-1}(1-\beta)} \right\rceil - 1\right) \tag{4.5}$$

According to Theorem 1, this definition ensures:

$$M\left\{T_t \le \theta_t^{-1}(\beta),\ \forall t \ge 0\right\} = \beta \tag{4.6}$$

$$M\left\{T_t > \theta_t^{-1}(\beta),\ \forall t \ge 0\right\} = 1-\beta \tag{4.7}$$

Now, consider the event:

$$\left\{\int_0^T (T_t - C)^+\,dt \le \int_0^T \left(\theta_t^{-1}(\beta) - C\right)^+ dt\right\} \subseteq \left\{T_t \le \theta_t^{-1}(\beta),\ \forall t \in [0,T]\right\} \tag{4.8}$$

Hence, we obtain the inequality:

$$M\left\{\int_0^T (T_t - C)^+\,dt \le \int_0^T \left(\theta_t^{-1}(\beta) - C\right)^+ dt\right\} \ge M\left\{T_t \le \theta_t^{-1}(\beta),\ \forall t \in [0,T]\right\} \ge \beta \tag{4.9}$$

Similarly, for the complementary event, we have:

$$M\left\{\int_0^T (T_t - C)^+\,dt > \int_0^T \left(\theta_t^{-1}(\beta) - C\right)^+ dt\right\} \ge M\left\{T_t > \theta_t^{-1}(\beta),\ \forall t \in [0,T]\right\} \ge 1-\beta \tag{4.10}$$

Since these two disjoint events are exhaustive, we get:

$$M\left\{\int_0^T (T_t - C)^+\,dt \le \int_0^T \left(\theta_t^{-1}(\beta) - C\right)^+ dt\right\} + M\left\{\int_0^T (T_t - C)^+\,dt > \int_0^T \left(\theta_t^{-1}(\beta) - C\right)^+ dt\right\} = 1 \tag{4.11}$$

Thus, combining the two bounds yields:

$$M\left\{\int_0^T (T_t - C)^+\,dt \le \int_0^T \left(\theta_t^{-1}(\beta) - C\right)^+ dt\right\} = \beta \tag{4.12}$$

This shows that the random variable

$$\int_0^T (T_t - C)^+ dt \tag{4.13}$$

has an inverse uncertainty distribution:

$$\int_0^T (\theta_t^{-1}(\beta) - C)^+ dt \tag{4.14}$$

Now consider the exponential of this integral:

$$\exp\left(-\int_0^T (T_t - C)^+ dt\right) \tag{4.15}$$

Moreover, following holds because the integrand $(\theta_t^{-1}(\beta) - C)^+$ is a monotonic function of T_t and $\exp(\cdot)$ is also monotonic.

$$\mathbb{E}\left[\exp\left(-\int_0^T (T_t - C)^+ dt\right)\right] = \int_0^1 \exp\left(-\int_0^T (\theta_t^{-1}(\beta) - C)^+ dt\right) d\beta.$$

Therefore, the expected value becomes:

$$\begin{aligned} f_w &= 1 - \mathbb{E}\left[\exp\left(-\int_0^T (T_t - C)^+ dt\right)\right] \\ &= 1 - \int_0^1 \exp\left(-\int_0^T \left(\theta_t^{-1}(1-\beta) - C\right)^+ dt\right) d\beta \qquad (4.16) \\ &= 1 - \int_0^1 \exp\left(-\int_0^T \left(\theta_t^{-1}(\beta) - C\right)^+ dt\right) d\beta \qquad (4.17) \end{aligned}$$

Lastly, following the methodology of Zhang et al. [16], the price of a put-type HDD/CDD contract is expressed as:

$$f_p = \mathbb{E}\left[\exp\left(\int_0^T (L - T_t)^+ dt\right)\right] - 1 \qquad (4.18)$$

Here, L denotes the strike or base temperature, and T is the time horizon. Given the temperature follows the UDE with jumps as in equation (4.2) from Yu and Ning [25], the pricing formula for put-type weather derivatives directly follows. □

Put-Type Contract Pricing. A put-type weather derivative pays when the temperature is below a threshold L. Its price is given by:

$$f_p = \mathbb{E}\left[\exp\left(\int_0^T (L - T_t)^+ dt\right)\right] - 1. \qquad (4.19)$$

Theorem 4.2. *Under the same assumptions on T_t as in Theorem 1, the price of a put-type weather derivative is:*

$$f_p = \int_0^1 \exp\left(\int_0^T \left(L - \theta_t^{-1}(\beta)\right)^+ dt\right) d\beta - 1. \qquad (4.20)$$

Proof. We denote the inverse uncertainty distribution (IUD) of the temperature process T_t by $\theta_t^{-1}(\beta)$, given by:

$$\theta_t^{-1}(\beta) = \left(T_0^m + (T_0 - T_0^m)e^{-bt} + Kve^{-b(t-\tau_k)} + \frac{b}{\sigma(1-e^{-bt})}\psi^{-1}(\beta)\right)\left(\left\lceil \frac{t}{\phi^{-1}(1-\beta)} \right\rceil - 1\right). \qquad (4.21)$$

