Int. J. Anal. Appl. (2025), 23:274

International Journal of Analysis and Applications

A Novel Class of Extreme Value Distributions Derived from the KM Transformation of the Generalized Extreme Value Distribution

P. Khamrot[1], **N. Deetae**[2,*]

[1]*Department of Mathematics, Faculty of Science and Agricultural Technology, Rajamangala University of Technology Lanna, Phitsanulok, Thailand*

[2]*Department of Statistics, Faculty of Science and Technology, Pibulsongkram Rajabhat University, Thailand*

[*]*Corresponding author:* natthinee@psru.ac.th

Abstract. Modeling extreme events is crucial in various disciplines such as environmental sciences, hydrology, finance, and engineering. This paper introduces the KM-transformed Generalized Extreme Value (KMGEV) distribution, a novel and flexible model that generalizes the classical Generalized Extreme Value (GEV) distribution using the KM transformation framework recently proposed by Kavya and Manoharan. We derive the key statistical properties of the KMGEV distribution, including the probability density function (PDF), cumulative distribution function (CDF), survival function, hazard rate function, and quantile function. Additionally, we explore order statistics and their expected values. Parameter estimation is carried out via Maximum Likelihood Estimation (MLE) methods. Through Monte Carlo simulations, we investigate the impact of the shape parameter on moments such as skewness and kurtosis. Graphical analysis highlights the flexibility of the KMGEV model, suggesting its potential in modeling a variety of extreme value phenomena.

1. Introduction

Researchers have applied extreme value theory in various fields to identify the most suitable model for parameter estimation [1–4]. To present a data model that is suitable for the data from the actual situation, in such a situation, extreme values are necessary for the occurrence of various events. Therefore, it is necessary to determine the probability of an event with extreme values occurring, which will be at the tail of the distribution. Examples include the maximum or minimum daily rainfall, which has been extensively studied in various regions [5–8], the maximum wind speed in a month, especially in coastal zones or for energy applications [9], and the maximum or

Received: Sep. 5, 2025.

2020 *Mathematics Subject Classification.* 60G70, 62G32.

Key words and phrases. extreme value theory; KM transformation; generalized extreme value distribution; maximum likelihood estimation; return level.

https://doi.org/10.28924/2291-8639-23-2025-274
ISSN: 2291-8639

minimum daily temperature, in relation to long-term climate variability [10, 11]. These examples can be analyzed more accurately using extreme value (EV) theory [12–14]. This theory examines the behavior of EVs to estimate parameters based on the probability of an event, in order to obtain the estimation of the EV that is consistent with the data set [15].

In EV theory, the choice of distribution models plays a crucial role in estimating parameters based on the nature of observed extreme events. Although the classical framework often utilizes the generalized extreme value (GEV) distribution and the generalized Pareto distribution (GPD), recent developments have focused on constructing new, more flexible distribution families. These include the exponentiated Kumaraswamy exponential distribution [16], the exponentiated Kumaraswamy inverse Weibull distribution for survival analysis [17], and the generalized transmuted-G family [18]. These models enhance the modeling of tail behavior and provide better fits for datasets where classical GEV or GPD structures may not capture the underlying complexity. Therefore, selecting an appropriate distribution requires an understanding of the data's tail behavior and the ability of the model to adapt to such features.

Parameter estimation plays an important role in studying probability distributions, as shown in the above literature, which applies or creates new distributions to suit the data and events that occur [19–21]. These new distributions often provide a better fit and insight into the underlying processes, allowing researchers to make more accurate predictions and informed decisions based on the observed data. By refining parameter estimates, analysts can enhance the reliability of models used across various fields, from finance to environmental science [22–24]. Researchers cannot completely adapt the existing probability distribution formats. Therefore, the current concept of the generalized distribution is of interest to researchers because it is more flexible by adding at least one parameter and is mostly related to other distributions in a specific manner [25–27].

The theoretical method for developing EV theory is to create a new distribution based on the basic GEV distribution, which extends the EV distribution that includes the Gumbel, Fréchet, and Weibull distributions into a family of distributions that can change continuously. Over the years, various distributions developed based on the GEV distribution have been investigated. For example, in 2016, Fernando Nascimento et al. [22] proposed an extended GEV distribution with an application to environmental data. They suggested new versions of the GEV distribution that include extra parameters, such as the double gamma generalized extreme value (GGEV), the exponential generalized extreme value (EGEV), and the transformed generalized extreme value (TGEV). Guloksuz et al. [25] proposed the Uniform-GEV distribution, a new extension of the GEV distribution, and applied it to a magnitude model of the Turkish earthquake data from 1970 to 2018. Jin Zhao et al. [26] proposed a new class of heavy-tailed distributions by studying the power transformation. In 2025, N. Deetae et al. [27] introduced a new heavy-tailed distribution called A New Extended Kumaraswamy Generalized Pareto Distribution with Rainfall Application, which combines the Kumaraswamy and Generalized Pareto distributions for improved parameter estimation of financial data.

In this research, a new distribution is presented as a combination of the GEV distribution and the KM transformation, called the KM-transformed generalized extreme value (KMGEV) distribution, to estimate the parameter value that is appropriate for the data by estimating the maximum likelihood and checking the distribution with a goodness-of-fit test. Therefore, the properties of the KMGEV distribution with CDF, PDF, hazard rate, quantile functions, and return function, along with data simulation models, will be shown to demonstrate the efficiency of the KMGEV distribution.

