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Generating Ruled Surfaces by Focal Curves and Their Characterizations in Minkowski 3-Space

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Abstract. The aim of this paper is to investigate distinct categories of ruled surfaces within three-dimensional Minkowski space. These surfaces are formed through the utilization of focal curves, generated by the Frenet vectors associated with these curves. The study aims to derive various geometric properties and significant findings concerning the curvatures of these surfaces. Furthermore, the paper includes computational examples that not only validate the theoretical results of the study but also provide visual representations through plots.

1. Introduction

The main aim of classical differential geometry is to understand the properties of different kinds of surfaces in three-dimensional Minkowski space $\mathbb{E}_1^3$, such as developable surfaces, ruled surfaces, minimal surfaces, and other similar surfaces. Ruled surfaces ($\mathcal{R}$-$\mathcal{S}$), parametrized by a one-dimensional family of straight lines, constitute a classical subject in differential geometry. Despite their historical roots, contemporary mathematicians are drawn to these surfaces, leading to a rich body of literature dedicated to their investigation. Beyond their historical context, ($\mathcal{R}$-$\mathcal{S}$) remain compelling due to their significant roles and applications in addressing design challenges within spatial mechanisms, physics, kinematics, and computer aided design (CAD).

Developable surfaces represent specific instances of ($\mathcal{R}$-$\mathcal{S}$). These surfaces exhibit a distinctive trait where the Gaussian curvature (GC) is consistently zero across the entire surface. Numerous investigations delve into the intriguing properties of these surfaces within both Euclidean and Minkowski spaces, offering various characterizations and insights (see [1–3]).

Several scholars have conducted research on ($\mathcal{R}$-$\mathcal{S}$) and their manifold properties. In [4–11], family

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of ($\mathcal{R}$-$\mathcal{S}$) and tube surfaces generated by various curves in both Euclidean and non-Euclidean spaces. It provides explicit formulas for the second mean curvature (*S-MC*) and the second Gaussian curvature (*S-GC*) of these ($\mathcal{R}$-$\mathcal{S}$). Special characteristics of these surfaces are elucidated, and the conditions under which they can be categorized as minimal, flat, II-minimal, and II-flat surfaces are determined. Furthermore, criteria for the base curve of these ($\mathcal{R}$-$\mathcal{S}$) to qualify as a geodesic curve, an asymptotic line, or a principal line are established

Li et al. [12–14] explored spacelike circular surfaces in $\mathbb{E}_1^3$, concentrating on their geometric and singularity properties. These surfaces were parametrized, and their Gaussian and mean curvatures were scrutinized, allowing for comparisons with ($\mathcal{R}$-$\mathcal{S}$) and an examination of singularities. Additionally, they derived conditions for spacelike roller coaster surfaces to exhibit flatness or minimality. Their findings were reinforced with illustrative examples, illuminating the intricate characteristics of spacelike circular surfaces in $\mathbb{E}_1^3$. The study delved into the singularities of non-developable ($\mathcal{R}$-$\mathcal{S}$) with spacelike ruling, employing the classical unfolding theorem in singularity theory. Their primary objective lay in comprehending the parameter-dependent aspects of mathematical objects, with a particular focus on the spherical indicatrix and evolute of spacelike ($\mathcal{R}$-$\mathcal{S}$).

Talat et al. [15–17] studied focal curves, associated with a given curve in a space, are typically characterized by properties related to nearby curve behavior. In differential geometry, they represent special curves offering insights into the original curve's geometric properties, like curvature or torsion. These curves play a significant role in understanding the local behavior and curvature properties of the original curve. In a study on focal curves in Minkowski 3-space focusing on the Darboux frame, integral equations were introduced as characterizations for space curves to be focal curves. They provided key results, including expressions for the focal curvatures and insights into their geometric properties, contributing to a better understanding and characterization of focal curves in $\mathbb{E}_1^3$.

Dillen et al. [18–21] discussed the properties and characteristics of timelike ($\mathcal{R}$-$\mathcal{S}$) in $\mathbb{E}_1^3$. The main results of the papers included the classification of timelike ($\mathcal{R}$-$\mathcal{S}$) based on their geometric properties, the determination of conditions for surfaces to be developable, and the analysis of the behavior of geodesics on these surfaces. Additionally, they explored the relationship between timelike ($\mathcal{R}$-$\mathcal{S}$) and other types of ($\mathcal{R}$-$\mathcal{S}$) in Minkowski 3-space and provided valuable insights into the study of timelike ($\mathcal{R}$-$\mathcal{S}$) and their applications in geometry and physics.

In our paper, we investigate the generation of ($\mathcal{R}$-$\mathcal{S}$) in Minkowski 3-space through the utilization of focal curves and Frenet vectors. Our study delves into the geometric properties of these surfaces, focusing on their curvatures and deriving key findings. By employing computational examples, we not only validate the theoretical results presented but also provide visual representations through plots. Through an exploration of the relationship between focal curves and ($\mathcal{R}$-$\mathcal{S}$), our paper offers valuable insights into the intricate characteristics of these surfaces and their applications in geometry and physics.

2. Preliminaries

In this section, we provide a concise overview of the geometry of ($\mathcal{R}$-$\mathcal{S}$) within the Minkowski 3-space, a necessary foundation for our investigation. The Minkowski 3-space is characterized by its natural Lorentz metric

$$\langle , \rangle = -du_1^2 + du_2^2 + du_3^2,$$

where (u_1, u_2, u_3) is an orthogonal coordinate system of $\mathbb{E}_1^3$. The vector $u = (u_1, u_2, u_3)$ within $\mathbb{E}_1^3$ can be categorized as spacelike if $\langle u, u\rangle > 0$ or $u = 0$, timelike if $\langle u, u\rangle < 0$, and lightlike (null) if $\langle u, u\rangle = 0$, with $u \neq 0$. Similarly, a parameterized curve $\beta(s) : I \subset \beta \longrightarrow \mathbb{E}_1^3$, where s is a pseudo arclength parameter, is termed spacelike if $\langle \beta'(s), \beta'(s)\rangle > 0$, timelike if $\langle \beta'(s), \beta'(s)\rangle < 0$, and lightlike if $\langle \beta'(s), \beta'(s)\rangle = 0$ or $\beta'(s) = 0$ for all $s \in I$. The vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3) \in \mathbb{E}_1^3$ are orthogonal if and only if $\langle u, v\rangle = 0$. Additionally, for any $u, v \in \mathbb{E}_1^3$, the Lorentzian cross product operation of u and v is defined by

$$u \times v = \begin{vmatrix} -e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

The norm of a vector $u \in \mathbb{E}_1^3$ is given by $\|u\| = \sqrt{|\langle u, u\rangle|}$.
Since $\beta(s)$ is a timelike curve, there exists the moving Serret-Frenet frame $\{e_1(s),\ e_2(s),\ e_3(s)\}$, where $e_1(s) = \beta^{'}(s)$ is the unit tangent, $e_2(s) = \beta^{''}(s)/\|\beta^{''}(s)\|$ is the unit principal normal, and $e_3(s) = e_1(s) \times e_2(s)$ is the unit binormal vector. The evolution of the Serret-Frenet frame's arc-length derivative is determined by:

$$\begin{pmatrix} e_1^{'}(s) \\ e_2^{'}(s) \\ e_3^{'}(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ \kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} e_1(s) \\ e_2(s) \\ e_3(s) \end{pmatrix}, \tag{2.1}$$

where $\kappa(s)$ and $\tau(s)$ are the curvature and the torsion of the curve $\beta(s)$, respectively.
For this frame the following are satisfying

$$\begin{aligned} &\langle e_1, e_1\rangle = -1, \langle e_2, e_2\rangle = \langle e_3, e_3\rangle = 1, \\ &\langle e_1, e_2\rangle = \langle e_1, e_3\rangle = \langle e_2, e_3\rangle = 0,\ \ \det(e_1, e_2, e_3) = 1. \end{aligned}$$

