Int. J. Anal. Appl. (2025), 23:294

International Journal of Analysis and Applications

Novel Results in Cone Bipolar Metric Spaces With Application in Initial Value Fractional Caputo Differential Equations

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Abstract. In this paper, we establish fixed point results in the setting of cone bipolar metric space. Some of the well-known results in the literature are extended and generalized by the demonstrated results. We give some examples based on our outcomes to strengthen our results. An application is presented based on integral equations and fractional differential equations that confirm our findings.

1. Introduction

Applications of fixed point theory are essential to many areas of mathematics. In fixed point theory, the goal of research activities has shifted to finding fixed points for generalized contraction mappings [1–3]. In recent times, numerous researchers have disseminated diverse articles on fixed point theory and its applications in diverse formats. The presence of fixed points of contraction mappings in bipolar metric spaces, which are essentially generalizations of the Banach contraction principle, has been a hot topic in fixed point theory in recent years.

Received: Sep. 12, 2025.

2020 *Mathematics Subject Classification.* 47H10; 54H25.

Key words and phrases. Iterative methods; Cone bipolar metric space; fixed point; covariant map; contravariant map; initial value problem; fractional Caputo differential equation.

https://doi.org/10.28924/2291-8639-23-2025-294

ISSN: 2291-8639

In 2016, Mutlu and Gürdal [4] introduced the concept of bipolar metric space and investigated some basic fixed point and coupled fixed point theorems for co-variant and contra-variant maps under contractive conditions; see [4, 6]. Furthermore, bipolar metric spaces have seen a great deal of significant study see [7–13]. Aydi et al. [14] established some fixed point theorems in a bipolar metric space in 2021. Mani et al. [15] proved a fixed point theorem in a probabilistic bipolar metric space. Mani et al. [16] have proved a fixed point theorem in a bipolar controlled metric space. Pasha et al [17] have presented a fixed point theorem in a bipolar parametric metric space. Mani et al. [18] have proved some fixed point theorems in a C^*-algebra valued bipolar metric space. Mani et al. [19] have proved some fixed point theorems in a bipolar metric space.

Cone metric spaces were first proposed by Huang and Zhang [20] in 2007, and they demonstrated certain fixed point theorems related to contractive mappings. Mani et al. [21] have proved a fixed point result in a cone b-metric space. Dey and Saha [22] have proved some fixed point theorems in a partial cone metric space. Shateri [23] has proved a common fixed point theorem in a partial cone metric space. Arif et al. [24] introduced an ordered implicit relation and proved some fixed point theorems in a cone metric space. Arif et al. [25] introduced an ordered implicit relation and proved some fixed point theorems in a cone A-metric space. In this paper, we introduce the concept of cone bipolar metric space and we prove some fixed point theorems in such spaces.

2. Preliminaries

We outline some fundamental definitions in this section.

Let $\mathcal{B}$ be a real Banach space and $\mathcal{W} \subseteq \mathcal{B}$. $\mathcal{W}$ is called a cone iff

- $\mathcal{W}$ is closed, nonempty, and $\mathcal{W} \neq \{0\}$;
- If $\bar{a}, \bar{c} \in \mathbb{R}$ and $\bar{a}, \bar{c} \geq 0$, then $\bar{a}\varpi + \bar{c}\eta \in \mathcal{W}$ for all $\varpi, \eta \in \mathcal{W}$;
- $\varpi \in \mathcal{W}$ and $-\varpi \in \mathcal{W} \implies \varpi = 0$.

$\varpi \leq \eta$ iff $\eta - \varpi \in \mathcal{W}$ denotes a partial ordering $\leq$ with respect to $\mathcal{W}$, where $\mathcal{W}$ is a cone $\mathcal{W} \subset \mathcal{B}$. While $\varpi \ll \eta$ stands for $\eta - \varpi \in int\mathcal{W}$, where $int\mathcal{W}$ indicates the interior of $\mathcal{W}$. Also, we write $\varpi < \eta$ to show that $\varpi \leq \eta$ and $\varpi \neq \eta$.

The cone $\mathcal{W}$ is called normal if there is a number $N > 0$ such that for all $\varpi, \eta \in \mathcal{B}$,

$$0 \leq \varpi \leq \eta \text{ implies } \|\varpi\| \leq N\|\eta\|.$$

The least positive number satisfying above is called the normal constant of $\mathcal{W}$.

The cone $\mathcal{W}$ is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{\varpi_\nu\}$ is a sequence such that

$$\varpi_1 \leq \varpi_2 \leq \cdots \leq \varpi_\nu \leq \cdots \leq \eta$$

for some $\eta \in \mathcal{B}$, then there is $\varpi \in \mathcal{B}$ such that $\|\varpi_\nu - \varpi\| \to 0 (\nu \to \infty)$.

Comparatively, if every decreasing sequence that is bounded from below be convergent, then the cone $\mathcal{W}$ is regular. A regular cone is a normal cone. In the following, we assume that $\mathcal{B}$ is a Banach space, $\leq$ is a partial ordering with respect to $\mathcal{W}$ and that $\mathcal{W}$ is a cone in $\mathcal{B}$ with $int\mathcal{W} \neq \emptyset$.

Definition 2.1. *Let Ξ and Γ be nonempty sets and $\varphi : \Xi \times \Gamma \to \mathcal{B}$ be a mapping satisfying,*

- *if $\varphi(\varpi, \eta) = 0$, then $\varpi = \eta$ for all $(\varpi, \eta) \in \Xi \times \Gamma$;*
- *if $\varpi = \eta$, then $\varphi(\varpi, \eta) = 0$ for all $(\varpi, \eta) \in \Xi \times \Gamma$;*
- *$\varphi(\varpi, \eta) = \varphi(\eta, \varpi)$ for all $(\varpi, \eta) \in \Xi \cap \Gamma$;*
- *$\varphi(\varpi, \eta) \leq \varphi(\varpi, \omega) + \varphi(\alpha, \omega) + \varphi(\alpha, \eta)$ for all $\varpi, \alpha \in \Xi$ and $\omega, \eta \in \Gamma$.*

The the triplet (Ξ, Γ, φ) is called a cone bipolar metric space(CBMS).

Cone bipolar metric spaces are obviously a generalization of bipolar metric spaces.

Example 2.1. *Let $\mathcal{B} = \mathbb{R}^2$, $\mathcal{W} = \{(\varpi, \eta) \in \mathcal{B} \mid \varpi, \eta \geq 0\} \subset \mathbb{R}^2$, $\Xi = [0, 1]$, $\Gamma = [1, 2]$ and $\varphi : \Xi \times \Gamma \to \mathcal{B}$ such that $\varphi(\varpi, \eta) = (|\varpi - \eta|, \alpha|\varpi - \eta|)$, where $\alpha \geq 0$ is a constant. Then (Ξ, Γ, φ) is a CBMS.*

Definition 2.2. (1) *Let (Ξ, Γ, φ) be a CBMS. Then the points of the sets Ξ, Γ and $\Xi \cap \Gamma$ are named as left, right and central points, respectively, and any sequence, which is consisted of only left (or right, or central) points is called a left (or right, or central) sequence in (Ξ, Γ, φ).*

(2) *Let $(\Xi_1, \Gamma_1, \varphi_1)$ and $(\Xi_2, \Gamma_2, \varphi_2)$ be two CBMSs and $\Omega : \Xi_1 \cup \Gamma_1 \to \Xi_2 \cup \Gamma_2$ be a function. If $\Omega(\Xi_1) \subseteq \Xi_2$ and $\Omega(\Gamma_1) \subseteq \Gamma_2$, then Ω is called a covariant map, or a map from $(\Xi_1, \Gamma_1, \varphi_1)$ to $(\Xi_2, \Gamma_2, \varphi_2)$ and this is written as $\Omega : (\Xi_1, \Gamma_1, \varphi_1) \rightrightarrows (\Xi_2, \Gamma_2, \varphi_2)$. If $\Omega : (\Xi_1, \Gamma_1, \varphi_1) \rightrightarrows (\Gamma_2, \Xi_2, \varphi_2)$ is a covariant map, then Ω is called a contravariant map from $(\Xi_1, \Gamma_1, \varphi_1)$ to $(\Xi_2, \Gamma_2, \varphi_2)$ and this is denoted as $\Omega : (\Xi_1, \Gamma_1, \varphi_1) \rightleftarrows (\Xi_2, \Gamma_2, \varphi_2)$.*

Definition 2.3. *Let (Ξ, Γ, φ) be a CBMS. A left sequence $\{\varpi_\nu\}$ converges to a right point η if and only if for every $\mathfrak{h} \in \mathcal{B}$ with $0 \ll \mathfrak{h}$ we can find an $\nu_0 \in \mathbb{N}$ such that $\varphi(\varpi_\nu, \eta) \ll \mathfrak{h}$ for all $\nu \geq \nu_0$. Similarly, a right sequence $\{\eta_\nu\}$ converges to a left point ϖ if and only if for every $\mathfrak{h} \in \mathcal{B}$ with $0 \ll \mathfrak{h}$ we can find an $\nu_0 \in \mathbb{N}$ such that whenever $\nu \geq \nu_0, \varphi(\varpi, \eta_\nu) \ll \mathfrak{h}$.*

Definition 2.4. *Let $(\Xi_1, \Gamma_1, \varphi_1)$ and $(\Xi_2, \Gamma_2, \varphi_2)$ be two CBMSs.*

