Int. J. Anal. Appl. (2025), 23:297

International Journal of Analysis and Applications

D-Stability and Structured Singular Values Analysis and Applications to Economic Models

Mutti-Ur Rehman[1,2,*], Abdukholik Mukhtorov[3], Nazimjon Askarov[3], Muqaddas Turdiyeva[4], Shahlokhon Gafurova[4], Zufar Qoryog'diyev[1], Ghulam Abbas Memon[5]

[1]*Center of Research and Innovation, Asia International University, 200100, Bukhara, Uzbekistan*
[2]*Department of Mathematics and Informatics, Al-Khwarizmi University, 220100, Khorezm, Uzbekistan*
[3]*I-SCAD, Labzak, 1A, 100140, Tashkent, Uzbekistan*
[4]*Department of Economics, Bukhara State University, M.Ikbal, 200117, Bukhara, Uzbekistan*
[5]*Department of Mathematics, Degree College Pir Jo Goth, Sindh, Pakistan*

[*]*Corresponding author:* mutti.abbasi@gmail.com, m.rehman@akhu.uz

Abstract. The stability analysis of transportation and economic models plays a fundamental role in understanding dynamic systems. The term stability means the ability of a system under consideration to return to an equilibrium state subject to perturbations. The analysis on D-stability extends the concept of stability by ensuring that the system is stable under a predefined set of various uncertainties. The computation of structured singular values plays a critical role in analyzing the dynamical system's robustness and performance. For transportation models, structured singular value analysis helps evaluate traffic demand fluctuations. On the other hand, in economic systems, structured singular values aid in understanding the impacts of interest rates or supply chain disruptions on the performance and stability of the system. This article discusses the interplay between stability analysis, D-stability analysis, and structured singular values, and then emphasizes their applications to transportation and economic models. The aim is to study how one can ensure a robust and resilient system design. Through practical and numerical examples, the study illustrates how these concepts may set up a mathematical foundation for advancing robust modeling practices in dynamic and uncertain environments.

1. Introduction

This article is concerned with the study of recent mathematical methods to analyze the spectral characterization of structured matrices appearing across economic and transportation models.

Received: Sep. 3, 2025.

2020 *Mathematics Subject Classification.* 15A18, 15A16.

Key words and phrases. stability; D-stability; singular values; structured singular values; transportation models; mathematical optimization.

https://doi.org/10.28924/2291-8639-23-2025-297

ISSN: 2291-8639

The primary focus is to study and analyze the interconnection between μ-values, stability, and D-stability of transportation and economic models. The μ-value is a positive homogeneous function quantifying the effect of uncertainty on a linear dynamical system. The μ-values for the class of complex or real-valued matrices, and an uncertainty set, that is, the set of block diagonal uncertainties, were first mentioned in a classical paper by J. C. Doyle. The determination of an exact certainty about the description of the physical world is almost impossible, even by using the most powerful mathematical techniques. For instance, a bus station is located about 5 kilometers away from a certain predefined position, and a packet of sugar taken from a shop is round about 1 kilogram in weight. The mathematical treatment and measurements of these physical quantities mainly depend upon the accuracy of the instrument capable of measurement. The same kind of principle is applicable for the modeling to analyze a physical system under consideration.

For the structured uncertainty and μ-value analysis, there exist several different sources of uncertainties. The mathematical quantity $\Delta_i(jw)$ represent uncertain elements across the dynamical system. The elements of $\Delta_i(jw)$ may be consider as values varying linearly in time. The block diagonal structure Δ_i are called as the complex uncertainties or perturbation corresponding to dynamical system. The block structures Δ_i are called as real block diagonal uncertainties for $\Delta_i \in [-1, 1]$. On the other hand, in the presence of real and complex uncertainties, the uncertainties $\Delta_i s$ are called as mixed type of uncertainties. To each of $\Delta_i = 0$, we label it as a nominal system, while on the other hand if $\Delta_i \neq 0$, the dynamical system is label as a perturbed system.

The computation of μ-value maybe considered as a straightforward generalization to singular values for coefficient matrices across the dynamical system. It does provides tools in order to analyze, synthesize the robustness and performance of dynamical systems. The μ-value tool is further helpful for stability analysis of structured or unstructured eigenvalue perturbation theory. This tool can also be used for the analysis of uncertain linear dynamical systems. The study and analysis of robust stability problems corresponding to feedback interconnection for a class of stable structured matrices $M(s)$ and $\Delta(s)$ for some given $s \in \mathbb{C}_+$ is also an important research topic in system theory.

The instability analysis of a dynamical system may be directly linked with the measurement of the mathematical quantity $(I_n - M\,\Delta)$, and it must remain singular, Here, I_n denotes an identity matrix having its dimension similar to given matrix M and an uncertainty Δ. This restrict the selection of uncertainty Δ with $\|\Delta\|_\infty$ which causes the closed loop system to be unstable. The quantity $\|\Delta\|_\infty < \alpha$, $\alpha \in \mathbb{R}$, $\alpha > 0$ for the stability analysis of dynamical system under consideration.

The small increment in α up to α_{max} may allow the dynamical system to remain unstable. The computation of α_{max} may yield the robust stability of under consideration dynamical system. The stability of dynamical system may depends upon the fact that mathematical quantity $(I - M\,\Delta)$ is such that M has atleast one of its eigenvalue to be exactly equal to zero. We describe a few specific cases below:

Case-1: The uncertainty Δ from set of block diagonal matrices acts as a stable transfer function matrix and it does describes either dynamical system is well-posed and internally stable or not. This is described by following Small Gain Theorem.

Theorem 1.1. *(Small Gain Theorem). The dynamical system is well-posed as well as stable for each Δ from the set of block diagonal matrices having $\bar{\sigma}(\Delta) \leq 1$ if and only if,*

$$\frac{1}{\alpha_{max}} = \|M\|_{\infty} := \max\left(\bar{\sigma}\left(M\right)\right) < 1.$$

The mathematical quantity $\bar{\sigma}(M)$ is the maximum singular value of a stable matrix M.

Case-2: The structured uncertainty Δ is a family of complex or real block diagonal matrices or a family of structured stable transfer matrices.

By Small Gain Theorem,

$$\frac{1}{\alpha_{max}} = \|M\|_{\infty} := \max\left(\bar{\sigma}(M)\right).$$

For structured Δ, $\bar{\sigma}(M)$ maybe presented as,

$$\bar{\sigma}(M) = \left(min\left\{\bar{\sigma}(\Delta) : det(I - M\,\Delta) = 0\right\}\right)^{-1}. \tag{1.1}$$

The concept of computing the singular value $\bar{\sigma}(M)$ needs to be generalized for the case of structured Δ which destabilizes the feedback system. The quantity $\mu(M)$ with respect to a structured Δ is defined as

$$\mu_{\Delta}(M) := \left(min\left\{\bar{\sigma}(\Delta) : det(I - M\,\Delta) = 0\right\}\right)^{-1}. \tag{1.2}$$

The robust margin α_{max} of feedback system with uncertainty Δ is,

$$\frac{1}{\alpha_{max}} = \max\left(\mu_{\Delta}(M)\right).$$

The quantity $\mu_{\Delta}(M)$ as defined in Equ. (2) known as to be μ-value of constant matrix M with respect to perturbation Δ.

The D-stability theory for real-valued square matrices [1] is a mathematical tool for the analysis of the models to study and perform analysis on the competitiveness in the markets. The D-stability theory is a versatile, and a fundamental concept from system theory. It generalizes the classical notions of matrix stability to define stability regions in the domain of complex plane $\mathbb{C}$. The D-stability theory further generalizes left-half plane for a class of continuous-time dynamical system. Furthermore, it generalized the unit circle for discrete-time dynamical systems. The D-stability theory discuss the stability of a dynamical systems subject to structured or unstructured perturbations.

In various economic models the D-stable matrices appears in input-output setting, for example, consider Leontief production model. The D-stability theory analyze the robustness and performance of equilibrium states across an economy models. The input-output matrix corresponding to an economic model is a D-stable matrix suggesting that the economy can withstand across various changes to factor productivity. The D-stable matrices does appears in economic growth models.

An interesting example could be multi-sector growth models. Mathematically, D-stable matrices represent the Jacobian of an equilibrium state corresponding to a macroeconomic systems in order to study stability. This further analyze the interactions among various sectors of the economy, for example, consumption, production, etc.

The structured D-stable matrices, and structured additive D-stable matrices have very many important properties and are of interests of research in many research directions but not limited to science and engineering. This particularly includes study and analysis of mathematical techniques in economics, dynamical analysis of population models, analysis of neural networks, electrical engineering, and the control engineering, we recommend interested reader to see [2–9] and the references therein. The interconnection between structured singular values, and D-stability analysis for a class of real-valued n-dimensional square matrix as presented in a great detail in [10]. A more simpler condition to analyze the strong D-stability was derived in [11].

The H-stability analysis, and H-semistability analysis being as two strong and effective notions of structured matrix stability were studied in [12]. Theoretical results were established for study of both necessary, and sufficient conditions to a given matrix to be structured H-stable or structured H-semistable. The study and analysis of H-stability does imply the stability for continuous-time linear dynamical systems. The H-stability has various important applications in engineering discipline, this includes but not limited to: Control engineering, electrical engineering, mechanical engineering. Furthermore, H-stability allow to study the modeling and analysis of economic problems. The H-stability implies that coefficient matrix to given linear dynamical system may have negative real parts for all of its eigenvalues.

The $D(\alpha)$ structured stability for choice of $\alpha = (\alpha_1, \alpha_1, \cdots, \alpha_p)$ was studied by Khalil and Kokotovic [13,14]. The study and analysis of $D(\alpha)$ stability is effective to very many important dynamical problems, for instance, time-invariant multi-parameter singular perturbations problems, specially, the dynamical systems represented by mathematical equation of the form $E(\epsilon)\dot{z} = Dz$. For the structured uncertainty and μ-value analysis, there exist several different sources of uncertainties. The mathematical quantity $\Delta_i(jw)$ represent uncertain elements across the dynamical system. The elements of $\Delta_i(jw)$ may be consider as values varying linearly in time. The block diagonal structure Δ_i are called as the complex uncertainties or perturbation corresponding to dynamical system. The block structures Δ_i are called as real block diagonal uncertainties for $\Delta_i \in [-1, 1]$. On the other hand, in the presence of real and complex uncertainties, the uncertainties $\Delta_i s$ are called as mixed type of uncertainties. To each of $\Delta_i = 0$, we label it as a nominal system, while on the other hand if $\Delta_i \neq 0$, the dynamical system is label as a perturbed system.

The computation of μ-value maybe considered as a straightforward generalization to singular values for coefficient matrices across the dynamical system. It does provides tools in order to analyze, synthesize the robustness and performance of dynamical systems. The μ-value tool is further helpful for stability analysis of structured or unstructured eigenvalue perturbation theory. This tool can also be used for the analysis of uncertain linear dynamical systems. The study and analysis of robust stability problems corresponding to feedback interconnection for a class of stable structured matrices $M(s)$ and $\Delta(s)$ for some given $s \in \mathbb{C}_+$ is also an important research topic in system theory.

The instability analysis of a dynamical system may be directly linked with the measurement of the mathematical quantity $(I_n - M\,\Delta)$, and it must remain singular, Here, I_n denotes an identity matrix having its dimension similar to given matrix M and an uncertainty Δ. This restrict the selection of uncertainty Δ with $\|\Delta\|_\infty$ which causes the closed loop system to be unstable. The quantity $\|\Delta\|_\infty < \alpha, \alpha \in \mathbb{R}, \alpha > 0$ for the stability analysis of dynamical system under consideration.

The small increment in α up to α_{max} may allow the dynamical system to remain unstable. The computation of α_{max} may yield the robust stability of under consideration dynamical system. The stability of dynamical system may depends upon the fact that mathematical quantity $(I - M\,\Delta)$ is such that M has atleast one of its eigenvalue to be exactly equal to zero. We describe a few specific cases below:

Case-1: The uncertainty Δ from set of block diagonal matrices acts as a stable transfer function matrix and it does describes either dynamical system is well-posed and internally stable or not. This is described by following Small Gain Theorem.

Theorem 1.2. *(Small Gain Theorem). The dynamical system is well-posed as well as stable for each Δ from the set of block diagonal matrices having $\bar{\sigma}(\Delta) \leq 1$ if and only if,*

$$\frac{1}{\alpha_{max}} = \|M\|_\infty := \max\left(\bar{\sigma}\,(M)\right) < 1.$$

The mathematical quantity $\bar{\sigma}(M)$ is the maximum singular value of a stable matrix M.

Case-2: The structured uncertainty Δ is a family of complex or real block diagonal matrices or a family of structured stable transfer matrices.

By Small Gain Theorem,

$$\frac{1}{\alpha_{max}} = \|M\|_\infty := \max\left(\bar{\sigma}(M)\right).$$

For structured Δ, $\bar{\sigma}(M)$ maybe presented as,

$$\bar{\sigma}(M) = \left(min\left\{\bar{\sigma}(\Delta) : det(I - M\,\Delta) = 0\right\}\right)^{-1}. \tag{1.3}$$

The concept of computing the singular value $\bar{\sigma}(M)$ needs to be generalized for the case of structured Δ which destabilizes the feedback system. The quantity $\mu(M)$ with respect to a structured Δ is defined as

$$\mu_\Delta(M) := \left(min\left\{\bar{\sigma}(\Delta) : det(I - M\,\Delta) = 0\right\}\right)^{-1}. \tag{1.4}$$

The robust margin α_{max} of feedback system with uncertainty Δ is,

$$\frac{1}{\alpha_{max}} = \max\left(\mu_\Delta(M)\right).$$

The quantity $\mu_\Delta(M)$ as defined in Equ. (2) known as to be μ-value of constant matrix M with respect to perturbation Δ.

The D-stability theory for real-valued square matrices [1] is a mathematical tool for the analysis of the models to study and perform analysis on the competitiveness in the markets. The D-stability theory is a versatile, and a fundamental concept from system theory. It generalizes the classical notions of matrix stability to define stability regions in the domain of complex plane $\mathbb{C}$. The D-stability theory further generalizes left-half plane for a class of continuous-time dynamical system. Furthermore, it generalized the unit circle for discrete-time dynamical systems. The D-stability theory discuss the stability of a dynamical systems subject to structured or unstructured perturbations.

In various economic models the D-stable matrices appears in input-output setting, for example, consider Leontief production model. The D-stability theory analyze the robustness and performance of equilibrium states across an economy models. The input-output matrix corresponding to an economic model is a D-stable matrix suggesting that the economy can withstand across various changes to factor productivity. The D-stable matrices does appears in economic growth models. An interesting example could be multi-sector growth models. Mathematically, D-stable matrices represent the Jacobian of an equilibrium state corresponding to a macroeconomic systems in order to study stability. This further analyze the interactions among various sectors of the economy, for example, consumption, production, etc.

The structured D-stable matrices, and structured additive D-stable matrices have very many important properties and are of interests of research in many research directions but not limited to science and engineering. This particularly includes study and analysis of mathematical techniques in economics, dynamical analysis of population models, analysis of neural networks, electrical engineering, and the control engineering, we recommend interested reader to see [2–9] and the references therein. The interconnection between structured singular values, and D-stability analysis for a class of real-valued n-dimensional square matrix as presented in a great detail in [10]. A more simpler condition to analyze the strong D-stability was derived in [11].

The H-stability analysis, and H-semistability analysis being as two strong and effective notions of structured matrix stability were studied in [12]. Theoretical results were established for study of both necessary, and sufficient conditions to a given matrix to be structured H-stable or structured H-semistable. The study and analysis of H-stability does imply the stability for continuous-time linear dynamical systems. The H-stability has various important applications in engineering discipline, this includes but not limited to: Control engineering, electrical engineering, mechanical engineering. Furthermore, H-stability allow to study the modeling and analysis of economic problems. The H-stability implies that coefficient matrix to given linear dynamical system may have negative real parts for all of its eigenvalues.

The $D(\alpha)$ structured stability for choice of $\alpha = (\alpha_1, \alpha_1, \cdots, \alpha_p)$ was studied by Khalil and Kokotovic [13,14]. The study and analysis of $D(\alpha)$ stability is effective to very many important dynamical problems, for instance, time-invariant multi-parameter singular perturbations problems, specially, the dynamical systems represented by mathematical equation of the form $E(\epsilon)\dot{z} = Dz$.

2. State-of-the-Art Methods to Compute μ-Values

In this section, we present review on some of well established mathematical techniques for the computation of μ-values. This includes to review three types of mathematical approaches: An exact computation, approximation techniques, and geometric based techniques to compute μ-values and its lower and upper bounds.

2.1. **Feedback systems analysis via structured perturbations.** A more generic technique to discuss and analyze the stability and instability of time varying linear dynamical systems which are subject to structured uncertainties consists upon generalized spectral theory of structured matrices [15]. This theory addresses a class of norm-bounded kind of uncertainty problem. For a given matrix $M \in \mathbb{C}^{n,n}$ *or* $\mathbb{R}^{n,n}$ having a block diagonal perturbation structure, a mathematical function associated with it is the computation of μ-value. The μ-value provides both necessary and sufficient conditions to structured perturbation problems subject to structured or unstructured uncertainties. A number of algebraic and geometric properties for μ function were developed over a long period of time while developing numerous mathematical techniques. These mathematical techniques were developed in order to determine the μ-value in some particular and an important cases. Further, for a better understanding, a detailed discussions was provided on the computation of μ-values.

