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Strong $\mathcal{D}$-Stability Analysis of Economic Models

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Abstract. The analysis of stability ensures that a dynamical system's output remains bounded for given bounded inputs and manages the system behavior. The concept of strong $\mathcal{D}$-stability extends the idea of stability requiring the system's poles to stay within a predefined $\mathcal{D}$-region in the complex plane $\mathbb{C}$ subject to uncertain parameter variations. This ensures robustness and performance of the dynamical system. We offer some new findings on stability analysis, strong $\mathcal{D}$-stability analysis, and their interconnections with structured singular values. The theoretical findings were gained by using several tools from linear algebra and system theory. The numerical experimentation shows the behavior of the spectrum and hence the stability of dynamical systems. For the computation and analysis of pseudo-spectrum, the Eigtool has been utilized.

1. Introduction

For a class of structured matrices, Arrow and McManus [1] first proposed the idea of $\mathcal{D}$-stability, followed by Enthoven and Arrow [2]. The fundamental idea was to make use of multiple tools from linear algebra and control system theory to develop mathematical methods to study and analyze stability of equilibrium in the competitive market dynamical models. The study and analysis of $\mathcal{D}$-stability theory plays a significant part in multi-parameter singular perturbation for stability of systems in higher dimensions, see [3–6]. The characterization of $\mathcal{D}$-stability is very hard to verify, particularly for matrices having dimensions $n \geq 2$, see [7, 8].

The single value that is structured (a.k.a μ-values) is a widely used mathematical tool for examining and evaluating the performance, adaptability, and stability of linear time invariant (LTI)

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systems. The μ-value was first introduced by Doyle [9] for the complex structured uncertainties. The μ-value's accurate computation is an NP-hard issue [10], and this motivates the development of numerical methods to approximate it from above and below.

A new algorithm [11] was presented for the computation of tight lower bounds of μ-values. This algorithm considers real, uncorrelated parametric perturbations. The main advantage of this algorithm is that it consists of simple matrix algebra and returns the actual values of worst-case parameters. A non-linear programming technique was developed [12] for the computation of lower bounds of real μ-values. A novel numerical approach was developed [13] for providing sufficient conditions to check whether a given matrix whose elements depend polynomially on uncertain parameters is $\mathcal{D}$-stable or not. A novel iterative method was developed in [14] to study and analyze the stability of LTI systems.

The close connection between μ-values and $\mathcal{D}$-stability for a class of real-valued square matrices was studied by [15]. It was shown that for a given real-valued square matrix to be $\mathcal{D}$-stable, its real structured singular values must be strictly less than 1. Several necessary and sufficient conditions were provided for an interconnection between μ-values and $\mathcal{D}$-stability. The novel theoretical results on the relationship between Schur stability, μ-values for n-dimensional matrices were established and presented in [16].

In [17,18], it was shown that $\mathcal{M}+G$ is a stable matrix with $\|G\| < \gamma$ for a given stable matrix $\mathcal{M}$, and $\gamma > 0$. This, as a result, shows that stability is the property that is resilient to minor allowable perturbations across the system. If we assume that the given matrix $\mathcal{M}$ is a $\mathcal{D}$-stable matrix, then for a positive diagonal matrix P, both $P\mathcal{M}$, and $P(\mathcal{M}+G)$ are stable matrices. The important point to discuss is that there is a possibility for the existence of a $\gamma > 0$ so that the matrix $\mathcal{M}+G$ is a $\mathcal{D}$-stable matrix when $\|G\| < \gamma$. Unfortunately, the answer is no in general. A given matrix $M \in \mathbb{R}^{n,n}$ is strongly $\mathcal{D}$-stable matrix if there is scalar $\gamma > 0$ in such a way that $\mathcal{M}+G$ is $\mathcal{D}$-stable matrix to each real-valued matrix G with $\|G\| < \gamma$. It is important to note here that each strongly $\mathcal{D}$-stable matrix is a $\mathcal{D}$-stable matrix. The simpler conditions to study and analyze $\mathcal{D}$-stability were developed in [19]. New findings about the relationships among $\mathcal{D}$-stability, strong $\mathcal{D}$-stability, and μ-values for a class of square matrices were given and examined in [20].

We have provided some fresh findings on stability computation in this study, and strong $\mathcal{D}$-stability of structured matrices. Further, we have analyzed the interlinks between strong $\mathcal{D}$-stability and structured singular values. The findings are obtained using a mix of methods from system theory, matrix analysis, and numerical linear algebra in the suggested approach.

Overview of article: In section 2, we present the preliminary concepts related to our study in this article. In part 3, we look back at several fundamental results on $\mathcal{D}$-stability, and strong $\mathcal{D}$-stability. Theoretical findings on stability analysis, $\mathcal{D}$-stability analysis, and their interconnection with μ-value are given in section 4. In Section 5, we present the numerical experimentation supporting theoretical results to discuss the stability of dynamical systems. For the stability of a dynamical system, we have analyzed the behavior of the spectrum and pseudo-spectrum of constant structured matrices appearing as the coefficient in the matrix formulation. Finally, we present the conclusion in Section 6.

2. Preliminaries

The notation $M \in \mathbb{K}^{n,n}$, with $\mathbb{K} = \mathbb{C}(\mathbb{R})$ means an n-dimensional complex-valued or a real-valued matrix. The symbol $\mu(\cdot)$ denotes the structured singular value or $\mu-$value of a matrix concerning a set of block-diagonal matrices. The notation $\sigma_2(\cdot)$ indicates the second largest singular value of a matrix. For $\Delta \in \mathbb{B}$, it indicates that the collection of block-diagonal matrices contains an admissible perturbation matrix. The symbol $\otimes$ denotes the entry-wise multiplication of two matrices. The time derivative of a matrix-valued function $\Delta(t)$ is shown by $\dot{\Delta}$. The spectral radius of a matrix is represented by the Greek symbol $\rho(\cdot)$.

$\mathbb{B}$ represents the set of block-diagonal matrices, which has the definition given below.

Definition 2.1. *The set of block-diagonal matrices is a set containing matrices across its main diagonal, that is,*

$$\mathbb{B} := \{diag(\delta_1 I_{r_1}, \cdots, \delta_s I_{r_S}, \Delta_1, \cdots, \Delta_F) : \ \delta_i \in \mathbb{K}, \ \Delta_j \in \mathbb{K}^{m_j,m_j}, \ \forall i = 1 : S, \ \forall j = 1 : F\}.$$

Remark 2.1. *In definition of $\mathbb{B}$, $\sum_i r_i + \sum_j m_j = n$.*

Definition 2.2. *For a given $M \in \mathbb{K}^{n,n}$, and $\mathbb{B}$ the μ-value is defined as*

$$\mu_{\mathbb{B}}(M) := \left(min\left(\|\Delta\|_2 : \Delta \in \mathbb{B}, \ det(I - M\Delta) = 0, \ \forall \ \Delta \in \mathbb{B}\right)\right)^{-1},$$

otherwise, $\mu_{\mathbb{B}}(M) = 0$, if $det(I - M\Delta) \neq 0, \ \forall \Delta \in \mathbb{B}$.