(1) *A map $\Omega : (\Xi_1, \Gamma_1, \varphi_1) \rightrightarrows (\Xi_2, \Gamma_2, \varphi_2)$ is said to be continuous at a point $\varpi_0 \in \Xi_1$, if for every $\mathfrak{h} \in \mathcal{B}$ with $0 \ll \mathfrak{h}$, there exists a $0 \ll \delta$ such that whenever $\eta \in \Gamma_1$ and $\varphi_1(\varpi_0, \eta) \ll \delta$, $\varphi_2(\Omega(\varpi_0), \Omega(\eta)) \ll \mathfrak{h}$. It is continuous at a point $\eta_0 \in \Gamma_1$ if for every $\mathfrak{h} \in \mathcal{B}$ with $0 \ll \mathfrak{h}$, there exists a $0 \ll \delta$ such that whenever $\varpi \in \Xi_1$ and $\varphi_1(\varpi, \eta_0) \ll \delta$, $\varphi_2(\Omega(\varpi), \Omega(\eta_0)) \ll \mathfrak{h}$. If Ω is continuous at all points $\varpi \in \Xi_1$ and $\eta \in \Gamma_1$, then it is called continuous.*

(2) *A contravariant map $\Omega : (\Xi_1, \Gamma_1, \varphi_1) \leftrightarrows (\Xi_2, \Gamma_2, \varphi_2)$ is continuous if and only if it is continuous as a covariant map $\Omega : (\Xi_1, \Gamma_1, \varphi_1) \rightrightarrows (\Gamma_2, \Xi_2, \varphi_2)$.*

Definition 2.5. *Let $(\Xi_1, \Gamma_1, \varphi_1)$ and $(\Xi_2, \Gamma_2, \varphi_2)$ be two CBMSs. If for a covariant map $\Omega : (\Xi_1, \Gamma_1, \varphi_1) \rightrightarrows (\Xi_2, \Gamma_2, \varphi_2)$, there exists $\lambda \in (0, 1)$ such that*

$$\varphi_2(\Omega(\varpi), \Omega(\eta)) \leq \lambda \varphi_1(\varpi, \eta) \text{ for all } \varpi \in \Xi_1, \eta \in \Gamma_1,$$

or if for a contravariant map $\Omega : (\Xi_1, \Gamma_1, \varphi_1) \leftrightarrows (\Xi_2, \Gamma_2, \varphi_2)$, there exists $\lambda \in (0, 1)$ such that,

$$\varphi_2(\Omega(\eta), \Omega(\varpi)) \leq \lambda \varphi_1(\varpi, \eta) \text{ for all } \varpi \in \Xi_1, \eta \in \Gamma_1,$$

then they are called a contraction.

Definition 2.6. *Let (Ξ, Γ, φ) be a CBMS.*

(1) *A sequence $(\{\varpi_n\}, \{\eta_n\})$ in the set $\Xi \times \Gamma$ is said to be a bisequence in (Ξ, Γ, φ).*
(2) *The bisequence $(\{\varpi_n\}, \{\eta_n\})$ is said to be convergent if $\{\varpi_n\}$ and $\{\eta_n\}$ converge. This bisequence is said to be biconvergent if $\{\varpi_n\}$ and $\{\eta_n\}$ both converge to a point $u \in \Xi \cap \Gamma$.*
(3) *A bisequence $(\{\varpi_n\}, \{\eta_n\})$ in (Ξ, Γ, φ) is a Cauchy bisequence if for any $0 \ll \epsilon$, we can find a number $\nu_0 \in \mathbb{N}$, such that for all positive integers $m, n \geq \nu_0$, $\varphi(\varpi_m, \eta_n) \ll \epsilon$.*

Definition 2.7. *If all of the Cauchy bisequences in a CBMS converge, the system is deemed complete.*

We prove some fixed point theorems in a CBMS in this study.

3. Main Results

Now, we demonstrate our first result.

Theorem 3.1. *A complete CBMS (Ξ, Γ, φ) is given, along with a contraction $\Omega : (\Xi, \Gamma, \varphi) \rightrightarrows (\Xi, \Gamma, \varphi)$. Let $\mathcal{W}$ be a normal cone with normal constant N. Then, there is a unique fixed point, or UFP, for the function $\Omega : \Xi \cup \Gamma \to \Xi \cup \Gamma$.*

Proof. Let $\varpi_0 \in \Xi$ and $\eta_0 \in \Gamma$. For each $\nu \in \mathbb{N}$, define $\Omega(\varpi_\nu) = \varpi_{\nu+1}$ and $\Omega(\eta_\nu) = \eta_{\nu+1}$. Then $(\{\varpi_\nu\}, \{\eta_\nu\})$ is a bisequence in (Ξ, Γ, φ). Let $\mathcal{M} := \varphi(\varpi_0, \eta_0) + \varphi(\varpi_0, \eta_1)$. Then, for all $\nu, \mathfrak{p} \in \mathbb{N}$,

$$\begin{aligned}\varphi(\varpi_\nu, \eta_\nu) &= \varphi(\Omega(\varpi_{\nu-1}), \Omega(\eta_{\nu-1})) \\ &\leq \lambda \varphi(\varpi_{\nu-1}, \eta_{\nu-1}) \\ &\vdots \\ &\leq \lambda^\nu \varphi(\varpi_0, \eta_0),\end{aligned}$$

and also,

$$\begin{aligned}\varphi(\varpi_\nu, \eta_{\nu+1}) &= \varphi(\Omega(\varpi_{\nu-1}), \Omega(\eta_\nu)) \\ &\leq \lambda \varphi(\varpi_{\nu-1}, \eta_\nu) \\ &\vdots \\ &\leq \lambda^\nu \varphi(\varpi_0, \eta_1).\end{aligned}$$

Moreover

$$\begin{aligned}\varphi(\varpi_{\nu+\mathfrak{p}}, \eta_\nu) &\leq \varphi(\varpi_{\nu+\mathfrak{p}}, \eta_{\nu+1}) + \varphi(\varpi_\nu, \eta_{\nu+1}) + \varphi(\varpi_\nu, \eta_\nu) \\ &\leq \varphi(\varpi_{\nu+\mathfrak{p}}, \eta_{\nu+1}) + \lambda^\nu \mathcal{M} \\ &\leq \varphi(\varpi_{\nu+\mathfrak{p}}, \eta_{\nu+2}) + \varphi(\varpi_{\nu+1}, \eta_{\nu+2}) + \varphi(\varpi_{\nu+1}, \eta_{\nu+1}) + \lambda^\nu \mathcal{M} \\ &\leq \varphi(\varpi_{\nu+\mathfrak{p}}, \eta_{\nu+2}) + (\lambda^{\nu+1} + \lambda^\nu) \mathcal{M} \\ &\vdots\end{aligned}$$

$$
\begin{aligned}
&\leq \varphi(\varpi_{\nu+\mathfrak{p}}, \eta_{\nu+\mathfrak{p}}) + (\lambda^{\nu+\mathfrak{p}-1} + \cdots + \lambda^{\nu+1} + \lambda^{\nu})\mathcal{M} \\
&\leq (\lambda^{\nu+\mathfrak{p}} + \cdots + \lambda^{\nu+1} + \lambda^{\nu})\mathcal{M} \\
&\leq \frac{\lambda^{\nu}\mathcal{M}}{1-\lambda} = \frac{\lambda^{\nu}(\varphi(\varpi_0, \eta_0) + \varphi(\varpi_0, \eta_1))}{1-\lambda},
\end{aligned}
$$

and similarly $\varphi(\varpi_{\nu}, \eta_{\nu+\mathfrak{p}}) \leq \frac{\lambda^{\nu}(\varphi(\varpi_0,\eta_0)+\varphi(\varpi_0,\eta_1))}{1-\lambda}$. Now,

$$
\begin{aligned}
\varphi(\varpi_{\nu}, \eta_{\mathfrak{m}}) &\leq \varphi(\varpi_{\nu}, \eta_{\nu+\mathfrak{p}}) + \varphi(\varpi_{\nu+\mathfrak{p}}, \eta_{\nu+\mathfrak{p}}) + \varphi(\varpi_{\nu+\mathfrak{p}}, \eta_{\mathfrak{m}}) \\
&\leq 2\frac{\lambda^{\nu}(\varphi(\varpi_0, \eta_0) + \varphi(\varpi_0, \eta_1))}{1-\lambda} + \lambda^{\nu+\mathfrak{p}}(\varphi(\varpi_0, \eta_0)),
\end{aligned}
$$

which implies that

$$
\|\varphi(\varpi_{\nu}, \eta_{\mathfrak{m}})\| \leq 2N\frac{\lambda^{\nu}(\|\varphi(\varpi_0, \eta_0)\| + \|\varphi(\varpi_0, \eta_1)\|)}{1-\lambda} + N\lambda^{\nu+\mathfrak{p}}(\|\varphi(\varpi_0, \eta_0)\|).
$$

Therefore, $\varphi(\varpi_{\nu}, \eta_{\mathfrak{m}}) \to 0 (\nu, \mathfrak{m} \to \infty)$. Hence $(\{\varpi_{\nu}\}, \{\eta_{\nu}\})$ is a Cauchy bisequence. Since (Ξ, Γ, φ) is complete, $(\{\varpi_{\nu}\}, \{\eta_{\nu}\})$ is converges, and thus it is biconverges to a point $\aleph \in \Xi \cap \Gamma$ and

$$
\{\Omega(\eta_{\nu})\} = \{\eta_{\nu+1}\} \to \aleph \in \Xi \cap \Gamma.
$$

Since Ω is continuous (any contraction is continuous), $\Omega(\eta_{\nu}) \to \Omega(\aleph)$, so $\Omega(\aleph) = \aleph$. Hence $\aleph$ is a fixed point of Ω. Let ℓ be another fixed point of Ω, then $\Omega(\ell) = \ell$ implies that $\ell \in \Xi \cap \Gamma$ and we have,

$$
\varphi(\aleph, \ell) = \varphi(\Omega(\aleph), \Omega(\ell)) \leq \lambda\varphi(\aleph, \ell),
$$

where $0 < \lambda < 1$, which implies $\varphi(\aleph, \ell) = 0$ and so $\aleph = \ell$. □

