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Strong Convergence of Cesàro Mean Sequences and Split Equilibrium Solutions via Hybrid Mappings in Hilbert Spaces

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Abstract. This paper introduces a novel accelerated shrinking projection algorithm for approximating Cesàro mean sequences and solving split equilibrium problems in real Hilbert spaces. The iterative scheme is constructed using finite families of commutative, normally m-generalized hybrid mappings, with a step size chosen independently of the spectral radius to facilitate computation. We prove that the generated sequence converges strongly to a common element in the intersection of the fixed point sets of the mappings, which also solves the associated split equilibrium problem. The proposed method yields new and extended strong convergence theorems for various classes of hybrid mappings, including normally generalized hybrid, m-generalized hybrid, and normally 2-generalized hybrid mappings. A numerical example is provided to demonstrate the superior convergence rate of our algorithm compared to existing methods. These results generalize and unify several known findings in this direction.

1. Introduction

Let H be a real Hilbert space and $C \subset H$ a nonempty closed convex set. Let $S : C \to H$ be a nonlinear mapping, and denote its fixed point set by

$$F(S) = \{x \in C : Sx = x\}.$$

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Definition 1.1. *A mapping $S : C \to H$ is called* normally m-generalized hybrid *[21] if there exist scalars $\alpha_k, \beta_k \in \mathbb{R}$ for $k = 0, 1, \ldots, m$, such that*

$$\sum_{k=0}^{m} \alpha_{k+1} > 0, \quad \sum_{k=0}^{m} (\alpha_{k+1} + \beta_{k+1}) \geq 0,$$

and for all $x, y \in C$,

$$\sum_{k=0}^{m} \alpha_{k+1} \|S^{m-k}x - Sy\|^2 + \sum_{k=0}^{m} \beta_{k+1} \|S^{m-k}x - y\|^2 \leq 0. \tag{1.1}$$

Remark 1.1. *The class of normally m-generalized hybrid mappings unifies and generalizes several known mappings:*

(1) *For $m = 1$, it reduces to the normally generalized hybrid mapping in the sense of Takahashi et al. [28].*
(2) *With $\alpha_k = -\beta_k = 1$, it recovers the m-generalized hybrid mapping of Maruyama et al. [24].*
(3) *For $m = 2$, it coincides with the 2-generalized hybrid and normally 2-generalized hybrid mappings [16, 21].*
(4) *The generalized hybrid mapping [19] is a special case of both normally generalized hybrid and 2-generalized hybrid mappings.*
(5) *This class includes nonspreading, hybrid, and nonexpansive mappings as subclasses.*

In 1975, Baillon [6] proved a nonlinear version of the classical mean ergodic theorem for nonexpansive mappings in Hilbert spaces. Specifically, he showed that the Cesàro sequence

$$S_n x = \frac{1}{n+1} \sum_{k=0}^{n} S^k x$$

converges weakly to a point in $F(S)$. This result laid the foundation for a wide range of iterative schemes in nonlinear functional analysis and fixed point theory.

Cesàro-type averaging has since become a powerful tool for regularizing and stabilizing iterative processes, especially when direct convergence of iterates is not guaranteed. It has been employed in the analysis of nonexpansive, hybrid, and generalized hybrid mappings, and has proven effective in both weak and strong convergence frameworks.

Beyond its theoretical significance, Cesàro averaging has found wide-ranging applications across various disciplines. In optimization and variational inequality problems, it serves to mitigate oscillations in subgradient and proximal algorithms, enhancing stability and convergence. In signal and image processing, Cesàro means are employed to stabilize iterative reconstruction methods, particularly in the context of ill-posed inverse problems. In equilibrium modeling such as in economics and game theory Cesàro type iterations are instrumental in approximating Nash equilibria and saddle points. Moreover, in machine learning and data science, ergodic averages underpin stochastic approximation techniques and online learning algorithms, contributing to robust performance in dynamic and noisy environments.

The ergodic principle also underpins convergence analysis in monotone operator theory, convex feasibility problems, and fixed point algorithms for nonexpansive mappings. Embedding Cesàro means into projection type methods such as shrinking projection algorithms has led to significant advances in strong convergence results, especially in the context of hybrid mappings and split structures.

Kocourek et al. [19] extended Baillon's work by considering generalized hybrid mappings and proving weak convergence of

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} S^k x$$

to a point in $F(S)$. Takahashi et al. [28] further generalized this for normally generalized hybrid mappings, while Hojo et al. [17] introduced a double Cesàro mean for commutative 2-generalized hybrid mappings:

$$S_n x = \frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^k T^l x,$$

and embedded it into a shrinking projection algorithm to obtain strong convergence. These developments have inspired a variety of algorithms for approximating fixed points and solving equilibrium problems.

To obtain strong convergence, Hojo et al. [18] defined a sequence $\{x_n\} \subset C$ by embedding the Cesàro mean into the following shrinking projection algorithm:

$$\begin{cases} x_1 = x \in C; \\ y_n = \gamma_n x_n + (1-\gamma_n)\frac{1}{n}\sum_{k=0}^{n} T^k x_n; \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}; \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where T is a 2-generalized hybrid mapping and $\{\gamma_n\} \subset [0,1]$ satisfies $0 \leq \gamma_n \leq a < 1$. They proved that the sequence $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $\{u_n\} \subset H$ and $u_n \to u$.

In 2018, Hojo et al. [16] proposed a modified shrinking projection algorithm for two commutative normally 2-generalized hybrid mappings $S, T : C \to H$, defined as:

$$\begin{cases} x_1 = x \in C; \\ y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^k T^l x_n; \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}; \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.3}$$

They proved that $\{x_n\}$ converges strongly to $z_0 = P_{F(S)\cap F(T)}x$, the metric projection of x onto the common fixed point set.

More recently, Kondo [20] considered two normally 2-generalized hybrid mappings S, T and proposed the following shrinking projection algorithm:

$$\begin{cases} x_1 = x \in C, \quad C_1 = C; \\ y_n = a_n x_n + b_n \frac{1}{n} \sum_{k=0}^{n} S^k x_n + c_n \frac{1}{n} \sum_{k=0}^{n} T^k x_n; \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}; \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.4}$$

He proved that the sequence $\{x_n\}$ converges strongly to a point $p \in F(S) \cap F(T)$.

On the other hand, the *split equilibrium problem (SEP)* involves finding a point

$$u \in C \quad \text{such that} \quad F(u,v) \ge 0 \quad \forall v \in C, \tag{1.5}$$

and

$$u^* = Bu \in Q \quad \text{such that} \quad G(u^*, w) \ge 0 \quad \forall w \in D, \tag{1.6}$$

where $C \subset H_1$ and $D \subset H_2$ are nonempty closed convex subsets of real Hilbert spaces, $B : H_1 \to H_2$ is a bounded linear operator, and $F : C \times C \to \mathbb{R}$, $G : D \times D \to \mathbb{R}$ are bifunctions.