Remark 2.2. *In the definition of μ-value of a given matrix M, the quantity $\|\cdot\|_2$ denotes the largest singular value of a matrix. The **min** is taken over $\Delta \in \mathbb{B}$.*

Definition 2.3. *If $Re(\lambda_i(M)) > 0, \ \forall i = 1 : n.$, then a matrix $M \in \mathbb{R}^{n,n}$ is a stable matrix.*

Definition 2.4. *If $Re(\lambda_i(PM)) > 0, \ \forall i = 1 : n$, where $P = diag(p_{ii})$ such that all the diagonal elements of P are strictly positive, then a matrix $M \in \mathbb{R}^{n,n}$ is referred to as a $\mathcal{D}$-stable.*

Definition 2.5. *A strongly $\mathcal{D}$-stable matrix is defined as $M \in \mathbb{R}^{n,n}$ if there is $\gamma > 0$ such that $Re(\lambda_i(P + M)) > 0, \ \forall i = 1 : n$, where $P = diag(p_{ii})$ such that every diagonal entry (real entries) of P are strictly positive, and $\|G\| < \gamma$.*

3. $\mathcal{D}$-stability and strong $\mathcal{D}$-stability

The computation of a $\mathcal{D}$-stable matrix for a given matrix was studied in [2, 5, 7, 22]. The computation of a $\mathcal{D}$-stable matrix has played an important role in the study of large-scale systems [6], and also for the multi-parameter singular perturbations, see [4,5]. In [7], 13 sufficient conditions were presented to investigate and evaluate the $\mathcal{D}$-stability of matrices.

The next theorem explains how $\mathcal{D}$-stability and μ-value computation are related.

Theorem 3.1. *Let $\mathcal{M} \in \mathbb{R}^{n,n}$. Then $\mathcal{M}$ is a $\mathcal{D}$-stable matrix iff $Re(\lambda_i(\mathcal{M})) > 0$, $\forall i$, and*

$$0 \leq \mu_{\mathbb{B}}\left((iI_n + \mathcal{M})^{-1}(iI_n - \mathcal{M})\right) < 1,\ i = \sqrt{-1},$$

with $\mathbb{B}$, the set of block-diagonal matrices.

For given $\mathcal{M} \in \mathbb{R}^{n,n}$, the $\mathcal{D}$-stability can also be defined by using the following theoretical results developed in [1].

Theorem 3.2. *The matrix $\mathcal{M} \in \mathbb{R}^{n,n}$ is $\mathcal{D}$-stable iff it is a stable matrix, and*

$$0 \leq \mu_{\mathbb{B}}\left((I_n + \mathcal{M})^{-1}(I_n - \mathcal{M})\right) < 1,$$

where I_n is an identity matrix having the same dimension as the given $\mathcal{M}$.

The following result [1] shows that computation of a $\mathcal{D}$-Stable matrix corresponds to a condition on the upper bound of complex functions that is scaled diagonally μ-value.

Theorem 3.3. *Given $\mathcal{M} \in \mathbb{R}^{n,n}$, $\mathcal{M}$ is a $\mathcal{D}$-stable matrix iff $\mathcal{M}$ is stable, and*

$$\min \lambda_{\max}(-\mathcal{M}^T P - P\mathcal{M}) < 0,$$

with min *taken over $P \in \gamma \cap \mathbb{B}$, where γ is the set of positive diagonal matrices.*

Remark 3.1. *Any $\mathcal{M} \in \mathbb{C}^{n,n}$, the μ-value can be calculated by resolving an optimization problem that is convex.*

Corollary 3.1. *For $n = 3$, it can be shown analytically that $\mathcal{M}$ is a $\mathcal{D}$-stable matrix.*

Corollary 3.2. *Consider that $\mathcal{M}$ is a stable matrix. Let*

$$\hat{\mathcal{M}} = (I_n + \mathcal{M})^{-1}(I_n - \mathcal{M}),$$

then M is $\mathcal{D}$-stable matrix iff

$$\bar{\sigma}(\mathcal{M}) < 1, \textit{or}\ \rho(\mathcal{M}) < 1,$$

where $\bar{\sigma}(\cdot)$and $\rho(\cdot)$ signify the greatest singular value and the spectral radius of a given matrix.

To examine and assess the integrity of $\mathcal{D}$-stability, [1] introduced the concept of the strong $\mathcal{D}$-stability. If $\mathcal{M} \in \mathbb{R}^{n,n}$ is a D-stable matrix and must continue to be so even if $\mathcal{M}$ is subject to a few minor perturbations, then it is considered a strongly $\mathcal{D}$-stable matrix. For instance, assume that given $M \in \mathbb{R}^{n,n}$ is D-stable, and let $\delta > 0$, a small positive parameter such that $\Delta\mathcal{M} \in \mathbb{R}^{n,n}$, having $\sigma_{\max}(\Delta\mathcal{M}) < \epsilon$, and $\mathcal{M} + \Delta\mathcal{M}$ to be a $\mathcal{D}$-stable matrix. The strong D-stability is known to imply D-stability, but the opposite may not be true, see [1].

The following theorem shows that given $\mathcal{M} \in \mathbb{R}^{n,n}$, it is a strongly $\mathcal{D}$-stable matrix if it is a stable matrix and a structured singular value is less than 1.

Theorem 3.4. *Let $\mathcal{M} \in \mathbb{R}^{n,n}$. Then $\mathcal{M}$ is strongly $\mathcal{D}$-stable matrix iff $\mathcal{M}$ is stable, and $\exists$ some small positive parameter ϵ such that*

$$0 \leq \mu_{\mathbb{B}}(\mathcal{M}) < 1,$$

where

$$\mathcal{M} = \begin{pmatrix} (iI_n + \mathcal{M})^{-1}(iI_n - \mathcal{M}) & 2i(iI_n + \mathcal{M})^{-1} \\ \epsilon(iI_n + \mathcal{M})^{-1} & -\epsilon(iI_n + \mathcal{M})^{-1} \end{pmatrix}.$$

The following results for $\mathcal{M} \in \mathbb{R}^{n,n}$ to be a strongly $\mathcal{D}$-stable matrix were proven in [3].