Remark 3.1. *If $\Xi = \Gamma$, then, the above result is reduced to Theorem 1 in [20].*

Example 3.1. *Let $\mathcal{B} = \mathbb{R}$ and $\mathcal{W} = \{\varpi \in \mathcal{B} | \varpi \geq 0\}$.*

Let $\Xi = \{\mathcal{U}_{\nu}(\mathbb{R}) : \mathcal{U}_{\nu}(\mathbb{R})$ is an $\nu \times \nu$ upper triangular matrix over $\mathbb{R}\}$, $\Gamma = \{\mathcal{L}_{\nu}(\mathbb{R}) : \mathcal{L}_{\nu}(\mathbb{R})$ is an $\nu \times \nu$ lower triangular matrix over $\mathbb{R}\}$ and the map $\varphi : \Xi \times \Gamma \to [0, \infty)$ be defined by

$$
\varphi(\mathcal{P}, \mathcal{Q}) = \sum_{\mathrm{i,j}=1}^{\nu} |\varsigma_{\mathrm{ij}} - \mathfrak{q}_{\mathrm{ij}}|,
$$

for all $\mathcal{P} = (\varsigma_{\mathrm{ij}})_{\nu\times\nu} \in \Xi$ and $\mathcal{Q} = (\mathfrak{q}_{\mathrm{ij}})_{\nu\times\nu} \in \Gamma$. Then (Ξ, Γ, φ) is a complete CBMS. Define $\mathcal{T} : (\Xi, \Gamma, \varphi) \rightrightarrows (\Xi, \Gamma, \varphi)$ by

$$
\mathcal{T}(\mathcal{P}) = \left(\frac{\varsigma_{\mathrm{ij}}}{4}\right)_{\nu\times\nu},
$$

for all $\mathcal{P} = (\varsigma_{\mathrm{ij}})_{\nu\times\nu} \in \mathcal{U}_{\nu}(\mathbb{R}) \cup \mathcal{L}_{\nu}(\mathbb{R})$. Now,

$$
\varphi(\mathcal{T}(\mathcal{P}), \mathcal{T}(\mathcal{Q})) = \frac{1}{4}\sum_{\mathrm{i,j}=1}^{\nu} |\varsigma_{\mathrm{ij}} - \mathfrak{q}_{\mathrm{ij}}|
$$

$$\leq \frac{1}{2}\sum_{i,j=1}^{\nu}|\varsigma_{ij}-\mathfrak{q}_{ij}|$$
$$=\lambda\varphi(\mathcal{P},\mathcal{Q}),$$

for all $\mathcal{P}=(\varsigma_{ij})_{\nu\times\nu}\in\Xi$ *and* $\mathcal{Q}=(\mathfrak{q}_{ij})_{\nu\times\nu}\in\Gamma$. *All the axioms of Theorem 3.1 are verified with* $\lambda=\frac{1}{2}$ *and* $\mathcal{T}$ *has a UFP* $0_{\nu\times\nu}\in\mathcal{U}_\nu(\mathbb{R})\cap\mathcal{L}_\nu(\mathbb{R})$ *where* $0_{\nu\times\nu}$ *is the null matrix.*

Example 3.2. *Let* $\mathcal{B}=\mathbb{R}$ *and* $\mathcal{W}=\{\varpi\in\mathcal{B}|\varpi\geq 0\}$. *Let* $\Xi=[0,1]$ *and* $\Gamma=\{0\}\cup\mathbb{N}-\{1\}$ *be equipped with* $\varphi(\varpi,\eta)=|\varpi-\eta|$ *for all* $\varpi\in\Xi$ *and* $\eta\in\Gamma$. *Then,* (Ξ,Γ,φ) *is a complete CBMS. Define* $\Omega:\Xi\cup\Gamma\rightrightarrows\Xi\cup\Gamma$ *by*

$$\Omega(\varpi)=\begin{cases}\frac{\varpi}{5}, & \text{if } \varpi\in(0,1],\\ 0, & \text{if } \varpi\in\{0\}\cup\mathbb{N}-\{1\},\end{cases}$$

$\forall\ \varpi\in\Xi\cup\Gamma$. *Let* $\varpi\in\Xi$ *and* $\eta\in\Gamma$, *then*

$$\varphi(\Omega\varpi,\Omega\eta)=|\frac{\varpi}{5}-0|$$
$$\leq\frac{1}{2}|\varpi-\eta|.$$

Consequently, Ω *has a UFP* $\varpi=0$ *as all of the axioms of Theorem 3.1 are satisfied.*

We demonstrate our second result.

Theorem 3.2. *Let* (Ξ,Γ,φ) *be a complete CBMS,* $\mathcal{W}$ *be a normal cone with normal constant N and given a contravariant contraction* $\Omega:(\Xi,\Gamma,\varphi)\leftrightarrows(\Xi,\Gamma,\varphi)$. *Then the function* $\Omega:\Xi\cup\Gamma\to\Xi\cup\Gamma$ *has a UFP.*

Proof. Let $\varpi_0\in\Xi$ and $\eta_0\in\Gamma$. For each $\nu\in\mathbb{N}$, define $\Omega(\varpi_\nu)=\eta_{\nu+1}$ and $\Omega(\eta_\nu)=\varpi_{\nu+1}$. Then $(\{\varpi_\nu\},\{\eta_\nu\})$ is a bisequence in (Ξ,Γ,φ).

Then for all $\nu,\mathfrak{p}\in\mathbb{Z}^+$,

$$\begin{aligned}\varphi(\varpi_\nu,\eta_\nu)&=\varphi(\Omega(\eta_{\nu-1}),\Omega(\varpi_{\nu-1}))\\&\leq\lambda\varphi(\varpi_{\nu-1},\eta_{\nu-1})\\&=\lambda\varphi(\Omega(\eta_{\nu-2}),\Omega(\varpi_{\nu-2}))\\&\leq\lambda^2\varphi(\varpi_{\nu-2},\eta_{\nu-2})\\&\vdots\\&\leq\lambda^\nu\varphi(\varpi_0,\eta_0).\end{aligned}$$

Also, we have

$$\begin{aligned}\varphi(\varpi_{\nu+1},\eta_\nu)&=\varphi(\Omega(\eta_\nu),\Omega(\varpi_{\nu-1}))\\&\leq\lambda\varphi(\varpi_{\nu-1},\eta_\nu)=\lambda\varphi(\Omega(\eta_{\nu-2}),\Omega(\varpi_{\nu-1}))\\&\leq\lambda^2\varphi(\varpi_{\nu-1},\eta_{\nu-2})]\\&\leq\lambda^\nu\varphi(\varpi_1,\eta_0).\end{aligned}$$

Therefore

$$\begin{aligned}
\varphi(\varpi_{\nu+\mathfrak{p}},\eta_\nu) &\leq \varphi(\varpi_{\nu+\mathfrak{p}},\eta_{\nu+1}) + \varphi(\varpi_{\nu+1},\eta_{\nu+1}) + \varphi(\varpi_{\nu+1},\eta_\nu)\\
&\leq \varphi(\varpi_{\nu+\mathfrak{p}},\eta_{\nu+1}) + \lambda^{\nu+1}\varphi(\varpi_0,\eta_0) + \lambda^{\nu}\varphi(\varpi_1,\eta_0)\\
&\leq \varphi(\varpi_{\nu+\mathfrak{p}},\eta_{\nu+2}) + \varphi(\varpi_{\nu+2},\eta_{\nu+2}) + \varphi(\varpi_{\nu+2},\eta_{\nu+1})\\
&+ \lambda^{\nu+1}\varphi(\varpi_0,\eta_0) + \lambda^{\nu}\varphi(\varpi_1,\eta_0)\\
&\leq \varphi(\varpi_{\nu+\mathfrak{p}},\eta_{\nu+2}) + (\lambda^{\nu+2}+\lambda^{\nu+1})\varphi(\varpi_0,\eta_0) + (\lambda^{\nu+1}+\lambda^{\nu})\varphi(\varpi_1,\eta_0)\\
&\vdots\\
&\leq \varphi(\varpi_{\nu+\mathfrak{p}},\eta_{\nu+\mathfrak{p}-1}) + (\lambda^{\nu+\mathfrak{p}-1}+....+\lambda^{\nu+1})\varphi(\varpi_0,\eta_0)\\
&+ (\lambda^{\nu+\mathfrak{p}-2}+....+\lambda^{\nu})\varphi(\varpi_1,\eta_0)\\
&\leq (\lambda^{\nu+\mathfrak{p}-1}+....+\lambda^{\nu+1})\varphi(\varpi_0,\eta_0)\\
&+ (\lambda^{\nu+\mathfrak{p}-1}+\lambda^{\nu+\mathfrak{p}-2}+....+\lambda^{\nu})\varphi(\varpi_1,\eta_0)\\
&\leq \frac{\lambda^{\nu+1}}{1-\lambda}\varphi(\varpi_0,\eta_0) + \frac{\lambda^{\nu}}{1-\lambda}\varphi(\varpi_1,\eta_0).
\end{aligned}$$

