Int. J. Anal. Appl. (2025), 23:288

International Journal of Analysis and Applications

Theoretic Properties of k^{th} Power Graphs of Finite Groups

Ojonugwa Ejima[1], **Ahmad R. Tasiu**[1], **Kazeem Olalekan Aremu**[1,2,*]

[1]*Department of Mathematics, Usmanu Danfodiyo University Sokoto, PMB 2346, Sokoto State, Nigeria*

[2]*Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, P.O. Box 94 Medunsa 0204, Ga-Rankuwa, Pretoria, South Africa*

[*]*Corresponding author:* aremukazeemolalekan@gmail.com

Abstract. In this paper, we investigate the structural and combinatorial properties of the kth power graph $\Gamma_k(G)$ associated with a finite group G, where $k \geq 2$. The graph $\Gamma_k(G)$ is defined by taking the elements of G as vertices and connecting two distinct vertices x and y by an edge if either $x = y^k$ or $y = x^k$. This construction generalizes the well-studied power graph of a group and provides new insight into the influence of exponentiation on group elements when viewed through graph-theoretical properties. We show that $\Gamma_k(G)$ is a subgraph of the power graph $\mathcal{P}(G)$ and analyze conditions under which $\Gamma_k(G)$ is connected, disconnected, or empty. Depending on the algebraic structure of G and the arithmetic properties of k, we show that $\Gamma_k(G)$ can exhibit a variety of structural forms, including being a tree, a union of disjoint stars, or a complete multipartite graph. For instance, when $G = \mathbb{Z}_n$ and $\gcd(k, n) = 1$, $\Gamma_k(G)$ decomposes into disjoint stars, while for certain non-cyclic groups, the graph becomes multipartite. Additionally, we provide formulas for computing the number of edges in $\Gamma_k(G)$ and discuss how subgroup structure and group automorphisms impact the topology of the graph.

1. Introduction

The concept of the *power graph* was first introduced by Kelarev and Quinn [1,2], who defined it as a directed graph associated with semigroups. Their studies particularly focused on the structure of power graphs in Archimedean semigroups. Building on this foundational work, Chakrabarty, Ghosh, and Sen [3] extended the idea to *undirected power graphs* of semigroups. They provided a classification of semigroups whose power graphs are connected and complete. Notably, they examined the power graph of the multiplicative semigroup $\mathbb{Z}_n$ and showed that the power graph of its subgroup U_n is complete if and only if $n = 1, 2, 4, p$ or $2p$, where p is a Fermat prime.
Further extending the concept, Chakrabarty et al. [3] considered power graphs of *finite groups* and computed the number of edges in such graphs. Their results were also applied to deduce the

Received: Sep. 3, 2025.

2020 *Mathematics Subject Classification.* 83-02, 05C25.

Key words and phrases. kth power graph; power graph; isomorphic graphs; finite groups; graph of finite groups.

https://doi.org/10.28924/2291-8639-23-2025-288

ISSN: 2291-8639

number of vertices in the corresponding graphs. In a related development, Cameron [4] established that if two finite groups G and H have isomorphic undirected power graphs, then their directed power graphs are also isomorphic. However, he also demonstrated that an isomorphism between the undirected power graphs of G and H does not necessarily preserve the orientation of edges, and this correspondence fails in the case of infinite groups.

In a separate study, Cameron and Ghosh [5] proved that non-isomorphic finite groups may still have isomorphic power graphs, although for finite abelian groups, isomorphic power graphs imply group isomorphism. They also showed that among finite groups, only the Klein four-group shares its power graph with its automorphism group. Mehranian, Gholami, and Ashrafi [6] explored power graphs using the Γ-join concept introduced by Cardoso et al. [7]. They obtained structural results for power graphs of $\mathbb{Z}_n$, showing that

$$P(\mathbb{Z}_n) = K_{\varphi(n)+1} + \Delta_n\left(K_{\varphi(d_1)}, K_{\varphi(d_2)}, \ldots, K_{\varphi(d_p)}\right),$$

where Δ_n is a graph with vertex set $V(\Delta_n) = \{d_i \mid 1 < d_i < n,\ d_i \mid n\}$ and edge set $E(\Delta_n) = \{d_i d_j \mid d_i \mid d_j\}$. They also established that

$$\mathrm{Aut}(P(\mathbb{Z}_n)) \cong S_{\varphi(n)+1} \times \prod_{1<d<n,\ d|n} S_{\varphi(d)}.$$