Block-diagonal uncertainties: The analysis consisting of computation of singular values establishes a framework in order to develop the multi-loop generalizations to the classical single-loop mathematical techniques. The singular value techniques have their own limitations, for example, the analysis of linear multi-variable feedback system with two multiplicative uncertainties does appears at the input stage, and the output stage of dynamical system.

The dynamical system may be isolated at two perturbations/uncertainties resulting a single perturbation having two-block-diagonal structures. Then the block-diagonal perturbation is written as a one full matrix perturbation.

The differential sensitivity analysis to singular values at a single point relative to perturbations at other points is an extension to singular values. But, this does not holds true for the case of large perturbations, and finally this yields to directional sensitivity information.

The matrix problem to determine both necessary and sufficient conditions in such a way that $\lambda_i(I + M\Delta) \neq 0, \ \forall i$. The partial solution corresponding to general block-diagonal perturbation, and its solution to three or fewer blocks is presented in [15].

A number of important properties corresponding to μ-value are provided in [15], these properties include:

P_1 : For any matrix A, $\mu_{\mathbb{B}_1}(A) \geq 0$.

P_2 : For any matrix A, and $\forall \ \alpha \in \mathbb{C}$, $\mu_{\mathbb{B}_1}(\alpha A) = |\alpha| \mu_{\mathbb{B}_1}(A)$.

P_3 : $\mu_{\mathbb{B}_1}(AB) \leq \sigma_1(A)\mu_{\mathbb{B}_1}(B)$, with $\sigma_1(A)$ being as largest singular value of matrix A.

P_4 : $\mu_{\mathbb{B}_1}(\Delta) = \sigma_1(\Delta), \ \forall \ \Delta \in \mathbb{B}_1$.

P_5 : Let $\Delta_0 = \{\lambda I : \lambda \in \mathbb{C}\}$, the block-diagonal structure, then $\mu_{\mathbb{B}_1}(A) = \rho(A)$, with $\rho(A)$ being as spectral radius of matrix A.

P_6 : For $\hat{\Delta} = \{\Delta_0 : \Delta \in \mathbb{C}^{n,n}\}$, then $\mu_{\hat{\Delta}}(A) = \sigma_1(\Delta)$.

P_7 : $Sup\, \mu_{\mathbb{B}_1}(A) = \|A\|_\infty$.

P_8 : For $\Delta = \{diag(\Delta_1, \Delta_2, \cdots, \Delta_n) : \Delta_i \in \mathbb{C}^{n,n}\}$, then $\mu_\Delta(A) = \mu_\Delta(D^{-1}MD)$, $D = \{diag(d_1, \cdots, d_n),\ |d_i| \geq 0\}$.

P_9 : For $\hat{\Delta}_0 = \{diag(\Delta_1, \Delta_2, \cdots, \Delta_n) : \Delta_i \in \mathbb{C}^{n,n}\}$, then $\rho(A) < \mu_{\hat{\Delta}_0}(A) < \sigma_1(A)$.

P_{10} : From P_8, and P_9, we obtain an inequality $\mu_\Delta(A) = \mu_\Delta(D^{-1}AD) \leq inf\, \sigma_1(D^{-1}AD)$. The **inf** must be taken over D.

In [15], for the differentiability properties of singular values, the necessary tools were developed for the analysis and approximation to the gradients of singular values. The following novel results on the computation of $\mu-$values were established and analyzed.

Theorem 2.1. *The structured singular value is exactly equal to* 1 *if and only if* $0 \in \nabla_2(M)$, *with* $\nabla_2(M)$ *defined in [15].*

Theorem 2.2. *The computation of largest singular value is exactly equal to its structured singular value if and only if* $0 \in Co\, \nabla_2$, *with* $Co\, \nabla_2$ *defined in [15], and* $n \leq 3$.

We refer [15] to see two numerical examples, and a detailed discussions for the computational and numerical testings.

2.2. **An iterative method to yield the lower bounds of μ.** A novel iterative method for the approximation of μ-values was developed in [16]. The numerical method consist of two-level algorithm. That is an inner algorithm and an outer algorithm. The main objective in the inner algorithm is the formulation and then solving a system of ordinary differential equations (ODE's). The system of ODE's is corresponding to an optimization problem which is induced on the manifold defined by the structure. In outer algorithm, an iteration technique, in our case a fast Newton's iterative scheme is utilized for the approximation of desired perturbation level which is denoted with ϵ. The inner-outer algorithm computes the μ-values from below instead of above.

2.2.1. *Inner-Algorithm.* Theorem 2.3 is to compute an admissible perturbation $\Delta \in \mathbb{B}^*$, with $\mathbb{B}^*$ being as the set of block-diagonal structured uncertainties consisting only pure type of complex perturbations. We further refer to see [16] for the definitions of sets of block diagonal matrices.

Theorem 2.3. *Let* $\Delta_{opt} = diag\,(\delta_1 I_{r_1}, \delta_2 I_{r_2}, \cdots, \delta_2 I_{rS};\ \Delta_1, \Delta_2, \cdots, \Delta_F) : \|\Delta_{opt}\|_2 = 1$, *be a local extremizer of structured spectral value set* $\Delta_\epsilon^{\mathbb{B}}(M)$. *Additionally, we suppose* $\epsilon M \Delta_{opt}$ *has a largest simple eigenvalue* $\lambda = |\lambda| e^{\iota\theta}$, $0 \leq \theta \leq 2\pi$ *with right and left eigenvectors denoted as* x *and* y. *These eigenvectors are then scaled so that* $s = e^{\iota\theta} y^* x > 0$. *Also, consider partition the eigenvectors given as*

$$x = \left(x_1^T, \cdots, x_S^T;\ x_{S+1}^T, \cdots, x_{S+F}^T\right)^T,$$

and

$$z = M^* y = \left(z_1^T, \cdots, z_S^T;\ z_{S+1}^T, \cdots, z_{S+F}^T\right).$$

We consider $z_k^* x_k \neq 0,\ \forall\ k = 1, 2, \cdots, S$ *and* $\|z_{S+h}\|_2 \cdot \|x_{s+h}\|_2 \neq 0,\ \forall\ h = 1, 2, \cdots, F$. *Then, this further yields that* $|\delta_k| = 1,\ \forall\ k = 1, 2, \cdots, S$, *and* $\|\Delta_h\|_2 = 2,\ \forall\ h = 1, 2, \cdots, F$.

An important conclusion [16] is to replace the total number of full blocks having a rank-1 structured matrices. This further enable us to work with matrix Frobenius norm rather than working with matrix 2-norm. The reason to work with rank-1 matrices is that the matrix 2-norm, and matrix Frobeneius norm turns out to be exactly the same. This finally helps to search for al local extremizer too.

Theorem 2.4. *[16] Let* $\Delta_{opt} = diag\left(\delta_1 I_{r_1}, \delta_2 I_{r_2}, \cdots, \delta_2 I_{rS}; \Delta_1, \Delta_2, \cdots, \Delta_F\right)$ *be a local extremizer. Suppose the non-degeneracy conditions for all full blocks holds. Then, for each block of* Δ_h *possesses singular value which is being an exactly equal to* 1*, and the corresponding singular vectors are*

$$u_h = \gamma_h \frac{z_{S+h}}{\|z_{S+h}\|_2}, \qquad v_h = \gamma_h \frac{x_{S+h}}{\|x_{S+h}\|_2}; \quad for \quad |\gamma_h| = 1\,.$$

Further, matrix valued function of the form

$$\Delta_* = diag\left(\delta_1 I_{r_1}, \delta_2 I_{r_2}, \cdots, \delta_2 I_{rs};\ u_1 v_1^*, \cdots, u_F v_F^*\right)$$

is a local extremizer, implying that $\rho(\epsilon M \Delta_{opt}) = \rho(\epsilon M \Delta_*)$.

The system of ordinary differential equation (gradient system) is constructed and was solved in [16]. The optimial solution to the gradient system of ordinary differential equations yields a local extremizer on manifold $\mathbb{B}^*$:

$$\dot{\Delta} = D_1 P_{\mathbb{B}^*}(zx^*) - D_2 \Delta\,,$$

with $\|x(t)\|_2 = 1$, and is corresponding to simple eigenvalue (with algebraic multiplicity 1) $\lambda(t)$ of the perturbed matrix $\epsilon M \Delta(t)$, $\epsilon > 0$. The diagonal structured matrices $D_1(t)$, $D_2(t)$ depends on $\Delta(t)$, and $P_{\mathbb{B}^*}(zx^*)$ is being an orthogonal projection of the structured matrices zx^* on $\mathbb{B}^*$. Finally, stationary points of $\dot{\Delta}(t)$ are computed with following Theorem 6.

Theorem 2.5. *[16] Let* $\Delta(t)$*, and assume that* $\lambda(t)$ *be the maximum and simple non-zero eigenvalue corresponding to the perturbed matrix* $\epsilon M \Delta$*. Let* $x(t),\ y(t)$ *are right and left eigenvectors. Further, we consider that* $z(t) = M^* y(t)$*, then*

$$\frac{d}{dt}\left|\lambda(t)\right|^2 = 0 \Leftrightarrow \dot{\Delta}(t) = 0 \Leftrightarrow \Delta(t) = D\, P_{\mathbb{B}^*}\left(z(t)\, x^*(t)\right),$$

for a diagonal structured matrix $D \in \mathbb{B}^*$*. Additionally, assume that if* $z(t)$ *has maximum modulus over the set* $\Delta_\epsilon^{\mathbb{B}^*}(M)$*, then* D *is a positive diagonal matrix.*

The new mathematical results for computation and analysis of lower bounds of μ-values corresponding to a class of mixed uncertainties, that is, both real and complex uncertainties and even for more general cases are presented and analyzed in [16].

2.2.2. *Outer Algorithm.* For the case of outer algorithm, the following result is given in [16] for the computation of a change in $\lambda(\epsilon)$ with respect to $\epsilon > 0$.

Theorem 2.6. *[16] Let $\Delta \in \mathbb{B}^*$, and $\lambda(\epsilon)$, $\epsilon > 0$ is a simple and largest eigenvalue corresponding to perturbed matrix $\epsilon M\Delta(\epsilon)$. Consider that $x(\epsilon)$ and $y(\epsilon)$ are right and left eigen-vectors corresponding to matrix function $\epsilon M\Delta(\epsilon)$. Consider that $z(\epsilon) = M^* y(\epsilon)$, then*

$$\frac{d|\lambda(\epsilon)|}{d\epsilon} = \frac{1}{|y(\epsilon)^* x(\epsilon)|}\left(\sum_{i=1}^{s} \left|z_i(\epsilon)^* x_i(\epsilon)\right| + \sum_{j=1}^{F} \left\|z_{s+j}(\epsilon)\right\| \|y_{s+j}(\epsilon)\|\right) > 0.$$

For numerical statistics, the numerical results obtained in [16] are compared with Matlab function `mussv`. The numerical testing confirms the better results for the approximation of lower bounds obtained in [16] than with `mussv`.

2.3. **A nonlinear programming technique based methodology to compute lower bounds of real μ-values.** A novel mathematical formulation for approximation of real μ-values as a mathematical problem for a non-linear programming problem was developed in [17]. Further, a mathematical optimization technique known as F-modified sub-gradient (F-MSG) was developed for the computation and analysis of the bounds (from below) of real μ-values. The F-MSG algorithm is applicable to a much greater class of non-convex programming models, and this is given as follows:

For given $\epsilon \geq 0$ a small non-negative parameter, and η denotes a compact hyper-rectangular domain given as

$$(\delta_1, \delta_2, \cdots, \delta_m, \lambda_1, \lambda_2) \in \eta \quad with \quad \delta_i, \lambda_j \in \mathbb{R}.$$

The mathematical quantity $\mu_\Delta(M)$ is obtained in following two steps.

Step-1:

$$\min_{(\delta_1,\cdots,\delta_m,\ \lambda_1,\lambda_2)} \max_i \left(|\delta_i| + (\lambda_1^r + \lambda_2^r)\right) \lambda$$

so that

$$\begin{cases} f_R(\delta_1, \cdots, \delta_m) - \lambda_1^p = 0, \\ f_I(\delta_1, \cdots, \delta_m) - \lambda_2^p = 0, \\ (\delta_1, \cdots, \delta_m, \lambda_1, \lambda_2) \in \eta. \end{cases}$$

Step-2:

$$\mu_\Delta(M) = \begin{cases} \dfrac{1}{\max|\delta_i|} & \text{if} \quad \|f_R, f_I\| < \epsilon, \\ 0 & \text{else.} \end{cases}$$

The penalty parameter $\lambda \geq 10^5$ is fixed. In step-I, the powers are chosen to be $r \in \{2, 4, 6, \cdots\}$ and $p \in \{1, 3, 5, \cdots\}$. Also, algorithm make it possible to study stability analysis of pendulum (inverted), and it approximate real μ-value lower bounds.

This algorithm further generalizes the modeling associated with the problem under consideration. For matrix function $M(s)$, the definition of μ-value can be generalized as

$$\mu_\Delta(M(s)) = Sup\ \mu_\Delta(M(\mathbf{i}\omega)),$$

where "Sup" is taken over $\omega \in \mathbb{R}_+$, $\mathbb{R}_+$ represent the closed interval $[0, \infty]$, and $\mathbf{i} = \sqrt{-1}$. The matrix function $M(s)$ is known as a transfer function, while ω denotes the frequency. The μ-value can be computed at various discrete levels of frequencies $[\omega_L, \omega_U] \subset \mathbb{R}_+$. In such case, the μ-value is obtained by maximizing over $\omega \in [\omega_L, \omega_U]$, means that,

$$\mu_\Delta(M(s)) = max\ \mu_\Delta(M(\mathbf{i}\omega)).$$

The solution to above given mathematical optimization problem is given with following F-MSG algorithm.

F-MSG Algorithm: The F-MSG algorithm was first developed by Karimbeyli. The aim was to develop a generalized version of modified sub-gradient algorithm. The F-MSG algorithm is with the help of following steps:

Initialization step: Take U_B such that $U_B >> |\delta_i|\ \ \forall i = 1 : m$, and sufficiently small $\epsilon_1,\ \epsilon_2 \geq 0$. Furthermore, take $L_B = 0$, and also let q a positive real number.

Step 1: Choose $n = 1$;

Step 2: Choose (v_1^n, c_1^n) with $v_1^n \in \mathbb{R}^{2,1}$, $c_1^n > 0$, and $\phi(1)$ so that $0 < \phi(1) < q$. Take $H_n = \frac{L_B+U_B}{2},\ j = 1$, and then go to **Step 3**.

Step 3: For (v_1^n, c_1^n), solve the constraint problem:

Determine $K \in \Omega$ so that $L(K, v_j^n, c_j^n) = f(K) + c_j^n\|h(K)\| - v_j^n h(K) \leq H_n$.

If solution to the problem does not exist, then jump to **Step 6**. If solution to the problem does exists, then jump to **Step 5**, otherwise move to **Step 4**.

Step 4: Update (v_1^n, c_1^n) wile taking

$$v_{j+1}^n = v_j^n - \alpha h(K_j), \quad c_{j+1}^n = c_j^n + (1+\alpha)p_j\|h(K_j)\|,$$

with p_j, a positive scalar step size chosen as to be

$$0 \leq p_j = \frac{\delta\alpha(H_n - L(K_j, v_j^n, c_j^n))}{(\alpha^2 + (1+\alpha)^2\|h(K_j)\|^2)},$$

with $\alpha > 0,\ 0 < \delta < 2$.

Step 5: Set up $U_B = f(K_j)$. If $\epsilon_2 > U_B - L_B$, $\mu_\Delta(M) = \frac{1}{U_B}$, and **STOP**; otherwise consider $n = n + 1$, and go back to **Step 2.**

Step 6: Set up $L_B = H_n$. If $\epsilon_2 > U_B - L_B$, $\ \mu_\Delta(M) = \frac{1}{U_B}$, and **STOP**; otherwise let $n = n + 1$, and jump back to **Step 2.**

In algorithm, the quantities L_B, and U_B denotes both lower and upper bounds of μ-values to mathematical optimization problem

$$min\ f(K).$$

The constraints are taken as $h(K) = 0,\ K \in \Omega$, with $K = (\delta_1, \cdots, \delta_m, \lambda_1, \lambda_2)$, $f(K) = max_i(\delta_i) + (\lambda_1^r + \lambda_2^r)\lambda$, $h(K) = [h_1(K)h_2(K)]^T$ with $h_1(K) = f_R(\delta_1, \delta_2, \cdots, \delta_m) - \lambda_1^p = 0$, $h_2(K) = f_1(\delta_1, \cdots, \delta_m) - \lambda_2^p = 0$, and Ω is a large compact hyper-rectangle with values of K.