Similarly, we obtain that

$$\begin{aligned}
\varphi(\varpi_\nu,\eta_{\nu+\mathfrak{p}}) &\leq \varphi(\varpi_\nu,\eta_\nu) + \varphi(\varpi_{\nu+1},\eta_\nu) + \varphi(\varpi_{\nu+1},\eta_{\nu+\mathfrak{p}})\\
&\leq \lambda^{\nu}\varphi(\varpi_0,\eta_0) + \lambda^{\nu}\varphi(\varpi_1,\eta_0) + \varphi(\varpi_{\nu+1},\eta_{\nu+\mathfrak{p}})\\
&\leq \lambda^{\nu}\varphi(\varpi_0,\eta_0) + \lambda^{\nu}\varphi(\varpi_1,\eta_0) + \varphi(\varpi_{\nu+1},\eta_{\nu+1}) + \varphi(\varpi_{\nu+2},\eta_{\nu+1})\\
&+ \varphi(\varpi_{\nu+2},\eta_{\nu+\mathfrak{p}})\\
&\leq [\lambda^{\nu}+\lambda^{\nu+1}]\varphi(\varpi_0,\eta_0) + [\lambda^{\nu}+\lambda^{\nu+1}]\varphi(\varpi_1,\eta_0) + \varphi(\varpi_{\nu+2},\eta_{\nu+\mathfrak{p}})\\
&\vdots\\
&\leq [\lambda^{\nu}+\lambda^{\nu+1}+\cdots+\lambda^{\nu+\mathfrak{p}-1}]\varphi(\varpi_0,\eta_0) + [\lambda^{\nu}+\lambda^{\nu+1}+\cdots+\lambda^{\nu+\mathfrak{p}-1}]\varphi(\varpi_1,\eta_0)\\
&+ \varphi(\varpi_{\nu+\mathfrak{p}},\eta_{\nu+\mathfrak{p}})\\
&\leq (\lambda^{\nu}+\lambda^{\nu+1}+\cdots+\lambda^{\nu+\mathfrak{p}-1}+\lambda^{\nu+\mathfrak{p}})[\varphi(\varpi_0,\eta_0)+\varphi(\varpi_1,\eta_0)]\\
&\leq \frac{\lambda^{\nu}}{1-\lambda}[\varphi(\varpi_0,\eta_0)+\varphi(\varpi_1,\eta_0)] + .
\end{aligned}$$

Now,

$$\begin{aligned}
\varphi(\varpi_\nu,\eta_{\mathfrak{m}}) &\leq \varphi(\varpi_\nu,\eta_{\nu+\mathfrak{p}}) + \varphi(\varpi_{\nu+\mathfrak{p}},\eta_{\nu+\mathfrak{p}}) + \varphi(\varpi_{\nu+\mathfrak{p}},\eta_{\mathfrak{m}})\\
&\leq 2\frac{\lambda^{\nu}[\varphi(\varpi_0,\eta_0)+\varphi(\varpi_1,\eta_0)]}{1-\lambda} + \lambda^{\nu+\mathfrak{p}}\varphi(\varpi_0,\eta_0),
\end{aligned}$$

which implies that

$$\|\varphi(\varpi_\nu,\eta_{\mathfrak{m}})\| \leq 3N\frac{\lambda^{\nu}\|[\varphi(\varpi_0,\eta_0)+\varphi(\varpi_1,\eta_0)]\|}{1-\lambda}.$$

Therefore, $\varphi(\varpi_\nu, \eta_{\mathfrak{m}}) \to 0(\nu, \mathfrak{m} \to \infty)$. Hence, $(\{\varpi_\nu\}, \{\eta_\nu\})$ is a Cauchy bisequence. Since (Ξ, Γ, φ) is complete, $(\{\varpi_\nu\}, \{\eta_\nu\})$ converges, and so $\{\varpi_\nu\} \to \aleph$ and $\{\eta_\nu\} \to \aleph$, where $\aleph \in \Xi \cap \Gamma$. Since Ω is continuous, $\{\varpi_n\} \to \aleph$ implies that

$$\{\eta_\nu\} = \{\Omega(\varpi_{\nu-1})\} \to \Omega(\aleph),$$

and combining this with $\{\eta_\nu\} \to \aleph$ gives $\Omega(\aleph) = \aleph$.

Let ℓ be another fixed point of Ω, then $\Omega(\ell) = \ell$ implies $\ell \in \Xi \cap \Gamma$. Then,

$$\begin{aligned}\varphi(\aleph, \ell) &= \varphi(\Omega(\aleph), \Omega(\ell)) \\ &\leq \lambda\varphi(\ell, \aleph) = \lambda\varphi(\aleph, \ell),\end{aligned}$$

which gives $\varphi(\aleph, \ell) = 0$. Hence, $\aleph = \ell$. □

Example 3.3. *Let $\mathcal{B} = \mathbb{R}$ and $\mathcal{W} = \{\varpi \in \mathcal{B} | \varpi \geq 0\}$. Let $\Xi = \{0, 1, 2, 7\}$ and $\Gamma = \{0, \frac{1}{4}, \frac{1}{2}, 3\}$ be equipped with $\varphi(\varpi, \eta) = |\varpi - \eta|$ for all $\varpi \in \Xi$ and $\eta \in \Gamma$. Then, (Ξ, Γ, φ) is a complete CBMS. Define $\Omega : \Xi \cup \Gamma \rightleftarrows \Xi \cup \Gamma$ by*

$$\Omega(\varpi) = \begin{cases} \frac{1}{4}, & \text{if } \varpi \in \{2, 7\}, \\ 0, & \text{if } \varpi \in \{0, \frac{1}{4}, \frac{1}{2}, 1, 3\}, \end{cases}$$

for all $\varpi \in \Xi \cup \Gamma$. Let $\varpi \in \Xi$ and $\eta \in \Gamma$, then we can easily get

$$\varphi(\Omega\varpi, \Omega\eta) \leq \frac{1}{2}\varphi(\varpi, \eta).$$

Theorem 3.2's axioms are thus satisfied, and Ω has a UFP $\varpi = 0$.

At last, we formulate a theorem that extends Kannan's fixed point result [26].

Theorem 3.3. *Let $\Omega : (\Xi, \Gamma, \varphi) \leftrightarrows (\Xi, \Gamma, \varphi)$, where (Ξ, Γ, φ) is a complete CBMS, $\mathcal{W}$ is a normal cone with normal constant N and let $\alpha \in (0, \frac{1}{2})$ such that $N\alpha < 1$ and the inequality*

$$\varphi(\Omega\eta, \Omega\varpi) \leq \alpha(\varphi(\varpi, \Omega\varpi) + \varphi(\Omega\eta, \eta)),$$

holds for all $\varpi \in \Xi$ and $\eta \in \Gamma$. Then the function $\Omega : \Xi \cup \Gamma \to \Xi \cup \Gamma$ has a unique fixed point.

Proof. Let $\varpi_0 \in \Xi$. For each $\nu \geq 0$, we define $\eta_\nu = \Omega\varpi_\nu$ and $\varpi_{\nu+1} = \Omega\eta_\nu$. Then we have,

$$\begin{aligned}\varphi(\varpi_\nu, \eta_\nu) &= \varphi(\Omega\eta_{\nu-1}, \Omega\varpi_\nu) \\ &\leq \alpha(\varphi(\varpi_\nu, \Omega\varpi_\nu) + \varphi(\Omega\eta_{\nu-1}, \eta_{\nu-1})) \\ &= \alpha(\varphi(\varpi_\nu, \eta_\nu) + \varphi(\varpi_\nu, \eta_{\nu-1}))\end{aligned}$$

for all integers $\nu \geq 1$. Then,

$$\varphi(\varpi_\nu, \eta_\nu) \leq \frac{\alpha}{1-\alpha}\varphi(\varpi_\nu, \eta_{\nu-1}).$$

Also,

$$\begin{aligned}\varphi(\varpi_\nu,\eta_{\nu-1}) &= \varphi(\Omega\eta_{\nu-1},\Omega\varpi_{\nu-1})\\ &\le \alpha(\varphi(\varpi_{\nu-1},\Omega\varpi_{\nu-1})+\varphi(\Omega\eta_{\nu-1},\eta_{\nu-1}))\\ &= \alpha(\varphi(\varpi_{\nu-1},\eta_{\nu-1})+\varphi(\varpi_\nu,\eta_{\nu-1})),\end{aligned}$$

so that

$$\varphi(\varpi_\nu,\eta_{\nu-1}) \le \frac{\alpha}{1-\alpha}\varphi(\varpi_{\nu-1},\eta_{\nu-1}).$$

If we say $\lambda := \frac{\alpha}{1-\alpha}$, then we have $\lambda \in (0,1)$ since $\alpha \in (0,\frac{1}{2})$. Now

$$\begin{aligned}\varphi(\varpi_\nu,\eta_\nu) &\le \lambda^{2\nu}\varphi(\varpi_0,\eta_0),\\ \varphi(\varpi_\nu,\eta_{\nu-1}) &\le \lambda^{2\nu-1}\varphi(\varpi_0,\eta_0).\end{aligned}$$

For each $\mathfrak{m} > \nu$,

$$\begin{aligned}\varphi(\varpi_\nu,\eta_\mathfrak{m}) &\le \varphi(\varpi_\nu,\eta_\nu)+\varphi(\varpi_{\nu+1},\eta_\nu)+\varphi(\varpi_{\nu+1},\eta_\mathfrak{m})\\ &\le (\lambda^{2\nu}+\lambda^{2\nu+1})\varphi(\varpi_0,\eta_0)+\varphi(\varpi_{\nu+1},\eta_\mathfrak{m})\\ &\vdots\\ &\le (\lambda^{2\nu}+\lambda^{2\nu+1}+\cdots+\lambda^{2\mathfrak{m}-1}+\lambda^{2\mathfrak{m}})\varphi(\varpi_0,\eta_0)\\ &= (\lambda^{2\nu}+\cdots+\lambda^{2\mathfrak{m}})\varphi(\varpi_0,\eta_0)\\ &\le \frac{\lambda^{2\nu}}{1-\lambda}\varphi(\varpi_0,\eta_0).\end{aligned}$$