Let the solution set of the SEP defined by (1.5)-(1.6) be denoted by

$$\Gamma := \{u \in EP(F) : Bu \in EP(G)\}.$$

It is easy to observe that if $H_1 = H_2$, $C = D$, and $B = I$ is the identity operator, then the SEP reduces to the classical equilibrium problem in the sense of Blum and Oettli [7], with solution set $EP(F) = EP(G)$. Furthermore, if $F(u,v) = \langle f(u), v - u\rangle$ and $G(u^*, w) = \langle g(u^*), w - u^*\rangle$, then the SEP reduces to the *split variational inequality problem (SVIP)* [11], which has been successfully applied to real-world problems such as image reconstruction, phase retrieval, signal processing, data compression, sensor networks, and inverse problems; see, for example, [8–10, 12, 13] and references therein.

Several researchers have proposed and analyzed iterative algorithms for finding common elements in the solution sets of fixed points of the aforementioned mappings and equilibrium problems. For instance, Alizardeh and Moradlou [5] established weak convergence theorems for approximating a common element in the solution set of an equilibrium problem and the fixed point set of a 2-generalized hybrid mapping S in Hilbert spaces. They considered the sequence $\{x_n\}$ generated by:

$$\begin{cases} x_1 = x \in E; \\ u_n \in E \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C; \\ y_n = (1-\beta_n)x_n + \beta_n S x_n; \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n S y_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.7}$$

They proved that $\{x_n\}$ converges weakly to $v = \lim_{n\to\infty} P_{F(S)\cap EP(f)} x_1$, where $P_{F(S)\cap EP(f)} x_1$ is the metric projection of x_1 onto the set $F(S) \cap EP(f)$, $E \subset H$ is a nonempty closed convex set, and $f : E \times E \to \mathbb{R}$ is a bifunction.

In another development, Zhao et al. [29] proposed a strongly convergent algorithm for solving the SEP and fixed point problem of a 2-generalized hybrid mapping S. The sequence $\{x_n\}$ is defined by:

$$\begin{cases} x_0 \in C, \quad D_1 = C_1; \\ u_n = T_{r_n}^{F_1}\left[I - \gamma A^*(I - T_{r_n}^{F_2})A\right]x_n; \\ v_n = (1-\beta_n)u_n + \frac{\beta_n}{n}\sum_{k=0}^{n-1} S^k u_n; \\ y_n = (1-\alpha_n)u_n + \frac{\alpha_n}{n}\sum_{k=0}^{n-1} S^k v_n; \\ D_{n+1} = \{x \in D_n : \|y_n - x\| \le \|x_n - x\|\}; \\ x_{n+1} = P_{D_{n+1}}x_0, \quad \forall n \in \mathbb{N}. \end{cases} \tag{1.8}$$

They proved that $\{x_n\}$ converges strongly to a common element in the solution set of the SEP and the fixed point set of S.

More recently, Haruna et al. [14, 15] considered finite families of commutative normally n-generalized hybrid mappings $S_1, S_2, \ldots, S_N : C \to C$, and defined the sequence $\{S_n x\}$ by:

$$S_n x = \frac{1}{(n+1)^N}\sum_{\tau_1=0}^{n}\sum_{\tau_2=0}^{n}\cdots\sum_{\tau_N=0}^{n} S_1^{\tau_1}S_2^{\tau_2}\cdots S_N^{\tau_N}x, \tag{1.9}$$

for all $n \in \mathbb{N}$, where $\tau_1, \tau_2, \ldots, \tau_N \in \mathbb{N} \cup \{0\}$. They showed that $\{S_n x\}$ converges weakly to a point $q \in \bigcap_{i=1}^{N} F(S_i)$, where

$$q = \lim_{\tau_1,\tau_2,\ldots,\tau_N \in D} PT_1^{\tau_1}T_2^{\tau_2}\cdots T_N^{\tau_N}x,$$

P is the metric projection of H onto $\bigcap_{i=1}^{N} F(T_i)$, and D is a directed set.

Motivated by these developments, this work proposes an accelerated algorithm that incorporates Cesàro averaging within a shrinking projection framework. The algorithm targets the common fixed point set of a finite family of commutative normally m-generalized hybrid mappings, while simultaneously solving a split equilibrium problem. The proposed scheme is designed to achieve strong convergence under mild assumptions, and contributes to the growing literature on ergodic methods in nonlinear analysis.
Key contributions include:

(1) The normally m-generalized hybrid mappings used here generalize those considered in [5], [16], [17], [18], [29] as special cases.
(2) Our algorithm approximates a common element in the intersection of fixed point sets of finitely many commutative nonlinear mappings that also solve split equilibrium problems. In contrast, algorithms in (1.2), (1.3), and (1.4) target only one or two mappings, and those in (1.7), (1.9) yield weak convergence.
(3) The step size is independent of the spectral radius, simplifying computation compared to Zhao et al. [29] (see (1.8)).
(4) Numerical results show that our algorithm converges faster than (1.8).

In light of these contributions, our results extend and generalize the foundational work of Alizardeh and Moradlou [5], Baillon [6], Haruna et al. [14, 15], Hojo et al. [16–18], and Zhao et al. [29].

2. Preliminaries

In this section, we present relevant definitions and existing results that will be used in establishing our main theorems.

Lemma 2.1 ([25]). *Let H be a real Hilbert space. Then for all $x, y \in H$ and $\alpha \in \mathbb{R}$, the following identities hold:*

(i) $\|x-y\|^2 = \|x-z\|^2 + \|z-y\|^2 + 2\langle x-z, z-y\rangle$;
(ii) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$.

Let $C \subset H$ be a nonempty closed convex set. It is well known that for each $v \in H$, there exists a unique point $u = P_C v \in C$ such that

$$\|v-u\| = \inf_{w\in C} \|v-w\|,$$

where $P_C : H \to C$ is called the *metric projection* of H onto C.

Lemma 2.2 ([25]). *Let $v \in H$ and $u \in C$. Then $u = P_C v$ if and only if*

$$\langle v-u, w-u\rangle \leq 0, \quad \forall w \in C.$$

Moreover, for all $z, w \in C$, the following inequality holds:

$$\|z-u\|^2 + \|u-w\|^2 \leq \|z-w\|^2. \tag{2.1}$$

The following identity, due to Maruyama et al. [24], will also be useful:

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x-y\|^2 - \alpha\gamma\|x-z\|^2 - \beta\gamma\|y-z\|^2, \tag{2.2}$$

for all $x, y, z \in H$ and $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha + \beta + \gamma = 1$.

To solve equilibrium problems, we assume the bifunction $g : C \times C \to \mathbb{R}$ satisfies the following conditions (see [7]):

(A1) $g(x,x) = 0 \quad \forall x \in C$;
(A2) g is monotone, i.e., $g(x,y) + g(y,x) \leq 0 \quad \forall x, y \in C$;
(A3) $\limsup_{t\to 0^+} g(x + t(z-x), y) \leq g(x,y) \quad \forall x, y, z \in C$;
(A4) For each fixed $x \in C$, the function $y \mapsto g(x,y)$ is convex and lower semicontinuous.

Lemma 2.3 ([7]). *Let $C \subset H$ be a nonempty closed convex set, and let $g : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions (A1)–(A4). Then for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$g(z,y) + \frac{1}{r}\langle z-x, z-y\rangle \geq 0, \quad \forall y \in C.$$

The following result, due to Combettes [13], characterizes the resolvent operator associated with g:

Lemma 2.4 ([13]). *Let $r > 0$ and $x \in H$. Define the operator $T_r : H \to C$ by*

$$T_r x = \left\{ z \in C : g(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \quad \forall y \in C \right\}.$$

Then the following properties hold:

(i) *T_r is single-valued;*
(ii) *T_r is firmly nonexpansive;*
(iii) *$F(T_r) = EP(g)$;*
(iv) *$EP(g)$ is closed and convex;*
(v) *For all $x \in H$ and $p \in F(T_r)$, one has*

$$\|p - T_r x\|^2 + \|T_r x - x\|^2 \leq \|p - x\|^2.$$

Finally, we recall a useful property of normally m-generalized hybrid mappings, which will play a key role in our analysis.