A timelike ($\mathcal{R}$-$\mathcal{S}$) in $\mathbb{E}_1^3$ is a differentiable one parameter set of straight lines L. Such a surface has a parameterization of the form:

$$\Psi(s, t) = \beta(s) + tX(s),\ t \in \mathbb{R}, \tag{2.2}$$

where $\beta(s)$ denotes its base curve, and X represents the unit vector along the ruling L of the surface. The rulings of a ($\mathcal{R}$-$\mathcal{S}$) are identified as asymptotic curves. If the tangent plane of the ($\mathcal{R}$-$\mathcal{S}$) remains constant along a specific ruling, the ($\mathcal{R}$-$\mathcal{S}$) is termed a developable surface (see [11–13]). The tangent planes on such surfaces is dictated by a singular parameter. Any other ($\mathcal{R}$-$\mathcal{S}$) are

categorized as skew surfaces.
The normative unit vector field of Ψ is determined by

$$U = \frac{\Psi_s \wedge \Psi_t}{\|\Psi_s \wedge \Psi_t\|}, \tag{2.3}$$

where $\Psi_s = \frac{\partial \Psi(s,t)}{\partial s}$ and $\Psi_t = \frac{\partial \Psi(s,t)}{\partial t}$.
The first (I) and second (II) fundamental forms of Ψ are, respectively given by

$$I = Eds^2 + 2Fdsdt + Gdt^2, \tag{2.4}$$

$$II = eds^2 + 2fdsdt + gdt^2, \tag{2.5}$$

where

$$E = \langle \Psi_s, \Psi_s \rangle, \;\; F = \langle \Psi_s, \Psi_t \rangle, \;\; G = \langle \Psi_t, \Psi_t \rangle, \tag{2.6}$$

$$e = \langle \Psi_{ss}, U \rangle, \;\; f = \langle \Psi_{st}, U \rangle, \;\; g = \langle \Psi_{tt}, U \rangle. \tag{2.7}$$

The (GC) K, the (MC) H and the distribution parameter λ of Ψ are expressed as:

$$K = \frac{eg - f^2}{EG - F^2}, \tag{2.8}$$

$$H = \frac{Eg + Ge - 2Ff}{2\left(EG - F^2\right)}, \tag{2.9}$$

$$\lambda = \frac{\det\left(\beta', X, X'\right)}{\|X'\|^2}. \tag{2.10}$$

In consideration of Brioschi's formula in $\mathbb{E}_1^3$, the expression for the $(S\text{-}GC)$ is as follows:

$$K_{II} = \frac{1}{\left(eg - f^2\right)^2} \left\{ \begin{vmatrix} -\frac{1}{2}e_{vv} + f_{sv} - \frac{1}{2}g_{ss} & \frac{1}{2}e_s & f_s - \frac{1}{2}e_t \\ f_t - \frac{1}{2}g_s & e & f \\ \frac{1}{2}g_t & f & g \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}e_t & \frac{1}{2}g_s \\ \frac{1}{2}e_t & e & f \\ \frac{1}{2}g_s & f & g \end{vmatrix} \right\}. \tag{2.11}$$

Furthermore, the $(S\text{-}MC)$ is expressed as:

$$H_{II} = H + \frac{1}{2\sqrt{\det(II)}} \sum_{i,j} \frac{\partial}{\partial u^i}\left[\sqrt{\det(II)}\, h^{ij} \frac{\partial}{\partial u^j}(\ln \sqrt{K})\right], \tag{2.12}$$

where (h_{ij}) represents the matrix corresponding to its inverse (h^{ij}); $i, j \in \{1,2\}$ and where the parameters u^1 and u^2 represent s and t respectively. To explore surfaces governed by focal and slant main curves, it is essential to introduce the subsequent definitions.
Let $\beta = \beta(s) : I \to \mathbb{E}_1^3$ be a unit-speed curve in Minkowski 3-space, where s represents the arc length parameter, the focal curve of β is formed by the centers of osculating spheres along the curve. These osculating spheres are tangent to the curve at each point. The normal hyperplanes to β at a given point comprise the set of centers for all spheres that are tangent to β at that specific

point. Consequently, the center of the osculating spheres at that point is situated within such a normal plane. Denoting the focal curve of β by $\mathcal{F}_\beta$, we can write

$$\mathcal{F}_\beta(s) = \beta(s) + \eta_1(s)\, e_2(s) + \eta_2(s)\, e_3(s), \tag{2.13}$$

where the coefficients η_1 and η_2 denote smooth functions of s, known as the first and second focal curvatures of $\mathcal{F}_\beta$ respectively. Moreover, these curvatures are defined as follows:

$$\eta_1 = \frac{1}{\kappa}, \quad \eta_2 = \frac{\eta_1'}{\tau}, \ \kappa \neq 0, \ \tau \neq 0.$$

Definition 2.1. *A regular surface in $\mathbb{E}_1^3$ is considered flat or developable when $K = 0$ and minimal when $H = 0$.*

Definition 2.2. *A non-developable surface in $\mathbb{E}_1^3$ is characterized as II-flat when $K_{II} = 0$ and designated as II-minimal if $H_{II} = 0$.*

It's important to highlight that a minimal surface has a (S-GC) that goes to zero, but a surface with a vanishing (S-GC) may not necessarily be minimal. The geodesic curvature ($\mathcal{GC}$), normal curvature ($\mathcal{NC}$), and geodesic torsion ($\mathcal{GT}$) of $\beta(s)$ are defined as follows: (for more details see [22–25]).

$$\begin{aligned} \kappa_g &= \left\langle U \wedge e_1, e_1' \right\rangle, \\ \kappa_n &= \left\langle U, \beta'' \right\rangle, \\ \tau_g &= \left\langle U \wedge U', e_1' \right\rangle. \end{aligned} \tag{2.14}$$

Definition 2.3. *In the case of a curve $\beta(s)$ existing on a surface, the subsequent statements remain true:*
(i) A curve $\beta(s)$ is a geodesic if and only if its ($\mathcal{GC}$) κ_g is zero.
(ii) A curve $\beta(s)$ is an asymptotic line if and only if its ($\mathcal{NC}$) κ_n is zero.
(iii) A curve $\beta(s)$ is a principal line if and only if its the ($\mathcal{GT}$) τ_g is zero.

3. Generated ruled surfaces in $\mathbb{E}_1^3$

Within this section, we conduct a geometric analysis of a ($\mathcal{R}$-$\mathcal{S}$), employing a focal curve as the fundamental basis for the surface. Our investigation encompasses three distinct cases: the primary involves the parametrization of the ($\mathcal{R}$-$\mathcal{S}$) through the utilization of the tangent of $\beta(s)$; the second involves the parametrization employing the principal normal of $\beta(s)$, and the third involves the parametrization utilizing the binormal of $\beta(s)$.