As $\nu,\mathfrak{m}\to\infty$, we get,

$$\varphi(\varpi_\nu,\eta_\mathfrak{m}) \to 0.$$

Similarly, for each $\mathfrak{m} < \nu$,

$$\begin{aligned}\varphi(\varpi_\nu,\eta_\mathfrak{m}) &\le \varphi(\varpi_\nu,\eta_{\mathfrak{m}+1})+\varphi(\varpi_{\mathfrak{m}+1},\eta_{\mathfrak{m}+1})+\varphi(\varpi_{\mathfrak{m}+1},\eta_\mathfrak{m})\\ &\le \varphi(\varpi_\nu,\eta_{\mathfrak{m}+1})+(\lambda^{2\mathfrak{m}+1}+\lambda^{2\mathfrak{m}+2})\varphi(\varpi_0,\eta_0)\\ &\vdots\\ &\le (\lambda^{2\mathfrak{m}+1}+\lambda^{2\mathfrak{m}+2}+\cdots+\lambda^{2\nu-2})\varphi(\varpi_0,\eta_0)+\varphi(\varpi_\nu,\eta_{\nu-1})\\ &\le (\lambda^{2\mathfrak{m}+1}+\lambda^{2\mathfrak{m}+2}+\cdots+\lambda^{2\nu-1})\varphi(\varpi_0,\eta_0)\\ &\le \frac{\lambda^{2\mathfrak{m}+1}}{1-\lambda}\varphi(\varpi_0,\eta_0).\end{aligned}$$

As $\nu,\mathfrak{m}\to\infty$, we get,

$$\varphi(\varpi_\nu,\eta_\mathfrak{m}) \to 0.$$

Therefore, $(\{\varpi_\nu\},\{\eta_m\})$ is a Cauchy bisequence. Since (Ξ,Γ,φ) is complete, $\{\varpi_\nu\}\to\aleph$ and $\{\eta_m\}\to\aleph$, where $\aleph\in\Xi\cup\Gamma$. We know that

$$\{\Omega\varpi_\nu\}=\{\eta_\nu\}\to\aleph.$$

On the other hand,

$$\begin{aligned}&\varphi(\Omega\aleph,\Omega\varpi_\nu)\\ &\leq\alpha(\varphi(\varpi_\nu,\Omega\varpi_\nu)+\varphi(\Omega\aleph,\aleph))=\alpha(\varphi(\varpi_\nu,\eta_\nu)+\varphi(\Omega\aleph,\aleph))\\ &\leq\alpha(\lambda^{2\nu}\varphi(\varpi_0,\eta_0)+\varphi(\Omega\aleph,\aleph))\end{aligned}$$

which implies that

$$\|\varphi(\Omega\aleph,\Omega\varpi_\nu)\|\leq N\alpha(\lambda^{2\nu}\|\varphi(\varpi_\nu,\eta_\nu)\|+\|\varphi(\Omega\aleph,\aleph)\|).$$

Taking $\nu\to\infty$ we deduce that $\|\varphi(\Omega\aleph,\aleph)\|=0$. Hence, $\Omega\aleph=\aleph$. Let ℓ be another fixed point of Ω. Then $\Omega\ell=\ell$ implies that $\ell\in\Xi\cap\Gamma$. Then,

$$\begin{aligned}\varphi(\aleph,\ell)=\varphi(\Omega\aleph,\Omega\ell)&\leq\alpha(\varphi(\ell,\Omega\ell)+\varphi(\Omega\aleph,\aleph))\\ &=\alpha(\varphi(\aleph,\aleph)+\varphi(\ell,\ell))=0.\end{aligned}$$

Consequently, $\aleph=\ell$. □

We now demonstrate our first common fixed point (CFP) finding.

Theorem 3.4. *Let (Ξ,Γ,φ) be a complete CBMS, $\mathcal{W}$ be a normal cone with normal constant N and let $\Omega,\mathcal{S}:(\Xi,\Gamma,\varphi)\rightleftarrows(\Xi,\Gamma,\varphi)$ be contravariant continuous mappings satisfying*

$$\varphi(\mathcal{S}\eta,\Omega\varpi)\leq\alpha\frac{\varphi(\varpi,\Omega\varpi)\varphi(\mathcal{S}\eta,\eta)}{\varphi(\varpi,\eta)}+\gamma\varphi(\varpi,\eta)+\imath[\varphi(\varpi,\Omega\varpi)+\varphi(\mathcal{S}\eta,\eta)],\tag{3.1}$$

for all $(\varpi,\eta)\in\Xi\times\Gamma$, with $\varpi\neq\eta$ and $0\leq\alpha+\gamma+2\imath<1$. Also, assume that $N\gamma<1$. Then $\Omega,\mathcal{S}:\Xi\cup\Gamma\to\Xi\cup\Gamma$ have a unique CFP.

Proof. Let $\varpi_0\in\Xi$. Then for each $\nu\in\mathbb{N}\cup\{0\}$, we define,

$$\Omega\varpi_\nu=\eta_\nu,\ \mathcal{S}\eta_\nu=\varpi_{\nu+1}.$$

Now by (3.1), we get,

$$\begin{aligned}\varphi(\varpi_{\nu+1},\eta_{\nu+1})=&\varphi(\mathcal{S}\eta_\nu,\Omega\varpi_{\nu+1})\\ \leq&\alpha\frac{\varphi(\varpi_{\nu+1},\Omega\varpi_{\nu+1})\varphi(\mathcal{S}\eta_\nu,\eta_\nu)}{\varphi(\varpi_{\nu+1},\eta_\nu)}+\gamma\varphi(\varpi_{\nu+1},\eta_\nu)\\ &+\imath[\varphi(\varpi_{\nu+1},\Omega\varpi_{\nu+1})+\varphi(\mathcal{S}\eta_\nu,\eta_\nu)]\\ =&\alpha\frac{\varphi(\varpi_{\nu+1},\eta_{\nu+1})\varphi(\varpi_{\nu+1},\eta_\nu)}{\varphi(\varpi_{\nu+1},\eta_\nu)}+\gamma\varphi(\varpi_{\nu+1},\eta_\nu)\\ &+\imath[\varphi(\varpi_{\nu+1},\eta_{\nu+1})+\varphi(\varpi_{\nu+1},\eta_\nu)]\\ =&\alpha\varphi(\varpi_{\nu+1},\eta_{\nu+1})+\gamma\varphi(\varpi_{\nu+1},\eta_\nu)+\imath\varphi(\varpi_{\nu+1},\eta_{\nu+1})\end{aligned}$$

$$+ \imath\varphi(\varpi_{\nu+1}, \eta_\nu),$$

which implies that

$$\varphi(\varpi_{\nu+1}, \eta_{\nu+1}) \leq \frac{\gamma + \imath}{1 - \alpha - \imath}\varphi(\varpi_{\nu+1}, \eta_\nu). \tag{3.2}$$

Also, we have

$$\begin{aligned}
\varphi(\varpi_{\nu+1}, \eta_\nu) =& \varphi(\mathcal{S}\eta_\nu, \Omega\varpi_\nu) \\
&\leq \alpha \frac{\varphi(\varpi_\nu, \Omega\varpi_\nu)\varphi(\mathcal{S}\eta_\nu, \eta_\nu)}{\varphi(\varpi_\nu, \eta_\nu)} + \gamma\varphi(\varpi_\nu, \eta_\nu) \\
&\quad + \imath[\varphi(\varpi_\nu, \Omega\varpi_\nu) + \varphi(\mathcal{S}\eta_\nu, \eta_\nu)] \\
&= \alpha \frac{\varphi(\varpi_\nu, \eta_\nu)\varphi(\varpi_{\nu+1}, \eta_\nu)}{\varphi(\varpi_\nu, \eta_\nu)} + \gamma\varphi(\varpi_\nu, \eta_\nu) \\
&\quad + \imath[\varphi(\varpi_\nu, \eta_\nu) + \varphi(\varpi_{\nu+1}, \eta_\nu)] \\
&= \alpha\varphi(\varpi_{\nu+1}, \eta_\nu) + \gamma\varphi(\varpi_\nu, \eta_\nu) + \imath\varphi(\varpi_\nu, \eta_\nu) \\
&\quad + \imath\varphi(\varpi_{\nu+1}, \eta_\nu),
\end{aligned}$$

which implies that

$$\varphi(\varpi_{\nu+1}, \eta_\nu) \leq \frac{\gamma + \imath}{1 - \alpha - \imath}\varphi(\varpi_\nu, \eta_\nu). \tag{3.3}$$

Since $\alpha + \gamma + 2\imath \in [0, 1)$, so, $\frac{\gamma+\imath}{1-\alpha-\imath} = \wp \in [0, 1)$. Hence, from (3.2) and (3.3), we can get that for any $\nu \in \mathbb{N}$,

$$\varphi(\varpi_{\nu+1}, \eta_{\nu+1}) \leq \wp^{2\nu+2}\varphi(\varpi_0, \eta_0), \quad \varphi(\varpi_{\nu+1}, \eta_\nu) \leq \wp^{2\nu+1}\varphi(\varpi_0, \eta_0).$$