Theorem 2.7 show that if the value of H_n is feasible to obtain, then the sequence K_j of the optimial solutions to problem in **Step 3** will start to converge to a solution of primal problem.

Theorem 2.7. *Consider Ω denotes a compact set and assume that f and h are the continuous functions defined on the set Ω. Let $p_j\|h(K_j)\| + c_j^n - \|v_j^n\| > \phi(j)$ holds. Then, $\|h(K_j)\| \to 0$ as $j \to \infty$, for each $H_n \geq \bar{H}$ if $\{K_j\},\ j = 1, 2, \cdots$.*

Theorem 2.8 gives the convergence for the F-MSG algorithm.

Theorem 2.8. *Consider (K_j, v_j, c_j) is an iteration obtained at steps 3 and 4 of the F-MSG algorithm for $H_n = \bar{H}$. Let $\{h(K_j)\}$ represents a bounded sequence, and each (v_{j+1}, c_{j+1}) is obtained for $\delta = 1$. Also, if steps 3-4 produces an infinite sequence $L_j = L(K_j, v_j, c_j)$ of Augmented Lagrangian, then $L_j \to \bar{H}$ as $j \to \infty$.*

2.4. **Geometrical formulation of bounds of μ-value.** The geometrical interpretation is presented in [18] for the computation and analysis of the lower bounds of μ-value. The set of block-diagonal structured matrices under consideration is with pure real types of uncertainties. This geometrical approach is to reset the parametric search space. This parametric search space is independent from the number of parameter repetition in the structured uncertainty matrix. An algorithm was presented which combines randomization and optimization methods to deal with the μ-value problem under consideration. This algorithm was successful to deal with two extremely challenging and hard higher-order real μ-analysis problems taken from the field of aerospace and system biology.

The geometrical analysis to subset of given uncertain parametric space must satisfy the singularity constraint in the μ-value lower bound:

$$det(I_n - M(\mathbf{i}\omega)\Delta),$$

where $det(\cdot)$ represent the determinant of a given matrix, and $\mathbf{i} = \sqrt{-1}$, denotes an imaginary unit while $\omega \in \mathbb{R}$, the frequency. The symbol Δ represents the diagonal matrix denoting the structured uncertainty from the set of block-diagonal structured matrices $\hat{\Delta}$.

The singularity condition can be written in both real and imaginary parts, means that,

$$f_R(\Delta) = Re\{det(I_n - M(\mathbf{i}\omega))\Delta\},$$

$$f_I(\Delta) = Img\{det(I_n - M(\mathbf{i}\omega))\Delta\},$$

with $\Delta \in \hat{\Delta} = \{diag(\delta_1 I_{r_1}, \cdots, \delta_p I_{r_p}) : \delta_i \in \mathbb{R}\}$, r_i is an element from set of the natural numbers, the positive real integers, and I_{r_i} is $r_i \times r_i$ denotes an identity matrix for $i = 1 : p$. The singularity condition may be formulated as follows, that is,

$$F_R = \{\Delta \in \hat{\Delta} : f_R(\Delta) = 0\},$$

$$F_I = \{\Delta \in \hat{\Delta} : f_I(\Delta) = 0\},$$

and then $\mu-$value is re-defined as

$$\mu_\Delta(M) = \frac{1}{min\ \bar{\sigma}(\Delta)}, \quad if \ \ F_R \cap F_I \neq \phi,$$

otherwise it is exactly equal to 0.

For given $\Delta = 0$, the $det(I_n - M\Delta) = 1$, and as a result $f_R(0) = 1$, and $f_I(0) = 0$. This further implies that the manifold F_I must passes through the origin. On the other hands, the manifold F_R must not passes through the origin. We refer to see [18] for a complete discussions to solution and the formulation of geometrical problems, examples, and applications.

2.5. **Gain-based lower bound technique for mixed μ-problems.** A new mathematical technique for computation and analysis of the lower bounds of both real and mixed μ-problem was presented in [19]. This mathematical technique make use of worse-case gain to the numerical approximation to real blocks, and the power algorithm to determine the number of complex blocks. The numerical testing provides tighter bounds for μ-values. The gain-based lower bounds algorithm (LBA) is capable to compute an exact real μ-values in relatively lower dimensions. Also, it has an ability to switch over to worst-case gain search for the class of larger-dimensional mathematical problems.

Gain-Based Algorithm (GBA): Let $M \in \mathbb{C}^{n_R, n_R}$, then the problem for computing tighter lower bounds for the case of pure real μ-values, means that, $\mu_{\Delta_R}(M_R)$, having $\Delta_R := \{\Delta = diag(\delta_1 I_{k_1}, \cdots, \delta_r I_{k_r}) : \delta_i \in \mathbb{R}\}$, M_R is the complex-valued structured matrix with R representing the real block structure being employed for the computation of μ-values. It is evident from Definition of μ-values that Δ_R must satisfies $det(I - M_R\Delta_R) = 0$, and in turn yields a lower bound $\frac{1}{\bar{\sigma}(\Delta_R)}$ which is bounded by $\mu_{\Delta_R}(M_R)$. This further implies the existence of $z \in \mathbb{C}^{n_R}$, $w \in \mathbb{C}^{n_R}$ satisfying $z = M_R w$, and $w = \Delta_R z$. These equations can be represented with the help of Linear Fractional Transformations (LFT) $F_u(M_R, \Delta_R)$ with z, w representing an output of M_R and Δ_R.

An algebraic equations: $\begin{bmatrix} z \\ e \end{bmatrix} = \bar{M}_R \begin{bmatrix} w \\ d \end{bmatrix}$, $w = \Delta_R z$, and

$$\bar{M}_R = \begin{bmatrix} M_R & i_k \\ i_k^T M_R & 1 \end{bmatrix},$$

with $d, e \in \mathbb{C}$, denoting the scalar disturbance and an error signals, respectively. Furthermore, these set of algebraic equations are also of well-posed nature. The external disturbance corresponding to an error signals relation is obtained by $e = F_u(\bar{M}_R, \Delta_R)d$ if $det(I - M_R\Delta_R) \neq 0$.

The GBA main aim is to solve following mathematical optimization problem:

$$max\ |F_u(\bar{M}_R\Delta_R)|,$$

with **max** is being taken over Δ_R, $\sigma(\bar{\Delta}_R) \leq \mu_l$, μ_l represents the lower bounds of μ-values. The mathematical optimization problem is non-convex and its objective is to search for the global maximizer. The search of such a global maximizer may be computationally expensive. Theorem

2.9 establishes two interesting linkages between $d-to-e$ gain and the distance of matrix $(I - M_R\Delta_R)$ to singularity.

Theorem 2.9. *If $\exists$ a Δ_R so that $det(I - M_R\Delta_R) \neq 0$, and $|F_u(\bar{M}_R, \Delta_R)| \geq \gamma \geq 0$, then we have that:*
1. $\exists$ a $\delta \in \mathbb{C}$, $|\delta| \leq \frac{1}{\gamma}$ so that $det(I - M_R\Delta_R - \delta i_k i_k^T) = 0$.
2. The smallest singular value $\sigma_{min}(I - M_R\Delta_R) \leq \frac{1}{\gamma}$.

Input: $M_R \in \mathbb{C}^{n,n}, \;\; \Delta_R, \;\; u_b, \; l_b$
Initilize $lb_{fac} = \frac{3}{4}$, and $cnt = 1$
while $cnt \leq N_{try}$ AND $l_b < u_b \; tol_{stop}$
$lb_{try} = lb + (u_b - l_b) \; lb_{fac},$
$k := mod(cnt - 1, n_R) + 1$

$$\bar{M}_R = \begin{bmatrix} M_R & i_k \\ i_k^T M_R & 1 \end{bmatrix}$$

$\Delta_{R,try} := arg \; max|F_u(\bar{M}_R, \Delta_R)|$
if $rcound(I - M_R\Delta_{R,try}) < tol_{real}$
$lb = \frac{1}{\bar{\sigma}(\Delta_{R,try})}$
$\Delta_R = \Delta_{R,try}$
$lb_{fac} := \frac{1}{2}$
else
$lb_{fac} := max(\frac{1}{32}, \frac{lb_{fac}}{2})$
end
$cnt = cnt + 1$
end
Return: $= \Delta_{R,lb}$

Algorithm 1: The GBA for real μ lower bound

2.6. **Computation of μ via Moment LMI relaxation technique.** A novel algorithm consist of Moment Linear Matrix Inequalities (LMI) is presented in [20] to determine μ-values from above subject to different types of uncertainties, for instance, a mixture of both real and complex uncertainties. The main idea was to re-formulate μ-values approximation as a non-convex polynomial optimization problem. In turn this yields convex mathematical optimization problems with help of the moment relaxation methodologies. The experimental results show numerical computation and behavior to the bounds of μ-values. It was observed that this yield more tighter lower bounds as compared with Matlab function `mussv`.

The uncertainty Δ possesses a block-diagonal structure, and is defined in [20]. The constraint $||\Delta|| < r$ may be re-written as follows

$$\begin{bmatrix} r^2 I & \Delta \\ \Delta^H & I \end{bmatrix} > 0.$$

Input: $M,\ \Delta,\ u_b,\ l_b$
Initilize $lb_{fac} = \frac{3}{4}$, and $cnt = 1$
while $cnt \leq N_{try}$ AND $l_b < u_b\ tol_{stop}$
$lb_{try} = lb + (u_b - l_b)\ lb_{fac}$,
$k := mod(cnt - 1, n_R) + 1$

$$\bar{M}_R = \begin{bmatrix} M_R & i_k \\ i_k^T M_R & 1 \end{bmatrix}$$

$\Delta_{R,try} := arg\ max|F_u(\bar{M}_R, \Delta_R)|$
if $rcound(I - M_R \Delta_{R,try}) < tol_{real}$
$lb = \frac{1}{\bar{\sigma}(\Delta_{R,try})}$
$\Delta = diag(\Delta_{R,try}, 0)$
$lb_{fac} := \frac{1}{2}$
else
$\bar{M}_C := F_u(M, \Delta_R)$
Power iteration on $\bar{M}_C$ to find $\Delta_{C,try}$
$\Delta_{try} := diag(\Delta_{R,try}, \Delta_{C,try})$
if $rcound(I - M\Delta_{try}) < tol_{complex}$ AND $\frac{1}{\bar{\sigma}(\Delta_{try})} \geq lb$
$lb := \frac{1}{\bar{\sigma}(\Delta_{try})}$, Return := Δ, lb

Algorithm 2: The GBA for mixed μ lower bound

Theorem 2.10 provides both necessary as well as the sufficient conditions for robust non-singularity of the perturbed matrix $(I - M\Delta)$.

Theorem 2.10. *For a given $r \geq 0$, non-negative real number, the perturbation matrix $I - M\Delta$ must have one of its eigenvalue to be exactly equal to zero for all possible uncertainties Δ if and only if the optimal solution to mathematical optimization problem (given bellow) is*

$$max\ \|x\|_2^2,\ \ s.t\ \ (I - M\Delta)x = 0,\ \ \begin{bmatrix} r^2 I & \Delta \\ \Delta^H & I \end{bmatrix} > 0.$$

*Here, the **max** is taken over $x \in \mathbb{C}^{s_2}$, and $\Delta \in \hat{\Delta}$, $\hat{\Delta}$ having the block-diagonal structure.*

Result: The μ-value of a given matrix M w.r.t uncertainty $\hat{\Delta}$ is given as

$$\mu_{\hat{\Delta}}(M) = \frac{1}{\sqrt{t^*}},$$

with t^* is an optimal solution to mathematical optimization problem:

$$t^* = min\ t\ \ s.t\ \ t \geq 0,\ \ \|x\|_2^2 \geq \bar{x},\ \ (I - M\Delta)x = 0,\ \begin{bmatrix} r^2 I & \Delta \\ \Delta^H & I \end{bmatrix} > 0.$$

Note that here $\bar{x} \geq 0$, can be chosen an arbitrary, and the **min** is taken over $t \in \mathbb{R},\ x \in \mathbb{C}^{s_2},\ \Delta \in \hat{\Delta}$.

Theorem 2.11 show how the convergence of t^* is the computation of μ-values.

Theorem 2.11. *The following results are true:*

1. $h \geq 1$, t^*_h *be the lower bound of* t^*, *means that,* $t^*_h \leq t^*$.

2. t^*_h *converges to* t^*, *from bellow, means that,* $t^*_h \leq t^*_{h+1} \leq t^*$, *and* $\lim_{h\to\infty} t^*_h = t^*$

3. For every $h \geq 1$, $det(I - M\Delta) = 0, \ \ \forall \Delta \in \hat{\Delta}$.

Also, $\mu_{\hat{\Delta}}(M) \leq \frac{1}{\sqrt{t^*_{h+1}}} \leq \frac{1}{\sqrt{t^*_h}}$, *and* $\lim_{h\to\infty} \frac{1}{\sqrt{t^*_h}} = \mu_{\hat{\Delta}}(M)$.

For further and a complete details on approximation to lower bound and numerical experimentation on μ-values, see [21], and the references therein.

3. Stability and D-stability analysis of transportation models

The Hitchcock-Koopmans transportation problem is a well-known mathematical optimization problem which focuses on minimization of the objective function. This objective function is basically the transportation cost obtained from multiple sources corresponding to multiple destinations. The novel results on the spectral properties of structured matrices appearing in Hitchcock-Koopmans transportation problems are given in [43]. The results on the computation of singular values are obtained with use of various mathematical tools from linear algebra and matrix analysis. Furthermore, some new results are derived on the interconnection between μ-values of pseudo-inverse and D-stable structured matrices corresponding to Hitchcock-Koopmans transportation models. The numerical testing show overall behavior of the singular values. The use of Matlab EigTool is for the computation of pseudo-spectrum corresponding to pseudo-inverse matrix for transportation model.

3.1. **Singular values for Hitchcock-Koopmans transportation problem.** The Hitchcock-Koopmans transportation problem was developed by Hitchcock in 1941. This method was also investigated by Koopmans in 1947 too. From application view point this problem was applied to simplex algorithm given by Dantzig in 1951. For given m numbers of origins and n numbers of destinations, the Hitchcock-Koopmans transportation problem main objective involve the study, analysis, and then to solve following mathematical optimization problem:

$$min\{c^T x : Mx = g, \ 1_m a = 1_n b, \ x \geq 0\},$$

with *min* is taken over x, Also,

$$a^T = [a_1, a_2, \cdots, a_m], \ b^T = [b_1, b_2, \cdots, b_m], \ x^T = [x_{11}, x_{12}, \cdots, x_{mn}], \ c^T = [c_{11}, c_{12}, \cdots, c_{mn}],$$

and $g^T = [a^T : b^T]$. The coefficient matrix M has the following form

$$M = \begin{pmatrix} 1_n \otimes I_m \\ I_n \otimes 1_n \end{pmatrix},$$

with I_m is m-dimensional identity matrix, 1_m denotes $1 \times m$ vector of all $1's$. Also $\otimes$ denotes kronecker product.

A mathematical relation between eigenvectors of matrix products M^+M and MM^T was developed in [33]. The Moore-Penrose inverse M^+ to a given matrix M was obtained in [34] as follows

$$M^+ = \frac{1}{mn}\left(m1_n^T \otimes ((m+n)I_m - J_m)n((m+n)I_n) - J_n \otimes 1_m^T\right),$$

with J_m as an $m \times m$ matrix of all $1's$, that is, $J_m = 1_m^T 1_m$. Furthermore, we have

$$M^+M = I - \frac{1}{mn}\left(nI_n - J_n\right) \otimes \left(mI_m - J_m\right),$$

with $I - M^+M$, a symmetric and idempotent matrix. The relation between eigenvectors of matrices J_n, J_m, and $I - M^+M$ was developed in the Theorem 3.2 [33].

Theorem 3.1 gives Moore-Penrose inverse for a given matrix M.

Theorem 3.1. *Consider n-dimensional real-valued matrix $M \in \mathbf{R}^{n,n}$. Then, the structured matrix M^+ is the Moore-Penrose inverse which must satisfy the matrix equation given as*

$$M = MM^+M.$$

Theorem 3.2. *Consider v_j denotes all eigenvectors of matrices J_m, and u_k, the eigenvectors corresponding to J_n. Then eigenvectors corresponding to eigenvalues with algebraic multiplicity 1 of matrix $I - M^+M$ are*

$$\hat{v}_i = u_j \otimes x_k,$$

with $j = 1 : n-1$; $k = 1 : m-1$; $i = m+n, \cdots, (m-1)(n-1)$.

Definition 3.1. *The singular values are the positive square roots of eigenvalue corresponding to matrix M^TM, where T represent the transpose of given square or rectangular matrix M.*

Theorem 3.3. *Consider n-dimensional real-valued matrix $M \in \mathbf{R}^{n,n}$. Then $\exists$ unitary matrices U, V, and a diagonal matrix Σ so that the matrix M may be decomposed as*

$$M = U\Sigma V^T.$$

Furthermore, the singular values (non-zero) of M appear along the main diagonal of Σ.