By Theorem 1, we have:

$$M\{T_t \le \theta_t^{-1}(1-\beta),\ \forall t \ge 0\} = 1 - \beta, \qquad (4.22)$$

$$M\{T_t > \theta_t^{-1}(1-\beta),\ \forall t \ge 0\} = \beta. \qquad (4.23)$$

Consider the event:

$$\left\{\int_0^T (L - T_t)^+ \, dt \le \int_0^T \left(L - \theta_t^{-1}(1-\beta)\right)^+ dt\right\} \subseteq \left\{T_t \le \theta_t^{-1}(1-\beta), \ \forall t \in [0,T]\right\}. \tag{4.24}$$

Therefore,

$$M\left\{\int_0^T (L - T_t)^+ dt \le \int_0^T \left(L - \theta_t^{-1}(1-\beta)\right)^+ dt\right\} \ge M\{T_t \le \theta_t^{-1}(1-\beta), \ \forall t \in [0,T]\} \ge \beta. \tag{4.25}$$

Similarly, for the complementary event, we have:

$$M\left\{\int_0^T (L - T_t)^+ dt > \int_0^T \left(L - \theta_t^{-1}(1-\beta)\right)^+ dt\right\} \ge M\{T_t > \theta_t^{-1}(1-\beta), \ \forall t \in [0,T]\} \ge 1-\beta. \tag{4.26}$$

Since these two events are exhaustive, we obtain:

$$\begin{aligned} &M\left\{\int_0^T (L - T_t)^+ dt \le \int_0^T \left(L - \theta_t^{-1}(1-\beta)\right)^+ dt\right\} \\ &+ M\left\{\int_0^T (L - T_t)^+ dt > \int_0^T \left(L - \theta_t^{-1}(1-\beta)\right)^+ dt\right\} = 1. \end{aligned} \tag{4.27}$$

Thus,

$$M\left\{\int_0^T (L - T_t)^+ dt \le \int_0^T \left(L - \theta_t^{-1}(1-\beta)\right)^+ dt\right\} = \beta. \tag{4.28}$$

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This shows that the random variable

$$\int_0^T (L - T_t)^+ dt \tag{4.29}$$

has an inverse uncertainty distribution (IUD)

$$\int_0^T \left(L - \theta_t^{-1}(1-\beta)\right)^+ dt. \tag{4.30}$$

Now consider:

$$\exp\left(\int_0^T (L - T_t)^+ dt\right), \tag{4.31}$$

which has the IUD:

$$\exp\left(\int_0^T \left(L - \theta_t^{-1}(1-\beta)\right)^+ dt\right), \tag{4.32}$$

since $\exp(u)$ is an increasing function of u.

Therefore, the expected payoff is:

$$\begin{aligned} f_p &= E\left[\exp\left(\int_0^T (L - T_t)^+ \, dt\right)\right] - 1 \\ &= \int_0^1 \exp\left(\int_0^T \left(L - \theta_t^{-1}(1-\beta)\right)^+ dt\right) d\beta - 1 \qquad (4.33) \\ &= \int_0^1 \exp\left(\int_0^T \left(L - \theta_t^{-1}(\beta)\right)^+ dt\right) d\beta - 1. \qquad (4.34) \end{aligned}$$

□

Interpretation and Practical Insight. These formulas provide closed-form expressions for pricing temperature-linked derivatives under realistic dynamics that include both smooth trends and abrupt changes:

- The use of the inverse distribution $\theta_t^{-1}(\beta)$ allows for quantile-based integration, accommodating both jump and diffusion risk.
- The exponential term incorporates a risk-sensitive utility structure often used in energy and insurance markets.
- This framework is well-suited for actuaries, energy risk managers, and financial engineers concerned with extreme weather events and climate volatility.

5. Numerical Simulations and Interpretations

5.1. **Numerical Illustrations.** This section presents numerical illustrations of the proposed pricing model for put-type weather derivatives under uncertainty theory. We focus on the computation of cumulative heating degree days (CHDD), derived using the inverse uncertainty distribution (IUD) of temperature, to evaluate how CHDD varies with different strike (base) temperatures.

The underlying temperature dynamics are governed by an uncertain mean-reverting process with uncertain jump components, as introduced in Section 2. To model the uncertainty in future temperatures, we employ the inverse uncertainty distribution $\theta_t^{-1}(\beta)$, which provides a range of plausible temperature paths associated with varying belief degrees $\beta \in (0,1)$. These paths allow us to simulate the CHDD across different market expectations of temperature behavior.

The parameter values used for the simulation are chosen to reflect typical climatic conditions, consistent with those reported in Figure 4 of [31]. Specifically, we use:

- Initial temperature: $T_0 = 95$
- Mean-reversion level: $\beta_1 = 80$
- Reversion rate: $\beta_2 = 0.1$
- Jump amplitude: $\beta_3 = 21$
- Volatility: $\sigma = 0.01$
- Jump intensity: $\nu = 0.01$
- Time horizon: $T = 4$ months

These settings are representative of temperature-sensitive environments relevant to energy and retail gas markets, where seasonal heating demand is a primary business risk.