2. KM Transformation and GEV Distribution

The KM transformation class of statistical models is a new transformation framework that was recently introduced by Kavya and Manoharan [28]. By modifying the existing baseline PDFs and CDFs, this transformation makes it possible to create more flexible distributions. In particular, the transformed distribution using the KM approach for a given baseline distribution with CDF and PDF is defined by

$$F(y) = \frac{e}{e-1}\left(1 - e^{-G(y)}\right), \quad y \in \mathbb{R}, \tag{2.1}$$

$$f(y) = \frac{e}{e-1} \cdot g(y)e^{-G(y)}, \quad y \in \mathbb{R}. \tag{2.2}$$

The primary advantage of the KM transformation is its ability to enhance the resulting distribution's skewness control and tail flexibility without overly complicating the model. When modeling EVs, this capability is especially helpful because conventional distributions, like the GEV distribution, might not be flexible enough to account for the different tail behaviors seen in empirical data.

For the GEV distribution with parameters μ, σ, and ξ, the baseline CDF and PDF are given by

$$G(z) = \exp\left(-\left(1 + \xi \cdot \frac{z-\mu}{\sigma}\right)^{-1/\xi}\right), \quad \text{for } 1 + \xi \cdot \frac{z-\mu}{\sigma} > 0, \tag{2.3}$$

$$g(z) = \frac{1}{\sigma}\left(1 + \xi \cdot \frac{z-\mu}{\sigma}\right)^{-1/\xi-1} \cdot \exp\left(-\left(1 + \xi \cdot \frac{z-\mu}{\sigma}\right)^{-1/\xi}\right). \tag{2.4}$$

Using the KM transformation with the GEV framework creates a new distribution that is a better fit for adaptation. which shows how important the GEV distribution is in EV theory. This situation drives the generalized form of the GEV distribution produced by the KM transformation, KM-transformed GEV distribution, to be developed.

Applying the transformation in Equations (2.1) and (2.2) to the baseline GEV distribution, we show the KMGEV distribution in the section following. Which derives its statistical features and shows how the KMGEV model provides better flexibility in modeling extreme events than the conventional GEV distribution.

3. KMGEV Distribution

Let $X \sim \text{KMGEV}(\mu, \sigma, \xi)$. The CDF, PDF, survival function, quantile function, and hazard rate function of the KMGEV distribution are given as follows

- CDF of KMGEV Distribution

$$F(x; \mu, \sigma, \xi) = \frac{e}{e-1}\left(1 - \exp\left(-\exp\left(-\left(1 + \xi \cdot \frac{x-\mu}{\sigma}\right)^{-1/\xi}\right)\right)\right).$$

  Let $z = 1 + \xi \cdot \frac{x-\mu}{\sigma}$, then

$$F(x) = \frac{e}{e-1}\left(1 - \exp\left(-\exp\left(-z^{-1/\xi}\right)\right)\right).$$

- PDF of KMGEV Distribution

$$f(x) = \frac{e}{e-1} \cdot \frac{1}{\sigma} \cdot u^{1+\xi} \cdot e^{-u} \cdot e^{-\exp(-u)}$$

  where

$$u = z^{-1/\xi}, \quad z = 1 + \xi \cdot \frac{x-\mu}{\sigma}.$$

  The PDF of the KMGEV distribution with certain parameter settings are shown in Figure 1, demonstrating how changing ξ affects the shape of the distribution.

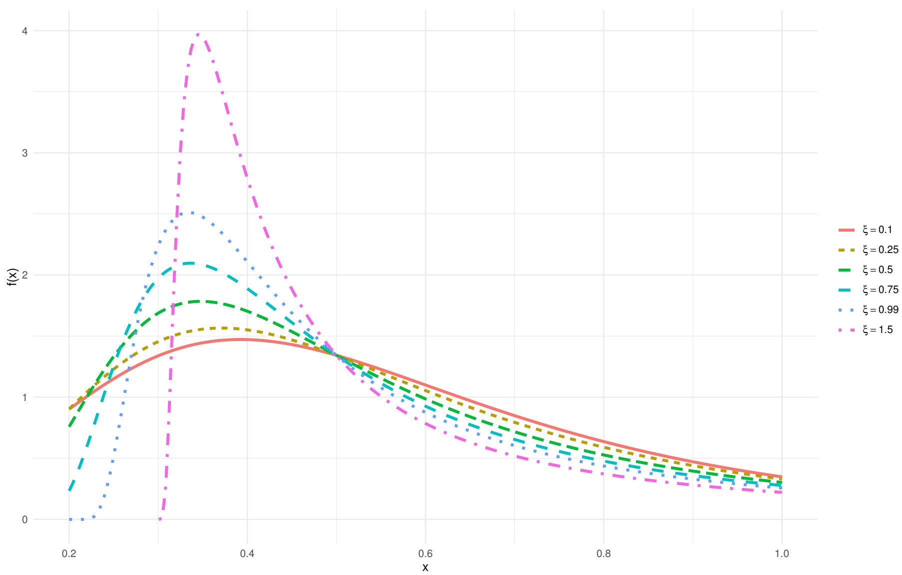

Figure 1. PDF of KMGEV Distribution for Various ξ Values

- Survival Function

$$S(x) = 1 - F(x) = \exp\left(-\exp\left(-z^{-1/\xi}\right)\right) - \left(\frac{1}{e-1}\right)\left(1 - \exp\left(-\exp\left(-z^{-1/\xi}\right)\right)\right).$$