Lemma 2.5 ([21]). *Let H be a real Hilbert space and $C \subset H$ a nonempty set. Let $T : C \to H$ be a normally m-generalized hybrid mapping with nonempty fixed point set $F(T)$. Then T is quasi-nonexpansive.*

Proof. Since T is normally m-generalized hybrid mapping, then there exist $\alpha_k, \beta_k \in \mathbb{R}$ $(k = 0, 1, \cdots, m)$ such that

$$\sum_{k=0}^{m} \alpha_{k+1} \|T^{m-k}x - Ty\|^2 + \sum_{k=0}^{m} \beta_{k+1} \|T^{m-k}x - y\|^2 \leq 0, \quad \forall y \in C. \tag{2.3}$$

By hypothesis, $F(T) \neq \emptyset$. Thus, let $x \in F(T)$ so that from (2.3) we get

$$\sum_{k=0}^{m} \alpha_{k+1} \|x - Ty\|^2 + \sum_{k=0}^{m} \beta_{k+1} \|x - y\|^2 \leq 0, \quad \forall y \in C.$$

This implies,

$$\sum_{k=0}^{m} \alpha_{k+1} \|x - Ty\|^2 \leq -\sum_{k=0}^{m} \beta_{k+1} \|x - y\|^2 \quad \forall y \in C.$$

Since $\sum_{k=0}^{m} \alpha_{k+1} > 0$, we get

$$\|x - Ty\|^2 \leq -\sum_{k=0}^{m} \frac{\beta_{k+1}}{\alpha_{k+1}} \|x - y\|^2 \quad \forall y \in C.$$

Using the fact that $\sum_{k=0}^{m} (\alpha_{k+1} + \beta_{k+1}) \geq 0$, we get

$$\|x - Ty\|^2 \leq \|x - y\|^2 \quad \forall y \in C.$$

Therefore, $\|x - Ty\| \leq \|x - y\| \quad \forall y \in C$. Hence, T is quasi-nonexpansive. This completes the proof. □

3. Main Results

In this section, we establish the strong convergence of a sequence generated by a new accelerated shrinking projection algorithm. The algorithm approximates a common element in the solution set of fixed points of finitely many commutative normally m-generalized hybrid mappings and the solution set of a split equilibrium problem in real Hilbert spaces.

We begin by stating the assumptions and describing the iterative scheme.

Assumption 3.1. Let the following conditions hold:

(A1) $C_1 \subset H_1$ and $C_2 \subset H_2$ are nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively.

(A2) $T : H_1 \to H_2$ is a bounded linear operator.

(A3) $F : C_1 \times C_1 \to \mathbb{R}$ and $G : C_2 \times C_2 \to \mathbb{R}$ are bifunctions satisfying the conditions of Lemma 2.3.

(A4) $J^F_{r_n} : H_1 \to C_1$ and $J^G_{r_n} : H_2 \to C_2$ denote the resolvent operators associated with F and G, respectively.

(A5) $S_1, S_2, \ldots, S_N : C_1 \to C_1$ and $T_1, T_2, \ldots, T_N : C_1 \to C_1$ are finite families of commutative, normally m-generalized hybrid mappings. The indices $\mu_1, \ldots, \mu_N$ and $\tau_1, \ldots, \tau_N$ belong to $\mathbb{N} \cup \{0\}$.

(A6) The sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} \alpha_n = 0$.

(A7) The solution set

$$\mathcal{S} := \left(\bigcap_{i=1}^{N} \mathrm{Fix}(S_i) \right) \cap \left(\bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \right) \cap \Gamma \neq \emptyset, \quad \text{where} \quad \Gamma := \{u \in EP(F) : Bu \in EP(G)\}.$$

Here, $EP(F)$ and $EP(G)$ denote the solution sets of the equilibrium problems associated with F and G, respectively.

Algorithm 3.1 Accelerated Shrinking Projection Algorithm

Let $u_0, u_1, v_1 \in C_1$, and set $C_1 = C$. Generate sequences $\{u_n\}, \{v_n\} \subset C_1$ as follows:

Step 1. Compute:

$$x_n = u_n + \theta_n(u_n - u_{n-1}), \quad \theta_n \in (0,1).$$

Define:

$$\begin{cases} S_n = \dfrac{1}{(n+1)^N} \sum_{\mu_1=0}^{n} \cdots \sum_{\mu_N=0}^{n} S_1^{\mu_1} \cdots S_N^{\mu_N}, \\ T_n = \dfrac{1}{(n+1)^N} \sum_{\tau_1=0}^{n} \cdots \sum_{\tau_N=0}^{n} T_1^{\tau_1} \cdots T_N^{\tau_N}. \end{cases} \tag{3.1}$$

Step 2. Compute:

$$y_n = \alpha_n v_n + \beta_n S_n x_n + \gamma_n T_n x_n.$$

Choose a step size δ_n such that for any fixed $\eta > 0$,

$$\begin{cases} 0 < \eta < \delta_n \leq \dfrac{\|(I - J^G_{r_n})Ty_n\|^2}{\|T^*(I - J^G_{r_n})Ty_n\|^2} - \eta, & \text{if } Ty_n = J^G_{r_n} Ty_n; \\ \delta_n = \delta > 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

Step 3. Compute:

$$v_{n+1} = J^F_{r_n}\left(y_n - \delta_n T^*(I - J^G_{r_n})Ty_n \right).$$

Update the constraint set:

$$C_{n+1} = \left\{ p \in C_n : \|v_{n+1} - p\|^2 \leq \alpha_n \|v_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \right\}.$$

Step 4. Compute:

$$u_{n+1} = P_{C_{n+1}} u_0.$$

Set $n := n + 1$ and return to **Step 1**.

Theorem 3.2. *Let Assumption 3.1 holds and the sequence $\{u_n\}$ be defined by the Algorithm 3.1. Then $\{u_n\}$ converges strongly to an element $v \in \mathcal{S} = \cap_{i=1}^{N} F(T_i) \cap \cap_{i=1}^{N} F(S_i) \cap \Gamma$.*

Proof. Let $u \in \mathcal{S}$ and $z_n = (y_n - \delta_n T^*(I - J_{r_n}^G)Ty_n)$. Since the mappings $S_1, S_2, \cdots, S_N$ and $T_1, T_2, \cdots, T_N$ are quasi-nonexpansive, then

$$\begin{aligned} \|S_n x_n - u\| &= \|\frac{1}{(n+1)^N} \sum_{\mu_1=0}^{n} \sum_{\mu_2=0}^{n} \cdots \sum_{\mu_N=0}^{n} S_1^{\mu_1} S_2^{\mu_2} \cdots S_N^{\mu_N} x_n - u\| \\ &\le \frac{1}{(n+1)^N} \sum_{\mu_1=0}^{n} \sum_{\mu_2=0}^{n} \cdots \sum_{\mu_N=0}^{n} \|S_1^{\mu_1} S_2^{\mu_2} \cdots S_N^{\mu_N} x_n - u\| \\ &\le \frac{1}{(n+1)^N} \sum_{\mu_1=0}^{n} \sum_{\mu_2=0}^{n} \cdots \sum_{\mu_N=0}^{n} \|x_n - u\| \\ &= \|x_n - u\|. \end{aligned}$$