The growing interest in power graphs has led to numerous further investigations in the literature (see [8–14]). This present work is inspired by Moghaddamfar et al. [15], who introduced the *proper power graph* $P^*(G)$ by removing the identity element from G, and characterized the conditions under which $P^*(G)$ is strongly regular, bipartite, or planar. They also identified classes of finite groups for which $P^*(G)$ contains cut-edges. In a similar spirit, we construct and analyze the *kth power graph* $\Gamma_k(G)$ of a finite group G. This graph is a simple undirected graph where two distinct elements x and y are adjacent if and only if $x = y^k$ or $y = x^k$, for a fixed integer $k \geq 2$. The main goal of this paper is to investigate which structural properties of the standard power graph $P(G)$ are preserved in the kth power graph $\Gamma_k(G)$ of a finite group.

1.1. **Preliminaries.** For the proof of our results and ease of understanding of this paper, we present some useful concepts, definitions, and known theorems. Refer to [16–18] for the definitions of these basic terms and results. The kth power of an element $x \in G$ is the composition of the element x by itself k times, that is, $x^k = x_1 x_2 \cdots x_k$. A *finite group* G is a group that has a finite number of elements; the number of these elements in G is called the *order* or the *cardinality* of the group. Let $(G_1, \times)$ and $(G_2, *)$ be groups. A function $\rho : G_1 \to G_2$ is a *homomorphism* if $\rho(x \times y) = \rho(x) * \rho(y)$ for all $x, y \in G_1$. In this case, the groups G_1 and G_2 are said to be *homomorphic groups*. Furthermore, if ρ is one-to-one, then ρ is called a *monomorphism*; if it is onto, then ρ is an *epimorphism*; and if the function is both one-to-one and onto, then it is called an *isomorphism*, and the corresponding groups are said to be *isomorphic groups* A graph Γ is a combinatorial structure formed by a set of vertices V and a set of edges E. The cardinality of $V(\Gamma)$ is called the *order* of Γ while the cardinality of $E(\Gamma)$ is called the *size* of Γ. The *degree* of a vertex x in a graph Γ, denoted by $\delta(x)$, is the number of edges that are incident to it. A graph Γ is said to be *planar* if there exists some geometric representation of Γ which can be drawn on a plane such that its edges intersect only at their endpoints. A *loop*, on the other hand, is an edge that joins a vertex to itself. A *simple connected graph* is an undirected graph without any loops or multiple edges. In graph theory, a *Hamiltonian path* is a path that contains each vertex exactly once. A *Hamiltonian circuit* is a circuit that contains each vertex exactly once, except for the first vertex which is also the last vertex. A simple graph Γ is called *bipartite* if its vertex set can be partitioned into two disjoint subsets $V = V_1 \cup V_2$, such that every edge e has the form $e = (a, b)$ where $a \in V_1$ and $b \in V_2$, i.e., no two vertices within the same subset V_1 or V_2 are adjacent. A *complete bipartite graph* is a bipartite graph in which every vertex in V_1 is adjacent to every vertex in V_2. The *diameter* of a graph is defined as the maximum distance $d(a, b)$ over all pairs of vertices a and b in the graph. A *star graph* is a tree consisting of one central vertex that is adjacent to all other vertices.

2. Main Results

In this section, we study the kth power graph, some of its basic properties, and some properties of power graphs preserved in kth power graphs of finite groups. Note that in this article, all graphs considered are simple; hence, they do not have loops. Below, we give the definition of the kth power graph of a finite group G.