- Hazard Rate Function

$$h(x) = \frac{f(x)}{S(x)} = \frac{\left(\frac{e}{e-1} \cdot \frac{1}{\sigma} \cdot u^{1+\xi} \cdot e^{-u} \cdot e^{-\exp(-u)}\right)}{S(x)}.$$

- Quantile Function

  Given $F(x) = p$, the quantile function $Q(p)$ is derived by solving for x as follows
$$p = \frac{e}{e-1}\left(1 - \exp\left(-\exp\left(-z^{-1/\xi}\right)\right)\right).$$
  Solving for z as
$$z^{-1/\xi} = -\ln\left(-\ln\left(1 - \frac{p(e-1)}{e}\right)\right).$$
  Hence,
$$z = \left[-\ln\left(-\ln\left(1 - \frac{p(e-1)}{e}\right)\right)\right]^{-\xi}$$
$$Q(p) = \mu + \frac{\sigma}{\xi}(z-1) = \mu + \frac{\sigma}{\xi}\left(\left[-\ln\left(-\ln\left(1 - \frac{p(e-1)}{e}\right)\right)\right]^{-\xi} - 1\right).$$

4. Moments of the KMGEV Distribution

In this section, we derive essential mathematical properties of the KMGEV distribution, including its PDF and moment generating function (MGF). The derivations are based on exponential and binomial expansions.

Let
$$u = \left[1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-1/\xi}, \quad \sigma > 0,\ \xi \in \mathbb{R}.$$
Then, the PDF of the KMGEV distribution is defined as
$$f(x;\mu,\sigma,\xi) = Wu^{1+\xi}e^{1-u-\exp(-u)}, \quad x > 0, \tag{4.1}$$
where the constant W is given by
$$W = \frac{1}{\sigma(e-1)}.$$
The k^{th} raw moment about zero is defined by
$$\mu_k' = E[X^k] = W\int_0^\infty x^k u^{1+\xi} e^{1-u-\exp(-u)}\,dx. \tag{4.2}$$
Using binomial expansion, the integral becomes
$$\mu_k' = W\sum_{j,m=0}^{k} \frac{(-1)^{k-m-1}k!^2\mu^{k-j}\sigma^k}{j!m!(k-j)!(k-m)!\xi^k}\int_0^\infty u^{-\xi m}e^{1-u-\exp(-u)}\,du. \tag{4.3}$$
Through further series manipulation, we obtain
$$\begin{aligned}\mu_k' &= W\tau\int_0^\infty u^{-\xi m-s-r}e^{-su}\,du \\ &= W\tau s^{\xi m-s+r-1}\Gamma(1-\xi m+s-r),\end{aligned} \tag{4.4}$$
where $\Gamma(\cdot)$ denotes the gamma function, and the coefficient τ is defined by
$$\tau = \sum_{j,m=0}^{k}\sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{r} \frac{(-1)^{r+k-m-1}k!^2 r!s!\mu^{k-j}\sigma^k}{(k-j)!(k-m)!(r-n)!(s-r)!j!m!n!^2r!\xi^k}. \tag{4.5}$$

The moments μ'_k can be obtained by substituting $k = 1, 2, 3, 4$ into the general moment expression. The MGF of $M_X(t) = E[e^{tX}]$ is expressed as

$$M_X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mu'_k = \sum_{k=0}^{\infty} \frac{t^k}{k!}\tau s^{\xi m-s+r-1}\Gamma(1-\xi m+s-r). \tag{4.6}$$

The k^{th} central moment μ_k is calculated from the raw moments via as

$$\begin{aligned}\mu_k =& E[(X-\mu'_1)^k] \\ =& \sum_{n=0}^{k}(-1)^n\binom{k}{n}(\mu'_1)^n\mu'_{k-n}.\end{aligned} \tag{4.7}$$

A Monte Carlo simulation was done to study how the KMGEV distribution behaves with fixed parameters $\mu = 0.5$, $\sigma = 0.3$, and different values of ξ ranging from -0.2 to 0.2. For each selected value of ξ, random samples were generated using the inverse transform sampling method based on the KMGEV quantile function. We used simulation sample sizes of $n = 25, 50, 100, 500$, and 1000. For each case, we calculated the mean, variance, skewness, and kurtosis to evaluate how the KMGEV changes as it ξ varies.