Result 3.1. *Suppose that $\mathcal{M} \in \mathbb{R}^{n,n}$. The matrix $\mathcal{M}$ is strongly $\mathcal{D}$-stable if*

$$\lambda_i(P\mathcal{M} + \mathcal{M}^T P) < 0, \ \forall i = 1, 2, \ldots, n.$$

Result 3.2. *Suppose that $\mathcal{M} \in \mathbb{R}^{n,n}$. If $\mathcal{M}$ is an M-matrix, then $\mathcal{M}$ is a strongly $\mathcal{D}$-stable matrix.*

Result 3.3. *Assume that $\mathcal{M} \in \mathbb{R}^{n,n}$. Consequently, if $\mathcal{M}$ is a triangular matrix, it is a strong $\mathcal{D}$-stable matrix, and $m_{ii} < 0, \ \forall i = 1, 2, \ldots, n$.*

Result 3.4. *Suppose that $\mathcal{M} \in \mathbb{R}^{n,n}$. If $\mathcal{M}$ is a sign-stable matrix with no zero entries, then $\mathcal{M}$ is a Strongly $\mathcal{D}$-stable matrix.*

Result 3.5. *Suppose that $\mathcal{M} \in \mathbb{R}^{n,n}$. If $\mathcal{M}$ is an oscillatory matrix, then $\mathcal{M}$ is a strongly $\mathcal{D}$-stable matrix.*

Result 3.6. *Suppose that $\mathcal{M} \in \mathbb{R}^{n,n}$. If there is a positive diagonal matrix P such that for every $x \in \mathbb{C}^n$, $x \neq 0$, $\mathcal{M}$ is a strongly $\mathcal{D}$-stable matrix then*

$$Re(\lambda_i(x^* P\mathcal{M}x)) < 0, \ \forall i = 1, 2, \ldots, n.$$

Result 3.7. *Let $\mathcal{M} \in \mathbb{R}^{2,2}$. Then $\mathcal{M}$ is strongly $\mathcal{D}$-stable matrix iff all j-th order principal minors are with $sign(-1)^j$.*

Consider the partition of $\mathcal{M} = \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{pmatrix}$, with $\mathcal{M}_{11}$ as the principal sub-matrix. Let $\mathcal{M}_{22}^{\mathbb{C}} := \mathcal{M}_{22} - \mathcal{M}_{21}\mathcal{M}_{11}^{-1}\mathcal{M}_{12}$ when $\mathcal{M}_{11}$ exists.

The following lemma shows that if $\mathcal{M}$ is strongly $\mathcal{D}$-stable, then $\mathcal{M}_{22}$ and $\mathcal{M}_{22}^{\mathbb{C}}$ are strongly $\mathcal{D}$-stable matrices.

Lemma 3.1. *Let $\mathcal{M} \in \mathbb{R}^{n,n}$ be a strongly $\mathcal{D}$-stable matrix. The matrices $\mathcal{M}_{22}$ and $\mathcal{M}_{22}^{\mathbb{C}}$ are also strongly $\mathcal{D}$-stable matrices. This means that $\hat{\mathcal{M}} = \mathcal{M}_{22}$ or $\mathcal{M}_{22}^{\mathbb{C}}$, $\exists\, \epsilon > 0$, a small positive parameter such that*

$$\prod_{i=1}^{n} \lambda_i \Big((I_n - (iI_n + \hat{\mathcal{M}} + E_n)^{-1}(iI_n - \hat{\mathcal{M}} - E_n))P \Big) \neq 0, \ \forall i,$$

where P is a positive diagonal matrix.

4. New results

In this segment, we introduce new theoretical findings on the stability analysis, $\mathcal{D}$-stability analysis, and strong $\mathcal{D}$-stability analysis of economic models. We employ several methods from system theory, matrix analysis, and linear algebra to construct and analyze results to study the dynamics of the economic model under consideration. In section 4.1, we provide results on stability analysis. We study stability in the sense that non-complex parts of all the eigenvalues of the matrix under consideration are strictly positive. In the subsequent section 4.2, we provide new results on strong $\mathcal{D}$-stability of economic models.

4.1. **Stability.**

Theorem 4.1. *The dynamical system*

$$\frac{dx(t)}{dt} = \left(\mathcal{M}^T S\mathcal{M} - S\right)x(t), \quad x^T(t)x(t) = 1, \quad t \in \mathbb{R}^+$$

is stable if

$$Re\left[\lambda_i(\mathcal{M}^T S\mathcal{M} - S)\right] > 0, \quad \forall i = 1 : n,$$

with S is a symmetric positive definite matrix.

Proof. Let $\beta \in \{1, 2, \ldots, m\}$, and let $x(t) \in \mathbb{R}^{n,1}, t \in \mathbb{R}^+$. Further, consider that $x[\beta] \neq 0$. To prove, we take $x(t) \neq 0$, meaning that

$$x[\beta]^T \left(\mathcal{M}^T S\mathcal{M} - S\right)x[\beta] = x^T(t)\left(\mathcal{M}^T S\mathcal{M} - S\right)x(t) > 0.$$

The above equality may be rewritten as

$$x^T(t)\mathrm{Re}[\lambda_i(t)]x(t) > 0, \quad \forall i, t \in \mathbb{R}^+.$$

This implies that

$$\mathrm{Re}[\lambda_i(t)]x^T(t)x(t) > 0.$$

Since, we know that for $t \in \mathbb{R}$, $x^T(t)x(t) = 1$, thus,

$$\mathrm{Re}[\lambda_i(t)] > 0, \quad \forall t \in \mathbb{R}^+.$$

Finally, we have that

$$\mathrm{Re}[\lambda_i(\mathcal{M}^T S\mathcal{M} - S)] > 0, \quad \forall i = 1 : n.$$

□

The subsequent Theorem 4.2 show that dynamical system is stable if non-complex part of all eigenvalues of modified matrix $(\tilde{\mathcal{M}}^T SM\tilde{\mathcal{M}} - \tilde{\mathcal{M}}^T S\tilde{\mathcal{M}})$ are strictly positive.

Theorem 4.2. *The dynamical system*

$$\frac{dx(t)}{dt} = \left(\mathcal{M}^T S\mathcal{M} - S\right)x(t), \quad x^T(t)x(t) = 1, \quad t \in \mathbb{R}^+$$

$$Re\left[\lambda_i\left(\tilde{\mathcal{M}}^T S\mathcal{M}\tilde{\mathcal{M}} - \tilde{\mathcal{M}}^T S\tilde{\mathcal{M}}\right)\right] > 0, \quad \forall i = 1 : n.$$

Furthermore, rank$(\tilde{\mathcal{M}}) = n$*, where n is the dimension of the coefficient matrix.*

Proof. The fact

$$\mathrm{Re}\left[\lambda_i\left(\tilde{\mathcal{M}}^T S\mathcal{M}\tilde{\mathcal{M}} - \tilde{\mathcal{M}}^T S\tilde{\mathcal{M}}\right)\right] > 0, \quad \forall i = 1 : n$$

ensures that

$$\mathrm{rank}(\tilde{\mathcal{M}}) = \mathrm{rank}\left((\mathcal{M}^T S\mathcal{M} - S)\tilde{\mathcal{M}}\right) = \mathrm{rank}\left(\tilde{\mathcal{M}}^T(\mathcal{M}^T S\mathcal{M} - S)\tilde{\mathcal{M}}\right).$$

As,

$$\mathrm{Re}\left[\lambda_i\left(\mathcal{M}^T S\mathcal{M} - S\right)\right] \neq 0, \quad \forall i = 1 : n,$$

yields that

$$\mathrm{rank}\left(\tilde{\mathcal{M}}^T(\mathcal{M}^T S\mathcal{M} - S)\tilde{\mathcal{M}}\right) = n,$$

where n is the rank of matrix $\tilde{\mathcal{M}}$. □

The following Theorem 4.3 shows that a dynamical system with coefficient matrix $\mathcal{M}^T S\mathcal{M} - S$ is stable if the real part of all eigenvalues of this matrix is strictly positive.