3.1. Generating T-ruled surfaces in $\mathbb{E}_1^3$.

Let $\beta = \beta(s)$ represent a specified timelike curve within Minkowski space $\mathbb{E}_1^3$, and $\mathcal{F}_\beta$ denote the focal curve of β. The parameterization of the ($\mathcal{R}$-$\mathcal{S}$) generated by the tangent of β with its base curve as $\mathcal{F}_\beta$ can be expressed as:

$$\Psi_1(s,t) = \mathcal{F}_\beta(s) + te_1(s), \quad \langle e_1(s), e_1(s) \rangle = -1. \tag{3.1}$$

The derivatives of Ψ_1 with respect to both s and t are outlined below:

$$\begin{aligned}
&\Psi_{1s}(s,t) = 2e_1 + t\kappa e_2 + (\eta_1\tau + \eta_2')e_3,\\
&\Psi_{1t}(s,t) = e_1.
\end{aligned} \tag{3.2}$$

From the previous calculation, the elements of the (I)- fundamental form of Ψ_1 are, respectively

$$E_{\Psi_1} = -4 + t^2\kappa^2 + (\eta_1\tau + \eta_2')^2, \quad F_{\Psi_1} = -2, \quad G_{\Psi_1} = -1, \tag{3.3}$$

and the unit normal vector of Ψ_1 is derived as:

$$U_{\Psi_1}(s,t) = \frac{(\eta_1\tau + \eta_2')e_2 - t\kappa e_3}{\sqrt{(\eta_1\tau + \eta_2')^2 + t^2\kappa^2}}, \ \langle U_{\Psi_1}, U_{\Psi_1}\rangle = 1. \tag{3.4}$$

From Eqs. (3.2), the second-order partial derivatives of Ψ_1 are as follows:

$$\begin{aligned}
&\Psi_{1ss} = t\kappa^2 e_1 + (2\kappa + t\kappa' - \eta_1\tau^2 - \eta_2'\tau)e_2 + (t\kappa\tau + \eta_1'\tau + \eta_1\tau' + \eta_2'')e_3,\\
&\Psi_{1st} = \kappa e_2, \quad \Psi_{1tt} = 0.
\end{aligned}$$

The computation of the second-order fundamental characteristics of Ψ_1 is conducted as outlined below:

$$\begin{cases}
e_{\Psi_1} = \dfrac{1}{\sqrt{(\eta_1\tau + \eta_2')^2 + t^2\kappa^2}}\Big(- t^2\kappa^2\tau - t\eta_1'\kappa\tau - t\eta_1\kappa\tau' - t\eta_2''\kappa + 2\eta_1\kappa\tau + t\eta_1\kappa'\tau \\
\qquad - \eta_1^2\tau^3 - \eta_1\eta_2'\tau^2 + 2\eta_2'\kappa + t\eta_2'\kappa' - \eta_1\eta_2'\tau^2 - \eta_2'^2\tau\Big),\\
f_{\Psi_1} = \dfrac{\eta_1\kappa\tau + \eta_2'\kappa}{\sqrt{(\eta_1\tau + \eta_2')^2 + t^2\kappa^2}}, \qquad g_{\Psi_1} = 0.
\end{cases} \tag{3.5}$$

Through simple computations, the (GC) of Ψ_1 can be determined as following:

$$\begin{aligned}
K_{\Psi_1} &= \frac{f^2}{E+4}\\
&= -\left(\frac{\kappa\Big(\eta_1\tau + \eta_2'\Big)}{(\eta_1\tau + \eta_2')^2 + t^2\kappa^2}\right)^2.
\end{aligned} \tag{3.6}$$

By using Eqs. (3.3) and (3.5), the $(MC)$of Ψ_1 is given by

$$\begin{aligned}
H_{\Psi_1} = &\frac{1}{2\Big((\eta_1\tau + \eta_2')^2 + t^2\kappa^2\Big)^{3/2}}\Big(t^2\kappa^2\tau + t\eta_1'\kappa\tau + t\eta_1\kappa\tau' + t\eta_2''\kappa + 2\eta_1\kappa\tau - t\eta_1\kappa'\tau\\
&+ \eta_1^2\tau^3 + \eta_1\eta_2'\tau^2 + 2\eta_2'\kappa - t\eta_2'\kappa' + \eta_1\eta_2'\tau^2 + \eta_2'^2\tau\Big).
\end{aligned} \tag{3.7}$$

From Eqs. (2.11) and (3.5), we obtain the $(S\text{-}GC)$ of Ψ_1 as:

$$K_{II_{\Psi_1}} = \frac{1}{f^3}\Big(f\Big(f_{st} - \frac{1}{2}e_{tt}\Big) + f_t\Big(f_s - \frac{1}{2}e_t\Big)\Big). \tag{3.8}$$

Also, the (*S-MC*) of Ψ_1 is given from:

$$\begin{aligned} H_{II_{\Psi_1}} &= H_{\Psi_1} + \frac{1}{2\sqrt{\det(II_{\Psi_1})}} \sum_{i,j} \frac{\partial}{\partial u^i}\Big[\sqrt{\det(II_{\Psi_1})} h^{ij} \frac{\partial}{\partial u^j} (\ln \sqrt{K_{\Psi_1}})\Big] \\ &= H_{\Psi_1} + \frac{1}{2\sqrt{|\det(II_{\Psi_1})|}}\Big[\frac{\partial}{\partial s}\Big(\sqrt{|\det(II_{\Psi_1})|} h^{11} \frac{\partial}{\partial s}\left(\ln \sqrt{|K_{\Psi_1}|}\right) \\ &\quad + \sqrt{|\det(II_{\Psi_1})|} h^{12} \frac{\partial}{\partial t}\left(\ln \sqrt{|K_{\Psi_1}|}\right)\Big) + \frac{\partial}{\partial t}\Big(\sqrt{|\det(II_{\Psi_1})|} h^{21} \frac{\partial}{\partial s}\left(\ln \sqrt{|K_{\Psi_1}|}\right) \\ &\quad + \sqrt{|\det(II_{\Psi_1})|} h^{22} \frac{\partial}{\partial t}\left(\ln \sqrt{|K_{\Psi_1}|}\right)\Big)\Big], \end{aligned}$$

where $\det(II_{\Psi_1}) = f^2$, then

$$H_{II_{\Psi_1}} = H_{\Psi_1} + \frac{1}{2f}\left[-2\frac{\partial^2}{\partial s \partial t} \ln\left(\frac{f}{\sqrt{E+4}}\right) + \frac{\partial}{\partial t}\left(e\frac{\partial}{\partial t}\ln\left(\frac{f}{\sqrt{E+4}}\right)\right)\right]. \tag{3.9}$$