Table 1. Monte Carlo simulation results for the KMGEV distribution with fixed $(\mu, \sigma) = (0.5, 0.3)$, and varying $\xi = [-0.2, 0.2]$.

n	ξ	Mean	Variance	Skewness	Kurtosis
25	-0.2	0.6442	0.1160	0.201	-0.682
	-0.1	0.4953	0.0757	0.823	0.106
	0.0	0.4724	0.0760	0.333	-0.683
	0.1	0.5480	0.1273	1.888	4.015
	0.2	0.6636	0.1709	0.965	-0.595
50	-0.2	0.5637	0.0979	0.490	-0.355
	-0.1	0.5149	0.0806	0.430	0.697
	0.0	0.4004	0.1302	-0.925	0.186
	0.1	0.5813	0.0889	1.914	4.712
	0.2	0.6231	0.1492	1.413	1.315
100	-0.2	0.5327	0.0875	0.355	0.110
	-0.1	0.5670	0.0851	0.915	0.553
	0.0	0.4421	0.1128	-0.890	0.331
	0.1	0.6238	0.2482	2.607	9.891
	0.2	0.6230	0.2465	2.625	8.785
500	-0.2	0.5351	0.0912	0.571	0.302
	-0.1	0.5462	0.0994	0.568	0.379
	0.0	0.4361	0.1219	-1.594	4.889
	0.1	0.6021	0.1584	1.687	3.827
	0.2	0.6286	0.2563	3.048	14.593
1000	-0.2	0.5322	0.0911	0.407	0.093
	-0.1	0.5510	0.1037	0.880	1.288
	0.0	0.4273	0.1259	-1.365	3.484
	0.1	0.6034	0.1584	1.746	5.552
	0.2	0.6011	0.2298	6.103	78.301

The simulation results in Table 1 demonstrate how varying the shape parameter ξ influences the moments of the KMGEV distribution under different sample sizes. For small values of ξ (e.g., $\xi = -0.2$), the distribution exhibits moderate skewness and low kurtosis. As ξ increases, both skewness and kurtosis increase significantly, particularly for $\xi = 0.1$ and $\xi = 0.2$, indicating heavier right tails and increasing asymmetry. These effects become more pronounced with larger sample sizes (e.g., $n = 500, 1000$), confirming the distribution's sensitivity to shape parameter changes and its flexibility in capturing diverse tail behaviors.

5. Order Statistics of KMGEV Distribution

Let $X_1, X_2, \ldots, X_n \sim \text{KMGEV}(\mu, \sigma, \xi)$ be independent and identically distributed (i.i.d.) random variables. The r^{th} order statistic, denoted $X_{(r)}$, has the following properties.

- PDF of $X_{(r)}$

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x)$$

  where $f(x)$ and $F(x)$ denote the PDF and CDF of the KMGEV distribution, respectively.
- CDF of $X_{(r)}$

$$F_{X_{(r)}}(x) = \sum_{k=r}^{n}\binom{n}{k}[F(x)]^k[1-F(x)]^{n-k}$$

- Expected Value of $X_{(r)}$

$$E[X_{(r)}] = \int_{-\infty}^{\infty} x \cdot f_{X_{(r)}}(x)dx$$

6. Confidence Interval of Return Level

In the analysis of EVs, the return level associated with the return period T is the quantile function that corresponds to the probability not exceeding $\nu = 1 - T^{-1}$ for the KMGEV distribution. The return level R_T^{KMGEV} is obtained by computing the quantile function $Q(p)$ at $p = \nu$ as follows

$$R_T^{\text{KMGEV}} = Q(\nu) = \mu + \frac{\sigma}{\xi}\left(\left[-\ln\left(-\ln\left(1 - \frac{\nu(e-1)}{e}\right)\right)\right]^{-\xi} - 1\right), \quad \nu = 1 - \frac{1}{T}.$$

The Delta method is employed to approximate the variance of the return level estimate R_T^{KMGEV} it provided by

$$\text{Var}\left(R_T^{\text{KMGEV}}\right) \approx \left(\nabla R_T^{\text{KMGEV}}\right)^\top V\left(\nabla R_T^{\text{KMGEV}}\right),$$

where V is the covariance matrix of the estimated parameters $(\mu, \sigma, \xi)^\top$, and $\nabla R_T^{\text{KMGEV}}$ is the gradient vector of partial derivatives:

$$\begin{aligned}\nabla R_T^{\text{KMGEV}} &= \left[\frac{\partial R_T^{\text{KMGEV}}}{\partial \mu}, \frac{\partial R_T^{\text{KMGEV}}}{\partial \sigma}, \frac{\partial R_T^{\text{KMGEV}}}{\partial \xi}\right]^\top \\ &= \left[1,\ \frac{1}{\xi}(z-1),\ -\frac{\sigma}{\xi^2}(z-1) - \frac{\sigma}{\xi}\cdot z \cdot \ln z \cdot \ln\left(-\ln\left(1 - \frac{\nu(e-1)}{e}\right)\right)\right]^\top,\end{aligned}$$

where

$$z = \left[-\ln\left(-\ln\left(1 - \frac{\nu(e-1)}{e} \right) \right) \right]^{-\xi}, \quad \nu = 1 - \frac{1}{T}.$$

Using the estimated variance, the Wald confidence interval for the return level R_T^{KMGEV} at a confidence level $1-\alpha$ is then constructed as:

$$R_T^{\text{KMGEV}} \pm Z_{\alpha/2} \cdot \sqrt{\operatorname{Var}\left(R_T^{\text{KMGEV}}\right)},$$

In this, $Z_{\alpha/2}$ represents the critical value derived from the standard normal distribution. This interval provides a probabilistic range within which the true return level is expected to lie with the specified confidence.