Corollary 1 [33]. Let $\{a_1, a_2, \cdots, a_{m+n-1}\}$ and $\{b_1, b_2, \cdots, b_{m+n-1}\}$ denotes the sets of eigenvectors corresponding to eigenvalues of structured matrices MM^T and M^TM. Let $\sigma_1, \sigma_2, \cdots, \sigma_{m+n-1}$ are singular values of given matrix M. Then,

$$MM^Ta_i = \sigma_i^2 a_i,\ i = 1 : m+n-1,$$

$$M^TMb_i = \sigma_i^2 b_i,\ i = 1 : m+n-1.$$

Corollary 2 [33]. The eigenvectors a_i for the eigenvalue 0 and $m+n$ are

$$J_m x_i = m x_i,\ J_n y_i = n y_i.$$

The eigenvectors for eigenvalues m and n are obtained as

$$J_m x_i = \vec{0},\ J_n y_i = \vec{0}.$$

Here, x_i are $m \times 1$ vectors and y_i are $n \times 1$ vectors.

Consider $M^+ = (M^*M)^{-1}M^*$ be the Moore-Penrose inverse matrix of the given structured matrix M, and $*$ represent complex conjugate transpose. Further, let $M^*(MM^*)^{-1}$ represents right inverse of the given matrix M. Theorem 3.4 is about to discuss an orthogonality corresponding to leading singular values.

Theorem 3.4. *[43] Let $(M^*M)^{-1}M^*$ and $M^*(MM^*)^{-1}$ be n-dimensional given matrices. Let $\{\sigma_i\}$, $\forall i = 1 : n$ denotes a sequence of singular values having $\{v_i\}$, and $\{\hat{v}_i\}$, $\forall i = 1 : n$ as left hand singular vectors and right singular vectors so that $\|v_i\|_2 = 1 = \|\hat{v}_i\|_2$, and $\{u_i\}$, and $\{\hat{u}_i\}$, $\forall i = 1 : n$ denotes both left hand singular vectors and right singular vectors for $\{\hat{\sigma}_i\}$ so that $\|u_i\|_2 = 1 = \|\hat{u}_i\|_2$. The leading singular vectors v_1 and u_1 are orthogonal to the singular values σ_1 and $\hat{\sigma}_1$. Then, any non-zero vector $\tilde{v} = \{\tilde{v}_1, \tilde{v}_2, \cdots, \tilde{v}_n\}$ is an orthogonal to a vector $e_n = \frac{1}{\sqrt{n}}(1, 1, \cdots, 1)^T$ and is not necessary to be a singular vectors corresponds to σ_1 and $\hat{\sigma}_1$.*

Proof. Consider the singular value problem

$$M\tilde{v} = \sigma\tilde{v},$$

with σ being as a singular value to given matrix M, and is corresponding to singular vector $\tilde{v}$. The vector $\tilde{v}$ is not essentially a singular vector to singular values σ_1 and $\hat{\sigma}_1$. The matrix times vector $M\tilde{v}$, may be reformulated as

$$M\tilde{v} = \begin{pmatrix} m_{11}\tilde{v}_1 + m_{12}\tilde{v}_2 + \cdots m_{1n}\tilde{v}_n \\ \cdots \\ \cdots \\ \cdots \\ m_{n1}\tilde{v}_1 + m_{n2}\tilde{v}_2 + \cdots m_{nn}\tilde{v}_n \end{pmatrix}.$$

Consider $\sum$ on the components of vector $M\tilde{v}$, this yields

$$\sum(M\tilde{v}) = \sum \begin{pmatrix} m_{11}\tilde{v}_1 + m_{12}\tilde{v}_2 + \cdots m_{1n}\tilde{v}_n \\ \cdots \\ \cdots \\ \cdots \\ m_{n1}\tilde{v}_1 + m_{n2}\tilde{v}_2 + \cdots m_{nn}\tilde{v}_n \end{pmatrix} = v_1 \sum(m_i 1) + \cdots + v_n \sum(m_i n) = v_1(\sigma_1) + \cdots + v_n(\sigma_1).$$

This further yields,

$$\sum(Mv) = \sigma_1 \sum(v).$$

By considering $\sum$ of right hand side of $Mv = \sigma v$, we obtain

$$\sum(\sigma v) = \sigma \sum(v).$$

Finally, from last two equations, we conclude that

$$(\sigma - \sigma_1) = \sum(v) = 0.$$

In turn this implies $\sigma \neq \sigma_1$, and $\sum(v) = 0$. This proves that $v = (v_1, v_2, \cdots, v_n)$ is orthogonal to vector $e_n = \frac{1}{\sqrt{n}}(1, 1, \cdots, 1)^T$. The vector e_n is not necessary a singular vector for singular values σ_1 and $\hat{\sigma}_1$. □

Theorem 3.5. *[43] Consider that $(M^*M)^{-1}M^*$ and $M^*(MM^*)^{-1}$ are n-dimensional matrices, and assume that σ_i and $\hat{\sigma}_i$ are leading order singular values to singular vectors v_i and $\hat{v}_i$, respectively. Further, assume that*

$$\hat{M} = \begin{pmatrix} (M^*M)^{-1}M^* + \alpha I_n & 2\alpha \hat{v}_2 v_1^T \\ 2\alpha v_1 \hat{v}_2^T & M^*(MM^*)^{-1} + \alpha I_n \end{pmatrix},$$

where I_n is an identity matrix, α be any scalar. The computation of singular values of matrix $\hat{M}$ not possesses the leading most singular values corresponding to structured matrices $(M^*M)^{-1}M^*$ and $M^*(MM^*)^{-1}$. The leading singular values σ_1 and $\hat{\sigma}_1$ does appear along the diagonal of $\hat{M}_1$ having

$$\hat{M}_1 = \begin{pmatrix} 3\alpha + \sigma_1 & 0 \\ 0 & \alpha - \hat{\sigma}_1 \end{pmatrix}.$$

Proof. Assume that the singular value problem can be written in following form

$$\hat{M} v_i = (\sigma_1 + \alpha) v_i, \ \forall i = 1 : n.$$

Consider that $\beta_1, \sigma_2 + \alpha, \sigma_3 + \alpha, \cdots, \sigma_n + \alpha$, and $\hat{\beta}_2, \hat{\sigma}_2 + \alpha, \hat{\sigma}_3 + \alpha, \cdots, \hat{\sigma}_n + \alpha$ with $\beta, \hat{\beta} \in \{3\alpha + \sigma_1, \alpha - \hat{\sigma}_1\}$ be the singular values corresponding to matrix

$$\begin{pmatrix} 3\alpha + \sigma_1 & 0 \\ 0 & \alpha - \hat{\sigma}_1 \end{pmatrix}.$$

For singular values σ_i, $\forall i = 2 : n$, singular vectors may be written as $[v_i \ \ 0]^T$, $\forall i = 2 : m$. On the other hand for the singular values $\hat{\sigma}_i$, $\forall i = 2 : n$, the singular vectors may be written as $[0 \ \ \hat{v}_i]^T$, $\forall i = 2 : n$. This further implies that singular vectors corresponding to $\hat{M}$ may be expressed as $[\beta_i v_i \ \ \hat{\beta}_i \hat{v}_i]^T$. Finally, we have that $[\beta_i \ \ \hat{\beta}_i]$, $\forall i = 2 : n$ are the singular vectors corresponding to singular values $\sigma_1 + 3\alpha$ and $\alpha - \hat{\sigma}$. □

Theorem 3.6 show theoretical results on the computation of singular values corresponding to a given matrix and these may not depend continuously on the entries of the same matrix.

Theorem 3.6. *[43] Consider n-dimensional matrices $M_1 = (M^*M)^{-1}M^*$ and $M_2 = M^*(MM^*)^{-1}$. Suppose that $\lim_{k\to\infty}(M_k) = M$, and $q = min\{m, n\}$. Furthermore, let $\sigma_1(M) \geq \sigma_2(M) \geq \cdots \geq \sigma_q(M)$, and $\sigma_1(M_k) \geq \cdots \geq \sigma_q(M_k)$ are the non-increasing singular values of structured matrices M and M_k, respectively for $k = 1, 2, \cdots$. Then, $\lim_{k\to\infty} \sigma_i(M_k) = \sigma_i(M)$ for all $i = 1 : q$.*

Proof. Consider that $k_1 < k_2 < \cdots$ is the sequence of some positive integers. Let $\epsilon > 0$, we have

$$max|\sigma_i(M_{kj}) - \sigma_i(M)| > \epsilon,$$

with the *max* taken over all $i = 1 : q$.

The singular value decomposition of M_{kj} is

$$M_{kj} = U_{kj}\Sigma_{kj}V_{kj}^*,$$

where U_{kj} and V_{kj} be the unitary matrices, and Σ_{kj} is a diagonal matrix with structure

$$\Sigma_{kj} = [\sigma_1(M_{kj}) \cdots \sigma_q(M_{kj})]^T.$$

Then,

$$\lim_{r\to\infty} \Sigma_{kjr} = \lim_{r\to\infty} U_{kjr}^* M_{kjr} V_{kjr} = \left(\lim_{r\to\infty} U_{kjr}^*\right)\left(\lim_{r\to\infty} M_{kjr}\right)\left(\lim_{r\to\infty} V_{kjr}\right) = U^*MV.$$

The matrix product U^*MV be a non-negative diagonal matrix. By the uniqueness of singular values of given structured matrix M it implies that $Diag\ \Sigma = [\sigma_1(M), \sigma_2(M), \cdots, \sigma_q(M)]^T$, which is clearly a contradiction with the following inequality

$$max|\sigma_i(M_{kj}) - \sigma_i(M)| > \epsilon.$$

□

Now, we give the numerical illustrations on approximation of singular values, and pseudo-inverse of transportation matrices.

Example 1. We take a 7×5 transportation matrix (a.k.a staircase matrix) from [35].

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

The graphical interpretation of singular values and pseudo-inverse are shown in Figure 1.

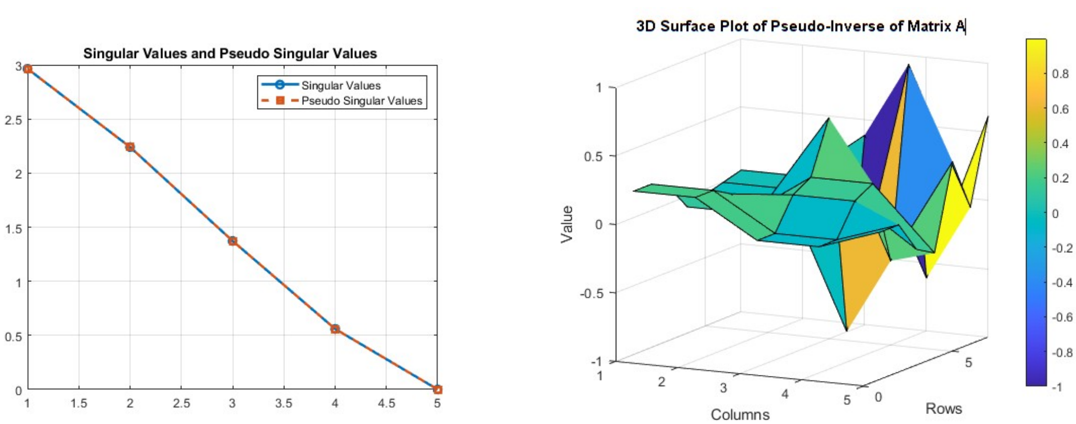


Figure 1. The graphical interpretation of singular values and pseudo-inverse of M in Example-1

Example 2. We take 7×5 transportation matrix (a.k.a distribution matrix) from [35].

$$M = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1.4 & 0.6 \end{pmatrix}.$$

The graphical interpretation of singular values and pseudo-inverse are shown in Figure 2.

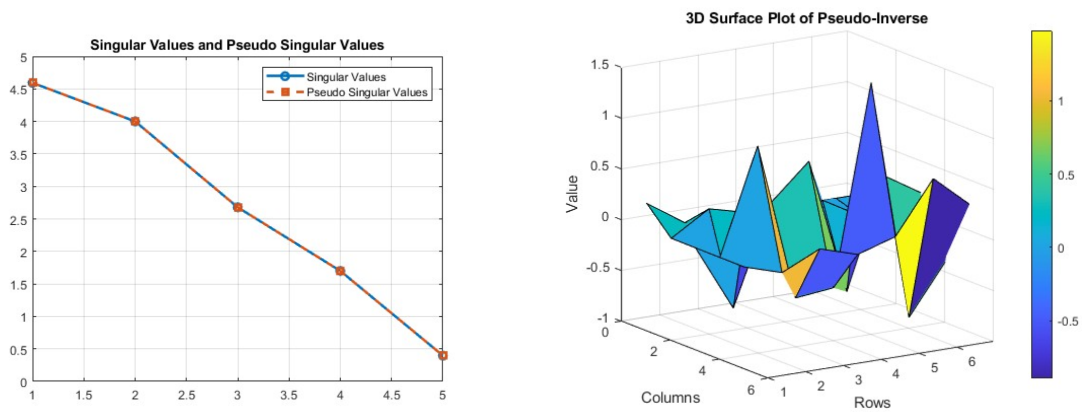


Figure 2. The graphical interpretation of singular values and pseudo-inverse of M in Example-2

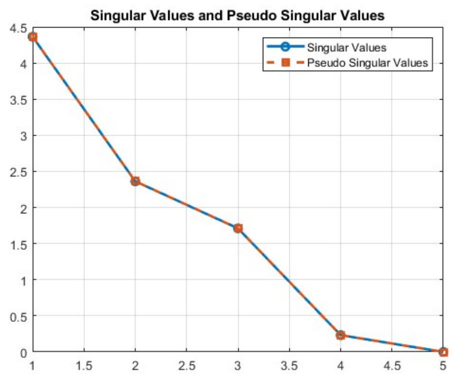
Example 3. We take 7×5 transportation matrix (a.k.a distribution matrix) from [35].

$$M = \begin{pmatrix} 1.6 & 0.8 & 1.6 & 0 & 0 & 0 & 0 \\ 2.4 & 1.2 & 2.4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 1.8 & 0.6 & 0 & 0 \\ 0 & 0 & 0.4 & 1.2 & 0.2 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 1.2 & 1.2 & 0.6 \end{pmatrix}.$$

The graphical interpretation of singular values and pseudo-inverse are shown in Figure 3.

3.2. **Structured singular values for Hitchcock-Koopmans transportation problem.** We present novel mathematical results on the computation of μ-values for structured matrices which does appear in Hitchcock-Koopmans transportation model. We use various mathematical tools from linear algebra, matrix theory, and control to provide a detailed analysis on μ-values.

Definition 3.2. *The given $M \in \mathbb{R}^{n,n}$ is a stable matrix if all real parts of its eigenvalues (or spectrum) are strictly positive, means that $Re(\lambda_i(M)) > 0$.*



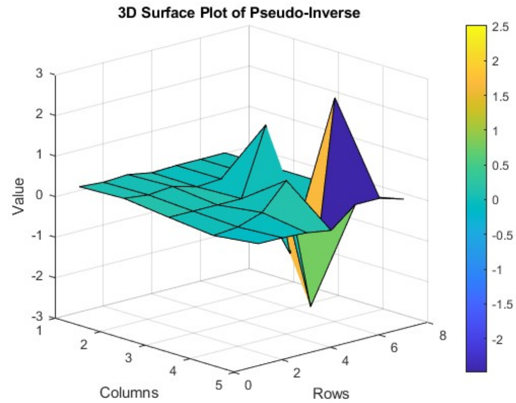


Figure 3. The singular values and pseudo-inverse of M in Example-3

Definition 3.3. *The given $M \in \mathbb{R}^{n,n}$ is structured D-stable matrix having all real parts of eigenvalues (or spectrum) of matrix product MD are strictly positive, means that $Re(\lambda_i(MD)) > 0$, where $D = diag(d_{ii}) > 0$, for all $i = 1, \cdots, n$.*

The μ-values for a given $M \in \mathbb{R}^{n,n}$ w.r.t set of block-diagonal structured matrices Δ, having

$$\Delta := \{diag\,(\delta_1 I_{r_1}, \delta_2 I_{r_2}, \cdots, \delta_S I_{r_S}; \Delta_1, \Delta_2, \cdots, \Delta_F) : \delta_i \in \mathbb{R}(\mathbb{C}),\ \Delta_j \in \mathbb{K}^{m_j, m_j},\ i = 1 : S;\ J = 1 : F\},$$

with $\mathbb{K} = \mathbb{R}(\mathbb{C})$, is the approximation of largest singular value of $\hat{\Delta} \in \Delta$. The μ-value is denoted with μ, and for $M \in \mathbb{C}^{n,n}$, and Δ, it is defined as (see [15]):

$$\mu_\Delta(M) := \left(min\{\|\hat{\Delta}\|_2 : det(I_n - M\hat{\Delta}) = 0,\ \forall\, \hat{\Delta} \in \Delta\}\right)^{-1},$$

with the **min** taken over $\hat{\Delta} \in \Delta$, and $\mu_\Delta(M) = 0$ if $det(I_n - M\hat{\Delta}) \neq 0,\ \forall\, \hat{\Delta} \in \Delta$.