The ($\mathcal{GC}$), ($\mathcal{NC}$), and ($\mathcal{GT}$) of the focal curve $\mathcal{F}_\beta(s)$ on Ψ_1 are, respectively

$$\begin{cases} (\kappa_g)_{\Psi_1} = \dfrac{-\kappa^2\tau}{\sqrt{(\eta_1\tau + \eta_2')^2 + t^2\kappa^2}}, \\ (\kappa_n)_{\Psi_1} = \dfrac{\kappa\left(\eta_1\tau + \eta_2'\right)}{\sqrt{(\eta_1\tau + \eta_2')^2 + t^2\kappa^2}}, \\ (\tau_g)_{\Psi_1} = \dfrac{-t\kappa^2\left(\eta_1\tau + \eta_2'\right)}{\left((\eta_1\tau + \eta_2')^2 + t^2\kappa^2\right)^2}\Big(t^2\kappa^3 - t\kappa\kappa' + \eta_1^2\tau(\kappa\tau - \tau') \\ \qquad - \eta_1'\eta_2'\tau + \kappa\eta_2'^2 - \eta_2'\eta_2'' - \eta_1\Big(\eta_1'\tau^2 + \eta_2'(\tau' - 2\kappa\tau) + \eta_2''\tau\Big)\Big). \end{cases} \tag{3.10}$$

Under the previous calculations, one can formulate the following theorem:

Theorem 3.1. *Let Ψ_1 be a ($\mathcal{R}$-$\mathcal{S}$) in Minkowski space $\mathbb{E}^3_1$, and consider a point $(s,0)$ on this surface. Then, at the point $(s,0)$, the ($\mathcal{R}$-$\mathcal{S}$) Ψ_1 possesses the following properties:*

- *The ($\mathcal{R}$-$\mathcal{S}$) Ψ_1 is not developable.*
- *The ($\mathcal{R}$-$\mathcal{S}$) Ψ_1 is II-flat.*
- *The ($\mathcal{R}$-$\mathcal{S}$) Ψ_1 is not maximal.*
- *The ($\mathcal{R}$-$\mathcal{S}$) Ψ_1 is not II-maximal.*

Through employing the earlier computation, ensuring that each attribute is explicitly defined and validated within the theorem's context.

Lemma 3.1. *If $\mathcal{F}_\beta$ is a helix, then Eq. (3.10) becomes:*

$$\begin{cases} (\kappa_g)_{\Psi_1} = \dfrac{-\kappa^2\tau}{\sqrt{(\eta_1\tau)^2 + t^2\kappa^2}}, \\ (\kappa_n)_{\Psi_1} = \dfrac{\eta_1\kappa\tau}{\sqrt{(\eta_1\tau)^2 + t^2\kappa^2}}, \\ (\tau_g)_{\Psi_1} = \dfrac{-t\eta_1\kappa^3\tau}{(\eta_1\tau)^2 + t^2\kappa^2}. \end{cases} \tag{3.11}$$

Lemma 3.2. *If $\mathcal{F}_r$ is a helix, then we obtain*

$$\kappa_g^2 + \kappa_n^2 = \kappa^2 \text{ and } \kappa_g \kappa_n = \tau_g.$$

Corollary 3.1. *By using Eq.(2.10), the distribution parameter λ_{Ψ_1} of Ψ_1 is given by*

$$\lambda_{\Psi_1} = \frac{\eta_1 \tau + \eta_2'}{\kappa}. \tag{3.12}$$

3.2. **Generating N-ruled surfaces in $\mathbb{E}_1^3$.** Let $\beta = \beta(s)$ represent a specified timelike curve in Minkowski space $\mathbb{E}_1^3$, and let $\mathcal{F}_\beta$ be the focal curve associated with β. The parametrization of the ($\mathcal{R}$-$\mathcal{S}$) formed by the principal normal of β, with $\mathcal{F}_\beta$ as its base curve, is articulated as follows:

$$\Psi_2(s,t) = \mathcal{F}_\beta(s) + t e_2(s), \quad \langle e_2(s), e_2(s) \rangle = 1. \tag{3.13}$$

The derivatives of Ψ_2 with respect to both s and t are outlined below:

$$\begin{aligned} \Psi_{2s}(s,t) &= (2 + t\kappa) e_1 + (\eta_1 \tau + \eta_2' + t\tau) e_3, \\ \Psi_{2t}(s,t) &= e_2. \end{aligned}$$

The constituent parts of the (I)- fundamental form of Ψ_2 are, in order,

$$E_{\Psi_2} = -(2 + t\kappa)^2 + (\eta_1 \tau + \eta_2' + t\tau)^2, \quad F_{\Psi_2} = 0, \quad G_{\Psi_2} = 1. \tag{3.14}$$

The unit normal vector of Ψ_2 is determined by

$$U_{\Psi_2}(s,t) = \frac{(\eta_1 \tau + \eta_2' + t\tau) e_1 + (2 + t\kappa) e_3}{\sqrt{\left|(2 + t\kappa)^2 - (\eta_1 \tau + \eta_2' + t\tau)^2\right|}} \ , \langle U_{\Psi_2}, U_{\Psi_2} \rangle = 1. \tag{3.15}$$

The second-order partial derivatives of Ψ_2 are outlined below:

$$\begin{aligned} \Psi_{2ss} &= t\kappa' e_1 + (2\kappa + t\kappa^2 - t\tau^2 - \eta_1 \tau^2 - \eta_2' \tau) e_2 + (t\tau' + \eta_1' \tau + \eta_1 \tau' + \eta_2'') e_3, \\ \Psi_{2st} &= \kappa e_1 + \tau e_3, \quad \Psi_{2tt} = 0, \end{aligned}$$

and the components of the (II)- fundamental form of Ψ_2 are calculated as follows

$$\begin{cases} e_{\Psi_2} = \dfrac{-(2 + t\kappa)(t\tau' + \eta_1' \tau + \eta_1 \tau' + \eta_2'') - t\kappa' (\eta_1 \tau + \eta_2' + t\tau)}{\sqrt{\left|(2 + t\kappa)^2 - (\eta_1 \tau + \eta_2' + t\tau)^2\right|}}, \\ f_{\Psi_2} = \dfrac{-(2 + t\kappa)\tau - (\eta_1 \tau + \eta_2' + t\tau)\kappa}{\sqrt{\left|(2 + t\kappa)^2 - (\eta_1 \tau + \eta_2' + t\tau)^2\right|}}, \qquad g_{\Psi_2} = 0. \end{cases} \tag{3.16}$$

By straightforward computation, the (GC) of Ψ_2 can be determined as following:

$$K_{\Psi_2} = \left(\frac{(2 + t\kappa)\tau + (\eta_1 \tau + \eta_2' + t\tau)\kappa}{(2 + t\kappa)^2 - (\eta_1 \tau + \eta_2' + t\tau)^2} \right)^2. \tag{3.17}$$

By using Eqs. (2.9), (3.14) and (3.16), the (MC) of Ψ_2 is given by

$$H_{\Psi_2} = \frac{-(2 + t\kappa)(t\tau' + \eta_1' \tau + \eta_1 \tau' + \eta_2'') - t\kappa' (\eta_1 \tau + \eta_2' + t\tau)}{2\Big((2 + t\kappa)^2 - (\eta_1 \tau + \eta_2' + t\tau)^2\Big)^{3/2}}.$$

From Eqs. (2.11), (3.14), and (3.16), we derive the second curvature of Ψ_2, which can be formulated as:

$$K_{II_{\Psi_2}} = \frac{f\left(f_{st} - \frac{1}{2}e_{tt}\right) + f_t\left(f_s - \frac{1}{2}e_t\right)}{f^3},$$

and the (S-MC) of Ψ_2 is given from:

$$H_{II_{\Psi_2}} = \frac{-e}{2E} + \frac{1}{2f}\left[-2\frac{\partial^2}{\partial s \partial t}\ln\left(\frac{f}{\sqrt{E}}\right) + \frac{\partial}{\partial t}\left(\frac{e}{f}\frac{\partial}{\partial t}\left(\ln\frac{f}{\sqrt{E}}\right)\right)\right]. \tag{3.18}$$