7. Log-Likelihood Function of the KMGEV Distribution

Let $X_1, X_2, \ldots, X_n$ be a random sample from the KMGEV distribution with parameters μ, σ, and ξ. The log-likelihood function for the observed data is given by

$$\ell(\mu, \sigma, \xi) = \sum_{i=1}^{n} \ln f(x_i; \mu, \sigma, \xi)$$

where $f(x; \mu, \sigma, \xi)$ is the PDF of the KMGEV distribution, defined as

$$f(x; \mu, \sigma, \xi) = \frac{e}{e-1} \cdot \frac{1}{\sigma} \cdot u^{1+\xi} \cdot e^{-u} \cdot e^{-\exp(-u)}$$

with

$$z_i = 1 + \xi \cdot \frac{x_i - \mu}{\sigma}, \quad u_i = z_i^{-1/\xi}$$

Substituting the PDF into the log-likelihood function, we obtain

$$\ell(\mu, \sigma, \xi) = n \cdot \ln\left(\frac{e}{e-1}\right) - n \cdot \ln \sigma + (1+\xi) \sum_{i=1}^{n} \ln u_i - \sum_{i=1}^{n} u_i - \sum_{i=1}^{n} \exp(-u_i)$$

where $u_i = z_i^{-1/\xi}, z_i = 1 + \xi \cdot \dfrac{x_i - \mu}{\sigma}$. The partial derivatives are as follows

- Derivative with respect to μ

$$\frac{\partial \ell}{\partial \mu} = \frac{1}{\sigma} \sum_{i=1}^{n} z_i^{-1} \left[\xi + \exp(-u_i) \right]$$

- Derivative with respect to σ

$$\frac{\partial \ell}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i - \mu) z_i^{-1} \left[\xi + \exp(-u_i) \right]$$

- Derivative with respect to ξ

  First, compute $\partial u_i / \partial \xi$

$$\frac{\partial u_i}{\partial \xi} = u_i \left(\frac{\ln z_i}{\xi^2} - \frac{x_i - \mu}{\sigma \xi z_i} \right)$$

Then,

$$\frac{\partial \ell}{\partial \xi} = \sum_{i=1}^{n} \ln u_i + \sum_{i=1}^{n} \left(\xi + \exp(-u_i)\right) \cdot \frac{\partial u_i}{\partial \xi}$$

8. Monte Carlo Simulation for Estimation Efficiency

8.1. **Simulation of Estimation Parameter.** This section presents a Monte Carlo simulation study designed to evaluate the efficiency and accuracy of the MLEs for the KMGEV distribution. Random samples are generated from the KMGEV distribution using the inverse CDF method. All simulations and subsequent analyses are conducted using the R programming language.

For each experimental condition, the simulation is replicated $N = 10,000$ times to ensure statistical reliability. Artificial datasets of varying sample sizes $n = 25, 50, 100, 500$, and $1,000$ are drawn from the KMGEV distribution via the inverse transform sampling method. For each sample, MLE method is employed to estimate the parameters μ, σ, and ξ. Optimization is performed using the L-BFGS-B algorithm [29–31] due to the complexity of the likelihood function.

Across all simulations, the mean estimates, biases, and Mean Squared Errors (MSEs) are computed for each parameter. These results, presented in Table 2, allow for a comprehensive evaluation of the estimators' performance under different sample sizes. The simulation outcomes provide empirical support for the consistency and efficiency of MLEs in the context of the KMGEV distribution.

Table 2. Monte Carlo simulation results for KMGEV distribution with $(\mu, \sigma, \xi) = (0.5, 0.3, 0.5)$.

n		μ	σ	ξ
25	MLE	0.4826	0.2775	0.4869
	Bias	-0.0174	-0.0225	-0.0131
	MSE	0.0129	0.0084	0.0767
50	MLE	0.4809	0.2823	0.4910
	Bias	-0.0191	-0.0177	-0.0090
	MSE	0.0096	0.0051	0.0382
100	MLE	0.4850	0.2889	0.4969
	Bias	-0.0150	-0.0111	-0.0031
	MSE	0.0086	0.0039	0.0216
500	MLE	0.4826	0.2899	0.4840
	Bias	-0.0174	-0.0101	-0.0160
	MSE	0.0092	0.0035	0.0115
1000	MLE	0.4893	0.2935	0.4925
	Bias	-0.0107	-0.0065	-0.0075
	MSE	0.0051	0.0019	0.0061

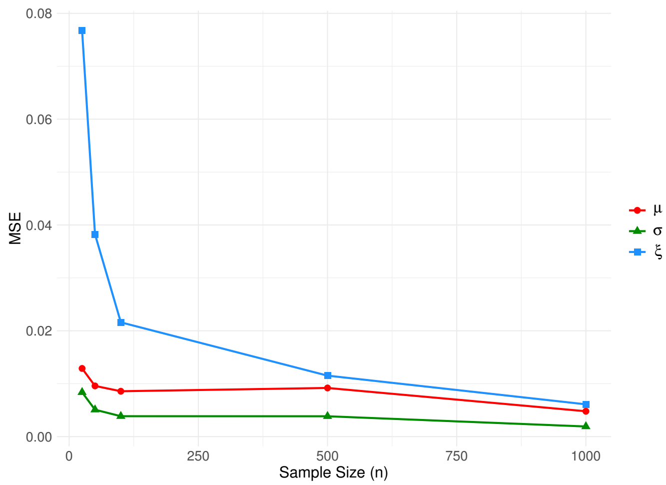

Figure 2. MSE of MLEs for KMGEV Parameters

The simulation results in Table 2 and Figure 2 demonstrate the efficiency of the MLE method for the KMGEV distribution. As sample size increases, the bias and MSE for all parameters (μ, σ, ξ) consistently decrease, confirming the consistency of MLE estimators.