Remark 3.1. *The block-diagonal structure Δ may be connected with multi-index of some given positive integers.*

Remark 3.2. *The number of the full blocks in block-diagonal structure Δ may be considered as a class of pure real blocks uncertainties, pure complex blocks uncertainties or a combination of both real and complex blocks uncertainties.*

These blocks can be chosen as rank-1 matrices.

Remark 3.3. *For a given $\alpha \in \mathbb{C}$, $\mu_\Delta(\alpha M) = |\alpha| \mu_\Delta(M)$.*

Corollary 3.1. *[15] The μ-values $\mu_\Delta(M)$ is exactly equal to the computation of spectral radius ρ of $V^T MW$, means that*

$$\mu_\Delta(M) = \rho(V^T MV),$$

where V, W are with the block-diagonal structures.

For a given $M \in \mathbb{C}^{n,n}$, the μ-value $\mu_\Delta(M)$ may be taken as the computation of spectral radius ρ of $M\hat{\Delta},\ \hat{\Delta} \in \Delta$.

Lemma 3.1. *For a given matrix M, and a set with the block-diagonal structure* Δ,

$$\mu_\Delta(M) = max\, \rho(M\hat{\Delta}),$$

with max chosen over $\hat{\Delta} \in \Delta$.

The following theorem 3.7 (see [10]) is a known mathematical result on the computation of D-stability for a given matrix. This does involve computation of strictly positive real part of all eigenvalues (or spectrum) of matrix product corresponding to a given matrix with a positive diagonal matrix.

Theorem 3.7. *Let all diagonal elements of a given matrix M are negative. Further, assume that there are no negative off diagonal elements.* $D = diag(d_{ii})$, *the matrices M, and DM are stable, then* $D = diag(d_{ii}) > 0,\ \forall i$.

The following theorem 3.8 is a characterization of D-stability. It further links the bridge between μ-values and D-stable matrices.

Theorem 3.8. *Let given matrix M is an n-dimensional matrix. Then matrix M is a D-stable matrix if and only if M be a stable matrix, and*

$$0 \leq \mu_\Delta(\boldsymbol{i}I_n + M)^{-1}(\boldsymbol{i}I_n - M) < 1.$$

Theorem 3.9. *[43] Let* $(MM^*)^{-1}M^*$ *is a n−dimensional complex-valued structure matrix. Then*

$$0 \leq \mu_\Delta((I_n + (MM^*)^{-1}M^*)^{-1}(I_n - (MM^*)^{-1}M^*)) < 1.$$

Proof. We make use of definition and basic concept of D-stability of a matrix to prove our results. For a given matrix M, the matrix $(MM^*)^{-1}M$ is D-stable, that is, $\lambda_k(I_n + (MM^*)^{-1}Mp) \neq 0, \forall k = 1 : n$, with $P = diag(p_{11}, p_{22}, \ldots, p_{nn}) > 0$. Consider that $P = (I_n + R)^{-1}(I_n - R), \forall R \in \Delta$, then we have

$$\lambda_k(I_n + (M^*M)^{-1}M^*(I_n + R)^{-1}(I_n - R)) \neq 0, \forall k = 1 : n, \forall R \in \Delta.$$

This further yields that

$$\lambda_k((I_n + (M^*M)^{-1}M^*) + (I_n - (M^*M)^{-1}M^*R)) \neq 0, \forall k = 1 : n, \forall R \in \Delta.$$

As,

$$\lambda_k(I_n + (M^*M)^{-1}M^*) \neq 0 \sim \lambda_k((I_n + (M^*M)^{-1}M^*) + (I_n - (M^*M)^{-1}M^*R)) \neq 0, \forall k = 1 : n, \forall R \in \Delta$$

Thus finally, we have that

$$0 \leq \mu_\Delta((I_n + (MM^*)^{-1}M^*)^{-1}(I_n - (MM^*)^{-1}M^*)) < 1.$$

□

Theorem 3.10. *[43] Consider that for given M, the matrix $(M^*M)^{-1}M^*$ is an n-dimensional complex-valued matrix. Then we have $0 \leq \mu_\Delta(A) < 1$ if $(M^*M)^{-1}M^*$ is D-stable matrix, where*

$$A = (\boldsymbol{i}I_n + P(M^*M)^{-1}M^* + M((M^*M)^{-1})^*)(\boldsymbol{i}I_n - P(M^*M)^{-1}M^* - M((M^*M)^{-1})^*P)$$

with $P = diag(p_{11}, p_{22}, \ldots, p_{nn})$, $p_{ii} > 0, \forall i = 1 : n$, $\boldsymbol{i} = \sqrt{-1}$.

Proof. For matrix M, the modified matrix $(M^*M)^{-1}M^*$ is a D-stable matrix if $Re(\lambda_k(P(M^*M)^{-1}M^* + M(M^*M^{-1})^*)) > 0, \forall k = 1 : n$, for more detail see [36]. In order to prove that $0 \leq \mu_\Delta(A) < 1$, we let a block-diagonal structure $\hat{\Delta} = (\mathbf{i}I_n - P)(\mathbf{i}I_n + P)^{-1}$ with $\hat{\Delta} \in \Delta$. For all positive definite diagonal structured matrices, P, the eigenvalues

$$\lambda_k(P(M^*M)^{-1}M^* + M((M^*M)^{-1})^*P + \mathbf{i}(\mathbf{i}I_n + \hat{\Delta})^{-1}(\mathbf{i}I_n - \hat{\Delta})) \neq 0 \forall k = 1 : n.$$

In turn this implies that $\lambda_k, \forall k = 1 : n$ and this further takes following form

$$\lambda_k((\mathbf{i}I_n + P(M^*M)^{-1}M^* + M((M^*M)^{-1})^*P) - (\mathbf{i}I_n - P(M^*M)^{-1}M^* - M((M^*M)^{-1})^*P)\hat{\Delta}) \neq 0, \forall \hat{\Delta} \in \Delta.$$

Thus we have that,

$$\lambda_k(I_n - (\mathbf{i}I_n + P(M^*M)^{-1}M^* + M((M^*M)^{-1})^*P))(\mathbf{i}I_n - P(M^*M)^{-1}M^* - M((M^*M)^{-1})^*P)\hat{\Delta}) \neq 0, \forall \hat{\Delta} \in \Delta.$$

The last expression for the eigenvalues $\lambda_k, \forall k = 1 : n$ further implies that $0 \leq \mu_\Delta(A) < 1$. □

Theorem 3.11. *[43] For given matrix M consider $(M^*M)^{-1}M^*$ an n-dimensional complex-valued matrix. Then we have that, $0 \leq \mu_\Delta(A) < 1$ if*

$$x^*(M((M^*M)^{-1})^*P^2 + P^2(M^*M)^{-1}M^*)x > 0$$

for $x \in \mathbb{C}^{n,1}$, and for all positive definite matrices $P = diag(p_{11}, p_{22}, \ldots, p_{nn}) > 0$, with

$$A = (\boldsymbol{i}I_n + (M^*M)^{-1}M^*)^{-1}(\boldsymbol{i}I_n - (M^*M)^{-1}M^*).$$

Proof. The μ-value is determination of the quantity $\alpha_{max} \geq 0$ in such a manner that for each P, matrix inequality

$$\frac{\|P(M^*M)^{-1}M^*x\|)}{\|Px\|} \geq \alpha_{max}.$$

For a given A, $0 \leq \mu_\Delta(A) < 1$ if $\|PAx\| < \|Px\|$ for each $x \in \mathbb{C}^{n,1}$, and for a positive diagonal matrix P. The above obtained inequality holds true if $\mathbf{i}I_n + (M^*M)^{-1}M^*$, that is,

$$\|PA(\mathbf{i}I_n + (M^*M)^{-1}M^*)x\| < \|P(\mathbf{i}I_n + (M^*M)^{-1}M^*)x\|.$$

Furthermore, we have,

$$\|PA(\mathbf{i}I_n + (M^*M)^{-1}M^*)x\|^2 < \|P(\mathbf{i}I_n + (M^*M)^{-1}M^*)x\|^2.$$

In turn this further yields that

$$x^*((\mathbf{i}I_n + (M^*M)^{-1}M^*)^*A^*P^*PA(\mathbf{i}I_n + (M^*M)^{-1}M^*))x$$

$$<$$

$$x^*((\mathbf{i}I_n + (M^*M)^{-1}M^*)^* P^* P(\mathbf{i}I_n + (M^*M)^{-1}M^*))x.$$

The above inequality reduces to

$$x^*((\mathbf{i}I_n + (M^*M)^{-1}M^*)^* A^* P^2 A(\mathbf{i}I_n + (M^*M)^{-1}M^*))x < x^*(\mathbf{i}I_n + (M^*M)^{-1}M^* P^2(\mathbf{i}I_n + (M^*M)^{-1}M^*))x.$$

As,

$$A = (\mathbf{i}I_n + (M^*M)^{-1}M^*)^{-1}(\mathbf{i}I_n - (M^*M)^{-1}M^*).$$

This maybe re-written as

$$x^*((\mathbf{i}I_n + (M^*M)^{-1}M^*)^*(\mathbf{i}I_n - (M^*M)^{-1}M^*)^*(\mathbf{i}I_n + (M^*M)^{-1}M^*)^{-1}P^2(\mathbf{i}I_n + (M^*M)^{-1}M^*)^{-1}$$

$$(\mathbf{i}I_n - (M^*M)^{-1}M^*)(\mathbf{i}I_n + (M^*M)^{-1}M^*)x - x^*((\mathbf{i}I_n + (M^*M)^{-1}M^*)^* P^2(\mathbf{i}I_n + (M^*M)^{-1}M^*))x < 0.$$

This yields

$$x^*((\mathbf{i}I_n - (M^*M)^{-1}M^*)^*(\mathbf{i}P^2 - P^2(M^*M)^{-1}M^*) - (\mathbf{i}I_n + (M^*M)^{-1}M^*)^*(\mathbf{i}P^2 + P^2(M^*M)^{-1}M^*))x < 0.$$

Thus finally, we have

$$x^*(M(M^*M)^{-1})^* P^2 + P^2(M^*M)^{-1}M^*)x > 0,$$

□

3.3. **Numerical Testing.** We present comparison on numerical testing for μ-values. We use the well-known numerical algorithms to approximate μ-values, this includes: The Matlab routine **mussv**, power algorithm (PA) [37], the Gain Based Algorithm (GBA) [19], the Poles migration Algorithm (PMA) [7], the Non-linear optimization Algorithm (NLA) [38], and an ODE's based Algorithm (LRA) given by first author [39]. We choose a class of structured matrices which appears in the transportation models. The Matlab EigTool is has been utilized for the graphical interpretation to the computation of pseudo-spectrum of the given structured matrices.

The graphical representations shown in 2-dimensional plane represents the spectrum and pseudo-spectrum of structured matrices. The black dots around the spectrum and pseudo-spectrum in each of such figures denotes the field of values (numerical range). The field of values (also known as numerical range) enclose all the eigenvalues, and pseudo-spectrum of each of the structured matrix. The visualization for the pseudo-spectrum in the complex plane represents various level sets of the resolvent $\|(M - zI_n)^{-1}\|$ for the given matrix M. The visualizations of the level sets paly an important and fundamental role for the analysis of the stability, and robustness in different applications, for instance, control systems, and fluid dynamics.

Example 1. We take a 4-dimensional real-valued matrix for Traveling Salesman Problem [40].

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 3 & 0 & 2 & 1 \end{pmatrix}.$$

The numerical testing and comparison on numerical computation to lower bounds of μ-values are as shown in following Table-1.

The lower bounds approximation to μ-values					
mussv	PA	GBA	PMA	NLA	LRA
3.9984	3.9992	3.9987	3.9990	3.9988	3.9985

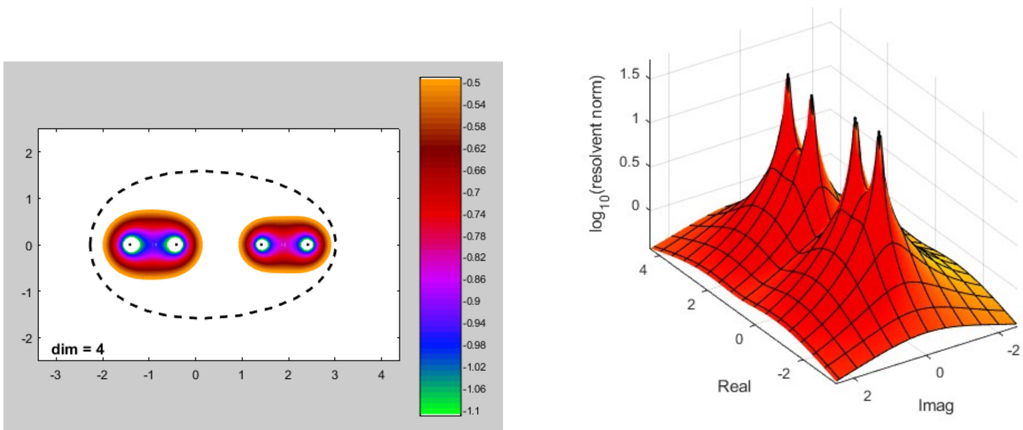


Figure 4. The pseudo-spectrum of matrix A given in Example-1.

Example 2. We take a 6-dimensional real-valued symmetric circulant matrix for Traveling Salesman Problem [41].

$$A = \begin{pmatrix} 0 & 4 & 1 & 6 & 1 & 4 \\ 4 & 0 & 4 & 1 & 6 & 1 \\ 1 & 4 & 0 & 4 & 1 & 6 \\ 6 & 1 & 4 & 0 & 4 & 1 \\ 1 & 6 & 1 & 4 & 0 & 4 \\ 4 & 1 & 6 & 1 & 4 & 0 \end{pmatrix}.$$

The numerical testing and comparison on numerical computation to lower bounds of μ-values are as shown in following Table-2.

The lower bounds approximation to μ-values					
mussv	PA	GBA	PMA	NLA	LRA
3.9984	3.9992	3.9987	3.9990	3.9988	3.9985

Example 3. We take a 4-dimensional real-valued matrix taken from [42].

$$A = \begin{pmatrix} 3 & 5 & 1 & -2 \\ 0 & -1 & 5 & 10 \\ 3 & 5 & 1 & 9 \\ -2 & 1 & 6 & 6 \end{pmatrix}.$$

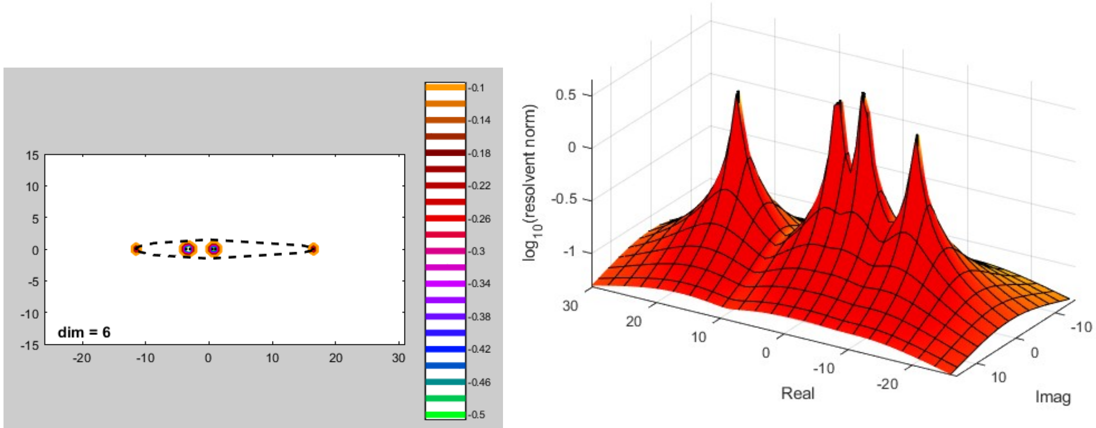


Figure 5. The pseudo-spectrum of A given in Example-2.

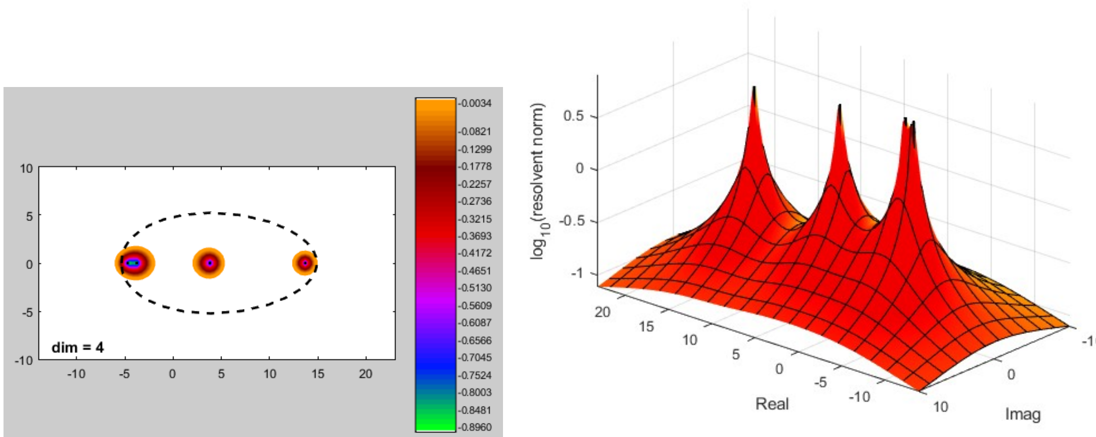


Figure 6. The pseudo-spectrum of A given in Example-3.