Furthermore, the ($\mathcal{GC}$), ($\mathcal{NC}$), and ($\mathcal{GT}$) of the focal curve $\mathcal{F}_\beta(s)$ on Ψ_2 are, respectively

$$\begin{cases} (\kappa_g)_{\Psi_2} = \dfrac{-\kappa(2+t\kappa) - (\eta_1\tau + \eta_2' + t\tau)\tau}{\sqrt{\left|(2+t\kappa)^2 - (\eta_1\tau + \eta_2' + t\tau)^2\right|}}, \\ (\kappa_n)_{\Psi_2} = 0, \\ (\tau_g)_{\Psi_2} = \dfrac{\left(2\tau - \eta_1\kappa\tau - \eta_2'\kappa\right)\left(\kappa(2+t\kappa) + (\eta_1\tau + \eta_2' + t\tau)\tau\right)}{(2+t\kappa)^2 - (\eta_1\tau + \eta_2' + t\tau)^2}. \end{cases} \tag{3.19}$$

Building upon prior calculations, we get

Theorem 3.2. *Let Ψ_2 be a ($\mathcal{R}$-$\mathcal{S}$) in Minkowski space $\mathbb{E}^3_1$, and consider a point $(s, 0)$ on this surface. Then, at the point $(s, 0)$, the ($\mathcal{R}$-$\mathcal{S}$) Ψ_2 has the following properties:*

- *The ($\mathcal{R}$-$\mathcal{S}$) Ψ_2 is not developable surface.*
- *The ($\mathcal{R}$-$\mathcal{S}$) Ψ_2 is II-flat.*
- *The ($\mathcal{R}$-$\mathcal{S}$) Ψ_2 is not maximal.*
- *The ($\mathcal{R}$-$\mathcal{S}$) Ψ_2 is not II-maximal.*

Utilizing the preceding calculation, where each characteristic is to be clearly defined and justified within the framework of the theorem.

Lemma 3.3. *At the point $(s, 0)$, if the base curve $\mathcal{F}_\beta$ is a helix, then the ($\mathcal{R}$-$\mathcal{S}$) Ψ_2 is maximal.*

Lemma 3.4. *At the point $(s, 0)$, if the base curve $\mathcal{F}_\beta$ is a helix, then the ($\mathcal{R}$-$\mathcal{S}$) Ψ_2 is II-maximal.*

Corollary 3.2. *Utilizing Eq. (2.10), we determine the distribution parameter λ_{Ψ_2} of Ψ_2 as follows:*

$$\lambda_{\Psi_2} = \frac{\kappa(\eta_1\tau + \eta_2') + 2\tau}{\kappa^2 - \tau^2}. \tag{3.20}$$

3.3. **Generating B-ruled surfaces in $\mathbb{E}^3_1$.** Let $\beta = \beta(s)$ denote a designated timelike curve within Minkowski space $\mathbb{E}^3_1$, and $\mathcal{F}_\beta$ represent the associated focal curve linked to β. The parametrization of the ($\mathcal{R}$-$\mathcal{S}$), formed by the binormal of β with $\mathcal{F}_\beta$ as its base curve, is expressed as follows:

$$\Psi_3(s,t) = \mathcal{F}_\beta(s) + te_3(s), \quad \langle e_3(s), e_3(s)\rangle = 1. \tag{3.21}$$

From the previous equation, we get

$$\Psi_{3s}(s,t) = 2e_1 - t\tau e_2 + (\eta_1\tau + \eta_2')e_3,$$
$$\Psi_{3t}(s,t) = e_3,$$

The elements of the (I)- fundamental form of Ψ_3 are, sequentially,

$$E_{\Psi_3} = -4 + t^2\tau^2 + (\eta_1\tau + \eta_2')^2, \quad F_{\Psi_3} = \eta_1\tau + \eta_2', \quad G_{\Psi_3} = 1. \tag{3.22}$$

Also, the unit normal vector of Ψ_3 is given from

$$U_{\Psi_3}(s,t) = \frac{t\tau e_1 - 2e_2}{\sqrt{\left|4 - t^2\tau^2\right|}}, \quad \langle U_{\Psi_3}, U_{\Psi_3}\rangle = 1. \tag{3.23}$$

The second-order partial derivatives of Ψ_3 are as follows:

$$\Psi_{3ss} = -t\kappa\tau e_1 + (2\kappa - t\tau' - \eta_1\tau^2 - \eta_2'\tau)e_2 + (-t\tau^2 + \eta_1'\tau + \eta_1\tau' + \eta_2'')e_3,$$
$$\Psi_{3st} = -\tau e_2, \quad \Psi_{3tt} = 0,$$

and the (II)- fundamental quantities of Ψ_3 are calculated as follows

$$\begin{cases} e_{\Psi_3} = \dfrac{2(2\kappa - t\tau' - \eta_1\tau^2 - \eta_2'\tau) + t^2\tau^2\kappa}{\sqrt{\left|4 - t^2\tau^2\right|}}, \\ f_{\Psi_3} = \dfrac{-2\tau}{\sqrt{\left|4 - t^2\tau^2\right|}}, \qquad g_{\Psi_3} = 0. \end{cases} \tag{3.24}$$

The (GC) of Ψ_3 can be determined as following:

$$K_{\Psi_3} = \left(\frac{2\tau}{4 - t^2\tau^2}\right)^2. \tag{3.25}$$

By using Eqs. (2.9), (3.22) and (3.24), the (MC) of Ψ_3 is given by

$$H_{\Psi_3} = \frac{4\kappa + t^2\tau^2\kappa - 2t\tau' + 2\eta_1\tau^2 + 2\eta_2'\tau}{2\left(\left|4 - t^2\tau^2\right|\right)^{3/2}}. \tag{3.26}$$

From Eqs. (2.11), (3.22), and (3.24), we get

$$K_{II_{\Psi_3}} = \frac{1}{f^3}\left(f\left(f_{st} - \frac{1}{2}e_{tt}\right) + f_t\left(f_s - \frac{1}{2}e_t\right)\right), \tag{3.27}$$

and

$$H_{II_{\Psi_3}} = \frac{e - 2Ff}{2(F^2 - E)} + \frac{1}{2f}\left[-2\frac{\partial^2}{\partial s\partial t}\ln\left(\frac{f}{\sqrt{F^2 - E}}\right) + \frac{\partial}{\partial t}\left(\frac{e}{f}\frac{\partial}{\partial t}\ln\left(\frac{f}{\sqrt{F^2 - E}}\right)\right)\right]. \tag{3.28}$$

Therefore, the $(\mathcal{GC})$, $(\mathcal{NC})$, and $(\mathcal{GT})$ of the focal curve $\mathcal{F}_\beta(s)$ on Ψ_3 are, respectively

$$\begin{cases} (\kappa_g)_{\Psi_3} = \dfrac{-t\tau^2}{\sqrt{\left|4 - t^2\tau^2\right|}}, \\ (\kappa_n)_{\Psi_3} = \dfrac{-2\kappa}{\sqrt{\left|4 - t^2\tau^2\right|}}, \\ (\tau_g)_{\Psi_3} = \dfrac{2t\tau^3}{4 - t^2\tau^2}. \end{cases} \tag{3.29}$$

Referring back to our earlier computations, we get

Theorem 3.3. *Let Ψ_3 be a ($\mathcal{R}$-$\mathcal{S}$) in Minkowski space $\mathbb{E}_1^3$, and consider a point $(s, 0)$ on this surface. Then, at the point $(s, 0)$, the ($\mathcal{R}$-$\mathcal{S}$) Ψ_3 has the following properties:*

- *The ($\mathcal{R}$-$\mathcal{S}$) Ψ_3 is not developable surface.*
- *The ($\mathcal{R}$-$\mathcal{S}$) Ψ_3 is II-flat.*
- *The ($\mathcal{R}$-$\mathcal{S}$) Ψ_3 is not maximal.*
- *The ($\mathcal{R}$-$\mathcal{S}$) Ψ_2 is not II-maximal.*

where each property is to be explicitly defined and justified within the context of the theorem.