Theorem 4.3. *The dynamical system*

$$\frac{dx(t)}{dt} = \left(\mathcal{M}^T S\mathcal{M} - S\right)x(t), \quad x^T(t)x(t) = 1, \quad t \in \mathbb{R}^+$$

is stable if

$$Re\left[\lambda_i\left(\mathcal{M}^T S\mathcal{M} - S\right)\right] > 0, \quad \forall i = 1 : n, \quad t \in \mathbb{R}^+.$$

Furthermore, let $(\lambda(t), x(t))$ *be an eigen-pair, and let*

$$\left|Re\left[\lambda_i\left(\mathcal{M}^T S\mathcal{M} - S\right)\right]\right| = \rho, \quad then \quad |x(t)| > 0, \quad \forall t \in \mathbb{R}^+,$$

and

$$(\mathcal{M}^T S\mathcal{M} - S)x(t) = \rho(\mathcal{M}^T S\mathcal{M} - S)|x(t)|.$$

Proof. Suppose that

$$\hat{x}(t) = (\mathcal{M}^T S\mathcal{M} - S)|x(t)| > 0.$$

The above inequality implies that

$$\hat{x}(t) = (\mathcal{M}^T S\mathcal{M} - S)|x(t)| \geq (\mathcal{M}^T S\mathcal{M} - S)|x(t)| = \mathrm{Re}\left(\lambda_i(\mathcal{M}^T S\mathcal{M} - \mathcal{M})\right)|x(t)|$$

Also,

$$\hat{x}(t) = \left|\mathrm{Re}\left(\lambda_i(\mathcal{M}^T S\mathcal{M} - S)\right)\right||x(t)| = \rho(\mathcal{M}^T S\mathcal{M} - S)|x(t)|.$$

Let

$$\hat{x}(t) = x(t) - \rho(\mathcal{M}^T S\mathcal{M} - S)|x(t)| \geq 0.$$

This further implies that,

$$\rho(\mathcal{M}^T S\mathcal{M} - S)|x(t)| = (\mathcal{M}^T S\mathcal{M} - S)|x(t)| \geq a$$

Also,

$$\rho(\mathcal{M}^T S\mathcal{M} - S) \geq 0, \quad \text{and } |x(t)| > 0, \quad \hat{x}(t) = 0.$$

If $\hat{x}(t) \neq 0$, we have

$$0 < \left(\mathcal{M}^T S\mathcal{M} - S\right)\hat{x}(t) = (\mathcal{M}^T S\mathcal{M} - S)x(t) - \rho(\mathcal{M}^T S\mathcal{M} - S)|x(t)|$$
$$= (\mathcal{M}^T S\mathcal{M} - S)x(t) - \rho(\mathcal{M}^T S\mathcal{M} - S)x(t).$$

Finally, we have

$$(\mathcal{M}^T S\mathcal{M} - S)x(t) \geq \rho(\mathcal{M}^T S\mathcal{M} - S)x(t).$$

This is true if we drop $x(t)$, which is not possible. This implies that $\hat{x}(t) \neq 0$. □

4.2. **Strong D-stability.**

Theorem 4.4. *Let $\mathcal{M} \in \mathbb{C}^{n,n}$ and $A \in \mathbb{C}^{n,n}$ (Hermitian matrix). Then $\mathcal{M} \in \mathbb{C}^{n,n}$ is strongly D-stable if*

$$\inf \sigma_2 \left[P \begin{pmatrix} Re\left((iI_n + log(\mathcal{M}))\,(iI_n - log(\mathcal{M}))^{-1}\right) & Im\left((iI_n + log(M))\,(iI_n - log(\mathcal{M}))^{-1}\right) \\ -Im\left((iI_n + log(\mathcal{M}))\,(iI_n - log(\mathcal{M}))^{-1}\right) & Re\left((iI_n + log(\mathcal{M}))\,(iI_n - log(\mathcal{M}))^{-1}\right) \end{pmatrix} P^{-1} \right] < 1,$$

where $\sigma_2(\cdot)$ denotes the 2nd largest singular value, and inf *is taken over all positive diagonal matrices P.*

Proof. Let $P = \mathrm{diag}(P_{ii}) > 0,\ \forall i = 1, 2, \ldots, n$, and let $\Delta \in \mathbb{B}$, where

$$\mathbb{B} = \{\mathrm{diag}(\delta_1, \delta_2, \ldots, \delta_m) : \delta_i \in \mathbb{C},\ |\delta_i| \leq 1,\ \forall i = 1, 2, \ldots, n\}.$$

$$\mathrm{rank}\begin{pmatrix} iI_n - P \begin{pmatrix} \mathrm{Re}\left((iI_n + \log(\mathcal{M}))\,(iI_n - \log(\mathcal{M}))^{-1}\right) & \mathrm{Im}\left((iI_n + \log(\mathcal{M}))\,(iI_n - \log(\mathcal{M}))^{-1}\right) \\ -\mathrm{Im}\left((iI_n + \log(\mathcal{M}))\,(iI_n - \log(\mathcal{M}))^{-1}\right) & \mathrm{Re}\left((iI_n + \log(\mathcal{M}))\,(iI_n - \log(\mathcal{M}))^{-1}\right) \end{pmatrix} \\ P^{-1} \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} P \end{pmatrix}$$
$$=$$
$$\mathrm{rank}\begin{pmatrix} iI_n - \begin{pmatrix} \mathrm{Re}\left((iI_n + \log(\mathcal{M}))\,(iI_n - \log(\mathcal{M}))^{-1}\right) & \mathrm{Im}\left((iI_n + \log(\mathcal{M}))\,(iI_n - \log(\mathcal{M}))^{-1}\right) \\ -\mathrm{Im}\left((iI_n + \log(\mathcal{M}))\,(iI_n - \log(\mathcal{M}))^{-1}\right) & \mathrm{Re}\left((iI_n + \log(\mathcal{M}))\,(iI_n - \log(\mathcal{M}))^{-1}\right) \end{pmatrix} \\ \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} \end{pmatrix}$$
$$=$$
$$\mathrm{rank}\begin{pmatrix} iI_n - \begin{pmatrix} \mathrm{Re}\left((iI_n + \log(\mathcal{M}))\,(iI_n - \log(\mathcal{M}))^{-1}\right) & \mathrm{Im}\left((iI_n + \log(\mathcal{M}))\,(iI_n - \log(\mathcal{M}))^{-1}\right) \\ -\mathrm{Im}\left((iI_n + \log(\mathcal{M}))\,(iI_n - \log(\mathcal{M}))^{-1}\right) & \mathrm{Re}\left((iI_n + \log(\mathcal{M}))\,(iI_n - \log(\mathcal{M}))^{-1}\right) \end{pmatrix} \\ \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} \end{pmatrix}.$$

Since, for all P, the positive diagonal matrices, we posses that

$$\lambda_k\left[(iI_n + log(\mathcal{M}))^{-1}(iI_n - log(\mathcal{M}))\Delta\right] \neq 0,\ \forall k = 1 : n.$$

Thus, finally we have

$$0 \leq \mu_{\mathbb{B}}\left[(iI_n + log(\mathcal{M}))^{-1}(iI_n - log(\mathcal{M}))\right] < 1,$$

this implies that $\mathcal{M} \in \mathbb{C}^{n,n}$ is strongly $\mathcal{D}$-stable matrix. □

The following Theorem 4.5 shows the strong $\mathcal{D}$-stability of a considered n-dimensional complex valued matrix. Further, we prove that matrix $(\log(\mathcal{M}) + (\log(\mathcal{M}) \otimes A)^* \Delta + \Delta (\log(\mathcal{M}) \otimes A))$ is a D-stable matrix.