The numerical testing and comparison on numerical computation to lower bounds of μ-values are as shown in following Table-3.

The lower bounds approximation to μ-values					
mussv	PA	GBA	PMA	NLA	LRA
16.4123	16.4176	16.4198	16.4183	16.4142	16.4130

4. Spectrum and Pseudo-Spectrum of D-Stable Structured Matrices in Economic Models

The main objective is to construct novel mathematical and theoretical results for interconnections among structured H-stability and the computation of μ-values. The structured matrices under consideration are squared real-valued or complex-valued.

The following Definition 4.1 is taken from [22].

Definition 4.1. *The given structured matrix $A \in \mathbb{R}^{n\times n}$ is known as a structured H-stable matrix if matrix multiplication HA have all real parts of eigenvalues to be strictly positive for each structured symmetric positive-definite matrix H.*

Theorem 4.1 shows that given matrix $A \in \mathbb{C}^{n\times n}$ is a H-stable matrix whenever H possesses a structured positive definite matrix.

Theorem 4.1. *[44] Consider $A, H \in \mathbb{C}^{n\times n}, H > 0$. If the matrix A be a structured H-stable matrix to each $H > 0$, then we have that $H > 0$, the positive definite structured matrix if A is structured H-stable.*

Proof. Consider that matrix $A \in \mathbb{C}^{n\times n}$ is a structured H-stable for each H. This implies that $\mathrm{Re}(\lambda_i(AH)) > 0, \forall i$. Furthermore, we have that $H > 0$ and it does causes $\mathrm{Re}(\lambda_i(AH)) > 0, \forall i$ for given $A \in \mathbb{C}^{n\times n}$. From this we conclude that for a given $A \in \mathbb{C}^{n\times n}$, $\mathrm{Re}(\lambda_i(AH)) > 0, \forall i$ and it holds true only if $H > 0$, the positive definite structured matrix. □

Theorem 4.2 yields an interesting theoretical results on the interaction between H-stability and μ-values.

Theorem 4.2. *[44] Consider that $A \in \mathbb{R}^{n\times n}$ is H-stable matrix. Then for each $H \in H^+, 0 \le \mu_{\mathbb{B}}\left(\frac{1}{A^2}\right) < 1$, having*

$$H^+ = \{H : \lambda_i(H) > 0, \forall i = 1 : n\},$$

and mathematical notation $\mathbb{B}$ represents set of perturbations with block diagonal structure.

Proof. The main objective is to prove that $0 \le \mu_{\mathbb{B}}\left(\frac{1}{A^2}\right) < 1$ holds true against each $H \in H^+$. Since, D-stability implies stability which further helps to prove required result. $A \in \mathbb{R}^{n\times n}$ is $D-$stable structured matrix iff the given matrix A is a stable matrix and

$$\lambda_i\begin{pmatrix} A & -H \\ H & A \end{pmatrix} \neq 0, \quad \forall i.$$

As, we know $\lambda_i\begin{pmatrix} A & -H \\ H & A \end{pmatrix} \neq 0,\ \forall i$. As a result this implies that

$$\lambda_i\left(A^2 - HA^{-1}HA\right) \neq 0, \forall i.$$

Thus finally, we have

$$\lambda_i\left(I_n - \frac{1}{A^2}H\right) \neq 0, \Rightarrow 0 \le \mu_{\mathbb{B}}\left(\frac{1}{A^2}\right) < 1.$$

□

Theorem 4.3. *[44] Consider that $A, H \in \mathbb{C}^{n\times n}, H \ge 0$, a Hermitian and positive semi-definite. If $\mathrm{Re}(\lambda_i(AH)) = \mathrm{Re}(\lambda_i(H))$ to each $H \ge 0$, then $\mathrm{Re}(\lambda_i(A)) > 0, \forall i$ and*

$$0 \le \mu_{\mathbb{B}}\left((\boldsymbol{i}I_n + A)^{-1}(\boldsymbol{i}I_n - A)\right) < 1,\ \boldsymbol{i} = \sqrt{-1}.$$

Proof. The main objective is to prove that $\mathrm{Re}(\lambda_i(A)) > 0, \forall i$ if $\mathrm{Re}(\lambda_i(AH)) = \mathrm{Re}(\lambda_i(H)), \forall i, \forall H \geq 0$. As, if $\mathrm{Re}(\lambda_i(A)) \geq 0, \forall i$ and $A \in \mathbb{C}^{n\times n}$ is $n \times n$-singular matrix, then we have that $\exists\, U$, a unitary matrix so that

$$U^* A U = \begin{pmatrix} M_{11} + \mathbf{i}N_{11} & \mathbf{i}N_{12} \\ \mathbf{i}N_{21} & \cdot \end{pmatrix},$$

with $M_{11} > 0$; for $U^*AU = \begin{pmatrix} \cdot & \cdot \\ \cdot & I_n \end{pmatrix} \geq 0$. In turn, this implies that

$$\mathrm{Re}(\lambda_i(AH)) = \mathrm{Re}(\lambda_i(H)), \forall i.$$

Next, the aim is to prove that

$$0 \leq \mu_{\mathbb{B}}\left((\mathbf{i}I_n + A)^{-1}(\mathbf{i}I_n - A)\right) < 1.$$

Let $A = e^H$, a stable matrix whereas the matrix $H \geq 0$, is positive semi-definite. Consider that $P > 0$ in such a way that $\lambda_i(\mathbf{i}I_n + e^H P) \neq 0, \forall i$ whereas $P = (\mathbf{i}I_n + \Delta)^{-1}(\mathbf{i}I_n - \Delta)$ for all $\Delta \in \mathbb{B}$. This implies that

$$\lambda_i\left(\mathbf{i}I_n + e^H (iI_n + \Delta)^{-1}(\mathbf{i}I_n - \Delta)\right) \neq 0 \quad \forall i, \forall \Delta \in \mathbb{B}.$$

Thus finally, we have that

$$\lambda_i\left((\mathbf{i}I_n + A)^{-1}(\mathbf{i}I_n - A)\Delta\right) \neq 0 \quad \forall i, \forall \Delta \in \mathbb{B}.$$

Thus,

$$0 \leq \mu_{\mathbb{B}}\left((\mathbf{i}I_n + A)^{-1}(\mathbf{i}I_n - A)\right) < 1.$$

□

The following Definitions are taken from classical papers [23] and [24], respectively.

Definition 4.2. *If the matrix product DA is stable, then the given structured matrix $A \in \mathbb{R}^{n\times n}$ is known as $D(\alpha)$- structured stable matrix to each positive α-scalar structured matrix D.*

Definition 4.3. *If $D[\alpha_k]$ is a scalar matrix for each $k = 1 : p$, means that,*

$$D = \mathrm{Diag}(d_{11}I[\alpha_1], ..., d_{11}I[\alpha_p]),$$

then diagonal structure matrix D is known as an α-scalar matrix.

Note: In above Definition 4.3, $D[\alpha_k]$ denotes a sub-matrix obtained from a number of rows and number of columns having the indices α_k whereas $\alpha = (\alpha_1, ..., \alpha_p)$ denotes the partition of indices set $\{1, ..., n\}$ and $1 \leq p \leq n$ is as a positive integer.

4.1. **The novel theoretical results for interconnection among $D(\alpha)$-stable structured matrices, and μ-values.** Theorem 4.4 is a bridge for class of structured matrices like $D(\alpha)$-stable matrices, and their μ-values.

Theorem 4.4. *[44] Consider matrix $A \in \mathbb{R}^{n\times n}$, then it is a $D(\alpha)$ structured stable if and only if*

$$\mathrm{Re}\Big(\lambda_i(\mathrm{Diag}(d_{kk}I[\alpha_k]))A + A^T(\mathrm{Diag}(d_{kk}I[\alpha_k]))\Big) > 0,\ \forall i = 1:n, \forall k = 1:p, \quad 1 \leq p \leq n,$$

and $0 \leq \mu_{\mathbb{B}}(M) < 1$, whereas the matrix M is obtained from A as

$$M = \Big(\boldsymbol{i}I + \mathrm{Diag}(d_{kk}I[\alpha_k])A + A^T(\mathrm{Diag}(d_{kk}I[\alpha_k]))\Big)^{-1}\Big(\boldsymbol{i}I - \mathrm{Diag}(d_{kk}I[\alpha_k])A - A^T(\mathrm{Diag}(d_{kk}I[\alpha_k]))\Big).$$

Proof. The main aim is to prove that matrix $A \in \mathbb{R}^{n\times n}$ is a $D(\alpha)$-stable matrix iff $0 \leq \mu_{\mathbb{B}}(M) < 1$. Consider that $\Delta \in \mathbb{B}$ possesses a block diagonal structure, that is

$$\Delta = (\mathbf{i}I_n - \mathrm{Diag}(d_{kk}I[\alpha_k]))\,(\mathbf{i}I_n + \mathrm{Diag}(d_{kk}I[\alpha_k]))^{-1},\ \forall k = 1:p, 1 \leq p \leq n.$$

Thus, we have that

$$\lambda_i\Big(I_n - \Big(\mathbf{i}I_n + \mathrm{Diag}(d_{kk}I[\alpha_k])A + A^T\mathrm{Diag}(d_{kk}I[\alpha_k])\Big)^{-1}\Big(\mathbf{i}I_n - \mathrm{Diag}(d_{kk}I[\alpha_k])A - A^T\mathrm{Diag}(d_{kk}I[\alpha_k])\Big)\Delta\Big)$$

is not exactly equal to zero, $\forall i, \forall k = 1:p, 1 \leq p \leq n; \forall \Delta \in \mathbb{B}$. The mathematical expression λ_i, $\forall i$ is exactly equivalent to condition $0 \leq \mu_{\mathbb{B}}(M) < 1$, we refer to see [25] and the references therein. □

Theorem 4.5 gives the necessary condition to given structured matrix $A \in \mathbb{C}^{n,n}$ is to be a $D(\alpha)$-stable structured matrix.

Theorem 4.5. *[44] Consider $A \in \mathbb{C}^{n,n}$. For given matrix A to be $D(\alpha)$-stable the necessary condition is to write $A = e^B$ (the hermitian $B \in \mathbb{C}^{n,n}$) is a stable structured matrix, and further* $0 \leq \mu_{\mathbb{B}}\Big((\boldsymbol{i}I + e^B)^{-1}(\boldsymbol{i}I - e^B)\Big) < 1$.

Proof. The main objective it to prove that A is $D(\alpha)$-stable matrix if $\lambda_k(\mathbf{i}I_n + e^B\, Diag(d_{kk}I[\alpha_k])) \neq 0,\ \forall\, k = 1:p,\ 1 \leq p \leq n$. Let $\Delta \in \mathbb{B}$ having a block-diagonal structure, and suppose that

$$Diag(d_{kk}I[\alpha_k]) = (\mathbf{i}I_n + \Delta)^{-1}(\mathbf{i}I_n - \Delta).$$

We further conclude that

$$\lambda_k(\mathbf{i}I + e^B(\mathbf{i}I + \Delta)^{-1}(\mathbf{i}I - \Delta)) \neq 0.$$

The definition of μ-values allows to compute

$$0 \leq \mu_{\mathbb{B}}\Big((\mathbf{i}I_n + e^B)^{-1}(\mathbf{i}I_n - e^B)\Big) < 1.$$

□

4.2. **The rank-1 perturbation to D-semistable structured matrices.** The main objective is to establish some novel theoretical results in order to study the interconnection between D-semistable matrices, and μ-values. The following Theorem 4.6 show matrix $A \in \mathbf{C}^{n,n}$ is D-semistable matrix iff it be a semi-stable matrix. Furthermore, all eigenvalues corresponding to rank-1 perturbation for given matrix A, that is, $A + v\omega^*$ has to be non-zero.

Theorem 4.6. *[44] Consider that $A \in \mathbb{C}^{n,n}$, it is a D-semistable matrix iff matrix A is a semi stable structured matrix, also $\lambda_k(A + v\omega^*) \neq 0, \forall k$, having $v\omega^*$, a rank-1 structure.*

Proof. Suppose given matrix A is D-semistable matrix. This means that that $\mathrm{Re}(\lambda_k(AD)) \geq 0$, for all k, for all $D \in \Omega$. Our main objective it to prove given matrix A has semi-stable structure. Meaning that $\mathrm{Re}(\lambda_k(A)) \geq 0$ for all k and $\lambda_k(A + v\omega^*) \neq 0, \forall k, v\omega \in \mathbb{C}^{n,1}$. As, $\mathrm{Re}(\lambda_k(AD)) \geq 0, \forall k, \forall D \in \Omega$. This in turn implies that $\mathrm{Re}(\lambda_k(AI_n)) \geq 0, \forall k$ or $\mathrm{Re}(\lambda_k(A)) \geq 0$. This further yields given matrix A possesses a semi-stable structure. Further, if A is being subject to rank-1 admissible perturbations, that is, $(A + v\omega^*)$, then we need to prove

$$\mathrm{Re}(\lambda_k(A + v\omega^*)) \neq 0, \quad \forall k.$$

Suppose for $v, \omega \in \mathbb{C}^{n,1}$, $z_l^* v \neq 0$ and $\omega^* z_r \neq 0$. Here we have, z_l and z_r are left eigenvectors, and right hand eigenvectors corresponds to a simple eigenvalue of structured matrix A. Furthermore, we assume $y \in \mathrm{Ker}(A + v\omega^*)$. This yields a system of linear equations (of homogeneous type)

$$z_l^* v\omega^* y = z_l^*(A + v\omega^*)y = 0.$$

This further implies that $\omega^* y = 0$ since $z_l^* v \neq 0$. Also, $(A + v\omega^*)y = 0$, implying $y = \alpha z_r$ for some $\alpha \in \mathbb{C}$. In turn this reduces to $\omega^* y = \alpha\omega^* z_r = 0 \Rightarrow \alpha = 0, y = 0$. Conversely, assume that if $\omega^* z_r = 0$, then we have that

$$(A + v\omega^*)z_r = 0.$$

Furthermore, $z_l^*(A + v\omega^*) = 0$ if $z_e^* v = 0$. □

Theorem 4.7 show that square complex-valued matrix is D-semistable type of matrix only if structured matrix A is a semi-stable, further μ-values are bounded from above by 1.

Theorem 4.7. *[44] Suppose $A \in \mathbb{C}^{n,n}$ then given matrix A is a D-semistable matrix iff A is a semi-stable matrix, and $0 \leq \mu_{\mathbb{B}}(M) < 1$. Also,*

$$M = (\boldsymbol{i}I_n + \widehat{A})^{-1}(\boldsymbol{i}I_n - \widehat{A}),$$

having $\widehat{A} = A + v\omega^*$ to $v, \omega \in \mathbb{C}^{n,1}$.

Proof. The structured matrix A is a D-semistable matrix iff A is a semi-stable matrix. Also, $\lambda_k(A + v\omega^* + \mathbf{i}P)$ not exactly equal to zero for all k, and for all $P \in \Omega$. We aim to prove that $\lambda_k(A + v\omega^* + \mathbf{i}P) \neq 0$ for all k, for all $P \in \Omega$. Let $\Delta = (\mathbf{i}I - P)(\mathbf{i}I + P)^{-1}$ have a block-diagonal structure, and $\Delta \in \mathbb{B}$. The diagonal matrix P have all positive entries on its main diagonal. Further, we have

$$P = (\mathbf{i}I_n + \Delta)^{-1}(\mathbf{i}I_n - \Delta).$$

Since, $\lambda_k(A + v\omega^* + \mathbf{i}P) \neq 0$, which implies that

$$\lambda_k\Big(A + v\omega^* + \mathbf{i}(\mathbf{i}I + \Delta)^{-1}(\mathbf{i}I - \Delta)\Big) \neq 0, \quad \forall \Delta \in \mathbb{B}.$$

Also,

$$\operatorname{rank}\Big(A + v\omega^* + \mathbf{i}(\mathbf{i}I_n + \Delta)^{-1}(\mathbf{i}I_n - \Delta)\Big) = \operatorname{rank}\left((\mathbf{i}I_n + A + v\omega^*) - (\mathbf{i}I_n - A + v\omega^*)\Delta\right), \quad \forall \Delta \in \mathbb{B}.$$

Thus finally, this yields

$$\lambda_k\Big(I_n - (\mathbf{i}I_n + A + v\omega^*)^{-1}(\mathbf{i}I_n - A + v\omega^*)\Delta)\Big) \neq 0, \quad \forall \Delta \in \mathbb{B},$$

which is the necessary condition that $0 \leq \mu_{\mathbb{B}}(A) < 1$. □

4.3. **Pseudo-spectrum.** The pseudo-spectrum for $M \in \mathbb{C}^{n,n}$ is the set of all the eigenvalues corresponding to M. The question may raise is either singularity of M does or doesn't appear to be robust in the sense that a small perturbation ϵ vary the answer from **yes** to **no**. We think either $\|M^{-1}\|$ is large enough or this is not the case? For an eigenvalue λ of given matrix M, one may ask a much better question: Does the matrix-norm $\|(\lambda I_n - M)^{-1}\|$ is large or not? these questions allows to write definitions given as below, see [26].