Lemma 3.5. *At the point $(s, 0)$, if the base curve $\mathcal{F}_\beta$ is a helix, then the ($\mathcal{R}$-$\mathcal{S}$) Ψ_3 is not developable.*

Lemma 3.6. *At the point $(s, 0)$, if the base curve $\mathcal{F}_\beta$ is a helix, then the ($\mathcal{R}$-$\mathcal{S}$) Ψ_3 is not maximal and not II-maximal.*

Corollary 3.3. *Referencing Eq. (2.10), we derive the distribution parameter λ_{Ψ_3} of Ψ_3 as:*

$$\lambda_{\Psi_3} = \frac{-2}{\tau}. \tag{3.30}$$

4. Applications

In this segment, our attention is directed towards the incorporation of computational illustrations featuring diverse ($\mathcal{R}$-$\mathcal{S}$), all of which maintain complete congruence with the outcomes derived in the course of this investigation.

Example 4.1. *Examine the subsequent ($\mathcal{R}$-$\mathcal{S}$) provided through the specified parameterizations (see Figure 1):*

$$\begin{cases} \Psi_1(s,t) = \mathcal{F}_\beta(s) + te_1(s), \\ \Psi_2(s,t) = \mathcal{F}_\beta(s) + te_2(s), \\ \Psi_3(s,t) = \mathcal{F}_\beta(s) + te_3(s), \end{cases}$$

where $\beta(s)$ represents a timelike curve defined as:

$$\beta(s) = \Big(\sqrt{2}\sinh(s),\ (\sqrt{2}\cosh(s),\ s \Big),$$

and its Frenet frame takes the form:

$$\begin{cases} e_1(s) = \Big(\sqrt{2}\cosh(s),\ \sqrt{2}\sinh(s), 1 \Big), \\ e_2(s) = \Big(\sinh(s), \cosh(s), 0 \Big), \\ e_3(s) = \Big(\cosh(s), \sinh(s),\ \sqrt{2} \Big). \end{cases}$$

From the previous equations, we obtain

$$\kappa = \sqrt{2}, \quad \tau = 1.$$

The focal curve of $\beta(s)$ is determined as

$$\mathcal{F}_\beta = \Big(\frac{3}{\sqrt{2}}\sinh(s), \frac{3}{\sqrt{2}}\cosh(s), s \Big),$$

with respect to this focal curve, the expressions for the ($\mathcal{R}$-$\mathcal{S}$) Ψ_1, Ψ_2, and Ψ_2 are reformulated as follows.

$$\begin{cases} \Psi_1(s,t) = \Big(\frac{1}{\sqrt{2}}\left(3\sinh(s)+2t\cosh(s)\right), \frac{1}{\sqrt{2}}\left(3\cosh(s)+2t\sinh(s)\right), s+t\Big), \\ \Psi_2(s,t) = \Big(\Big(\frac{3}{\sqrt{2}}+t\Big)\sinh(s), \Big(\frac{3}{\sqrt{2}}+t\Big)\cosh(s), s\Big), \\ \Psi_3(s,t) = \Big(t\cosh(s)+\frac{3}{\sqrt{2}}\sinh(s), t\sinh(s)+\frac{3}{\sqrt{2}}\cosh(s), s+\sqrt{2}t\Big). \end{cases} \tag{4.1}$$

Given that the computations for the three surfaces adhere to a uniform methodology for determining the values of geometric invariants, we will consider one of them, let's say Ψ_1, as a representative model for these surfaces. Consequently, from Eq. (4.1), we get

$$\begin{cases} \Psi_{1s} = \Big(\frac{1}{\sqrt{2}}\left(3\cosh(s)+2t\sinh(s)\right), \frac{1}{\sqrt{2}}\left(3\sinh(s)+2t\cosh(s)\right), 1\Big), \\ \Psi_{1t} = \Big(\sqrt{2}\cosh(s), \sqrt{2}\sinh(s), 1\Big), \quad \Psi_{1st} = \Big(\sqrt{2}\sinh(s), \sqrt{2}\cosh(s), 0\Big), \\ \Psi_{1ss} = \Big(\frac{1}{\sqrt{2}}\left(3\sinh(s)+2t\cosh(s)\right), \frac{1}{\sqrt{2}}\left(3\cosh(s)+2t\sinh(s)\right), 0\Big), \\ \Psi_{1tt} = \Big(0,0,0\Big). \end{cases}$$

The spacelike unit normal vector to the surface $\Psi_1(s,t)$ is expressed as follows:

$$U_{\Psi_1} = \frac{-1}{\sqrt{1+4t^2}}\Big(\sinh(s)+2t\cosh(s), \cosh(s)+2t\sinh(s), 2\sqrt{2}\,t\Big). \tag{4.2}$$

Subsequently, the components of the (I) and (II) fundamental forms of $\Psi_1(s,t)$ are derived as follows:

$$\begin{cases} E_{\Psi_1} = \frac{-7}{2}+2t^2, \quad F_{\Psi_1} = -2, \quad G_{\Psi_1} = -1, \\ e_{\Psi_1} = \frac{-1}{\sqrt{2+8t^2}}\Big((3+4t^2)\cosh(2s)+8t\sinh(2s)\Big), \\ f_{\Psi_1} = \frac{-\sqrt{2}}{\sqrt{1+4t^2}}\left(\cosh(2s)+2t\sinh(2s)\right), \\ g_{\Psi_1} = 0. \end{cases} \tag{4.3}$$

The (GC) K_{Ψ_1} and the (MC) H_{Ψ_1} are respectively, defined by:

$$\begin{cases} K_{\Psi_1} = \dfrac{4\left(\cosh(2s)+2t\sinh(2s)\right)^2}{(1+4t^2)^2}, \\ H_{\Psi_1} = \dfrac{(-5+4t^2)\cosh(2s)-8t\sinh(2s)}{\sqrt{2}\,(1+4t^2)^{\frac{3}{2}}} \end{cases} \tag{4.4}$$

The (S-GC) of Ψ_1 is:

$$K_{II_{\Psi_1}} = \frac{\Big(1+4t^2\Big)^2\Big((-\frac{1}{2}e_{tt}+f_{st})f^2+(f_s-\frac{1}{2}e_t)ff_t\Big)}{4\left(\cosh(2s)+2t\sinh(2s)\right)^4}, \tag{4.5}$$