Similarly,

$$\begin{aligned} \|T_n x_n - u\| &= \|\frac{1}{(n+1)^N} \sum_{\tau_1=0}^{n} \sum_{\tau_2=0}^{n} \cdots \sum_{\tau_N=0}^{n} T_1^{\tau_1} T_2^{\tau_2} \cdots T_N^{\tau_N} x_n - u\| \\ &\le \frac{1}{(n+1)^N} \sum_{\tau_1=0}^{n} \sum_{\tau_2=0}^{n} \cdots \sum_{\tau_N=0}^{n} \|T_1^{\tau_1} T_2^{\tau_2} \cdots T_N^{\tau_N} x_n - u\| \\ &\le \frac{1}{(n+1)^N} \sum_{\tau_1=0}^{n} \sum_{\tau_2=0}^{n} \cdots \sum_{\tau_N=0}^{n} \|x_n - u\| \\ &= \|x_n - u\|. \end{aligned}$$

Also,

$$\begin{aligned} &\|z_n - u\|^2 = \|y_n - \delta_n T^*(I - J_{r_n}^G)Ty_n) - u\|^2 \\ &= \|y_n - u\|^2 - 2\delta_n \langle y_n - u, T^*(I - J_{r_n}^G)Ty_n)\rangle + \delta_n^2 \|T^*(I - J_{r_n}^G)Ty_n)\|^2 \\ &= \|y_n - u\|^2 - 2\delta_n \langle Ty_n - Tu, (I - J_{r_n}^G)Ty_n)\rangle + \delta_n^2 \|T^*(I - J_{r_n}^G)Ty_n)\|^2 \\ &\le \|y_n - u\|^2 - \delta_n \|(I - J_{r_n}^G)Ty_n)\|^2 + \delta_n^2 \|T^*(I - J_{r_n}^G)Ty_n)\|^2 \\ &= \|y_n - u\|^2 - \delta_n \Big[\|(I - J_{r_n}^G)Ty_n)\|^2 - \delta_n \|T^*(I - J_{r_n}^G)Ty_n)\|^2\Big]. \end{aligned} \tag{3.3}$$

We see from

$$\begin{aligned} &\delta_n \le \frac{\|(I - J_{r_n}^G)Ty_n)\|^2}{\|T^*(I - J_{r_n}^G)Ty_n)\|^2} - \eta \\ \iff \quad &\eta \|T^*(I - J_{r_n}^G)Ty_n)\|^2 \le \|(I - J_{r_n}^G)Ty_n)\|^2 - \delta_n \|T^*(I - J_{r_n}^G)Ty_n)\|^2. \end{aligned}$$

Also, the fact that $\eta < \delta_n$ implies

$$\begin{aligned} &\eta^2 \|T^*(I - J_{r_n}^G)Ty_n)\|^2 < \eta\delta_n \|T^*(I - J_{r_n}^G)Ty_n)\|^2 \\ &\quad \le \delta_n \Big[\|(I - J_{r_n}^G)Ty_n)\|^2 - \delta_n \|T^*(I - J_{r_n}^G)Ty_n)\|^2\Big]. \end{aligned} \tag{3.4}$$

Using (3.4) in (3.3) we get

$$\|z_n - u\|^2 \le \|y_n - u\|^2 - \eta^2 \|T^*(I - J_{r_n}^G)Ty_n)\|^2. \tag{3.5}$$

Thus, using (3.5)

$$\begin{aligned} \|v_{n+1} - u\|^2 &= \|J_{r_n}^F z_n - J_{r_n} u\|^2 \\ &\le \|z_n - u\|^2 \\ &\le \|y_n - u\|^2 - \eta^2 \|T^*(I - J_{r_n}^G)Ty_n)\|^2 \qquad (3.6) \\ &\le \|y_n - u\|^2. \qquad (3.7) \end{aligned}$$

But,

$$\begin{aligned} \|y_n - u\|^2 &= \|\alpha_n v_n + \beta_n S_n x_n + \gamma_n T_n x_n - u\|^2 \\ &\le \alpha_n \|v_n - u\|^2 + \beta_n \|S_n x_n - u\|^2 + \gamma_n \|T_n x_n - u\|^2 \\ &\le \alpha_n \|v_n - u\|^2 + (\beta_n + \gamma_n)\|x_n - u\|^2 \\ &= \alpha_n \|v_n - u\|^2 + (1 - \alpha_n)\|x_n - u\|^2. \qquad (3.8) \end{aligned}$$

It follows from (3.7) and (3.8) that

$$\|v_{n+1} - u\|^2 \le \alpha_n \|v_n - u\|^2 + (1 - \alpha_n)\|x_n - u\|^2. \tag{3.9}$$

Therefore, $u \in C_{n+1}$. Hence C_n is nonempty, closed and convex subset of G since $\mathcal{S} \subset C_{n+1} \subset C_n$ for all $n \ge 1$, it follows from $u_n = P_{C_n} u_0$ and Lemma 2.2 that

$$\|u_n - u_0\| = \|P_{C_n} u_0 - u_0\| \le \|u - u_0\|, \quad \forall u \in \mathcal{S}.$$

Thus, the set $\{\|u_n - u_0\|\}$ is bounded. Hence the sequences $\{u_n\}, \{v_n\}, \{x_n\}$ and $\{y_n\}$ are all bounded. Furthermore, $u_n = P_{C_n} u_0$ implies that

$$\begin{aligned} 0 &\le \langle u_0 - u_n, u_n - u_{n+1} \rangle \\ &= \frac{1}{2}\Big[\|u_{n+1} - u_0\|^2 - \|u_{n+1} - u_n\|^2 - \|u_n - u_0\|^2\Big]. \end{aligned}$$

Thus, we see that

$$\begin{aligned} 0 &\le \|u_{n+1} - u_0\|^2 - \|u_{n+1} - u_n\|^2 - \|u_n - u_0\|^2 \qquad (3.10) \\ &\Rightarrow \|u_n - u_0\| \le \|u_{n+1} - u_0\|. \end{aligned}$$

Therefore, $\{\|u_n - u_0\|\}$ is monotone increasing. Hence, $\lim_{n\to\infty} \|u_n - u_0\|$ exists. It follows from (3.10) that

$$\|u_{n+1} - u_n\|^2 \le \|u_{n+1} - u_0\|^2 - \|u_n - u_0\|^2.$$

Taking limit as $n \to \infty$ we get

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{3.11}$$

In similar passion, we may take for any $n > m \ge 1$, $u_m = P_{C_m} u_0$ so that

$$\|u_n - u_0\|^2 = \|u_n - u_m + u_m - u_0\|^2$$

$$
\begin{aligned}
&= \|u_n - u_m\|^2 + \|u_m - u_0\|^2 + 2\langle u_n - u_m, u_m - u_0\rangle \\
&\geq \|u_n - u_m\|^2 + \|u_m - u_0\|^2.
\end{aligned}
$$