Definition 4.4. *Consider that M is a given n-dimensional matrix, and let $\epsilon > 0$, a small perturbation. The pseudo-spectrum $\sigma_\epsilon(M)$ is the collection of all the eigenvalues $\lambda \in \mathbb{C}$ such that*

$$\|(\lambda I_n - M)^{-1}\| > \frac{1}{\epsilon}.$$

Remark 4.1. *The $\lambda \in \sigma(M)$, $\sigma(M)$ represents spectrum of given structured matrix M, $\|(\lambda I_n - M)^{-1}\| = \infty$.*

The second definition of pseudo-spectrum is:

Definition 4.5. *Let M is a n-dimensional matrix, and assume that $\epsilon > 0$, a small perturbation. The ϵ-pseudospectrum $\sigma_\epsilon(M)$ is the set of eigenvalues $\lambda \in \mathbb{C}$ so that*

$$\lambda \in \sigma(M + E),$$

for some E with $\|E\| < \epsilon$.

The third definition of pseudo-spectrum is given as bellow:

Definition 4.6. *Let M is a n-dimensional matrix, and assume that $\epsilon > 0$, a small perturbation. The ϵ-pseudospectrum $\sigma_\epsilon(M)$ is the collection of all the eigenvalues $\lambda \in \mathbb{C}$ so that*

$$\|(\lambda I_n - M)v\| < \epsilon$$

for some $v \in \mathbb{C}^{n,1}$, $\|v\| = 1$.

The following Theorem results establishes an equivalence relation for all above definitions of pseudo-spectrum.

Theorem 4.8. *Let M is a n-dimensional complex-valued matrix, then all three definitions of pseudo-spectrum are equivalent.*

4.4. **Numerical Testing.** We draw a comparison on the numerical computation of μ-values bounds from below. The well-known Matlab routine **mussv** provides best feedback on approximation of lower and upper bounds of μ-values. Our methodology provides tighter outcomes on numerical approximation of μ-values bounds approximated from below. The class of structured matrices under consideration are taken from an economic models. The new results as compared to existing techniques are are much sharper because:
(i) We make use of computation of singular values rather than computing only the eigenvalues of structured matrices.
(ii) The singular values are obtained via singular value decomposition (SVD). We have used Matlab command **svd** to approximate μ-values of given structured matrix.
(iii) The computation to the largest singular value of structured matrices helps to compute the sharper lower bounds of μ-values.
(iv) The computational cost is not high as compared with exiting mathematical techniques.
A number of numerical examples are provided for economy models. The numerical testing are performed to compute and compare μ-values bounds from below. The results on the numerical computation on bounds of μ-values guarantees the effectiveness of proposed technique as compare with existing techniques.

Examples 1 and 2 are taken from [27]. In these examples the main objective was to determine the capital demands which is a complicated procedure in economics. Each factory and company faces some problems and they seek for an optimal but practically valid solutions. From mathematical view point, this involve the demand of study of **cost matrix** which is obtain by calculating capital demands.

Example 1. We take a real-valued **cost matrices** which is taken from [27].

$$A_1 = \begin{bmatrix} 4 & 6 \\ 5 & 7 \end{bmatrix}, \ A_2 = \begin{bmatrix} 10 & 30 & 20 \\ 80 & 70 & 90 \\ 40 & 50 & 60 \end{bmatrix}.$$

Figure 1 present the behavior of spectrum, singular values, μ-values, and pseudo-spectrum of matrices A_1, and A_2.

Example 2. We take $30, 40, 50, 60, 70, 80, 100$ and 120 dimensional real-valued matrices which are generated with MATLAB command **"rand"**.

A few comparison between the proposed methodology [16], and **mussv** as implemented in the MATLAB Control Toolbox.

n	$\mu^l_{New} = \mu^l_{PD}$	$\mu^l_{New} > \mu^l_{PD}$	$\mu^l_{New} < \mu^l_{PD}$
05	63	26	11
10	66	24	10
25	39	50	11
50	37	57	6
100	34	63	3

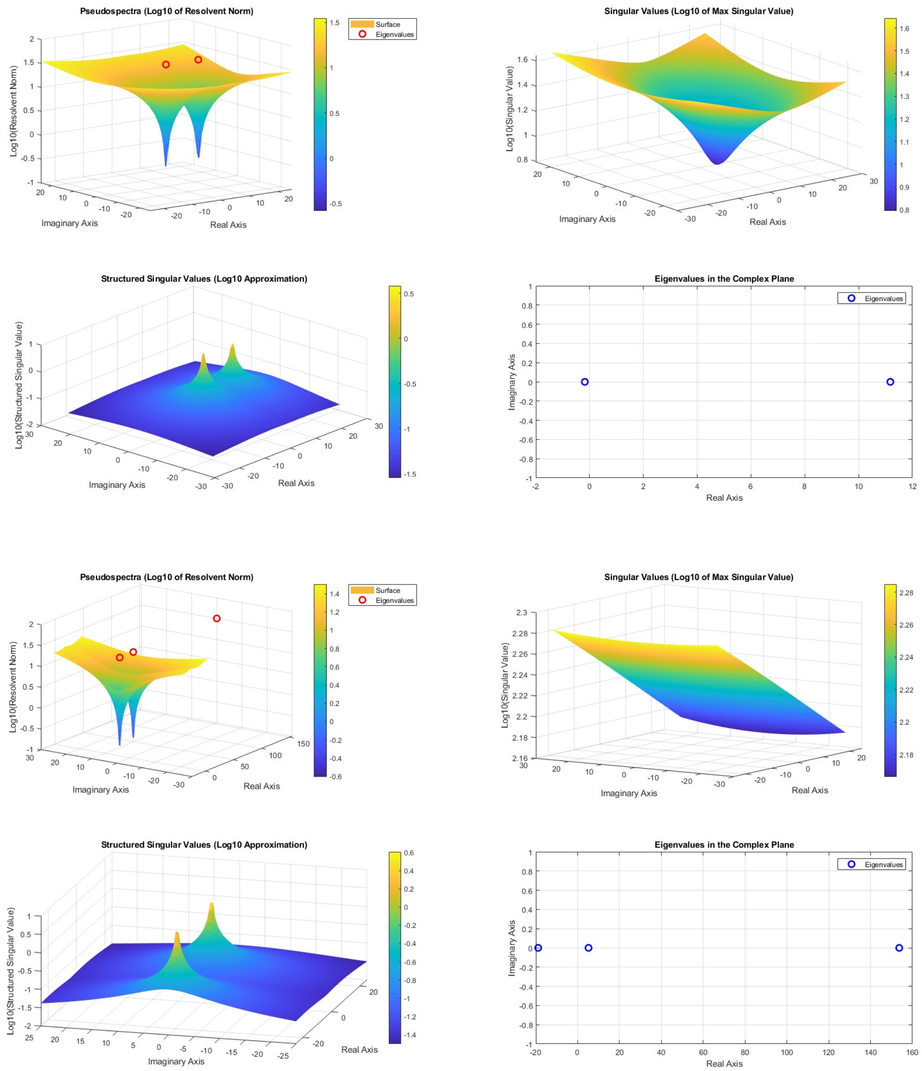


Figure 7. The pseudo-spectrum of A_1 (left), and A_2 (right) from Example 1.

We present a comparison for an algorithm which is discussed in this article, and the algorithm presented by Doyle and Packard. In first column, we give dimension of the randomly generated examples. In second column, we give number of cases (total number of 100). The lower bounds μ^{l}_{New} are computed with the new method while lower bound μ^{l}_{PD} computed by the MATLAB function **mussv** are taken within a tolerance limit of 10^{-3}; while across third column, we report a number of cases where the new lower bound improves as compare to one computed by **mussv**. Finally, in the forth column, we report the number of cases with lower bounds approximated by **mussv** is larger than the lower bound approximated with the new method.

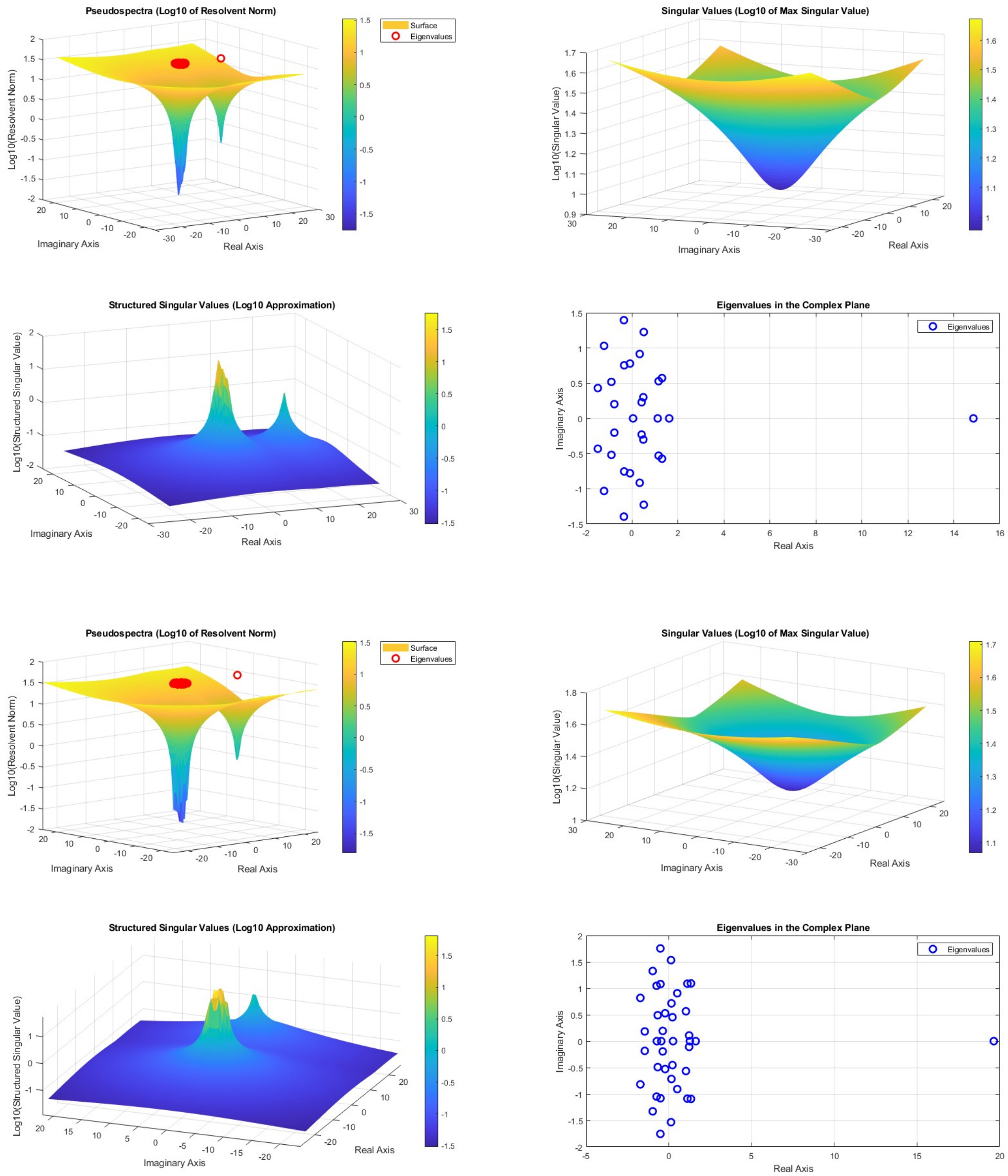


FIGURE 8. The pseudo-spectrum of the matrices with dimensions 30 (left), and 40 (right), respectively from Example 2.

n	δ_{max}	δ_{min}	$\langle\delta\rangle$	$var(\delta)$
05	0.8115	-1.1650	0.0277	0.0454
10	1.0082	-0.8674	0.0364	0.0382
25	1.6358	-0.5046	0.1506	0.1070
50	0.8504	-0.0016	0.1775	0.0586
100	6.1290	-0.0782	0.5793	1.2956

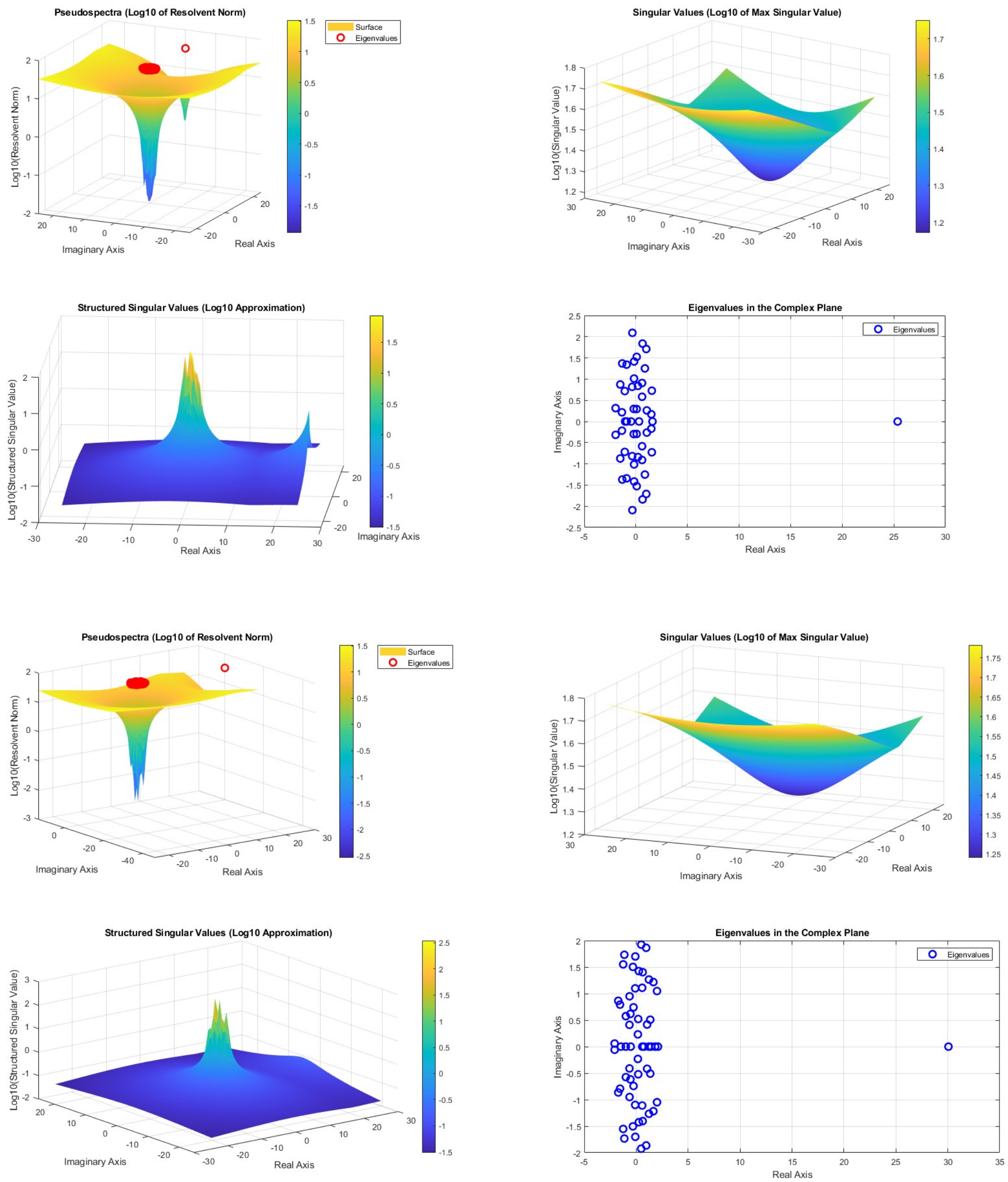


FIGURE 9. The pseudo-spectrum of the matrices with dimensions 50 (left), and 60 (right), respectively from Example 2.