Definition 2.1. *Given a finite group G, the kth power graph $\Gamma(G)$ of G has as its vertex set the elements of G, and two distinct vertices x and y are adjacent if and only if $x = y^k$ or $y = x^k$, where $2 \leq k \leq n$ and $n \in \mathbb{Z}_+$.*

Remark 2.1. *As a consequence of Definition 2.1, the 2nd power graph of a finite group G is a simple graph whose vertices are the elements of G, and two distinct vertices x and y are adjacent if and only if $x = y^2$ or $y = x^2$.*

Remark 2.2. *Let G be a finite group. Then the kth power graph of G is a subgraph of the power graph of G.*

Proposition 2.1. *Let G be a group and let x be an element of G such that $x \neq x^{-1}$. Then the girth of the kth power graph $\Gamma(G)$ of G is equal to 3, if the order of x, denoted $o(x)$, is k, where $2 \leq k \leq n$ and $n \in \mathbb{Z}_+$.*

Proof. If $o(x) = k$, then $x^k = e$, where e is the identity element of G. It follows from Definition 2.1 that x is adjacent to e in $\Gamma(G)$. Also, since $x \in G$, it has an inverse $x^{-1} \in G$, and by hypothesis, $x \neq x^{-1}$. Using a known property of group elements, x and x^{-1} have the same order, so $o(x^{-1}) = o(x) = k$. Thus, x^{-1} is also adjacent to e and to x in $\Gamma(G)$. Therefore, the subgraph induced by the set $\{e, x, x^{-1}\}$ forms a triangle, i.e., isomorphic to K_3. Hence, the girth of $\Gamma(G)$ is 3. □

Theorem 2.1. *Let G be a finite group of order n, and let $k > \log_2 n$. Then the identity element $e \in G$ is isolated in $\Gamma_k(G)$.*

Proof. For e to be adjacent to any element x, we must have $x^k = e$, i.e., x is of order dividing k. Since $k > \log_2 n$, and the order of any non-identity element $x \in G$ is at most n, the number of such elements with $x^k = e$ becomes negligible or zero for large k. In fact, no such element exists if $\gcd(k, o(x)) > 1$ does not divide $o(x)$. For sufficiently large k, all $x^k \neq e$, so e is isolated. □

Proposition 2.2. *Let $H \leq G$ be a subgroup of a finite group G. Then $\Gamma_k(H)$ is an induced subgraph of $\Gamma_k(G)$.*

Proof. Since adjacency in $\Gamma_k(G)$ depends solely on the power relation $x = y^k$, and since powers of elements in H remain in H, any such adjacency among elements of H will occur identically in both $\Gamma_k(G)$ and $\Gamma_k(H)$. Thus, $\Gamma_k(H) \subseteq \Gamma_k(G)$ as an induced subgraph. □

Theorem 2.2. *Let $G = \mathbb{Z}_n$. Then $\Gamma_k(G)$ is a union of disjoint stars if and only if $\gcd(k, n) = 1$.*

Proof. If $\gcd(k, n) = 1$, then the map $f : \bar{x} \mapsto \bar{x}^k$ is a permutation of $\mathbb{Z}_n$. Each element has a unique kth root. Hence, for each $x \in G$, there is a unique y such that $y^k = x$, forming a directed tree of depth one centered at x. When converted to an undirected graph, these trees become stars. If $\gcd(k, n) \neq 1$, the map $f(x) = x^k$ is not bijective, and multiple elements may map to the same power, creating overlapping connections, not forming disjoint stars. □

Proposition 2.3. *Let G be a finite group, and let $\Gamma_k(G)$ denote its kth power graph. Then the number of edges is given by:*

$$|E(\Gamma_k(G))| = \frac{1}{2} \cdot |\{(x, y) \in G \times G : x = y^k, x \neq y\}|.$$

Proof. Every adjacency in $\Gamma_k(G)$ comes from either $x = y^k$ or $y = x^k$, for $x \neq y$. Each such pair is counted twice in the set $\{(x, y) : x = y^k\}$, so we divide by 2 to get the number of undirected edges. □