The bias of μ and σ remains small across all sample sizes, while the estimation of ξ shows slightly larger MSE in small samples (e.g., $n = 25$) due to the sensitivity of shape parameter estimation. For $n \geq 500$, all estimators exhibit minimal bias and variance, validating the practical reliability of the MLE approach for KMGEV distribution.

8.2. **Simulation of Return Level Estimation.** In the analysis of EV data, estimating the return level for a specified return period T is of significant importance, especially in applications such as hydrology, finance, and environmental risk assessment. The return level represents the quantile corresponding to a non-exceedance probability $\nu = 1 - 1/T$, which quantifies the magnitude of an event expected to be exceeded once every T periods.

To evaluate the accuracy and precision of return level estimation under the KMGEV distribution, a Monte Carlo simulation study was conducted. The simulation aims to assess the performance of point estimators and confidence intervals constructed via the Delta method. The study fixes the parameters at $(\mu, \sigma, \xi) = (0.5, 0.3, 0.5)$, and considers return periods $T = 5, 10, 20, 50, 100$. For each T, the simulation is replicated $N = 10,000$ times of size $n = 100$ are generated using inverse transform sampling. In each simulation replication, sample estimates of μ and σ were used to compute the estimated return level R_T^{KMGEV}, while ξ was assumed known for simplicity. The Delta method was employed to construct 95% confidence intervals for R_T^{KMGEV}, and key statistics including bias, mean squared error (MSE), and interval coverage were recorded. Table 3 summarizes the simulation results, providing insights into the estimation efficiency and the reliability of confidence intervals as T increases.

Table 3. Simulation results for return level estimation of the KMGEV distribution with fixed parameters $(\mu, \sigma, \xi) = (0.5, 0.3, 0.5)$.

T	Mean	Bias	MSE	CI Lower	CI Upper
5	2.2559	1.3419	8.7712	-23.1882	27.7000
10	3.6266	2.2825	28.6724	-45.8880	53.1411
20	5.3816	3.4358	27.4083	-25.0413	35.8045
50	8.9647	5.8284	89.1273	-54.5492	72.4786
100	13.5244	9.0473	213.9352	-98.1397	125.1885

The results in Table 3 reveal several important trends regarding the estimation of return levels under the KMGEV distribution using the Delta method. First, the mean of the estimated return levels increases with the return period T, as expected, since larger T values correspond to more extreme quantiles. However, the bias and MSE of the estimates also increase notably with T, indicating that estimating return levels in the extreme tail becomes more difficult and less reliable as T grows. For instance, at $T = 5$, the bias is relatively modest (1.3419), while at $T = 100$, the bias reaches 9.0473, with an MSE exceeding 200. Additionally, the 95% confidence intervals constructed via the Delta method are observed to be increasingly wide for larger T. For example, the interval at $T = 5$ spans approximately 50 units, while at $T = 100$, the interval width exceeds 220 units. Such wide intervals highlight the high uncertainty in estimating return levels in the far tail of the distribution.

Furthermore, the simulation suggests that the Delta method, while straightforward, may not provide sufficiently narrow intervals for large T. This underscores the potential need for alternative methods such as bootstrap-based confidence intervals, which may offer improved finite-sample performance in capturing the true return level. In conclusion, while return level estimation via the Delta method is effective for moderate return periods, caution is warranted for large T, where both point estimation and interval estimation become substantially less precise.

8.3. **Application in Cryptographic Pseudo-Random Number Generation.** The flexibility of the KMGEV distribution in modeling tail behavior has implications for entropy generation in cryptographic systems. In particular, the ability to control skewness and tail thickness via the shape parameter ξ enables designers to fine-tune the unpredictability of pseudo-random outputs.

Since entropy quantifies the uncertainty in random variables, distributions like KMGEV with heavy tails and asymmetric structures offer a rich source of high-entropy values. These are essential in applications such as key generation, random masking, and probabilistic encryption schemes, where predictable patterns could lead to security vulnerabilities.

Cryptographic protocols often rely on pseudo-random number generators (PRNGs) that must produce unpredictable outputs with high entropy. The quantile function of KMGEV can be used in the inverse transform method, where a uniformly generated value $U \sim \text{Uniform}(0, 1)$ is transformed into a KMGEV-distributed value $X = Q(U)$.

The advantage of KMGEV is its ability to produce samples with tail-controlled variability, providing enhanced entropy characteristics. By adjusting the shape parameter ξ, designers can tune the randomness to either minimize or emphasize extreme deviations, depending on the cryptographic need. This makes KMGEV-based PRNGs particularly suitable for secure key generation, random masking, and threshold secret sharing schemes.