Theorem 4.5. *Let $\mathcal{M} \in \mathbb{C}^{n,n}$, and $A \in \mathbb{C}^{n,n}$ (Hermitian matrix). Then $\mathcal{M}$ is strongly $\mathcal{D}$-stable if $\mathcal{M}$ is stable, and for some $\alpha > 0$,*

$$(log(\mathcal{M}) + (log(\mathcal{M}) \otimes A)^* \Delta + \Delta (log(\mathcal{M}) \otimes A))$$

is a $\mathcal{D}$-stable matrix, with $\Delta \in \mathbb{B}$, and $\otimes$ denotes the entry-wise product of two matrices.

Proof. Let $\Delta = \Delta(t) \in \mathbb{B}$ be a valid perturbation from a collection of block-diagonal matrices $\mathbb{B}$. The structure of $\mathbb{B}$ is such that

$$\mathbb{B} = \{\mathrm{diag}(\delta_1(t), \delta_2(t), \ldots, \delta_n(t)) : \delta_i(t) \in \mathbb{C}, |\delta_i(t)| \leq 1\}.$$

Assume that $\lambda(t) = |\lambda(t)|e^{i\theta}$, $0 \leq \theta \leq 2\pi$ is greatest eigenvalue with algebraic multiplicity 1 corresponding to matrix-valued function $(\log(\mathcal{M}) \otimes A)^* \Delta + \Delta (\log(\mathcal{M}) \otimes A)$, $\Delta \in \mathbb{B}, \varepsilon > 0$. Consider that $x(t), y(t)$ are right and left eigenvectors corresponding to $\lambda(t)$. Further, assume that

$$Z = (\log(\mathcal{M}) \otimes A)^* \Delta + \Delta (\log(\mathcal{M}) \otimes A) \cdot y(t).$$

The outcome of Kato's eigenvalue perturbation applied to $\lambda(t)$ is

$$\frac{d}{dt}|\lambda(t)|^2\bigg|_{t=0} = 2\varepsilon \frac{|\lambda(t)|}{\lambda} \mathrm{Re}\left(Z^* \dot{\Delta}(t) x\right),$$

where $\lambda = e^{i\theta} x > 0, \varepsilon > 0$, and $x = x(t), Z = Z(t)$. As, $\mathrm{Re}\left(Z^* \dot{\Delta}(t) x\right) > 0$, and in turn this further means that

$$(\log(\mathcal{M}) \otimes A)^* \Delta + \Delta (\log(\mathcal{M}) \otimes A) > 0, \varepsilon > 0,$$

means a positive definite matrix. This allows us to have

$$(\log(\mathcal{M}) + (\log(\mathcal{M}) \otimes A)^* \Delta + \Delta (\log(\mathcal{M}) \otimes A))$$

is a $\mathcal{D}$-stable matrix. □

The subsequent Theorem 4.6 demonstrates that a real-valued n-dimensional matrix $\mathcal{M}$, a Jacobi matrix, is a strongly $\mathcal{D}$-stable matrix if its j-th order principal minors are with $sign(-1)^j$.

Theorem 4.6. *Let $\mathcal{M} \in \mathbb{R}^{n \times n}$ be a Jacobi matrix. Then $\mathcal{M}$ is strongly $\mathcal{D}$-stable matrix if all j-th order principal minors have $sig(-1)^j$.*

Proof. For given $\mathcal{M} \in \mathbb{R}^{n \times n}$, we construct $\hat{\mathcal{M}} = P\mathcal{M}$ where $P = \mathrm{diag}(p_1, p_2, \ldots, p_n)$ with

$$p_1 = 1,\ p_2 = \prod_{i=2}^{k} \frac{m_{(i-1),i}}{m_{i,(i-1)}}, \quad p_k = p_{k+1} \frac{m_{k,(k+1)}}{m_{(k+1),k}}, \quad k = 2, 3, \ldots, n.$$

The matrix $\hat{\mathcal{M}}$ is also a Jacobi matrix with jth principal minors having $\mathrm{sig}(-1)^j$. This means that matrix $\hat{\mathcal{M}}+\hat{\mathcal{M}}^T$ is such that $\mathrm{Re}(\lambda_i(\hat{\mathcal{M}}+\hat{\mathcal{M}}^T))>0\ \forall i$, where T represents a matrix's transposition. This also suggests that $\lambda_i(P\mathcal{M}+\mathcal{M}^TP)>0$. Thus, $\exists$ a matrix Q such that $\|Q\|<\alpha$ so that

$$\lambda_i\Big((P(\mathcal{M}+Q)+\mathcal{M}+Q)^T P\Big)>0,$$

means that $\mathcal{M}$ is a strongly D-stable matrix. □

Theorem 4.7. *Let $\mathcal{M}\in\mathbb{C}^{n,n}$. Then $\mathcal{M}=e^A$ (A is Hermitian matrix) is strongly D-stable, and $\hat{\mathcal{M}}=PA$ for a diagonal matrix P, and $\hat{\mathcal{M}}$ satisfies*

$$\hat{m}_{ii}<-\sum_{\substack{j=1\\ j\neq i}}^{n}|\hat{m}_{ij}|,\quad 1\le i\le n.$$

Proof. The inequality

$$\hat{m}_{ii}<-\sum_{\substack{j=1\\ j\neq i}}^{n}|\hat{m}_{ij}|,\quad 1\le i\le n,$$

yields

$$\mu_{\mathbb{B}}(Pe^A)<0.$$

This implies that $\mu_{\mathbb{B}}(e^A)<0$. For given $M\in\mathbb{R}^{n\times n}$ to be strongly $\mathcal{D}$-stable, we have

$$\mu_{\mathbb{B}}(Pe^A)+\mu_{\mathbb{B}}(PQ)<0,$$

where $Q\in\mathbb{R}^{n,n}$ with $\|Q\|<\alpha$. In turn, this further implies that

$$\mathrm{Re}\left[\lambda_i\left(P(e^A+Q)\right)\right]\le\mu_{\mathbb{B}}\left(P(e^A+Q)\right)\le\mu_{\mathbb{B}}(Pe^A)+\mu_{\mathbb{B}}(PQ)<0.$$

This is further equivalent to

$$p_{ii}(m_{ii}+q_{ii})<-\left(\sum_{j\neq i}|p_{ii}m_{ij}|+\sum_{j\neq i}|p_{ii}q_{ij}|\right)\le-\sum_{j\neq i}|p_{ii}(m_{ij}+q_{ij})|<0.$$

Thus, finally, we have that

$$e^A+Q,$$

a $\mathcal{D}$-Stable matrix, and hence $\mathcal{M}=e^A$ is a strongly $\mathcal{D}$-stable matrix. □

5. Numerical experimentation

In this segment, we provide numerical experiments on spectrum computation, i.e., the pseudo-spectrum, μ-values, singular values, and eigenvalues of structured matrices that appear in economic models. The graphical representations of the pseudo-spectrum denote the level sets for the resolvent norm $\|(zI_n-\mathcal{M})^{-1}\|$ for the given matrix $\mathcal{M}$. Here I_n denotes an $n\times n$ identity matrix.