Thus,

$$\|u_n - u_m\|^2 \leq \|u_n - u_0\|^2 - \|u_m - u_0\|^2 \to 0 \text{ as } n, m \to \infty.$$

Therefore, the sequence $\{u_n\}$ is Cauchy. Hence, by completeness of H we see that

$$u_n \to v \in H \text{ (say) as } n \to \infty. \tag{3.12}$$

Using Algorithm 3.1 and (3.11), we see that

$$\lim_{n\to\infty} \|u_n - x_n\| = \lim_{n\to\infty} \theta_n \|u_n - u_{n-1}\| = 0. \tag{3.13}$$

It follows from (3.12) and (3.13) that $x_n \to v$. Also, it follows from (3.11) and (3.13) that

$$\lim_{n\to\infty} \|u_{n+1} - x_n\| = 0. \tag{3.14}$$

Since $u_{n+1} = P_{C_{n+1}} u_0 \in C_{n+1} \subset C_n$, then

$$\|v_{n+1} - u_{n+1}\|^2 \leq \alpha_n \|v_n - u_{n+1}\|^2 + (1-\alpha_n)\|x_n - u_{n+1}\|^2.$$

With (3.14) and the fact that $\lim_{n\to\infty} \alpha_n = 0$, we get that

$$\lim_{n\to\infty} \|v_{n+1} - u_{n+1}\| = 0. \tag{3.15}$$

Using (3.13), (3.14) and (3.15), we get

$$\lim_{n\to\infty} \|v_{n+1} - x_n\| = 0, \quad \lim_{n\to\infty} \|v_{n+1} - u_n\| = 0. \tag{3.16}$$

Combining (3.6), (3.8) and Cauchy-Schwartz inequality we get

$$
\begin{aligned}
&\eta^2 \|T^*(I - J_{r_n}^G)Ty_n\|^2 \leq \|y_n - u\|^2 - \|v_{n+1} - u\|^2 \\
&\leq \alpha_n \|v_n - u\|^2 + (1-\alpha_n)\|x_n - u\|^2 - \|v_{n+1} - u\|^2 \\
&= \alpha_n (\|v_n - u\|^2 - \|x_n - u\|^2) + (\|x_n - u\|^2 - \|v_{n+1} - u\|^2) \\
&= \alpha_n (\|v_n - u\|^2 - \|x_n - u\|^2) + (\|x_n - v_{n+1}\|^2 + 2\langle x_n - v_{n+1}, v_{n+1} - u\rangle) \\
&\leq \alpha_n (\|v_n - u\|^2 - \|x_n - u\|^2) \\
&+ \|x_n - v_{n+1}\|(\|x_n - v_{n+1}\| + 2\|v_{n+1} - u\|).
\end{aligned} \tag{3.17}
$$

Using (3.16) with the fact that $\lim_{n\to\infty} \alpha_n = 0$ and $\eta > 0$ on (3.17), we obtain

$$\lim_{n\to\infty} \|T^*(I - J_{r_n}^G)Ty_n\| = 0. \tag{3.18}$$

Thus, we see that

$$\lim_{n\to\infty} \|z_n - y_n\| = \lim_{n\to\infty} \delta_n \|T^*(I - J_{r_n}^G)Ty_n\| = 0. \tag{3.19}$$

From (3.3) and Cauchy-Schwartz inequality, we get

$$
\begin{aligned}
&\delta_n \|(I - J_{r_n}^G)Ty_n\|^2 \leq \|y_n - u\|^2 - \|z_n - u\|^2 + \delta_n^2 \|T^*(I - J_{r_n}^G)Ty_n\|^2 \\
&= \|y_n - z_n\|^2 + 2\langle y_n - z_n, z_n - u\rangle + \delta_n^2 \|T^*(I - J_{r_n}^G)Ty_n\|^2
\end{aligned}
$$

$$\leq \|y_n - z_n\|(\|y_n - z_n\| + 2\|z_n - u\|) + \delta_n^2\|T^*(I - J_{r_n}^G)Ty_n\|^2. \tag{3.20}$$

Taking limit of (3.20) as $n \to \infty$, it follows from (3.18), (3.19) and the fact that $\delta_n = \delta > 0$ that

$$\lim_{n\to\infty}\|(I - J_{r_n}^G)Ty_n\| = 0. \tag{3.21}$$

on the other hand, using the fact that $J_{r_n}^F$ is firmly nonexpansive and $J_{r_n}^F u = u$ we get

$$\begin{aligned} 2\|v_{n+1} - u\|^2 &= 2\|J_{r_n}^F z_n - J_{r_n}^F u\|^2 \\ &\leq 2\langle z_n - u, v_{n+1} - u\rangle \\ &= \|z_n - u\|^2 + \|v_{n+1} - u\|^2 - \|v_{n+1} - z_n\|^2. \end{aligned}$$

Thus,

$$\begin{aligned} &\|v_{n+1} - z_n\|^2 \leq \|y_n - u\|^2 - \|v_{n+1} - u\|^2 \\ &\leq \alpha_n\|v_n - u\|^2 + (1 - \alpha_n)\|x_n - u\|^2 - \|v_{n+1} - u\|^2 \\ &= \alpha_n(\|v_n - u\|^2 - \|x_n - u\|^2) + (\|x_n - u\|^2 - \|v_{n+1} - u\|^2) \\ &= \alpha_n(\|v_n - u\|^2 - \|x_n - u\|^2) + (\|x_n - v_{n+1}\|^2 + 2\langle x_n - v_{n+1}, v_{n+1} - u\rangle) \\ &\leq \alpha_n(\|v_n - u\|^2 - \|x_n - u\|^2) \\ &+ \|x_n - v_{n+1}\|(\|x_n - v_{n+1}\| + 2\|v_{n+1} - u\|). \end{aligned} \tag{3.22}$$

By taking limit of (3.22) as $n \to \infty$, we get

$$\lim_{n\to\infty}\|v_{n+1} - z_n\| = 0. \tag{3.23}$$

It follows from (3.16), (3.19) and (3.23) that

$$\lim_{n\to\infty}\|v_{n+1} - y_n\| = \lim_{n\to\infty}\|u_n - y_n\| = \lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{3.24}$$

Also,

$$\begin{aligned} &\|y_n - u\|^2 = \|\alpha_n v_n + \beta_n S_n x_n + \gamma_n T_n x_n - u\|^2 \\ &\leq \alpha_n\|v_n - u\|^2 + \beta_n\|S_n x_n - u\|^2 + \gamma_n\|T_n x_n - u\|^2 \\ &- \beta_n\gamma_n\|S_n x_n - T_n x_n\|^2 \\ &\leq \alpha_n\|v_n - u\|^2 + (1 - \alpha_n)\|x_n - u\|^2 - \beta_n\gamma_n\|S_n x_n - T_n x_n\|^2. \end{aligned}$$

This implies

$$\begin{aligned} &\beta_n\gamma_n\|S_n x_n - T_n x_n\|^2 \leq \alpha_n\|v_n - u\|^2 + (1 - \alpha_n)\|x_n - u\|^2 - \|y_n - u\|^2 \\ &= \alpha_n(\|v_n - u\|^2 - \|x_n - u\|^2) + (\|x_n - u\|^2 - \|y_n - u\|^2) \\ &= \alpha_n(\|v_n - u\|^2 - \|x_n - u\|^2) + (\|x_n - y_n\|^2 + 2\langle x_n - y_n, y_n - u\rangle) \\ &\leq \alpha_n(\|v_n - u\|^2 - \|x_n - u\|^2) + \|x_n - y_n\|(\|x_n - y_n\| + 2\|y_n - u\|). \end{aligned}$$