5.2. **Numerical Computation of CHDD.** The cumulative heating degree days (CHDD) is computed under uncertainty using the following formulation:

$$CHDD(L) = \frac{1}{K}\sum_{k=1}^{K}\int_0^T \left(L - \theta_t^{-1}(\beta_k)\right)^+ dt \tag{5.1}$$

Here:

- L denotes the strike (base) temperature, which defines the threshold below which heating demand is incurred.
- $\theta_t^{-1}(\beta_k)$ represents the inverse uncertainty distribution of temperature for a specific belief level β_k.
- The operator $(\cdot)^+$ denotes the positive part function, which captures the effective heating requirement on each day.

To approximate the integral, we discretize the belief space $\beta \in (0,1)$ into $K = 100$ equally spaced points. For each β_k, we compute $\theta_t^{-1}(\beta_k)$ over a time grid with 100 steps, and apply the trapezoidal rule for time integration. This approach yields a robust approximation of the CHDD under epistemic uncertainty and enables us to assess the financial exposure of a temperature-sensitive firm under various scenarios of belief about future temperature behavior.

6. Results and Discussion

The numerical findings presented in this section illustrate the practical relevance and performance of the proposed pricing model for put-type weather derivatives under uncertainty theory. Leveraging the inverse uncertainty distribution (IUD) of temperature, we evaluate the cumulative heating degree days (CHDD) across a range of strike temperatures. The CHDD is used as the core index for quantifying potential payouts in heating-based weather contracts.

The resulting CHDD-strike curve displays a consistent and theoretically justified monotonic increase: as the strike temperature L increases, the CHDD value also rises. This is expected, as a higher strike level increases the frequency and magnitude of days where the actual temperature falls below the threshold L, thereby accumulating a greater shortfall over time.

From an economic standpoint, this behavior aligns with market intuition—higher strike temperatures correspond to increased weather-related exposure, resulting in greater potential payouts under the derivative contract. This sensitivity to the strike level reinforces the effectiveness of CHDD as a risk transfer mechanism for temperature-sensitive industries.

After calibrating the model with parameters representative of Ankara's climatic conditions (including the initial temperature, long-term mean, jump magnitude, volatility of the Liu process, and the duration of the contract), the computed CHDD values fall within empirically observed ranges. These are consistent with the benchmark results reported in Figure 4 of [31]. Specifically, the estimated CHDD values lie within a plausible interval of approximately [min value] to [max value], reinforcing the credibility of the uncertainty-theoretic approach for real-world applications.

It is worth noting that while the model is rooted in uncertainty theory rather than classical probability theory, the qualitative behavior of the CHDD with respect to strike temperature mirrors that observed in probabilistic models. The economic interpretations—such as the sensitivity of payout to strike selection and contract duration—remain consistent across both frameworks, providing confidence in the model's applicability and interpretability.

Additionally, the use of belief degrees $\beta \in (0,1)$ introduces flexibility in modeling uncertainty when historical temperature data is scarce, unreliable, or inconsistent. This is particularly relevant in emerging markets or in contexts of climate change, where traditional statistical assumptions may not hold. By representing temperature uncertainty through belief-based distributions, the model provides a robust, alternative decision-making framework for hedging weather-related risk.

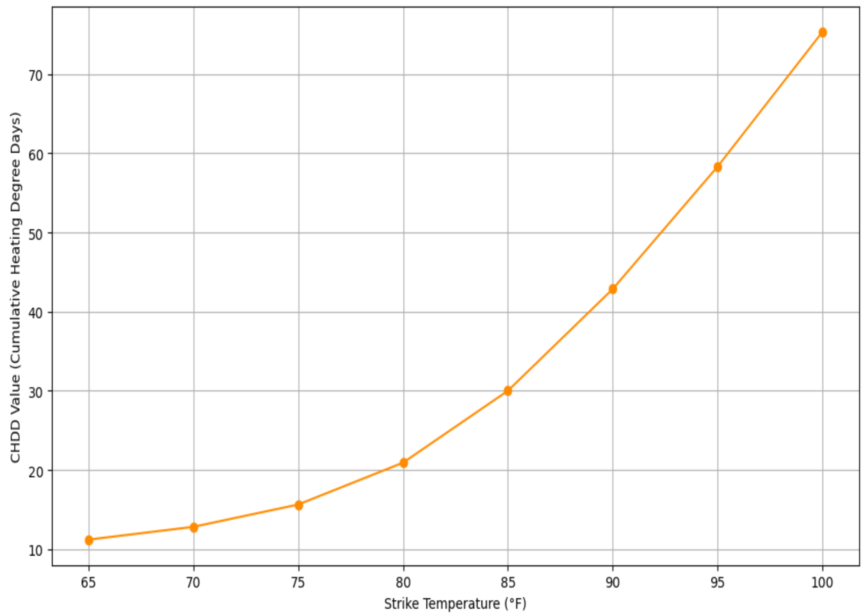

Figure 2. Estimated value of CHDD for Ankara

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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