Example 1. We take 4×4 the composite coefficient matrix to the workhorse New Keynesian model [21]

$$\mathcal{M}_1 = \begin{bmatrix} 1.5057 & -0.5355 & -3.2133 & -0.0607 \\ -0.0648 & 1.4821 & 0 & 0.0648 \\ -0.0039 & 0.0576 & 1.7171 & 0.0043 \\ 0.5847 & -1.9717 & -5.4962 & 5.4962 \end{bmatrix}.$$

For the offered matrix $\mathcal{M}_1$, Figure 1 shows the spectrum, singular values, structured singular values, and pseudo-spectrum calculations.

We plot the eigen-mode corresponding to the eigenvalues of $\mathcal{M}_1$ in Figure 2 (left-hand figure). In Figure 2 (left-hand figure), the top plot shows an envelope produced by plotting the absolute value of an eigen-mode. The cyan line shows the real part. The plot at the bottom level of Figure 2 (left-hand figure) shows the absolute value of the eigen-mode, and it is plotted on a log scale. The condition number is shown in the top plot. The larger value of the condition number implies the greater sensitivity of the eigenvalue to perturbations.

In Figure 2 (Right-hand figure), we plot the value of the inverse of the resolvent norm. We further show the real part of the pseudo-mode in magenta. In pseudo-mode, the right singular vector for the matrix $zI - \mathcal{M}$'s least singular value has been displayed.

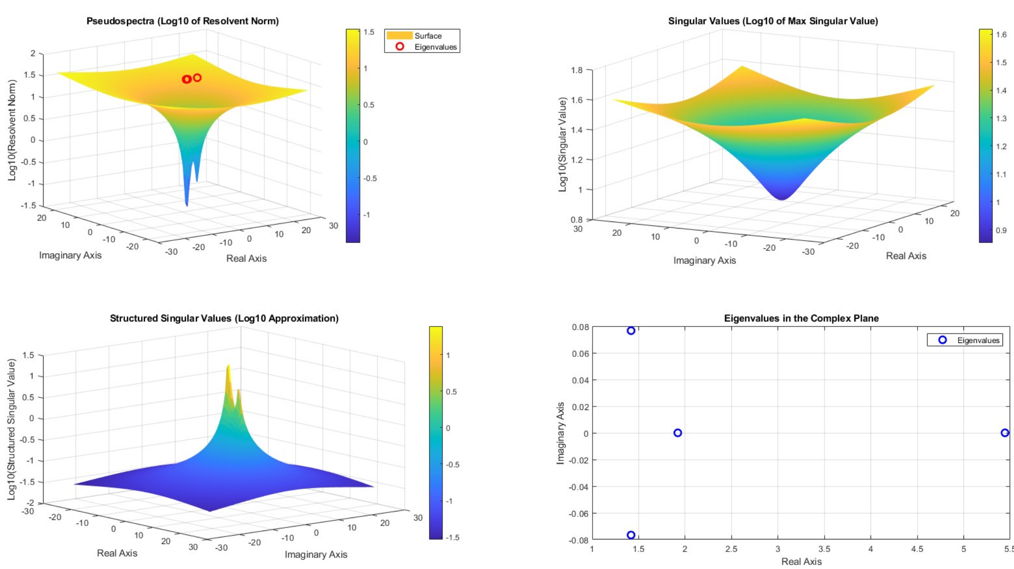

Figure 1. Spectral properties of matrix $\mathcal{M}_1$

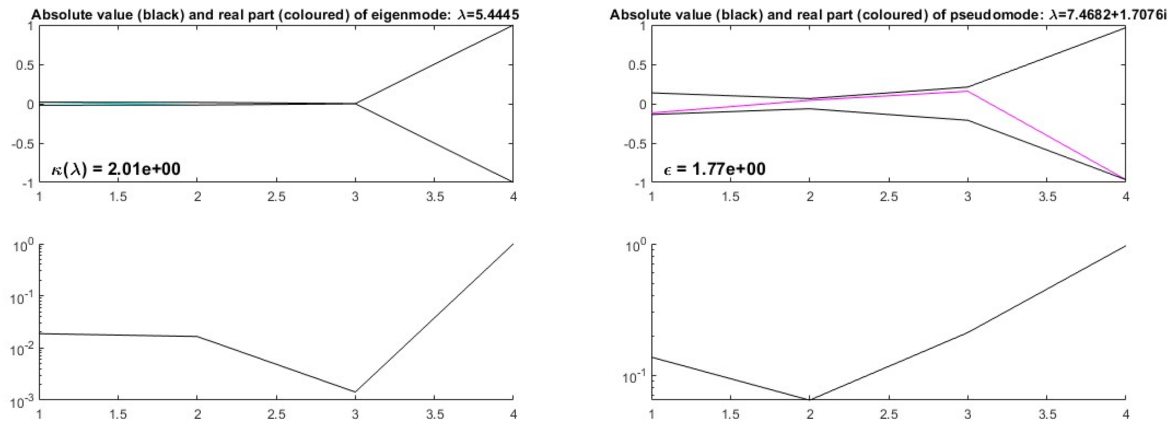

FIGURE 2. Eigenmode (left) and inverse of resolvent norm (right) of matrix $\mathcal{M}_1$

Example 2. We take 4×4 the composite coefficient matrix to the workhorse New Keynesian model [21]

$$\mathcal{M}_2 = \begin{bmatrix} -0.5253 & 0.2655 & 2.4934 & 0.0179 \\ 0.0222 & -0.4768 & 0 & -0.0222 \\ 0.0006 & -0.0367 & -0.7254 & -0.0009 \\ -0.2740 & 1.0036 & 4.2124 & -0.1464 \end{bmatrix}.$$

For the provided matrix $\mathcal{M}_2$, Figure 3 shows the spectrum, singular values, structured singular values, and pseudo-spectrum calculations.

We plot the eigen-mode corresponding to the eigenvalues of $\mathcal{M}_2$ in Figure 4 (left-hand figure). In Figure 4 (Left-hand figure), the top plot shows an envelope produced by plotting the absolute value of an eigen-mode. The cyan line shows the real part. The plot at the bottom level of Figure 4 (left-hand figure) shows the absolute value of the eigen-mode, and it is plotted on a log scale. The condition number is shown in the top plot. The larger value of the condition number implies the greater sensitivity of the eigenvalue to perturbations.

In Figure 4 (right-hand figure), we plot the value of the inverse of the resolvent norm. We further show the real part of the pseudo-mode in magenta. The right singular vector for the least singular value of the matrix $zI - \mathcal{M}$ has been shown in pseudo-mode.