Algorithm 8.1 KMGEV-based Pseudo-Random Number Generator

Require: Number of samples n, parameters μ, $\sigma > 0$, and ξ

Ensure: Array $X[1..n]$ containing KMGEV-distributed random variables

```
for i ← 1 to n do
    Generate U_i ~ Uniform(0,1)
    Compute z ← −ln(−ln(1 − U_i(e−1)/e))
    Compute Q ← μ + (σ/ξ)(z^{−ξ} − 1)
    Set X[i] ← Q
end for
return X
```

9. Conclusion

This study introduced the KM-transformed Generalized Extreme Value (KMGEV) distribution, a novel and flexible class of distributions derived via the KM transformation applied to the classical Generalized Extreme Value (GEV) distribution. By leveraging the KM transformation framework, the KMGEV distribution achieves enhanced flexibility in modeling skewness and tail behavior, which are crucial in extreme value analysis. The key statistical properties of the KMGEV distribution, including the probability density function, cumulative distribution function, survival function, hazard rate, and quantile function, were systematically derived. Additionally, the behavior of order statistics and the moment generating function were discussed, underscoring the distribution's theoretical robustness. Parameter estimation was carried out using both the Maximum Likelihood Estimation (MLE). Simulation studies revealed that the MLE approach provides efficient and unbiased estimates as sample size increases, with corresponding reductions in mean squared error. Monte Carlo simulations further examined the return level estimation under the KMGEV model, using the Delta method to construct confidence intervals. Results indicated that while point estimation remains stable for moderate return periods, the uncertainty grows substantially for large return periods, as evidenced by widening confidence intervals and increased estimation bias.

One limitation of the present work is the reliance on the Delta method for constructing confidence intervals, which may underestimate uncertainty in small samples or extreme quantiles. Additionally, parameter estimation for the shape parameter ξ was simplified in simulations, and real-world applications may require more robust numerical optimization techniques. Future work may focus on applications in real-world data, further generalizations, and the derivation of closed-form expressions for order statistics and return levels.

Acknowledgment

This research was supported by Pibulsongkram Rajabhat University under the Fundamental Fund Year 2025 Capital Code 208599.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] S. El Adlouni, T.B.M.J. Ouarda, X. Zhang, R. Roy, B. Bobée, Generalized Maximum Likelihood Estimators for the Nonstationary Generalized Extreme Value Model, Water Resour. Res. 43 (2007), 2005WR004545. https://doi.org/10.1029/2005wr004545.

[2] E. Gilleland, M. Ribatet, A.G. Stephenson, A Software Review for Extreme Value Analysis, Extremes 16 (2012), 103–119. https://doi.org/10.1007/s10687-012-0155-0.

[3] P. Busababodhin, A. Kaewmun, Extreme Values Statistics, J. King Mongkut's Univ. Technol. North Bangkok, 25 (2015), 55–65.

[4] D. Chikobvu, R. Chifurira, Modelling of Extreme Minimum Rainfall Using Generalised Extreme Value Distribution for Zimbabwe, S. Afr. J. Sci. 111 (2015), 8. https://doi.org/10.17159/sajs.2015/20140271.

[5] L. Gao, B. Tao, Y. Miao, et al. A Global Data Set for Economic Losses of Extreme Hydrological Events During 1960-2014, Water Resour. Res. 55 (2019), 5165–5175. https://doi.org/10.1029/2019WR025135.

[6] F.C. Onwuegbuche, A.B. Kenyatta, S.B. Affognon, et al. Application of Extreme Value Theory in Predicting Climate Change Induced Extreme Rainfall in Kenya, Int. J. Stat. Probab. 8 (2019), 85–94.

[7] S.B. Sunday, N.S. Agog, P. Magdalene, et al. Modeling Extreme Rainfall in Kaduna Using the Generalised Extreme Value Distribution, Sci. World J. 15 (2020), 73–77. https://doi.org/10.47514/swj/15.03.2020.010.

[8] C. Jones, D.E. Waliser, K.M. Lau, W. Stern, Global Occurrences of Extreme Precipitation and the Madden–Julian Oscillation: Observations and Predictability, J. Clim. 17 (2004), 4575–4589. https://doi.org/10.1175/3238.1.

[9] B. Memon, M.H. Baloch, A.H. Memon, S.H. Qazi, R. Haider, et al., Assessment of Wind Power Potential Based on Raleigh Distribution Model: An Experimental Investigation for Coastal Zone, Eng. Technol. Appl. Sci. Res. 9 (2019), 3721–3725. https://doi.org/10.48084/etasr.2381.

[10] C.S. Withers, S. Nadarajah, Evidence of Trend in Return Levels for Daily Rainfall in New Zealand, J. Hydrol. (N. Z.) 39 (2000), 155–166. https://www.jstor.org/stable/43944839.

[11] J.L. Martel, M. Alain, B. Francois, Global and Regional Projected Changes in 100-yr Subdaily, Daily, and Multiday Precipitation Extremes Estimated From Three Large Ensembles of Climate Simulations, J. Clim. 33 (2020), 1089–1103. https://doi.org/10.1175/JCLI-D-18-0764.s1.

[12] J. Pickands III, Statistical Inference Using Extreme Order Statistics, Ann. Stat. 3 (1975), 131–199. https://doi.org/10.1214/aos/1176343003.

[13] C. Rohrbeck, E.F. Eastoe, A. Frigessi, J.A. Tawn, Extreme Value Modelling of Water-Related Insurance Claims, Ann. Appl. Stat. 12 (2018), 246–282. https://doi.org/10.1214/17-aoas1081.

[14] D.J. Dupuis, S. Engelke, L. Trapin, Modeling Panels of Extremes, Ann. Appl. Stat. 17 (2023), 498–517. https://doi.org/10.1214/22-aoas1639.