Corollary 2.1. *If the kth power map in G is injective, then $|E(\Gamma_k(G))| = 0$.*

Proof. If $x = y^k$ implies $y = x$, then there are no pairs $x \neq y$ satisfying $x = y^k$. So the edge set is empty. □

Theorem 2.3. *Let G be a finite group and $\phi \in Aut(G)$. Then ϕ is a graph automorphism of $\Gamma_k(G)$.*

Proof. Suppose $x \sim y$ in $\Gamma_k(G)$. Then either $x = y^k$ or $y = x^k$. Since ϕ is an automorphism of the group, it preserves group operations and powers: $\phi(x) = \phi(y^k) = (\phi(y))^k$. Thus, $\phi(x) \sim \phi(y)$. Hence, ϕ induces a graph automorphism of $\Gamma_k(G)$. □

Remark 2.3. *This theorem shows that the automorphism group of G embeds naturally into the automorphism group of $\Gamma_k(G)$.*

Proposition 2.4. *Let $\Gamma_k(G)$ be the kth power graph of finite group with vertex set $G \setminus \{e\}$. Then the number of edges in $\Gamma_k(G)$ is given by*

$$|E(\Gamma_k(G))| = \frac{1}{2} \sum_{y \in G \setminus \{e\}} \delta(y).$$

where

$$\delta(y) = \left| \{x \in G \setminus \{e\} : x = y^k\} \right|.$$

Proof. For each element $y \in G \setminus \{e\}$, the value $\delta(y)$ counts the number of elements $x \in G \setminus \{y\}$ such that $x = y^k$, that is, the number of edges originating from y via the kth power map.

Since the graph is undirected and edges are defined by the relation $x = y^k$ or $y = x^k$, every edge $\{x, y\}$ is counted twice in the total sum—once as $x = y^k$ and once as $y = x^k$. Therefore, the total number of undirected edges is obtained by dividing the total directional counts by 2:

$$|E(\Gamma_k(G))| = \frac{1}{2} \sum_{y \in G \setminus \{e\}} \delta(y).$$

□

Theorem 2.4. *Let G be a finite group. The kth power graph $\Gamma(G)$ is a tree if the following conditions hold:*

(1) $k \notin G$ and all non-identity elements $x \in G$ have order k for some $k \in \mathbb{Z}$.

(2) Every element $x \in G$ is its own inverse, i.e., $x = x^{-1}$.

Proof. Let G be a finite group and $\Gamma(G)$ its kth power graph.
(1) If all non-identity elements have order k and $k \notin G$, then these elements can only be adjacent to the identity element. Since $k \notin G$, it is not in the vertex set and cannot form any additional adjacency. Hence, $\Gamma(G)$ forms a star-like structure, which is a tree.
(2) If every $x \in G$ satisfies $x = x^{-1}$, then the only possible adjacencies (by Definition 2.1) would be x to itself and the identity. But since $\Gamma(G)$ is simple, loops are not allowed, so no edge can connect x to itself. Only adjacency with the identity is allowed, again forming a star, which is a tree. □

Theorem 2.5. *Let G be a finite non-cyclic group. Then the kth power graph $\Gamma(G)$ of G is a complete n-partite graph, for some $n \geq 2$.*

Proof. Suppose G is a finite non-cyclic group. Then G cannot be generated by a single element. Hence, there are at least $n \geq 2$ distinct generating subsets in G, with no kth power connections between elements from different subsets. These subsets form disjoint partitions in $\Gamma(G)$, and by definition of adjacency in power graphs, every element in one partition can be adjacent to elements in others if they satisfy the kth power relation. Hence, $\Gamma(G)$ forms a complete n-partite graph. □

Corollary 2.2. *Let G be a finite non-cyclic group of order $n \geq 2$. The complete n-partite kth power graph $\Gamma(G)$ is determined by the number of conjugacy classes in G.*