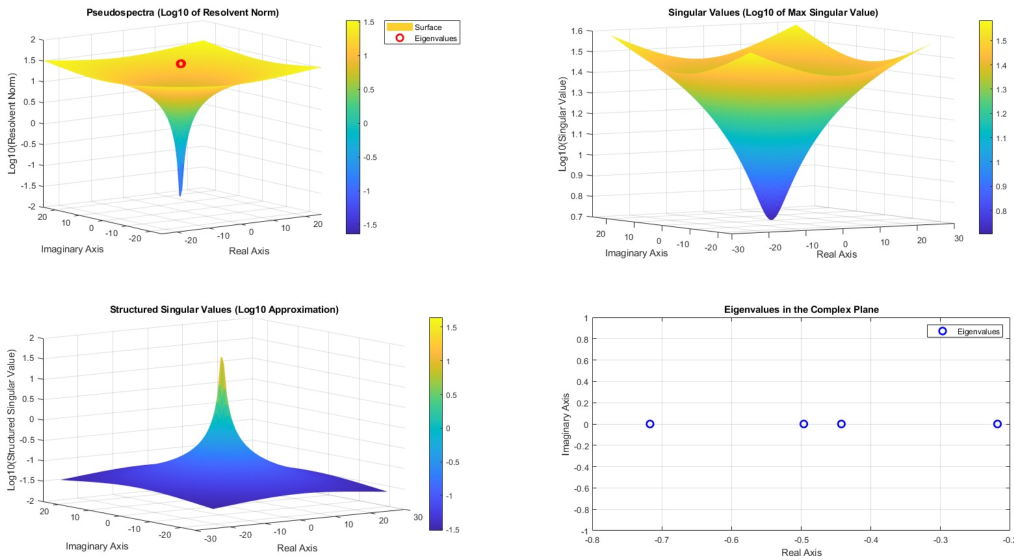

Figure 3. Spectral properties of matrix $\mathcal{M}_2$

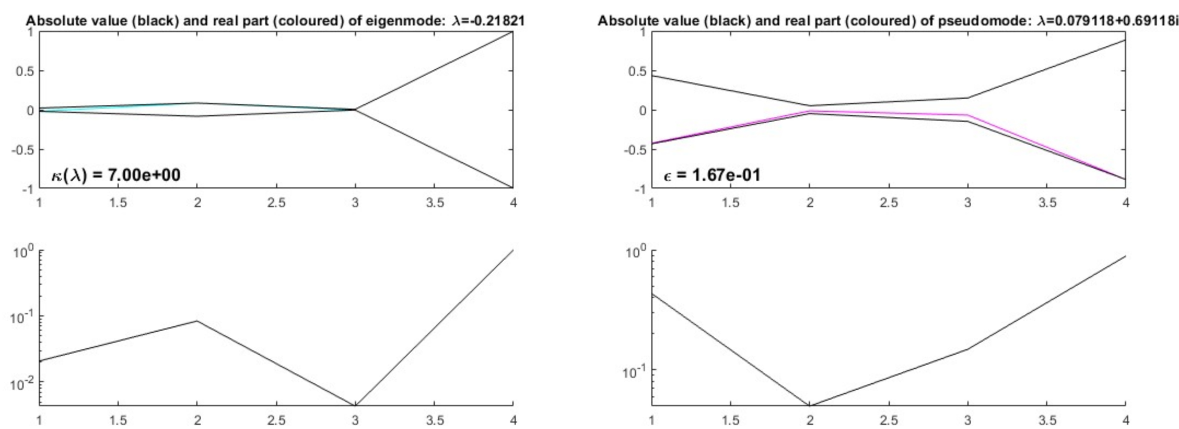

Figure 4. Eigenmode (left) and inverse of resolvent norm (right) of matrix $\mathcal{M}_2$

Example 3. We take 4×4 the composite coefficient matrix to the workhorse New Keynesian model [21]

$$\mathcal{M}_3 = \begin{bmatrix} 0.0282 & 0.0203 & -0.0405 & -0.0132 \\ -0.0263 & 0.0048 & 0.0312 & -0.0055 \\ -0.0026 & 0.0004 & 0.0029 & 0.0006 \\ 0.2050 & 0 & -0.2117 & 0 \end{bmatrix}.$$

Figure 5 shows the spectrum, singular values, structured singular values, and pseudo-spectrum calculations for the provided matrix $\mathcal{M}_3$.

We plot the eigen-mode corresponding to the eigenvalues of $\mathcal{M}_3$ in Figure 6 (left-hand figure). In Figure 6 (Left-hand figure), the top plot shows an envelope produced by plotting the absolute value of an eigen-mode. The cyan line shows the real part. The plot at the bottom level of Figure 6 (left-hand figure) shows the absolute value of the eigen-mode, and it is plotted on a log scale. The condition number is shown in the top plot. The larger value of the condition number implies the greater sensitivity of the eigenvalue to perturbations.

In Figure 6 (right-hand figure), we plot the value of the inverse of the resolvent norm. We further show the real part of the pseudo-mode in magenta. The right singular vector for the lowest singular value of the matrix $zI - \mathcal{M}$ has been shown in pseudo-mode.