Taking limit as $n \to \infty$ keeping in mind that $\beta_n, \gamma_n \in (0,1)$ and $\lim_{n\to\infty} \alpha_n = 0$ we get

$$\lim_{n\to\infty} \|S_n x_n - T_n x_n\| = 0. \tag{3.25}$$

Using (3.25) and the fact that

$$\|y_n - T_n x_n\| \le \alpha_n \|v_n - S_n x_n\| + \beta_n \|S_n x_n - T_n x_n\|,$$

implies,

$$\lim_{n\to\infty} \|y_n - T_n x_n\| = 0. \tag{3.26}$$

It follows from (3.24), (3.25) and (3.26) that

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0, \quad \lim_{n\to\infty} \|x_n - S_n x_n\| = 0. \tag{3.27}$$

It is known that for any subsequence $\{x_{n_j}\}$ of $\{x_n\}$, $x_{n_j} \rightharpoonup v$, since $x_n \to v$. Thus, using (3.27) we conclude that both $S_{n_j} x_{n_j} \rightharpoonup v$ and $T_{n_j} x_{n_j} \rightharpoonup v$. We now show that $v \in \mathcal{S}$. Since S_1 is normally m-generalized hybrid mapping, then by (1.1) there exist $\alpha_k, \beta_k \in \mathbb{R}$ $(k = 0, 1, \cdots, m)$ such that

$$\sum_{k=0}^{m} \alpha_{k+1} \|S_1^{m-k} x - S_1 y\|^2 + \sum_{k=0}^{m} \beta_{k+1} \|S_1^{m-k} x - y\|^2 \le 0, \quad \forall x, y \in C_1. \tag{3.28}$$

Using Lemma 2.1(1) on (3.28) we get for all $x, y \in C_1$ that

$$\begin{aligned} \sum_{k=0}^{m} \alpha_{k+1} \Big(\|S_1^{m-k} x - y\|^2 + \|y - S_1 y\|^2 \quad &+ \quad 2\langle S_1^{m-k} x - y, y - S_1 y \rangle \Big) \\ &+ \quad \sum_{k=0}^{m} \beta_{k+1} \|S_1^{m-k} x - y\|^2 \le 0. \end{aligned}$$

This implies, for all $x, y \in C_1$ that

$$\begin{aligned} (\sum_{k=0}^{m} \alpha_{k+1} \quad &+ \quad \sum_{k=0}^{m} \beta_{k+1}) \|S_1^{m-k} x - y\|^2 & (3.29) \\ &+ \quad \sum_{k=0}^{m} \alpha_{k+1} \Big(\|y - S_1 y\|^2 + 2\langle S_1^{m-k} x - y, y - S_1 y \rangle \Big) \le 0. & (3.30) \end{aligned}$$

Since $\sum_{k=0}^{m} (\alpha_k + \beta_k) \ge 0$, then we obtain from (3.29) that

$$\sum_{k=0}^{m} \alpha_{k+1} \Big(\|y - S_1 y\|^2 + 2\langle (S_1^{m-k} x - x) + (x - y), y - S_1 y \rangle \Big) \le 0, \quad \forall x, y \in C. \tag{3.31}$$

Thus, inequality (3.31) becomes

$$\sum_{k=0}^{m} \alpha_{k+1} \Big(\|y - S_1 y\|^2 + 2\langle x - y, y - S_1 y \rangle \Big) + 2 \sum_{k=0}^{m-1} \alpha_{k+1} \langle S_1^{m-k} x - x, y - S_1 y \rangle \le 0, \tag{3.32}$$

for all $x, y \in C_1$. Since $\{x_n\}$ is bounded, then $\|x_n - v\| \leq K$ for some $K > 0$. Thus, $\|S_1^{\mu_1} S_2^{\mu_2} \cdots S_N^{\mu_N} x_n - v\| \leq \|S_2^{\mu_2} \cdots S_N^{\mu_N} x_n - v\| \leq \cdots \leq \|S_N^{\mu_N} x_n - v\| \leq \|x_n - v\| \leq K$. Therefore, the set $\{S_1^{\mu_1} S_2^{\mu_2} \cdots S_N^{\mu_N} x_n\}$ is bounded in C_1. Hence, by replacing x with $S_1^{\mu_1} S_2^{\mu_2} \cdots S_N^{\mu_N} x_n$ in (3.32) we get that for all $y \in C_1$,

$$\begin{aligned} &\sum_{k=0}^{m} \alpha_{k+1}\Big(\|y - S_1 y\|^2 + 2\langle S_1^{\mu_1} S_2^{\mu_2} \cdots S_N^{\mu_N} x_n - y, y - S_1 y\rangle\Big) \\ &+ \quad 2\sum_{k=0}^{m-1} \alpha_{k+1}\langle S_1^{\mu_1+m-k} S_2^{\mu_2} \cdots S_N^{\mu_N} x_n - S_1^{\mu_1} S_2^{\mu_2} \cdots S_N^{\mu_N} x_n, y - S_1 y\rangle \leq 0. \end{aligned} \tag{3.33}$$

Let the inequality (3.33) be summed up with respect to $\mu_1 = 0, 1, \cdots, n$

$$\begin{aligned} &\sum_{k=0}^{m} \alpha_{k+1}\Big((n+1)\|y - S_1 y\|^2 + 2\langle \sum_{\mu_1=0}^{n} S_1^{\mu_1} S_2^{\mu_2} \cdots S_N^{\mu_N} x_n - (n+1)y, y - S_1 y\rangle\Big) \\ &+ \quad 2\sum_{k=0}^{m-1} \alpha_{k+1}\langle (S_1^{n+m-k} S_2^{\mu_2} \cdots S_N^{\mu_N} x_n + \cdots + S_1^{n+1} S_2^{\mu_2} \cdots S_N^{\mu_N} x_n) \\ &- \quad (S_1^{m-(k+1)} S_2^{\mu_2} \cdots S_N^{\mu_N} x_n + \cdots + S_2^{\mu_2} \cdots S_N^{\mu_N} x_n), y - S_1 y\rangle \leq 0, \end{aligned} \tag{3.34}$$

for all $y \in C_1$. Again, let the inequality (3.34) be summed up with respect to $\mu_2 = 0, 1, \cdots, n$

$$\begin{aligned} &\sum_{k=0}^{m} \alpha_{k+1}\Big((n+1)^2\|y - S_1 y\|^2 + 2\langle \sum_{\mu_1=0}^{n} \sum_{\mu_2=0}^{n} S_1^{\mu_1} S_2^{\mu_2} \cdots S_N^{\mu_N} x_n \\ &- \quad (n+1)^2 y, y - S_1 y\rangle\Big) \\ &+ \quad 2\sum_{\mu_2=0}^{n} \sum_{k=0}^{m-1} \alpha_{k+1}\langle (S_1^{n+m-k} S_2^{\mu_2} \cdots S_N^{\mu_N} x_n + \cdots + S_1^{n+1} S_2^{\mu_2} \cdots S_N^{\mu_N} x_n) \\ &- \quad (S_1^{m-(k+1)} S_2^{\mu_2} \cdots S_N^{\mu_N} x_n + \cdots + S_2^{\mu_2} \cdots S_N^{\mu_N} x_n), y - S_1 y\rangle \leq 0, \end{aligned} \tag{3.35}$$