Corollary 2.3. *Let $\mathbb{Z}_n$ be the additive group of integers modulo n, and let $\varphi(n)$ be Euler's totient function. Then the kth power graph $\Gamma(\mathbb{Z}_n)$ is a complete $\varphi(n)$-partite graph.*

Proof. From the definition of Euler's totient function $\varphi(n)$, it counts the number of integers k in the range $1 \leq k \leq n$ such that $\gcd(n, k) = 1$. The elements of $\mathbb{Z}_n$ that are relatively prime to n form the group of units $\mathbb{Z}_n^*$. These elements determine $\varphi(n)$ equivalence classes with no internal kth power adjacency, leading to a complete $\varphi(n)$-partite structure in the graph. □

Theorem 2.6. *Let G be a finite cyclic group of order n. The kth power graph $\Gamma(G)$ of G is an empty graph if and only if the power(s) of the generating element $a \in G$ are not equal to k.*

Proof. Let G be a finite cyclic group of order n, generated by a. Suppose the kth power graph $\Gamma(G)$ is empty. Then, by Definition 2.1, no two distinct vertices $x, y \in G$ satisfy $x = y^k$ or $y = x^k$. This implies that no element in G has its kth power equal to another element in G, particularly, none of the powers a^q satisfy $a^q = a^k$. Hence, k is not a power of the generator a. Conversely, suppose that no power of the generator a equals k, i.e., $a^q \neq a^k$ for all $q \in \{1, 2, \dots, n\}$. Then no two distinct elements $x, y \in G$ satisfy the adjacency condition $x = y^k$ or $y = x^k$. Thus, $\Gamma(G)$ contains no edges and is therefore an empty graph. □

Theorem 2.7. *If G is a finite group, then both $\mathcal{P}(G)$ and $\Gamma_k(G)$ are connected graphs.*

Proof. In $\mathcal{P}(G)$, the identity element $e \in G$ is adjacent to every element since any $g \in G$ satisfies $g = e^k$ for $k = 1$, and similarly, $e = g^{\mathrm{ord}(g)}$. Thus, all elements are connected via e. In $\Gamma_k(G)$, for any $x \in G$, if $x = y^k$, then x and y are connected. Since the power map $x \mapsto x^k$ is surjective on many subgroups (especially when $\gcd(k, |G|) = 1$), all elements are still connected through chains involving powers. Hence, $\Gamma_k(G)$ is connected. □

Theorem 2.8. *Let $C \leq G$ be a cyclic subgroup. Then the subgraph of $\mathcal{P}(G)$ induced by C is a complete graph. The same is nearly true in $\Gamma_k(G)$, provided* $\gcd(k, |C|) = 1$.

Proof. In a cyclic subgroup $C = \langle g \rangle$, any element is of the form g^m, so any two elements x, y satisfy $x = y^r$ for some integer r, hence are adjacent in $\mathcal{P}(G)$. For $\Gamma_k(G)$, if $x = g^m$, then $x^k = g^{mk}$. When $\gcd(k, |C|) = 1$, the map $m \mapsto mk \mod |C|$ is a bijection, ensuring connections between many pairs. Thus, $\Gamma_k(G)$ retains a near-complete subgraph. □

Theorem 2.9. *The power graph $\mathcal{P}(G)$ of a finite group is always connected, but $\Gamma_k(G)$ may have more than one component.*

Example 2.1. *Let $G = \mathbb{Z}_8$ and $k = 2$. Then $\mathcal{P}(G)$ is connected. However, in $\Gamma_2(G)$, some elements such as 1 and 3 are not perfect squares modulo 8, and thus may not be connected.*

Proposition 2.5. *The identity element $e \in G$ is a dominating vertex in both $\mathcal{P}(G)$ and $\Gamma_k(G)$.*

Proof. For any $x \in G$, we have $e = x^{\mathrm{ord}(x)}$, and $x = e^1$. So e is adjacent to all elements in both graphs. □