For all $\mathfrak{m}, \nu \in \mathbb{N}$, we have two cases:
Case 1. If $\mathfrak{m} > \nu$,

$$\begin{aligned}
\varphi(\varpi_\nu, \eta_\mathfrak{m}) \leq& \varphi(\varpi_\nu, \eta_\nu) + \varphi(\varpi_{\nu+1}, \eta_\nu) + \varphi(\varpi_{\nu+1}, \eta_\mathfrak{m}) \\
\leq& \wp^{2\nu}\varphi(\varpi_0, \eta_0) + \wp^{2\nu+1}\varphi(\varpi_0, \eta_0) + \varphi(\varpi_{\nu+1}, \eta_\mathfrak{m}) \\
\leq& (\wp^{2\nu} + \wp^{2\nu+1})\varphi(\varpi_0, \eta_0) + \varphi(\varpi_{\nu+1}, \eta_{\nu+1}) \\
&+ \varphi(\varpi_{\nu+2}, \eta_{\nu+1}) + \varphi(\varpi_{\nu+2}, \eta_\mathfrak{m}) \\
\leq& (\wp^{2\nu} + \wp^{2\nu+1})\varphi(\varpi_0, \eta_0) + \wp^{2\nu+2}\varphi(\varpi_0, \eta_0) \\
&+ \wp^{2\nu+3}\varphi(\varpi_0, \eta_0) + \varphi(\varpi_{\nu+2}, \eta_\mathfrak{m}) \\
&\vdots \\
\leq& \wp^{2\nu}(1 + \wp + \wp^2 + \wp^3 + ...)\varphi(\varpi_0, \eta_0) \\
=& \wp^{2\nu}(\frac{1}{1 - \wp})\varphi(\varpi_0, \eta_0).
\end{aligned}$$

Since $\wp < 1$, $\lim_{\nu,\mathfrak{m}\to\infty} \varphi(\varpi_\nu, \eta_\mathfrak{m}) = 0$.
Case 2. If $\mathfrak{m} < \nu$, we have,

$$\begin{aligned}\varphi(\varpi_\nu, \eta_\mathfrak{m}) \leq & \varphi(\varpi_{\mathfrak{m}+1}, \eta_\mathfrak{m}) + \varphi(\varpi_{\mathfrak{m}+1}, \eta_{\mathfrak{m}+1}) + \varphi(\varpi_\nu, \eta_{\mathfrak{m}+1}) \\ \leq & \wp^{2\mathfrak{m}+1}\varphi(\varpi_0, \eta_0) + \wp^{2\mathfrak{m}+2}\varphi(\varpi_0, \eta_0) + \varphi(\varpi_\nu, \eta_{\mathfrak{m}+1}) \\ \leq & (\wp^{2\mathfrak{m}+1} + \wp^{2\mathfrak{m}+2})\varphi(\varpi_0, \eta_0) + \varphi(\varpi_{\mathfrak{m}+2}, \eta_{\mathfrak{m}+1}) \\ & + \varphi(\varpi_{\mathfrak{m}+2}, \eta_{\mathfrak{m}+2}) + \varphi(\varpi_\nu, \eta_{\mathfrak{m}+2}) \\ & \vdots \\ \leq & (\wp^{2\mathfrak{m}+1} + \wp^{2\mathfrak{m}+2} + \wp^{2\mathfrak{m}+3} + \wp^{2\mathfrak{m}+4} + \ldots)\varphi(\varpi_0, \eta_0) \\ = & \wp^{2\mathfrak{m}+1}(\frac{1}{1-\wp})\varphi(\varpi_0, \eta_0).\end{aligned}$$

Again, since $\wp < 1$, $\lim_{\nu,\mathfrak{m}\to\infty} \varphi(\varpi_\nu, \eta_\mathfrak{m}) = 0$.
Therefore, $(\{\varpi_\nu\}, \{\eta_m\})$ is a Cauchy bisequence. Since (Ξ, Γ, φ) is complete, $\{\varpi_\nu\} \to \varpi^*$ and $\{\eta_\mathfrak{m}\} \to \varpi^*$, where $\varpi^* \in \Xi \cup \Gamma$. Also, $\{\mathcal{S}(\eta_\nu)\} = \{\varpi_{\nu+1}\} \to \varpi^* \in \Xi \cap \Gamma$. Since $\mathcal{S}$ is continuous, $\mathcal{S}(\eta_\nu) \to \mathcal{S}\varpi^*$. Therefore, $\mathcal{S}\varpi^* = \varpi^*$.
Similarly, $\{\Omega(\varpi_\nu)\} = \{\eta_\nu\} \to \varpi^* \in \Xi \cap \Gamma$. Now, the continuity of Ω implies that $\{\Omega(\varpi_\nu)\} \to \Omega\varpi^*$. Therefore, $\Omega\varpi^* = \varpi^*$.

Let $\eta^* \in \Xi \cap \Gamma$ such that $\mathcal{S}\eta^* = \Omega\eta^* = \eta^* \in \Xi \cap \Gamma$. Then, we get,

$$\begin{aligned}\varphi(\eta^*, \varpi^*) = & \varphi(\mathcal{S}\eta^*, \Omega\varpi^*) \\ \leq & \alpha\frac{\varphi(\varpi^*, \Omega\varpi^*)\varphi(\mathcal{S}\eta^*, \eta^*)}{\varphi(\varpi^*, \eta^*)} + \gamma\varphi(\varpi^*, \eta^*) + \imath[\varphi(\varpi^*, \Omega\varpi^*) + \varphi(\mathcal{S}\eta^*, \eta^*)] \\ = & \alpha\frac{\varphi(\varpi^*, \varpi^*)\varphi(\eta^*, \eta^*)}{\varphi(\varpi^*, \eta^*)} + \gamma\varphi(\varpi^*, \eta^*) + \imath[\varphi(\varpi^*, \varpi^*) + \varphi(\eta^*, \eta^*)].\end{aligned}$$

Therefore, $\|\varphi(\eta^*, \varpi^*)\| \leq N\gamma\|\varphi(\varpi^*, \eta^*)\|$. Hence, $\varpi^* = \eta^*$. □

Now, we present our second CFP result.

Theorem 3.5. *Let (Ξ, Γ, φ) be a complete CBMS, $\mathcal{W}$ be a normal cone with normal constant N and $\Omega, \mathcal{S} : (\Xi, \Gamma, \varphi) \rightleftarrows (\Xi, \Gamma, \varphi)$ be contravariant continuous mappings satisfying*

$$\varphi(\mathcal{S}\eta, \Omega\varpi) \leq \alpha\frac{\varphi(\varpi, \Omega\varpi)\varphi(\varpi, \mathcal{S}\eta) + \varphi(\mathcal{S}\eta, \eta)\varphi(\eta, \Omega\varpi)}{\varphi(\varpi, \mathcal{S}\eta) + \varphi(\eta, \Omega\varpi)}, \tag{3.4}$$

for all $(\varpi, \eta) \in \Xi \times \Gamma$ such that $\varpi \neq \mathcal{S}\eta$ or $\eta \neq \Omega\varpi$ and $0 < \alpha < 1$. Then $\Omega, \mathcal{S} : \Xi \cup \Gamma \to \Xi \cup \Gamma$ have a unique CFP.

Proof. Let $\varpi_0 \in \Xi$. Then for each $\nu \in \mathbb{N} \cup \{0\}$, we define,

$$\Omega\varpi_\nu = \eta_\nu, \mathcal{S}\eta_\nu = \varpi_{\nu+1}.$$

Now by (3.4), we get,

$$\begin{aligned}\varphi(\varpi_{\nu+1},\eta_{\nu+1}) =&\varphi(\mathcal{S}\eta_\nu,\Omega\varpi_{\nu+1})\\ \leq&\alpha\frac{\varphi(\varpi_{\nu+1},\Omega\varpi_{\nu+1})\varphi(\varpi_{\nu+1},\mathcal{S}\eta_\nu)+\varphi(\mathcal{S}\eta_\nu,\eta_\nu)\varphi(\eta_\nu,\Omega\varpi_{\nu+1})}{\varphi(\varpi_{\nu+1},\mathcal{S}\eta_\nu)+\varphi(\eta_\nu,\Omega\varpi_{\nu+1})}\\ =&\alpha\frac{\varphi(\varpi_{\nu+1},\eta_{\nu+1})\varphi(\varpi_{\nu+1},\varpi_{\nu+1})+\varphi(\varpi_{\nu+1},\eta_\nu)\varphi(\eta_\nu,\eta_{\nu+1})}{\varphi(\varpi_{\nu+1},\varpi_{\nu+1})+\varphi(\eta_\nu,\eta_{\nu+1})}\\ =&\alpha\varphi(\varpi_{\nu+1},\eta_\nu),\end{aligned}$$

which implies that

$$\varphi(\varpi_{\nu+1},\eta_{\nu+1})\leq\alpha\varphi(\varpi_{\nu+1},\eta_\nu). \tag{3.5}$$

Also, we have

$$\begin{aligned}\varphi(\varpi_{\nu+1},\eta_\nu) =&\varphi(\mathcal{S}\eta_\nu,\Omega\varpi_\nu)\\ \leq&\alpha\frac{\varphi(\varpi_\nu,\Omega\varpi_\nu)\varphi(\varpi_\nu,\mathcal{S}\eta_\nu)+\varphi(\mathcal{S}\eta_\nu,\eta_\nu)\varphi(\eta_\nu,\Omega\varpi_\nu)}{\varphi(\varpi_\nu,\mathcal{S}\eta_\nu)+\varphi(\eta_\nu,\Omega\varpi_\nu)}\\ =&\alpha\frac{\varphi(\varpi_\nu,\eta_\nu)\varphi(\varpi_\nu,\varpi_{\nu+1})+\varphi(\varpi_{\nu+1},\eta_\nu)\varphi(\eta_\nu,\eta_\nu)}{\varphi(\varpi_\nu,\varpi_{\nu+1})+\varphi(\eta_\nu,\eta_\nu)}\\ =&\alpha\varphi(\varpi_\nu,\eta_\nu),\end{aligned}$$

which implies that

$$\varphi(\varpi_{\nu+1},\eta_\nu)\leq\alpha\varphi(\varpi_\nu,\eta_\nu). \tag{3.6}$$