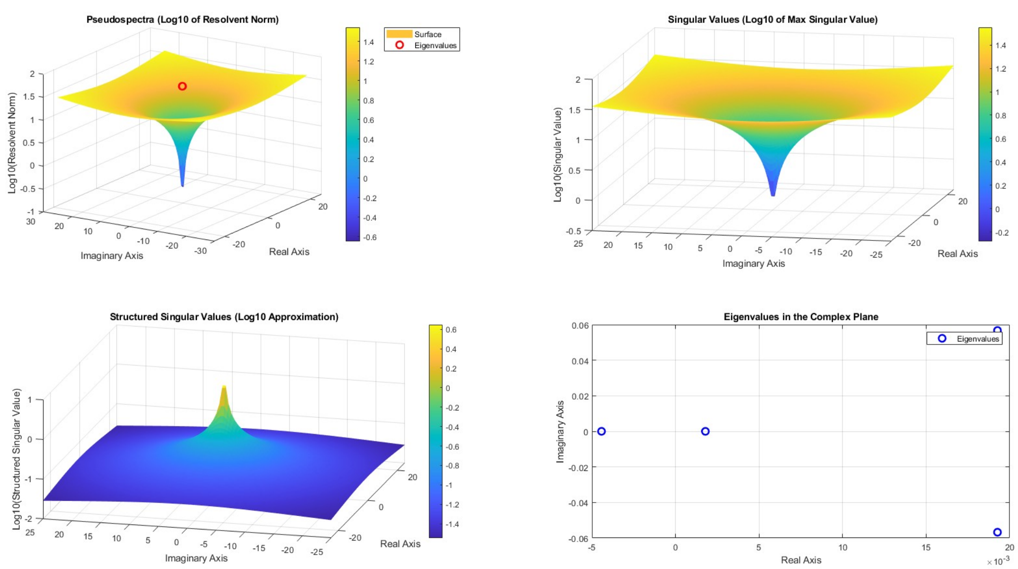

FIGURE 5. Spectral properties of matrix $\mathcal{M}_3$

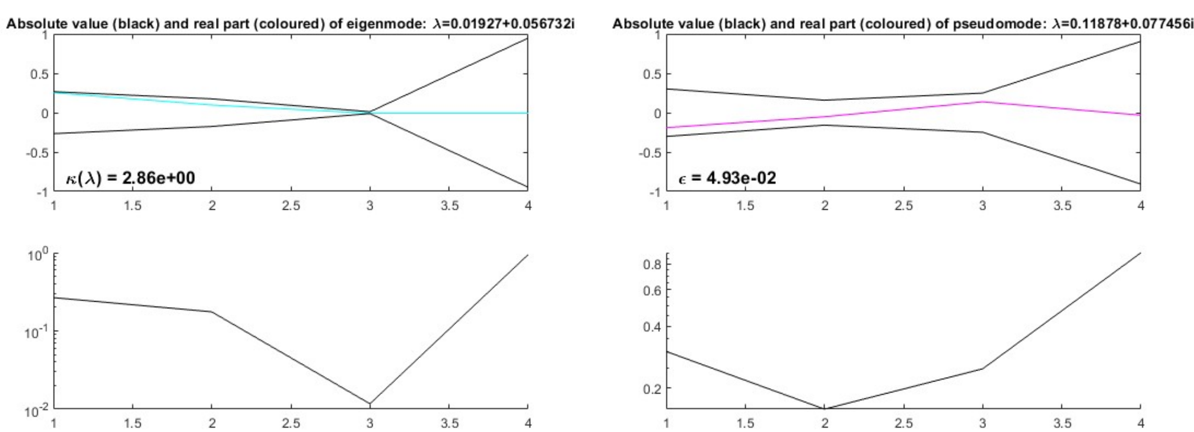

FIGURE 6. Eigenmode (left) and inverse of resolvent norm (right) of matrix $\mathcal{M}_3$

6. Conclusion

In this paper, we have studied stability analysis and strong $\mathcal{D}$-stability analysis of economic models. We have studied and analyzed the stability considering that each of the matrix's eigenvalues has a strictly positive real part, while the real components of each matrix-product eigenvalue of a given positive diagonal matrix also remain strictly positive. The new theoretical and computational results are developed by using various elements from numerical linear algebra, matrix analysis, and system theory. The numerical experimentation shows how the spectrum of structured matrices behaves. This includes the analysis of the behavior of the spectrum, singular values, pseudo-spectrum, and the results on lower bounds of μ-values allows us to discuss and analyze the stability of economic models.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] J. Chen, M.K. Fan, C. Yu, On D-Stability and Structured Singular Values, Syst. Control. Lett. 24 (1995), 19–24. https://doi.org/10.1016/0167-6911(94)00036-u.

[2] A.C. Enthoven, K.J. Arrow, A Theorem on Expectations and the Stability of Equilibrium, Econometrica 24 (1956), 288–293. https://doi.org/10.2307/1911633.

[3] E.H. Abed, Strong *D*-Stability, Syst. Control. Lett. 7 (1986), 207–212. https://doi.org/10.1016/0167-6911(86)90116-7.

[4] E.H. ABED, A.L. TITS, On the Stability of Multiple Time-Scale Systems, Int. J. Control. 44 (1986), 211–218. https://doi.org/10.1080/00207178608933591.

[5] H.K. Khalil, P.V. Kokotovic, *D*-Stability and Multi-Parameter Singular Perturbation, SIAM J. Control. Optim. 17 (1979), 56–65. https://doi.org/10.1137/0317006.

[6] D.D. Siljak, Stability of Large-Scale Systems Under Structural Perturbations, in: IEEE Transactions on Systems, Man, and Cybernetics, Institute of Electrical and Electronics Engineers (IEEE), 1972, pp. 657-663. https://doi.org/10.1109/TSMC.1972.4309194.

[7] C.R. Johnson, Sufficient Conditions for *D*-Stability, J. Econ. Theory 9 (1974), 53–62. https://doi.org/10.1016/0022-0531(74)90074-x.

[8] C. Yu, M.K. Fan, Decentralized Integral Controllability and D-Stability, Chem. Eng. Sci. 45 (1990), 3299–3309. https://doi.org/10.1016/0009-2509(90)80221-y.

[9] J. Doyle, Analysis of Feedback Systems with Structured Uncertainties, IEE Proc. D Control. Theory Appl. 129 (1982), 242–250. https://doi.org/10.1049/ip-d.1982.0053.

[10] R. Braatz, P. Young, J. Doyle, M. Morari, Computational Complexity of *mu* Calculation, IEEE Trans. Autom. Control. 39 (1994), 1000–1002. https://doi.org/10.1109/9.284879.

[11] R.L. Dailey, A New Algorithm for the Real Structured Singular Value, in: 1990 American Control Conference, IEEE, 1990, pp. 3036-3040. https://doi.org/10.23919/ACC.1990.4791276.

[12] A. Yazıcı, A. Karamancıoğlu, R. Kasimbeyli, A Nonlinear Programming Technique to Compute a Tight Lower Bound for the Real Structured Singular Value, Optim. Eng. 12 (2010), 445–458. https://doi.org/10.1007/s11081-010-9120-4.

[13] D. Piga, A. Benavoli, A Unified Framework for Deterministic and Probabilistic *D*-Stability Analysis of Uncertain Polynomial Matrices, IEEE Trans. Autom. Control. 62 (2017), 5437–5444. https://doi.org/10.1109/tac.2017.2699281.

[14] N. Guglielmi, M. Rehman, D. Kressner, A Novel Iterative Method to Approximate Structured Singular Values, SIAM J. Matrix Anal. Appl. 38 (2017), 361–386. https://doi.org/10.1137/16m1074977.

[15] J. Chen, M.K. Fan, C. Yu, On *D*-Stability and Structured Singular Values, Syst. Control. Lett. 24 (1995), 19–24. https://doi.org/10.1016/0167-6911(94)00036-u.

[16] M. Rehman, J. Alzabut, M. Tayyab, F. Amir, Interconnection Between Schur Stability and Structured Singular Values, Contemp. Math. 6 (2024), 63–72. https://doi.org/10.37256/cm.6120255707.

[17] R.W. Brockett, Finite Dimensional Linear Systems, Wiley, 1970.

[18] T. Kato, Perturbation Theory for Linear Operators, Springer, 1966. https://doi.org/10.1007/978-3-642-66282-9.

[19] J. Lee, T.F. Edgar, Real Structured Singular Value Conditions for the Strong D-Stability, Syst. Control. Lett. 44 (2001), 273–277. https://doi.org/10.1016/s0167-6911(01)00147-5.

[20] M. Rehman, T.H. Rasulov, F. Amir, *D*-Stability, Strong *D*-Stability and μ-Values, Lobachevskii J. Math. 45 (2024), 1227–1233. https://doi.org/10.1134/s1995080224600754.

[21] E. Martínez-García, A Matter of Perspective: Mapping Linear Rational Expectations Models into Finite-Order VAR Form, Globalization Institute Working Paper 389, Federal Reserve Bank of Dallas, (2020). https://doi.org/10.24149/gwp389.

[22] K.J. Arrow, M. McManus, A Note on Dynamic Stability, Econometrica 26 (1958), 448–454. https://doi.org/10.2307/1907624.