for all $y \in C$. Continue summing inequality (3.35) until with respect to $\mu_N = 0, 1, \cdots, n$ we get for all $y \in C$ that

$$\begin{aligned} &\sum_{k=0}^{m} \alpha_{k+1}\Big((n+1)^N\|y - S_1 y\|^2 + 2\langle \sum_{\mu_1=0}^{n} \sum_{\mu_2=0}^{n} \cdots \sum_{\mu_N=0}^{n} S_1^{\mu_1} S_2^{\mu_2} \cdots S_N^{\mu_N} x_n \\ &- \quad (n+1)^N y, y - S_1 y\rangle\Big) \\ &+ \quad 2\sum_{\mu_2=0}^{n} \cdots \sum_{\mu_N=0}^{n} \sum_{k=0}^{m-1} \alpha_{k+1}\langle (S_1^{n+m-k} S_2^{\mu_2} \cdots S_N^{\mu_N} x_n + \cdots + S_1^{n+1} S_2^{\mu_2} \cdots S_N^{\mu_N} x_n) \\ &- \quad (S_1^{m-(k+1)} S_2^{\mu_2} \cdots S_N^{\mu_N} x_n + \cdots + S_2^{\mu_2} \cdots S_N^{\mu_N} x_n), y - S_1 y\rangle \leq 0. \end{aligned} \tag{3.36}$$

Dividing inequality (3.36) through by $(n+1)^N$ keeping in mind that $S_n = \frac{1}{(n+1)^N} \sum_{\mu_1=0}^{n} \sum_{\mu_2=0}^{n} \cdots \sum_{\mu_N=0}^{n}$

$S_1^{\mu_1}S_2^{\mu_2}\cdots S_N^{\mu_N}$ we get

$$\begin{aligned}
&\sum_{k=0}^{m}\alpha_{k+1}\Big(\|y-S_1y\|^2+2\langle S_nx_n-y,y-S_1y\rangle\Big)\\
+\quad&\sum_{\mu_2=0}^{n}\cdots\sum_{\mu_N=0}^{n}\sum_{k=0}^{m-1}\frac{2\alpha_{k+1}}{(n+1)^N}\langle(S_1^{n+m-k}S_2^{\mu_2}\cdots S_N^{\mu_N}x_n+\cdots+S_1^{n+1}S_2^{\mu_2}\cdots S_N^{\mu_N}x_n)\\
-\quad&(S_1^{m-(k+1)}S_2^{\mu_2}\cdots S_N^{\mu_N}S_n+\cdots+S_2^{\mu_2}\cdots S_N^{\mu_N}x_n),y-S_1y\rangle\leq 0,\quad\forall y\in C. \qquad (3.37)
\end{aligned}$$

Now replacing n with n_j in (3.37) and allowing $j\to\infty$ (remembering that $S_{n_j}x_{n_j}\rightharpoonup v$) we get

$$\sum_{k=0}^{m}\alpha_{k+1}\Big(\|y-S_1y\|^2+2\langle v-y,y-S_1y\rangle\Big)\leq 0,\quad\forall y\in C_1.$$

Using the fact that $\sum_{k=0}^{m}\alpha_{k+1}>0$, we get

$$\|y-S_1y\|^2+2\langle v-y,y-S_1y\rangle\leq 0,\quad\forall y\in C_1. \qquad (3.38)$$

Applying (1) of Lemma 2.1 on (3.39), we get

$$\|y-S_1y\|^2+\|v-S_1y\|^2-\|v-y\|^2-\|y-S_1y\|^2\leq 0,\quad\forall y\in C_1.$$

Thus,

$$\|v-S_1y\|^2\leq\|v-y\|^2,\quad\forall y\in C_1. \qquad (3.39)$$

Since $S_1,S_2,\cdots,S_N$ are commutative in nature, we can replace S_1 with either of $S_2,\cdots,S_N$. Thus, we get from (3.39) that

$$\begin{aligned}
\|v-S_2y\|^2 &\leq \|v-y\|^2 \qquad (3.40)\\
&\vdots\\
\|v-S_Ny\|^2 &\leq \|v-y\|^2,\quad\forall y\in C_1. \qquad (3.41)
\end{aligned}$$

By setting $y=v$ in (3.39), (3.40), $\cdot$, (3.41), we get $v=S_1v=S_2v=\cdots=S_Nv$. Therefore, $v\in\cap_{i=1}^{N}F(S_i)$. Following similar argument, we see that $v\in\cap_{i=1}^{N}F(T_i)$.

Next we show that $v\in\Gamma$. We see from (3.24) that $y_{n_j}\rightharpoonup v$. Since T is bounded and linear then $Ty_{n_j}\rightharpoonup Tv$. Since $J_{r_n}^G$ is nonexpansive and hence demiclosed, then $Tv\in F(J_{r_n}^G)$. We also have from (3.19) that $z_{n_j}\rightharpoonup v$. Since $J_{r_n}^F$ is nonexpansive and $\lim_{n\to\infty}\|J_{r_n}^F z_n-z_n\|=\|v_{n+1}-z_n\|=0$, then we have $v\in F(J_{r_n}^F)$. Therefore, $v\in\Gamma$. Hence, $v\in\mathcal{S}$. □

4. Applications

In this section, we apply Theorem 3.2, and obtain some new strong convergence theorems for finite commutative normally generalized hybrid mappings, m-generalized hybrid mapping and normally 2-generalized hybrid mappings in Hilbert spaces. These results extend and generalize the corresponding ones in Alizardeh and Moradlou [5], Baillon [6], Haruna et al. ([14], [15]), Hojo et al. ([16], [17], [18]) and Zhao et al. [29].

Theorem 4.1. *Let the mappings* $S_1, S_2, \cdots, S_N : C_1 \to C_1$ *and* $T_1, T_2, \cdots, T_N : C_1 \to C_1$ *be finite commutative normally generalized hybrid mappings such that Assumption 3.1 holds and the sequence* $\{u_n\}$ *be defined by the Algorithm 3.1. Then* $\{u_n\}$ *converges strongly to an element* $v \in \mathcal{S} = \cap_{i=1}^{N} F(T_i) \cap \cap_{i=1}^{N} F(S_i) \cap \Gamma$.

Proof. By setting $m = 1$, we see that the class of normally m-generalized hybrid mappings reduces to that of normally generalized hybrid. Hence, by applying Theorem 3.2, we get the desired results. This completes the proof. □

Theorem 4.2. *Let the mappings* $S_1, S_2, \cdots, S_N : C_1 \to C_1$ *and* $T_1, T_2, \cdots, T_N : C_1 \to C_1$ *be finite commutative m-generalized hybrid mappings such that Assumption 3.1 holds and the sequence* $\{u_n\}$ *be defined by the Algorithm 3.1. Then* $\{u_n\}$ *converges strongly to an element* $v \in \mathcal{S} = \cap_{i=1}^{N} F(T_i) \cap \cap_{i=1}^{N} F(S_i) \cap \Gamma$.