Hence, from the previous two inequalities (3.5) and (3.6), we can get that for any $\nu\in\mathbb{N}$,

$$\varphi(\varpi_{\nu+1},\eta_{\nu+1})\leq\alpha^{2\nu+2}\varphi(\varpi_0,\eta_0),\quad \varphi(\varpi_{\nu+1},\eta_\nu)\leq\alpha^{2\nu+1}\varphi(\varpi_0,\eta_0).$$

For all $\mathfrak{m},\nu\in\mathbb{N}$, we have two cases:

Case 1. If $\mathfrak{m}>\nu$,

$$\begin{aligned}\varphi(\varpi_\nu,\eta_\mathfrak{m}) \leq&\varphi(\varpi_\nu,\eta_\nu)+\varphi(\varpi_{\nu+1},\eta_\nu)+\varphi(\varpi_{\nu+1},\eta_\mathfrak{m})\\ \leq&\alpha^{2\nu}\varphi(\varpi_0,\eta_0)+\alpha^{2\nu+1}\varphi(\varpi_0,\eta_0)+\varphi(\varpi_{\nu+1},\eta_\mathfrak{m})\\ \leq&(\alpha^{2\nu}+\alpha^{2\nu+1})\varphi(\varpi_0,\eta_0)+\varphi(\varpi_{\nu+1},\eta_{\nu+1})\\ &+\varphi(\varpi_{\nu+2},\eta_{\nu+1})+\varphi(\varpi_{\nu+2},\eta_\mathfrak{m})\\ \leq&(\alpha^{2\nu}+\alpha^{2\nu+1})\varphi(\varpi_0,\eta_0)+\alpha^{2\nu+2}\varphi(\varpi_0,\eta_0)\\ &+\alpha^{2\nu+3}\varphi(\varpi_0,\eta_0)+\varphi(\varpi_{\nu+2},\eta_\mathfrak{m})\\ &\vdots\\ \leq&(\alpha^{2\nu}+\alpha^{2\nu+1}+\alpha^{2\nu+2}+\alpha^{2\nu+3}+\cdots)\varphi(\varpi_0,\eta_0)\\ =&\alpha^{2\nu}(\frac{1}{1-\alpha})\varphi(\varpi_0,\eta_0).\end{aligned}$$

Since $\alpha < 1$, $\lim_{\nu,\mathfrak{m}\to\infty} \varphi(\varpi_\nu, \eta_\mathfrak{m}) = 0$.
Case . If $\mathfrak{m} < \nu$,

$$\begin{aligned}\varphi(\varpi_\nu, \eta_\mathfrak{m}) \leq& \varphi(\varpi_{\mathfrak{m}+1}, \eta_\mathfrak{m}) + \varphi(\varpi_{\mathfrak{m}+1}, \eta_{\mathfrak{m}+1}) + \varphi(\varpi_\nu, \eta_{\mathfrak{m}+1})\\ \leq& \alpha^{2\mathfrak{m}+1}\varphi(\varpi_0, \eta_0) + \alpha^{2\mathfrak{m}+2}\varphi(\varpi_0, \eta_0) + \varphi(\varpi_\nu, \eta_{\mathfrak{m}+1})\\ \leq& (\alpha^{2\mathfrak{m}+1} + \alpha^{2\mathfrak{m}+2})\varphi(\varpi_0, \eta_0) + \varphi(\varpi_{\mathfrak{m}+2}, \eta_{\mathfrak{m}+1})\\ &+ \varphi(\varpi_{\mathfrak{m}+2}, \eta_{\mathfrak{m}+2}) + \varphi(\varpi_\nu, \eta_{\mathfrak{m}+2})\\ &\vdots\\ \leq& (\alpha^{2\mathfrak{m}+1} + \alpha^{2\mathfrak{m}+2} + \alpha^{2\mathfrak{m}+3} + \alpha^{2\mathfrak{m}+4} + ...)\varphi(\varpi_0, \eta_0)\\ =& \wp^{2\mathfrak{m}+1}(\frac{1}{1-\wp})\varphi(\varpi_0, \eta_0).\end{aligned}$$

Again, since $\alpha < 1$, $\lim_{\nu,\mathfrak{m}\to\infty} \varphi(\varpi_\nu, \eta_\mathfrak{m}) = 0$.
Therefore, $(\{\varpi_\nu\}, \{\eta_m\})$ is a Cauchy bisequence. Since (Ξ, Γ, φ) is complete, $\{\varpi_\nu\} \to \varpi^*$ and $\{\eta_\mathfrak{m}\} \to \varpi^*$, where $\varpi^* \in \Xi \cup \Gamma$. Also, $\{\mathcal{S}(\eta_\nu)\} = \{\varpi_{\nu+1}\} \to \varpi^* \in \Xi \cap \Gamma$. Since $\mathcal{S}$ is continuous, $\mathcal{S}(\eta_\nu) \to \mathcal{S}\varpi^*$. Therefore, $\mathcal{S}\varpi^* = \varpi^*$.
Similarly, $\{\Omega(\varpi_\nu)\} = \{\eta_\nu\} \to \varpi^* \in \Xi \cap \Gamma$. Now, the continuity of Ω implies that $\{\Omega(\varpi_\nu)\} \to \Omega\varpi^*$. Therefore, $\Omega\varpi^* = \varpi^*$.

Let $\eta^* \in \Xi \cap \Gamma$ such that $\mathcal{S}\eta^* = \Omega\eta^* = \eta^* \in \Xi \cap \Gamma$. Then, we get,

$$\begin{aligned}\varphi(\eta^*, \varpi^*) =& \varphi(\mathcal{S}\eta^*, \Omega\varpi^*)\\ \leq& \alpha\frac{\varphi(\varpi^*, \Omega\varpi^*)\varphi(\varpi^*, \mathcal{S}\eta^*) + \varphi(\mathcal{S}\eta^*, \eta^*)\varphi(\eta^*, \Omega\varpi^*)}{\varphi(\varpi^*, \mathcal{S}\eta^*) + \varphi(\eta^*, \Omega\varpi^*)}\\ =& \alpha\frac{\varphi(\varpi^*, \varpi^*)\varphi(\varpi^*, \eta^*) + \varphi(\eta^*, \eta^*)\varphi(\eta^*, \varpi^*)}{\varphi(\varpi^*, \eta^*) + \varphi(\eta^*, \varpi^*)}\\ =& 0.\end{aligned}$$

Therefore, $\varpi^* = \eta^*$. □

4. Applications

4.1. Application to integral equations. As an application of Theorem 3.1, we examine the existence and uniqueness of a solution to an integral equation in this section.

Theorem 4.1. *Let us consider the integral equation*

$$\varpi(\vartheta) = \mathfrak{b}(\vartheta) + \int_{\Lambda_1\cup\Lambda_2} \mathcal{G}(\vartheta, \psi, \varpi(\psi))d\psi, \ \vartheta \in \Lambda_1 \cup \Lambda_2,$$

where Λ_1, Λ_2 *is a partition of* $[0.1]$. *Suppose*

(1) $\mathcal{G} : ([\Lambda_1 \cup \Lambda_2]^2) \times [0, \infty) \to [0, \infty)$ *and* $b \in C(\Lambda_1 \cup \Lambda_2)$,

(2) *there is a continuous function* $\theta : [\Lambda_1 \cup \Lambda_2]^2 \to [0,\infty)$ *and* $\lambda \in (0,1)$ *such that*

$$|\mathcal{G}(\vartheta,\psi,\varpi(\psi)) - \mathcal{G}(\vartheta,\psi,\eta(\psi)| \leq \lambda\theta(\vartheta,\psi)(|\varpi(\psi)-\eta(\psi)|,$$

for all $\vartheta,\psi \in \Lambda_1 \cup \Lambda_2$,

(3) $\sup_{\vartheta\in\Lambda_1\cup\Lambda_2} \int_{\Lambda_1\cup\Lambda_2} \theta(\vartheta,\psi)d\psi \leq 1.$

Then the aforementioned integral equation has a unique solution in $C(\Lambda_1) \cup C(\Lambda_2)$.

Proof. Let $\mathcal{B} = \mathbb{R}$ and $\mathcal{W} = \{\varpi \in \mathcal{B} | \varpi \geq 0\}$. Let $\Xi = C(\Lambda_1)$ and $\Gamma = C(\Lambda_2)$.
Consider $\varphi : \Xi\times\Gamma \to \mathcal{B}$ by $\varphi(\varpi,\eta) = \sup_{\vartheta\in\Lambda_1\cup\Lambda_2} |\varpi(\vartheta)-\eta(\vartheta)|$ for all $(\varpi,\eta) \in \Xi\times\Gamma$. Then (Ξ,Γ,φ) is a complete CBMS. Define $\Omega : C(\Lambda_1)\cup C(\Lambda_2) \rightrightarrows C(\Lambda_1)\cup C(\Lambda_2)$ by

$$\Omega(\varpi(\vartheta)) = \mathfrak{b}(\vartheta) + \int_{\Lambda_1\cup\Lambda_2} \mathcal{G}(\vartheta,\psi,\varpi(\psi))d\psi, \ \ \vartheta \in \Lambda_1\cup\Lambda_2.$$