Theorem 2.10. *If G is torsion-free, then both $\mathcal{P}(G)$ and $\Gamma_k(G)$ are sparse and often disconnected.*

Remark 2.4. *If no nontrivial element is a power of another, then edges in both graphs are rare, leading to sparse structures. Thus, this degeneracy is preserved.*

Theorem 2.11. *The power graph $\mathcal{P}(G)$ is always connected, but $\Gamma_k(G)$ may be disconnected.*

Proof. In $\mathcal{P}(G)$, every element $g \in G$ is adjacent to the identity element e, since $g = e^m$ for some m. Thus, all vertices are connected via the identity, making the graph connected. In contrast, $\Gamma_k(G)$ only connects elements via exact kth power relations. Elements that are not kth powers or do not have kth roots in G may become isolated, leading to disconnected components. □

Example 2.2. *Let $G = \mathbb{Z}_6$ and $k = 2$. The element $\bar{3}$ has no square root in $\mathbb{Z}_6$, hence is isolated in $\Gamma_2(G)$, making the graph disconnected.*

Proposition 2.6. *The clique number of $\Gamma_k(G)$ is not necessarily equal to that of $\mathcal{P}(G)$. That is,*

$$\omega(\Gamma_k(G)) < \omega(\mathcal{P}(G)) \text{ may occur.}$$

Proof. In $\mathcal{P}(G)$, the elements of any cyclic subgroup form a clique, since powers of a generator relate all such elements.

However, $\Gamma_k(G)$ contains only edges corresponding to the kth power map. As such, many pairwise connections in $\mathcal{P}(G)$ are missing in $\Gamma_k(G)$, reducing the maximum clique size. □

Example 2.3. *Let $G = \mathbb{Z}_8$. The subgroup $\langle\bar{1}\rangle$ contains all $\bar{1}, \bar{2}, \ldots, \bar{7}$, which form a clique in $\mathcal{P}(G)$. In $\Gamma_2(G)$, only squares such as $\bar{1}^2 = \bar{2}$, $\bar{3}^2 = \bar{1}$, etc., yield adjacencies, significantly reducing the clique size.*

Proposition 2.7. *The edge density of $\Gamma_k(G)$ is strictly less than that of $\mathcal{P}(G)$, in general.*

Proof. This follows from the definitions: $\Gamma_k(G)$ only includes edges derived from kth power relations, whereas $\mathcal{P}(G)$ includes all power relations. Thus, $\Gamma_k(G) \subseteq \mathcal{P}(G)$ as subgraphs (on the same vertex set), with strictly fewer edges unless $k = 1$. □

Proposition 2.8. *The domination number $\gamma(\Gamma_k(G))$ may differ from $\gamma(\mathcal{P}(G))$.*

Proof. In $\mathcal{P}(G)$, a dominating set may consist of group generators or powers reaching all elements. But in $\Gamma_k(G)$, not all elements are reachable via kth power maps. As a result, more vertices may be required to dominate all others. □

Proposition 2.9. *The planarity of $\mathcal{P}(G)$ does not imply planarity of $\Gamma_k(G)$, and vice versa.*

Proof. Since the structure of cycles and adjacency changes significantly in $\Gamma_k(G)$, planarity, which is sensitive to such structure, is not preserved. For instance, reducing edges may remove K_5 or $K_{3,3}$ minors and make a graph planar, or fragmentation may increase the number of subgraphs that interfere with planarity. □

3. Conclusion

In this research, we explored some algebraic properties of finite groups to study the kth power graph $\Gamma_k(G)$ of a finite group G; a graph where two distinct vertices are adjacent if one is the kth power of the other, where $k \geq 2$. we showed that $\Gamma_k(G)$ is a subgraph of the power graph $\mathcal{P}(G)$, and identified conditions under which the graph is connected, disconnected, or even empty. We established that $\Gamma_k(G)$ can be a tree or a union of disjoint stars depending on the nature of G and the value of k. Furthermore, when G is non-cyclic or when $\gcd(k, n) = 1$ for $G = \mathbb{Z}_n$, the resulting graph can be complete multipartite or a union of stars, respectively. We also analyzed the edge count in $\Gamma_k(G)$, providing combinatorial formulas, and examined the influence of automorphisms and subgroup structure on the graph's topology. These results not only generalize known properties of power graphs but also reveal new and rich connections between algebraic and graph-theoretic structures through the kth powers of finite groups.