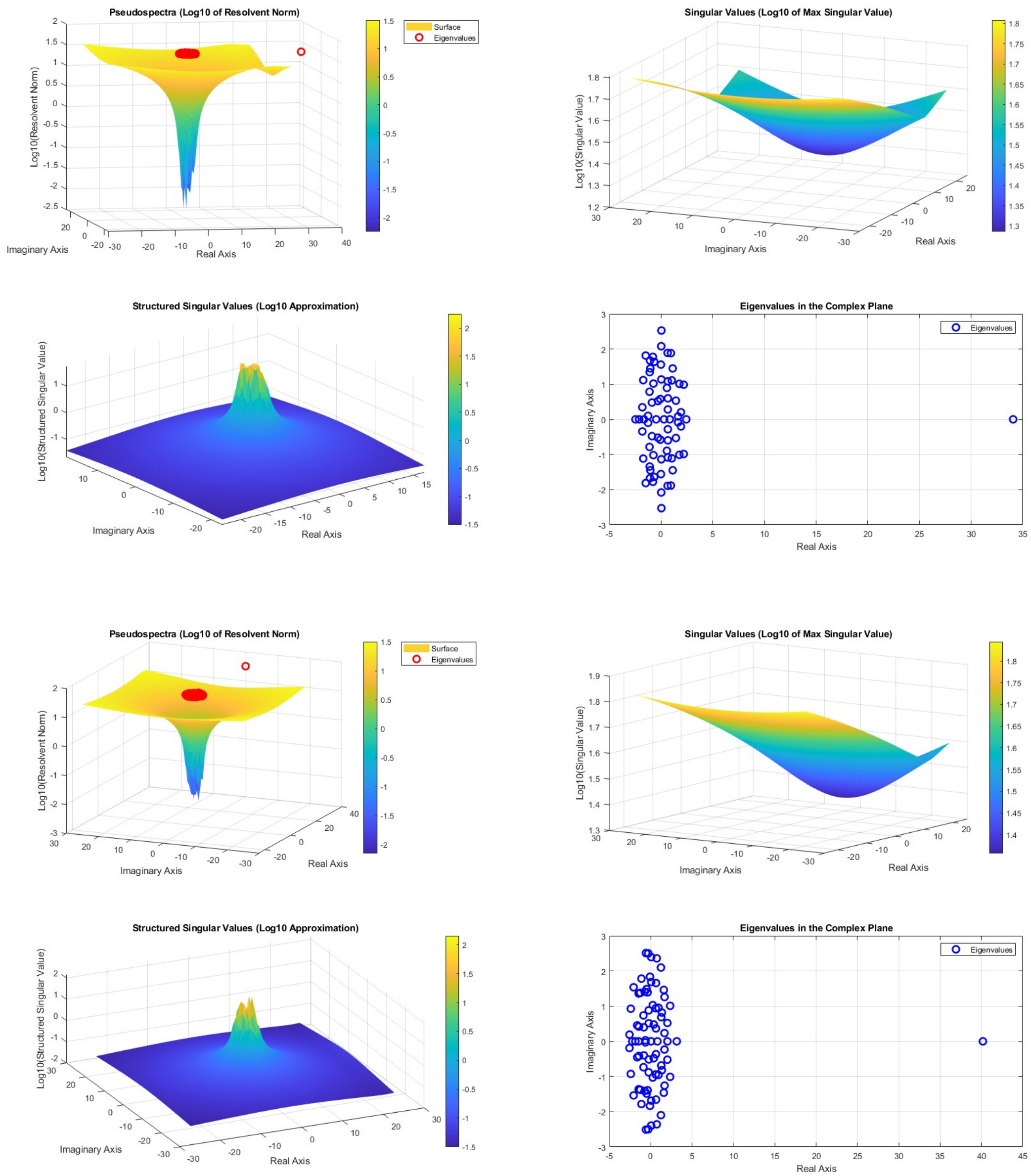


Figure 10. The pseudo-spectrum of the matrices with dimensions 70 (left), and 80 (right), respectively from Example 2.

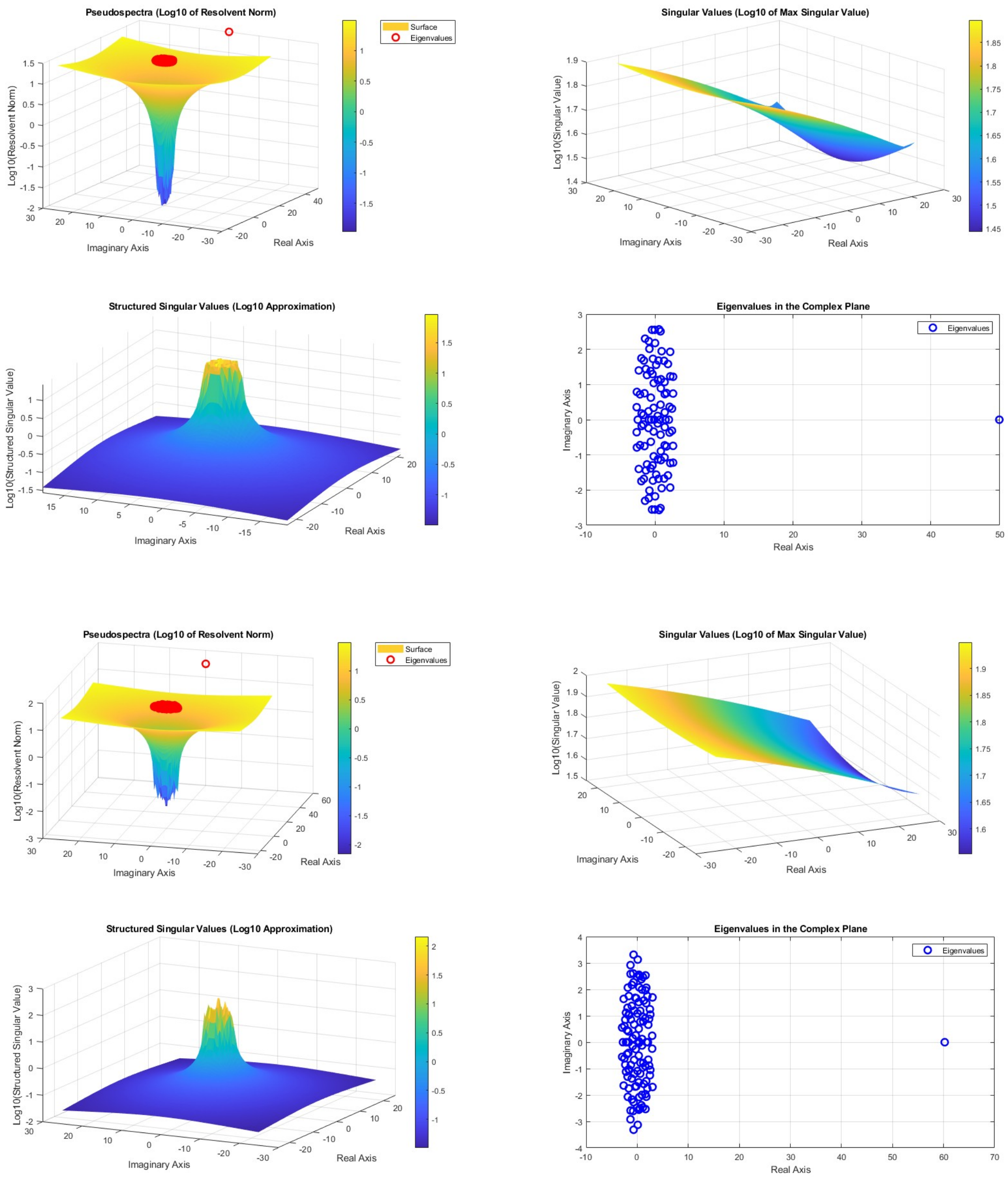


Figure 11. The pseudo-spectrum of the matrices with dimensions 100 (left), and 120 (right), respectively from Example 2.

5. Conclusion

The interconnection between the analysis on stability, D-stability, and μ-values provides a much deeper understanding about the behavior of dynamical systems. The stability analysis provides a basic and foundational assurance to system behavior subject to normal conditions. The analysis on D-stability offers insights into resilience subject to diagonal kind of uncertainties. The computation of μ-values on the other hand, bridge the links between these concepts by quantifying the system's vulnerability subject to structured and unstructured uncertainties. The interconnection between these quantities formulate a cohesive framework allowing to study and analyze the stability properties of complex systems. In this article, we have presented:

(1) State-of-the-art methods for analytical and numerical treatments on the computation of μ-values;

(2) Stability analysis and D-stability analysis of Hitchcock-Koopmans transportation model;

(3) Spectrum and pseud-spectrum of D-stability of structured matrices appearing across economic models.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] K.J. Arrow, M. McManus, A Note on Dynamic Stability, Econometrica 26 (1958), 448–454. https://doi.org/10.2307/1907624.

[2] A.Y. Aleksandrov, A.V. Platonov, V.N. Starkov, N.A. Stepenko, Mathematical Modeling and Study of Stability of Biological Communities, SOLO, St. Petersburg, (2006).

[3] G.V. Kanovei, On One Necessary Condition of the D-Stability of Matrices Having No Less Than Two Zero Elements on the Main Diagonal, Autom. Remote. Control. 62 (2001), 704–708. https://doi.org/10.1023/a:1010210504451.

[4] G.V. Kanovei, D.O. Logofet, D-Stability of 4×4 matrices, Zh. Vychisl. Mat. i Mat. Fiz, 38 (1999), 1429–1435.

[5] H.K. Khalil, P.V. Kokotovic, D-Stability and Multi-Parameter Singular Perturbation, SIAM J. Control Optim. 17 (1979), 56–65. https://doi.org/10.1137/0317006.

[6] G.V. Kanovei, D.O. Logofet, Relations, Properties, and Invariant Transformations of D- and aD-Stable Matrices, Vestnik Moskov. Univ. Ser. I. Mat. Mekh. 6 (2001), 40–43.

[7] J. Magni, C. Döll, C. Chiappa, B. Frapard, B. Girouart, Mixed-μ-Analysis for Flexible Systems. Part I: Theory 1, IFAC Proc. Vol. 32 (1999), 8003–8008. https://doi.org/10.1016/s1474-6670(17)57365-6.

[8] M.S. Yu, D.O. Logofet, Stability of Biological Communities, Nauka, Moscow, (1978).

[9] A.Y. Yu, Equilibrium and Stability in the Models of Population Dynamics. Nauka, Moscow, (1983).

[10] A.C. Enthoven, K.J. Arrow, A Theorem on Expectations and the Stability of Equilibrium, Econometrica 24 (1956), 288–293. https://doi.org/10.2307/1911633.

[11] B.M. Norboevna, R.T. Husenovich, The Method of Using Problematic Education in Teaching Theory of Matrix to Students, Academy 4 (2020), 68–71.

[12] D. Carlson, A New Criterion for H-Stability of Complex Matrices, Linear Algebr. Appl. 1 (1968), 59–64. https://doi.org/10.1016/0024-3795(68)90048-7.

[13] H.K. Khalil, P.V. Kokotovic, Control of Linear Systems with Multiparameter Singular Perturbations, Automatica 15 (1979), 197–207. https://doi.org/10.1016/0005-1098(79)90070-0.

[14] D.O. Logofet, Svicobians of compartment models and the *DaD*-stability of svicobians, Dokl. Akad. Nauk, Ross. Akad. Nauk. 360 (1998), 167–170.

[15] J. Doyle, Analysis of Feedback Systems with Structured Uncertainties, IEE Proc. D Control. Theory Appl. 129 (1982), 242–250. https://doi.org/10.1049/ip-d.1982.0053.

[16] N. Guglielmi, M. Rehman, D. Kressner, A Novel Iterative Method to Approximate Structured Singular Values, SIAM J. Matrix Anal. Appl. 38 (2017), 361–386. https://doi.org/10.1137/16m1074977.

[17] R.L. Dailey, A New Algorithm for the Real Structured Singular Value, 1990 Am. Control. Conf. (1990), 3036–3040. https://doi.org/10.23919/acc.1990.4791276.

[18] A. Yazıcı, A. Karamancıoğlu, R. Kasimbeyli, A Nonlinear Programming Technique to Compute a Tight Lower Bound for the Real Structured Singular Value, Optim. Eng. 12 (2010), 445–458. https://doi.org/10.1007/s11081-010-9120-4.

[19] P. Seiler, G. Balas, A. Packard, A Gain-Based Lower Bound Algorithm for Real and Mixed μ Problems, in: Proceedings of the 45th IEEE Conference on Decision and Control, IEEE, 2006. https://doi.org/10.1109/CDC.2006.377123.

[20] A. Sideris, R.S.S. Pena, Robustness Margin Calculation with Dynamic and Real Parametric Uncertainty, in: 1988 American Control Conference, IEEE, 1988, pp. 1201-1206. https://doi.org/10.23919/ACC.1988.4789903.

[21] Piga, Dario, Computation of the structured singular value via moment lmi relaxations, IEEE Trans. Autom. Control. 61 (2015), 520–525. https://doi.org/10.1109/TAC.2015.2438452.

[22] K.J. Arrow, M. McManus, A Note on Dynamic Stability, Econometrica 26 (1958), 448–454. https://doi.org/10.2307/1907624.

[23] H.K. Khalil, P.V. Kokotovic, D-Stability and Multi-Parameter Singular Perturbation, SIAM J. Control. Optim. 17 (1979), 56–65. https://doi.org/10.1137/0317006.

[24] D. Hershkowitz, N. Mashal, $P^a lpha$-Matrices and Lyapunov Scalar Stability, Electron. J. Linear Algebr. 4 (1998), 39–47. https://doi.org/10.13001/1081-3810.1024.

[25] M. Rehman, T.H. Rasulov, F. Amir, D-Stability, Strong D-Stability and μ-Values, Lobachevskii J. Math. 45 (2024), 1227–1233. https://doi.org/10.1134/s1995080224600754.

[26] L.N. Trefethen, M. Embree, Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators, Princeton University Press, 2005.

[27] D.L.K. Oanh, S.H. Pan, Review of Matrix Theory with Applications in Economics and Finance, Adv. Decis. Sci. 26 (2022), 54–74. https://doi.org/10.47654/v26y2022i3p54-74.

[28] A. Packard, J. Doyle, The Complex Structured Singular Value, Automatica 29 (1993), 71–109. https://doi.org/10.1016/0005-1098(93)90175-s.

[29] M. Newlin, R. Smith, A Generalization of the Structured Singular Value and Its Application to Model Validation, IEEE Trans. Autom. Control. 43 (1998), 901–907. https://doi.org/10.1109/9.701088.

[30] R. Castellanos, A. Messina, H. Sarmiento, Robust Stability Analysis of Large Power Systems Using the Structured Singular Value Theory, Int. J. Electr. Power Energy Syst. 27 (2005), 389–397. https://doi.org/10.1016/j.ijepes.2005.02.001.

[31] K. Weischedel, T.J. McAvoy, Feasibility of Decoupling in Conventionally Controlled Distillation Columns, Ind. Eng. Chem. Fundam. 19 (1980), 379–384. https://doi.org/10.1021/i160076a010.

[32] Y. Arkun, C. Morgan, On the Use of the Structured Singular Value for Robustness Analysis of Distillation Column Control, Comput. Chem. Eng. 12 (1988), 303–306. https://doi.org/10.1016/0098-1354(88)85042-7.

[33] H. Bulut, Algebraic Characterizations of the Singular Value Decompositions in the Transportation Problem, J. Math. Anal. Appl. 154 (1991), 13–21. https://doi.org/10.1016/0022-247x(91)90066-9.

[34] R.E. Cline, L.D. Pyle, The Generalized Inverse in Linear Programming—An Intersection Projection Method and the Solution of a Class of Structured Linear Programming Problems, SIAM J. Appl. Math. 24 (1973), 338–351. https://doi.org/10.1137/0124036.

[35] F.L. Hitchcock, The Distribution of a Product from Several Sources to Numerous Localities, J. Math. Phys. 20 (1941), 224–230. https://doi.org/10.1002/sapm1941201224.

[36] M. Embree, N. Lloyd, Trefethen. Pseudospectra Gateway, http://www.comlab.ox.ac.uk/pseudospectra.

[37] A. Packard, M. Fan, J. Doyle, A Power Method for the Structured Singular Value, in: Proceedings of the 27th IEEE Conference on Decision and Control, IEEE, pp. 2132-2137. https://doi.org/10.1109/CDC.1988.194710.

[38] M. Halton, M.J. Hayes, P. Iordanov, State-space μ Analysis for an Experimental Drive-by-wire Vehicle, Int. J. Robust Nonlinear Control. 18 (2008), 975–992. https://doi.org/10.1002/rnc.1322.

[39] N. Guglielmi, M. Rehman, D. Kressner, A Novel Iterative Method to Approximate Structured Singular Values, SIAM J. Matrix Anal. Appl. 38 (2017), 361–386. https://doi.org/10.1137/16m1074977.

[40] R.H. Warren, Classes of Matrices for the Traveling Salesman Problem, Linear Algebr. Appl. 139 (1990), 53–62. https://doi.org/10.1016/0024-3795(90)90387-r.

[41] F. Greco, I. Gerace, The Symmetric Circulant Traveling Salesman Problem, InTech, 2008. https://doi.org/10.5772/5581.

[42] D. Rückert, M. Stamminger, An Efficient Solution to Structured Optimization Problems Using Recursive Matrices, Comput. Graph. Forum 38 (2019), 33–39. https://doi.org/10.1111/cgf.13758.

[43] M. Rehman, B. Aminov, M.N. Alshehri, M.M. Mohammed, A.O. Mustafa, et al., Spectral Properties of Structured Matrices in Transportation Problems, Eur. J. Pure Appl. Math. 18 (2025), 5637. https://doi.org/10.29020/nybg.ejpam.v18i1.5637.

[44] M. Rehman, S.H. Alshabhi, A.O. Mustafa, M.M. Mohammed, S. Aljohani, et al., Spectrum and Pseudspectrum of *D*-Stable Matrices of Economy Models, J. Math. Comput. Sci. 38 (2024), 298–312. https://doi.org/10.22436/jmcs.038.03.02.