Now,

$$\begin{aligned}
\varphi(\Omega\varpi,\Omega\eta) &= \sup_{\vartheta\in\Lambda_1\cup\Lambda_2} |\Omega\varpi(\vartheta)-\Omega\eta(\vartheta)| \\
&= \sup_{\vartheta\in\Lambda_1\cup\Lambda_2} \Bigg| \mathfrak{b}(\vartheta) + \int_{\Lambda_1\cup\Lambda_2} \mathcal{G}(\vartheta,\psi,\varpi(\psi))d\psi \\
&- \left(\mathfrak{b}(\vartheta) + \int_{\Lambda_1\cup\Lambda_2} \mathcal{G}(\vartheta,\psi,\eta(\psi))d\psi\right)\Bigg| \\
&\leq \sup_{\vartheta\in\Lambda_1\cup\Lambda_2} \int_{\Lambda_1\cup\Lambda_2} |\mathcal{G}(\vartheta,\psi,\varpi(\psi)) - \mathcal{G}(\vartheta,\psi,\eta(\psi))|d\psi \\
&\leq \sup_{\vartheta\in\Lambda_1\cup\Lambda_2} \int_{\Lambda_1\cup\Lambda_2} \lambda\theta(\vartheta,\psi)(|\varpi(\psi)-\eta(\psi)|)d\psi \\
&\leq \lambda(\sup_{\vartheta\in\Lambda_1\cup\Lambda_2} |\varpi(\vartheta)-\eta(\vartheta)|) \sup_{\vartheta\in\Lambda_1\cup\Lambda_2} \int_{\Lambda_1\cup\Lambda_2} \theta(\vartheta,\psi)d\psi \\
&\leq \lambda\varphi(\varpi,\eta).
\end{aligned}$$

Because of this, Theorem 3.1's axioms are all confirmed, and as a result, there is only one solution to the aforementioned integral equation. □

Example 4.1. *Consider the following non-linear integral equation.*

$$\varpi(\vartheta) = |\sin\vartheta| + \frac{1}{13}\int_0^1 \psi\varpi(\psi)d\psi.$$

Then, it has a solution in $\Xi\cup\Gamma$.

Proof. Let $\Omega\colon \Xi\cup\Gamma \to \Xi\cup\Gamma$ be defined by

$$\Omega(\varpi(\vartheta)) = |\sin\vartheta| + \frac{1}{13}\int_0^1 \psi\varpi(\psi)d\psi$$

and set $\mathcal{G}(\vartheta,\psi,\varpi(\psi)) = \frac{1}{13}\psi\varpi(\psi)$ and $\mathcal{G}(\vartheta,\psi,\eta(\psi)) = \frac{1}{13}\psi\eta(\psi)$. Then

$$\begin{aligned}|\mathcal{G}(\vartheta,\psi,\varpi(\psi)) - \mathcal{G}(\vartheta,\psi,\eta(\psi))| &= |\frac{1}{13}\psi\varpi(\psi) - \frac{1}{13}\psi\eta(\psi)| \\ &= \frac{\psi}{13}|\varpi(\psi) - \eta(\psi)| \leq \frac{\psi}{2}|\varpi(\psi) - \eta(\psi)|.\end{aligned}$$

Furthermore, see that $\int_0^1 \psi d\psi = \frac{(1)^2}{2} - \frac{(0)^2}{2} = \frac{1}{2} \leq 1$. Then, it is easy to see that all other conditions of the above application are easy to examine and the above problem has a solution in $\Xi \cup \Gamma$. □

4.2. **Application to fractional differential equations.** We review a number of significant definitions from the theory of fractional calculus. Here, is the Reiman-Liouville fractional derivative of order $\delta > 0$ for a function $\psi \in C[0,1]$:

$$\mathcal{D}^{\delta}\psi(\xi) = \frac{1}{\Gamma(\nu-\delta)}\frac{d^{\nu}}{d\xi^{\nu}}\int_0^{\xi}\frac{\psi(\varrho)d\varrho}{(\xi-\varrho)^{\delta-\nu+1}},$$

provided that the right hand side is pointwise defined on $[0,1]$, with $\nu = [\delta]+1$, where Γ represents the Euler gamma function and $[\delta]$ represents the integer part of the number δ.

Take a look at the following fractional differential equation:

$$\begin{aligned}&{}^{C}\mathcal{D}^{\delta}\psi(\xi) = \mathfrak{f}(\xi,\psi(\xi)), \quad 0 \leq \xi \leq T, \quad 0 < \delta \leq 1; \\ &\psi(0) = 0,\end{aligned} \tag{4.1}$$

where ${}^{C}\mathcal{D}^{\delta}$ denotes the Caputo fractional derivative of order δ, i.e.,

$${}^{C}\mathcal{D}^{\delta}\psi(\xi) = \frac{1}{\Gamma(\nu-\delta)}\int_0^{\zeta}\frac{\psi^{(\nu)}(\varrho)d\varrho}{(\xi-\varrho)^{\delta-\nu+1}},$$

and $\mathfrak{f}$ is a continuous function from $[0,T]\times\mathbb{R}$ to $\mathbb{R}$.

Let $\mathcal{B} = \mathbb{R}$ and $\mathcal{W} = \{\varpi \in \mathcal{B} | \varpi \geq 0\}$.

The set of all continuous functions defined on $[0,T]$ with values in the interval $[0,\infty)$ is denoted by $\Xi = (C([0,T]),[0,\infty))$, and the set of all continuous functions defined on $[0,T]$ with values in the interval $(-\infty,0]$ is denoted by $\Gamma = (C([0,T]),(-\infty,0])$.

Consider $\varphi : \Xi \times \Gamma \to \mathcal{B}$ to be defined by,

$$\varphi(\psi,\psi') = \sup_{\xi\in[0,T]} |\psi(\xi) - \psi'(\xi)|$$

for all $(\psi,\psi') \in \Xi\times\Gamma$. Then (Ξ,Γ,φ) is a complete CBMS.

Theorem 4.2. *Assume that we have the nonlinear fractional differential equation* (4.1). *Suppose that the following conditions are satisfies:*

(1) *there exists $\lambda \in (0,1)$ so that for all $\xi \in [0,1]$ and for all $(\psi,\psi') \in \Xi\times\Gamma$ we have*

$$|\mathfrak{f}(\xi,\psi(\xi)) - \mathfrak{f}(\xi,\psi'(\xi)| \leq \lambda|\psi(\xi) - \psi'(\xi)|;$$

(2)

$$\frac{\lambda T^{\delta}}{\Gamma(\delta+1)} < 1.$$

Then the fractional differential equation (4.1) *has a unique solution in* $\Xi \cup \Gamma$.

Proof. The given fractional differential equation (4.1) is equivalent to the succeeding integral equation (Lemma 1, page 47 of [5])

$$\psi(\xi) = \psi(0) + \frac{1}{\Gamma(\delta)} \int_0^{\xi} (\xi - \varrho)^{\delta-1} \mathfrak{f}(\varrho, \psi(\varrho)) d\varrho.$$

Define $\Omega \colon \Xi \cup \Gamma \to \Xi \cup \Gamma$ by

$$\Omega\psi(\xi) = \psi(0) + \frac{1}{\Gamma(\delta)} \int_0^{t} (\xi - \varrho)^{\delta-1} \mathfrak{f}(\varrho, \psi(\varrho)) d\varrho.$$

Now

$$\begin{aligned}
|\Omega\psi(\xi) - \Omega\psi^{'}(\xi)| &= |\psi(0) - \psi'(0)| + \frac{1}{\Gamma(\delta)} \bigg| \int_0^{\xi} (\xi - \varrho)^{\delta-1} \mathfrak{f}(\varrho, \psi(\varrho)) d\varrho - \\
&\quad \int_0^{\xi} (\xi - \varrho)^{\delta-1} \mathfrak{f}(\varrho, \psi^{'}(\varrho)) d\varrho \bigg| \\
&\leq |\psi(0) - \psi'(0)| + \frac{1}{\Gamma(\delta)} \int_0^{\xi} (\xi - \varrho)^{\delta-1} \bigg| \mathfrak{f}(\varrho, \psi(\varrho)) - \mathfrak{f}(\varrho, \psi^{'}(\varrho)) \bigg| d\varrho \\
&\leq |\psi(0) - \psi'(0)| + \frac{\lambda}{\Gamma(\delta)} \int_0^{\xi} (\xi - \varrho)^{\delta-1} \bigg| \psi(\varrho) - \psi^{'}(\varrho) \bigg| d\varrho \\
&\leq \frac{\lambda T^{\delta}}{\Gamma(\delta+1)} \varphi(\psi, \psi^{'}).
\end{aligned}$$

Taking the supremum on the left side, we get

$$\varphi(\Omega\psi, \Omega\psi^{'}) \leq \lambda' \varphi(\psi, \psi^{'}).$$

Hence, all the axioms of Theorem 3.1 are satisfied and consequently, the fractional differential equation (4.1) has a unique solution. □

5. Conclusion

Cone bipolar metric space, a generalization of several well-known metric structures like metric spaces, cone metric spaces, and bipolar metric spaces, was used in this study. Additionally, we proved several fixed point theorems for the covariant and contravariant contractions in this structure. The shown results extend and generalise some of the well-known results in the literature. Our results are validated by applications we provide in analyzing the existence and uniqueness of solutions to a fractional differential equation and an integral equation. Ishtiaq et al. [27] introduced neutrosophic cone metric spaces and proved some fixed point theorems. It is an interesting open problem to introduce neutrosophic cone bipolar metric spaces. Younis et al. [28] proved some fixed point results in graphical extended b-metric spaces. It is an interesting open problem to introduce graphical cone b-bipolar metric spaces and establish the fixed point results generalising proven results of the past.

Author Contribution: Conceptualisation - RR, GM, BR, Methodology - RR, ZM, GM, Analysis -BR, OAA, KHK, Software - KHK, OAA, Supervision - RR, ZM, Project Management - RR, GM, Writing Orignisal Draft - RR, GM, BR, Writing -Review and Editing - GM, BR, RR, ZM, KHK, OAA

Funding: The authors extend their appreciation to Prince Sattam bin Abdulaziz University for funding this research work through the project number (2025/01/33096).

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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