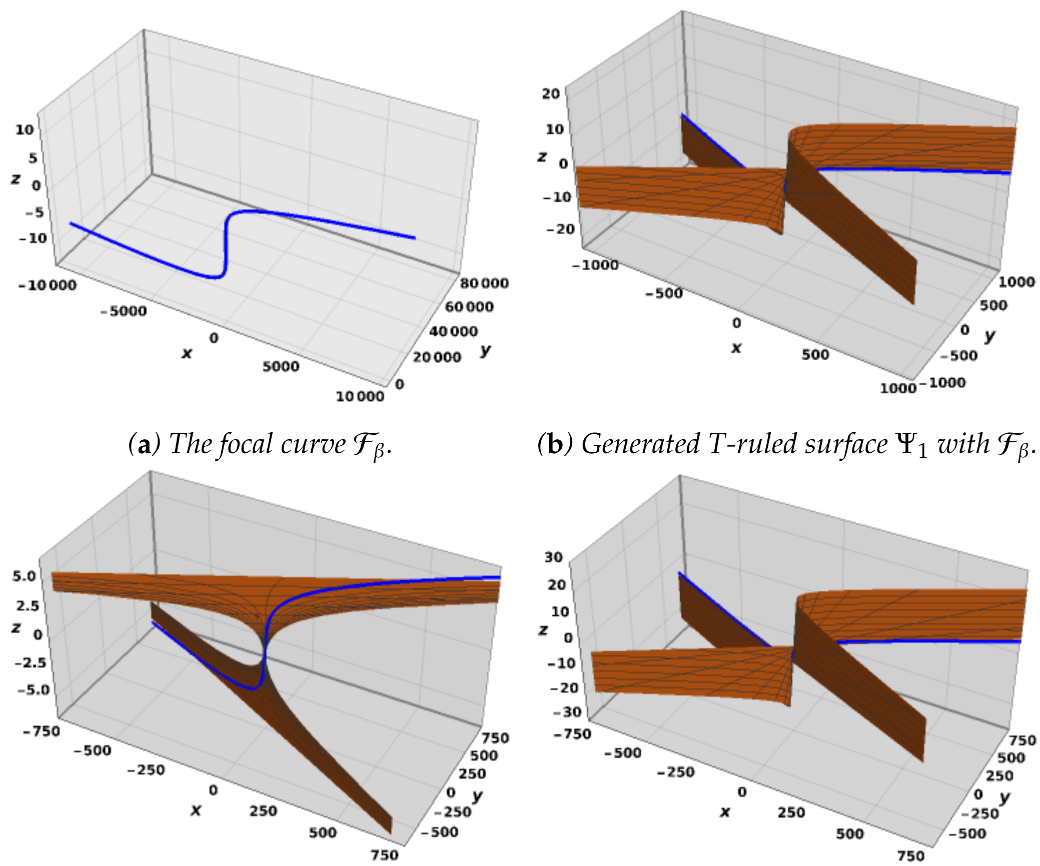
and the (S-MC) is given from:

$$H_{II_{\Psi_1}} = H_{\Psi_1} + \frac{1}{2f}\left(-2\frac{\partial}{\partial s\partial t}\ln\left(\frac{\sqrt{2}f}{\sqrt{1+4t^2}}\right) + \frac{\partial}{\partial t}\left(\frac{e}{f}\frac{\partial}{\partial t}\ln\left(\frac{\sqrt{2}f}{\sqrt{1+4t^2}}\right)\right)\right). \tag{4.6}$$

At the point $(s, 0)$, *we get*

$$K_{\Psi_1} = 4\cosh^2(2s), \quad H_{\Psi_1} = \frac{-5\cosh(2s)}{\sqrt{2}},$$
$$K_{II_{\Psi_1}} = 0, \qquad H_{II_{\Psi_1}} = \frac{-5\cosh(2s)}{\sqrt{2}},$$

which means the ruled surface Ψ_1 *at the point* $(s, 0)$ *is not developable, II-flat, not maximal, and not II-maximal.*



(**a**) *The focal curve* $\mathcal{F}_\beta$. (**b**) *Generated T-ruled surface* Ψ_1 *with* $\mathcal{F}_\beta$.

(**c**) *Generated N-ruled surface* Ψ_2 *with* $\mathcal{F}_\beta$. (**d**) *Generated B-ruled surface* Ψ_3 *with* $\mathcal{F}_\beta$.

Figure 1. Generated ruled surfaces with focal curve (the blue curve represents the focal curve) with $s \in [-2\pi, 2\pi]$ and $t \in [-5, 5]$.

Example 4.2. *Consider the (*$\mathcal{R}$*-*$\mathcal{S}$*) produced by* e_1, e_2, *and* e_3 *of a timelike curve* $\beta(s)$ *serving as the foundational curve for each of these (*$\mathcal{R}$*-*$\mathcal{S}$*) (see Figure 2).*

$$\begin{cases} \Phi_1(s,t) = \mathcal{F}_\beta(s) + te_1(s), \\ \Phi_2(s,t) = \mathcal{F}_\beta(s) + te_2(s), \\ \Phi_3(s,t) = \mathcal{F}_\beta(s) + te_3(s), \end{cases}$$

where $\beta(s)$ is a timelike curve given by:

$$\beta(s)=\Big(\sqrt{3}s,\ \sqrt{2}\sin(s),\ \sqrt{2}\cos(s)\Big),$$

and its Frenet frame takes the form:

$$\begin{cases} e_1(s)=\Big(\sqrt{3},\ \sqrt{2}\cos(s),-\sqrt{2}\sin(s)\Big),\\ e_2(s)=\Big(0,-\sin(s),-\cos(s)\Big),\\ e_3(s)=\Big(\sqrt{2},\ \sqrt{3}\cos(s),-\sqrt{3}\sin(s)\Big). \end{cases}$$

From the preceding equations, we get

$$\kappa=\sqrt{2},\quad \tau=\sqrt{3}.$$

The focal curve of $\beta(s)$ is determined as

$$\mathcal{F}_\beta=\Big(\sqrt{3}s,\frac{1}{\sqrt{2}}\sin(s),\frac{1}{\sqrt{2}}\cos(s)\Big),$$

with respect to this focal curve, the expressions for the ($\mathcal{R}$-$\mathcal{S}$) Φ_1, Φ_2, and Φ_2 are reformulated as follows.

$$\begin{cases} \Phi_1(s,t)=\Big(\sqrt{3}\,(s+t)\,,\frac{1}{\sqrt{2}}\,(2t\cos(s)+\sin(s))\,,\frac{1}{\sqrt{2}}\,(\cos(s)-2t\sin(s))\Big),\\ \Phi_2(s,t)=\Big(\sqrt{3}s,\frac{1}{2}\Big(\sqrt{2}-2t\Big)\sin(s),\frac{1}{2}\Big(\sqrt{2}-2t\Big)\cos(s)\Big),\\ \Phi_3(s,t)=\Big(\sqrt{3}s+\sqrt{2}t,\ \sqrt{3}t\cos(s)+\frac{1}{\sqrt{2}}\sin(s),\ \frac{1}{\sqrt{2}}\cos(s)-\sqrt{3}t\sin(s)\Big). \end{cases} \tag{4.7}$$