Proof. Since the class of normally m-generalized hybrid mappings reduces to that of m-generalized hybrid by setting $k = k - 1$ and $\alpha_k = \beta_k = -1$, then by applying Theorems 3.2, we get the desired results. This completes the proof. □

Theorem 4.3. *Let the mappings* $S_1, S_2, \cdots, S_N : C_1 \to C_1$ *and* $T_1, T_2, \cdots, T_N : C_1 \to C_1$ *be finite commutative normally 2-generalized hybrid mappings such that Assumption 3.1 holds and the sequence* $\{u_n\}$ *be defined by the Algorithm 3.1. Then* $\{u_n\}$ *converges strongly to an element* $v \in \mathcal{S} = \cap_{i=1}^{N} F(T_i) \cap \cap_{i=1}^{N} F(S_i) \cap \Gamma$.

Proof. By setting $m = 2$, the class of normally m-generalized hybrid mappings reduces to that of normally 2-generalized hybrid. Hence, by applying Theorems 3.2, we get the desired results. This completes the proof. □

5. Numerical Example

In this section, we give a numerical example and used it to compare our result with some existing ones under consideration. Let $H = \mathbb{R}$, $C = [-1, 1]$, $k = 0,1$, $N = 2$ and $\mu_1 = \mu_2 = \tau_1 = \tau_2 = 0,1$. Define $F, G : C \times C \to \mathbb{R}$ by $F(x, y) = 2y^2 + xy - 3x^2$ and $G(x, y) = y^2 + xy - 2x^2$, we see that both F and G satisfy conditions $(A1) - (A4)$. Also, following similar technique as in [5], we see that the resolvents $J_r^F(x) = \frac{x}{5r+1}$ and $J_r^G(x) = \frac{x}{3r+1}$ for $r > 0$. Since $F(J_{r_n}^F) = F(J_{r_n}^G) = 0$, then by Lemma 2.4 we get $EP(F) = EP(G) = 0$. Define $S_1, T_1 : C \to C$ by $S_1(x) = T_1(x) = \frac{x}{2}$ and $S_2, T_2 : C \to C$ by $S_2(x) = T_2(x) = \frac{2}{3}x$ for all $x \in C$, we see that $S_1 S_2 x = S_2 S_1 x = \frac{x}{3}$ and $T_1 T_2 x = T_2 T_1 x = \frac{x}{3}$. Hence the mappings are commutative normally 2-generalized hybrid mappings with $F(S_1) = F(S_2) = F(T_1) = F(T_2) = 0$. With the choice of $\theta_n = \frac{n}{2n+3}$, $a_n = \alpha_n = \frac{1}{3} + \frac{1}{2n}$, $b_n = \beta_n = \frac{1}{3} - \frac{1}{3n}$, $c_n = \gamma_n = \frac{1}{3} - \frac{1}{6n}$, $\delta_n = \delta = \frac{1}{3} > 0$ and $r = \frac{1}{2}$, we see that the algorithms (1.4), (1.8) and (3.1) respectively become,

$$\begin{cases} x_0 \in [-1,1]; \\ y_n = (\frac{1}{3} + \frac{1}{2n})x_n + \frac{1}{n}(\frac{1}{3} - \frac{1}{3n})(x_n + Sx_n) + \frac{1}{n}(\frac{1}{3} - \frac{1}{6n})(x_n + Tx_n); \\ C_{n+1} = [\epsilon_n, \infty), \text{ where } \epsilon_n := \frac{x_n + y_n}{2}; \\ x_{n+1} = \mathrm{P}_{C_{n+1}u_{n+1}}, \quad \forall n \in \mathbb{N}. \end{cases} \tag{5.1}$$

$$\begin{cases} x_0 \in [-1,1]; \\ u_n = \frac{3}{8}x_n, \\ v_n = (\frac{2}{3} + \frac{1}{3n})u_n + \frac{1}{n}(\frac{1}{3} - \frac{1}{3n})(u_n + Su_n); \\ y_n = (\frac{2}{3} - \frac{1}{2n})u_n + \frac{1}{n}(\frac{1}{3} + \frac{1}{2n})(v_n + Sv_n); \\ D_{n+1} = [\epsilon_n, \infty), \text{ where } \epsilon_n := \frac{x_n + y_n}{2}; \\ x_{n+1} = P_{D_{n+1}}x_0, \quad \forall n \in \mathbb{N}. \end{cases} \tag{5.2}$$

and

$$\begin{cases} u_0, u_1 \in [-1,1], v_1 = 1 \\ x_n = u_n + \frac{n}{2n+3}(u_n - u_{n-1}; \\ y_n = (\frac{1}{3} + \frac{1}{2n})v_n + \frac{1}{(n+1)^2}(\frac{1}{3} - \frac{1}{3n})(x_n + S_1x_n + S_2x_n + S_1S_2x_n) \\ \qquad + \frac{1}{(n+1)^2}(\frac{1}{3} - \frac{1}{6n})(x_n + T_1x_n + T_2x_n + T_1T_2x_n); \\ v_{n+1} = \frac{3}{8}y_n; \\ C_{n+1} = [\epsilon_n, \infty), \text{ where } \epsilon_n = \frac{v_{n+1}^2 - x_n^2 + (\frac{2}{3} + \frac{1}{3n})(x_n^2 - v_n^2)}{2(v_{n+1} - x_n + (\frac{2}{3} + \frac{1}{3n})(x_n - v_n))}; \\ x_{n+1} = P_{C_{n+1}}u_0. \end{cases} \tag{5.3}$$

Hence, the sequence generated by algorithms (5.1),(5.2) and (5.3) converges to 0 as shown graphically below. The codes use in generating the graphs are written using MATLAB R2015a.

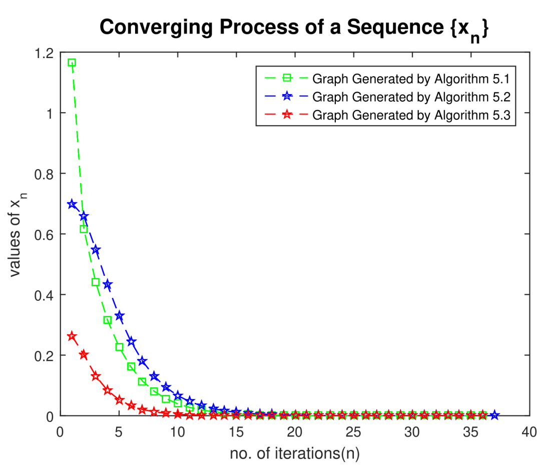

Figure 1. Converging Process of the Sequence $\{x_n\}$ Generated by Algorithms (5.1), (5.2) and (5.3) with initial point $x_0 = 0.7$ and $x_1 = 0.5$.

6. Conclusion

We approximate Cesàro mean sequences generated by finite commutative normally m-generalized hybrid mappings using a new accelerated shrinking projection algorithms. Also, we chose a step size to be independent of the spectral radius for easy computation. We then prove that the algorithm converges strongly to a common element in solution set of fixed point of the said mappings which also solve some split equilibrium problem in the space. As an application, we established new strong convergence theorems for finite commutative normally generalized hybrid, m-generalized hybrid and normally 2-generalized hybrid mappings in the space. We finally give a numerical example that shows how our algorithm out performs the existing ones under consideration in terms of convergence rate as seen from the graph.

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