[15] J. Rodrigues, A. Silva, The Exponentiated Kumaraswamy-Exponential Distribution, Br. J. Appl. Sci. Technol. 10 (2015), 1–12. https://doi.org/10.9734/bjast/2015/16935.

[16] J. Rodrigues, A. Silva, G. Hamedani, The Exponentiated Kumaraswamy InverseWeibull Distribution with Application in Survival Analysis, J. Stat. Theory Appl. 15 (2016), 8–24. https://doi.org/10.2991/jsta.2016.15.1.2.

[17] Z.M. Nofal, A.Z. Afify, H.M. Yousof, G.M. Cordeiro, The Generalized Transmuted-G Family of Distributions, Commun. Stat. - Theory Methods 46 (2016), 4119–4136. https://doi.org/10.1080/03610926.2015.1078478.

[18] G.M. Cordeiro, M. de Castro, A New Family of Generalized Distributions, J. Stat. Comput. Simul. 81 (2011), 883–898. https://doi.org/10.1080/00949650903530745.

[19] G.M. Cordeiro, M. Alizadeh, G. Ozel, B. Hosseini, E.M.M. Ortega, et al., The Generalized Odd Log-Logistic Family of Distributions: Properties, Regression Models and Applications, J. Stat. Comput. Simul. 87 (2016), 908–932. https://doi.org/10.1080/00949655.2016.1238088.

[20] R. Silva, F. Gomes-Silva, M. Ramos, G.M. Cordeiro, P. Marinho, et al., The Exponentiated Kumaraswamy-G Class: General Properties and Application, Rev. Colomb. Estad. 42 (2019), 1–33. https://doi.org/10.15446/rce.v42n1.66205.

[21] F. Nascimento, M. Bourguignon, J. Leao, Extended Generalized Extreme Value Distribution with Applications in Environmental Data, Hacet. J. Math. Stat. 46 (2015), 1847–1864. https://doi.org/10.15672/hjms.20159514081.

[22] N. Deetae, Analysis and Mathematical Modeling for Flood Surveillance from Rainfall by Extreme Value Theory for Agriculture in Phitsanulok Province, Thailand, Eur. J. Pure Appl. Math. 15 (2022), 1797–1807. https://doi.org/10.29020/nybg.ejpam.v15i4.4558.

[23] N. Deetae, P. Khamrot, K. Jampachaisri, Modelling Extreme Rainfall Using Extended Generalized Extreme Value Distribution, Int. J. Anal. Appl. 23 (2025), 73. https://doi.org/10.28924/2291-8639-23-2025-73.

[24] S. Nadarajah, F. Haghighi, An Extension of the Exponential Distribution, Statistics 45 (2011), 543–558. https://doi.org/10.1080/02331881003678678.

[25] C.T. Guloksuz, N. Celik, An Extension of Generalized Extreme Value Distribution: Uniform-GEV Distribution and Its Application to Earthquake Data, Thail. Stat. 18 (2020), 491–506.

[26] J. Zhao, Z. Ahmad, E. Mahmoudi, E.H. Hafez, M.M. Mohie El-Din, A New Class of Heavy-Tailed Distributions: Modeling and Simulating Actuarial Measures, Complexity 2021 (2021), 5580228. https://doi.org/10.1155/2021/5580228.

[27] N. Deetae, P. Khamrot, K. Jampachaisri, A New Extended Kumaraswamy Generalized Pareto Distribution with Rainfall Application, IEEE Access 13 (2025), 68259–68269. https://doi.org/10.1109/access.2025.3561150.

[28] P. Kavya, M. Manoharan, Some Parsimonious Models for Lifetimes and Applications, J. Stat. Comput. Simul. 91 (2021), 3693–3708. https://doi.org/10.1080/00949655.2021.1946064.

[29] R.H. Byrd, P. Lu, J. Nocedal, A Limited Memory Algorithm for Bound Constrained Optimization, SIAM J. Sci. Comput. 16 (1995), 1190–1208.

[30] J. Liu, J. Leu, ETCN-NNC-LB: Ensemble TCNs with L-BFGS-B Optimized No Negative Constraint-Based Forecasting for Network Traffic, IEEE Trans. Netw. Serv. Manag. 22 (2025), 3692–3704. https://doi.org/10.1109/tnsm.2025.3563978.

[31] A. Bemporad, An L-BFGS-B Approach for Linear and Nonlinear System Identification Under ℓ_1 and Group-Lasso Regularization, IEEE Trans. Autom. Control. 70 (2025), 4857–4864. https://doi.org/10.1109/tac.2025.3541018.

[32] R.W. Katz, M.B. Parlange, P. Naveau, Statistics of Extremes in Hydrology, Adv. Water Resour. 25 (2002), 1287–1304. https://doi.org/10.1016/s0309-1708(02)00056-8.

[33] A. Asgharzadeh, H.S. Bakouch, M. Habibi, A Generalized Binomial Exponential 2 Distribution: Modeling and Applications to Hydrologic Events, J. Appl. Stat. 44 (2016), 2368–2387. https://doi.org/10.1080/02664763.2016.1254729.

[34] V. Choulakian, M.A. Stephens, Goodness-of-Fit Tests for the Generalized Pareto Distribution, Technometrics 43 (2001), 478–484. https://doi.org/10.1198/00401700152672573.