We will take Φ_1, as an exemplar model for the previous surfaces. From Eq. (4.7), we obtain

$$\begin{cases} \Phi_{1s}=\Big(\sqrt{3},\frac{1}{\sqrt{2}}\,(\cos(s)-2t\sin(s))\,,\frac{-1}{\sqrt{2}}\,(2t\cos(s)+\sin(s))\Big),\\ \Phi_{1t}=\Big(\sqrt{3},\ \sqrt{2}\cos(s),-\sqrt{2}\sin(s)\Big),\ \ \Phi_{1st}=\Big(0,-\sqrt{2}\sin(s),-\sqrt{2}\cos(s)\Big),\\ \Phi_{1ss}=\Big(0,\frac{-1}{\sqrt{2}}\,(2t\cos(s)+\sin(s))\,,\frac{-1}{\sqrt{2}}\,(\cos(s)-2t\sin(s))\Big),\\ \Phi_{1tt}=\Big(0,0,0\Big). \end{cases} \tag{4.8}$$

The spacelike unit normal vector to the ruled surface $\Phi_1(s,t)$ is expressed as follows:

$$U_{\Phi_1}=\frac{1}{\sqrt{\frac{3}{2}+2t^2}}\Big(-2t,-\sqrt{\frac{3}{2}}\Big[2t\cos(s)-\sin(s)\Big],\sqrt{\frac{3}{2}}\Big[\cos(s)+2t\sin(s)\Big]\Big). \tag{4.9}$$

Subsequently, the components of the (I) and (II) fundamental forms of $\Phi_1(s,t)$ are derived as follows:

$$\begin{cases} E_{\Phi_1}=\frac{-5}{2}+2t^2,\quad F_{\Phi_1}=-2,\quad G_{\Phi_1}=-1,\\ e_{\Phi_1}=\frac{-1+4t^2}{\sqrt{2+\frac{8}{3}t^2}},\quad f_{\Phi_1}=\frac{-\sqrt{6}}{\sqrt{3+4t^2}},\quad g_{\Phi_1}=0. \end{cases} \tag{4.10}$$

The (GC) K_{Φ_1} and the (MC) H_{Φ_1} are respectively, defined by:

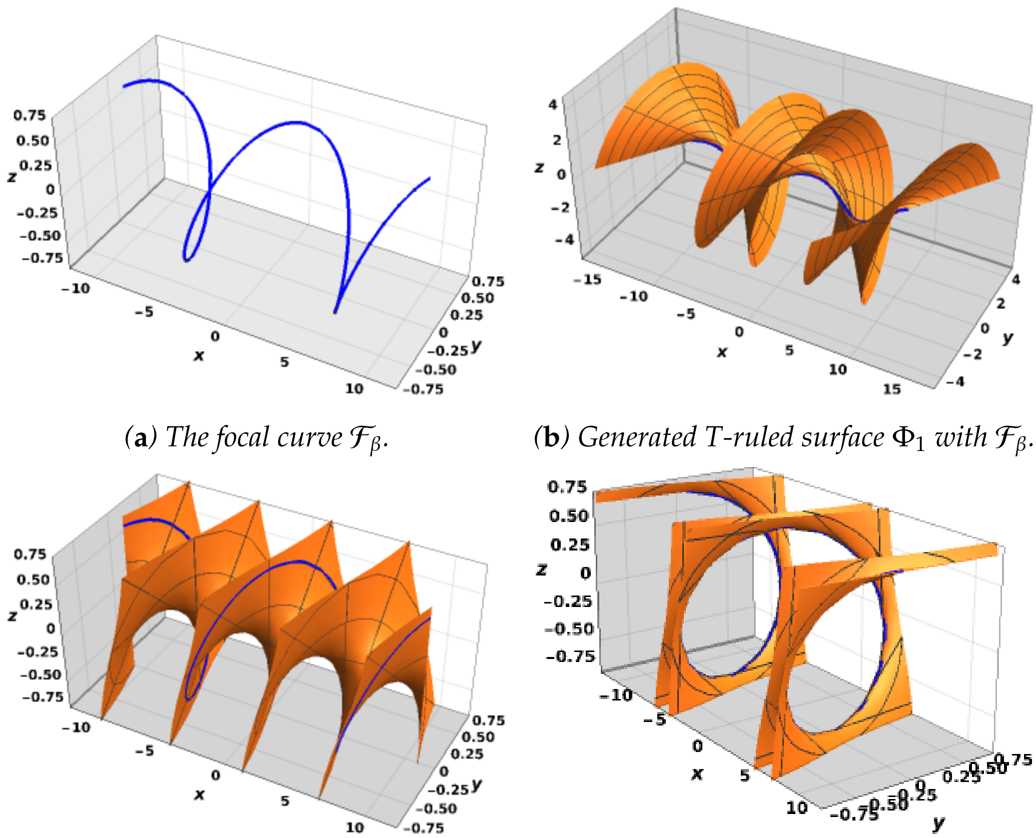
$$K_{\Phi_1}=\frac{-12}{\Big(3+4t^2\Big)^2},\quad H_{\Phi_1}=-\frac{\sqrt{\frac{3}{2}}\Big(7+4t^2\Big)}{\Big(3+4t^2\Big)^{\frac{3}{2}}}. \tag{4.11}$$

Besides, the (S-GC) $(K_{II})_{\Phi_1}$ *is*

$$(K_{II})_{\Phi_1} = \left(\frac{3+4t^2}{6}\right)^2 \left(-\frac{1}{2}e_{tt}f^2 - \frac{1}{2}e_t f_t f\right), \tag{4.12}$$

and the (S-MC) $(H_{II})_{\Phi_1}$ *is:*

$$H_{II_{\Phi_1}} = \frac{4f-e}{2(4+E)} + \frac{1}{2f}\left(-2\frac{\partial}{\partial s \partial t}\ln\sqrt{K} + \frac{\partial}{\partial t}\left(\frac{e}{f}\right)\frac{\partial}{\partial t}\ln\sqrt{K}\right). \tag{4.13}$$



(**a**) *The focal curve* $\mathcal{F}_\beta$. (**b**) *Generated T-ruled surface* Φ_1 *with* $\mathcal{F}_\beta$.

(**c**) *Generated N-ruled surface* Φ_2 *with* $\mathcal{F}_\beta$. (**d**) *Generated B-ruled surface* Φ_3 *with* $\mathcal{F}_\beta$.

Figure 2. Generated ruled surfaces with focal curve (the blue curve represents the focal curve) with $s \in [-2\pi, 2\pi]$ and $t \in [-5, 5]$.

Conclusion

Our study on generating ($\mathcal{R}$-$\mathcal{S}$) in Minkowski 3-space using focal curves and Frenet vectors has provided valuable insights into the geometric properties and curvatures of these surfaces. By exploring the relationship between focal curves and ($\mathcal{R}$-$\mathcal{S}$), we have deepened our understanding of their characteristics and applications in various fields. The computational examples presented in this paper have not only validated our theoretical results but have also offered visual representations that enhance the comprehension of the topic. Overall, this research contributes to the ongoing exploration of ($\mathcal{R}$-$\mathcal{S}$) and their significance in geometry and physics within the context of Minkowski 3-space.

Abbreviations

In this document, the following abbreviations are employed:

GC	Gaussian curvature.
MC	Mean curvature.
$\mathcal{GC}$	Geodesic curvature.
$\mathcal{NC}$	Normal curvature.
$\mathcal{GT}$	Geodesic torsion.
$\mathcal{R}$-$\mathcal{S}$	Ruled surface(s).
S-MC	Second mean curvature.
S-GC	Second Gaussian curvature.

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