Acknowledgment: The authors sincerely thank the anonymous reviewers for their careful reading, constructive comments, and fruitful suggestions that substantially improved this article.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] A.V. Kelarev, S.J. Quinn, A Combinatorial Property and Power Graphs of Groups, Contrib. Gen. Algebra 12 (2000), 229–235.

[2] A. Kelarev, S. Quinn, Directed Graphs and Combinatorial Properties of Semigroups, J. Algebra 251 (2002), 16–26. https://doi.org/10.1006/jabr.2001.9128.

[3] I. Chakrabarty, S. Ghosh, M.K. Sen, Undirected Power Graphs of Semigroups, Semigroup Forum 78 (2009), 410–426. https://doi.org/10.1007/s00233-008-9132-y.

[4] P.J. Cameron, The Power Graph of a Finite Group, II, J. Group Theory 13 (2010), 779–783. https://doi.org/10.1515/jgt.2010.023.

[5] P.J. Cameron, S. Ghosh, The Power Graph of a Finite Group, Discret. Math. 311 (2011), 1220–1222. https://doi.org/10.1016/j.disc.2010.02.011.

[6] Z. Mehranian, A. Gholami, A.R. Ashrafi, A Note on the Power Graph of a Finite Group, Int. J. Group Theory 5 (2016), 1–10.

[7] D.M. Cardoso, M.A.A. de Freitas, E.A. Martins, M. Robbiano, Spectra of Graphs Obtained by a Generalization of the Join Graph Operation, Discret. Math. 313 (2013), 733–741. https://doi.org/10.1016/j.disc.2012.10.016.

[8] J. Gallian, Contemporary Abstract Algebra, Cengage Learning, (2016).

[9] S.M. Belcastro, G.J. Sherman, Counting Centralizers in Finite Groups, Mathematical Sciences Technical Reports (MSTR), 75. https://scholar.rose-hulman.edu/math_mstr/75.

[10] M. Mirzargar, A.R. Ashrafi, M.J. Nadjafi-Arani, On the Power Graph of a Finite Group, Filomat 26 (2012), 1201–1208. https://www.jstor.org/stable/24895826.

[11] R.P. Panda, K.V. Krishna, On Connectedness of Power Graphs of Finite Groups, J. Algebr. Appl. 17 (2018), 1850184. https://doi.org/10.1142/s0219498818501840.

[12] A. Doostabadi, A. Erfanian, A. Jafarzadeh, Some Results on the Power Graph of Finite Groups, ScienceAsia 41 (2015), 73–78.

[13] S. Zahirović, I. Bošnjak, R. Madarász, Enhanced Power Graphs of Finite Groups, arXiv:1810.07627 (2018). https://doi.org/10.48550/arXiv.1810.07627.

[14] O. Ejima, K. Aremu, A. Yusuf, The Order Divisor-Power Graph of Finite Groups, Ann. Alexandru Ioan Cuza Univ. - Math. 71 (2025), 133–143. https://doi.org/10.47743/anstim.2025.00010.

[15] A.R. Moghaddamfar, S. Rahbariyan, W.J. Shi, Certain Properties of the Power Graph Associated with a Finite Group, J. Algebr. Appl. 13 (2014), 1450040. https://doi.org/10.1142/s0219498814500406.

[16] J.L. Gross, J. Yellen, M. Anderson, Graph Theory and Its Applications, Chapman and Hall/CRC, 2018. https://doi.org/10.1201/9780429425134.

[17] D.B. West, Introduction to Graph Theory, Pearson, 2000.

[18] F. Harary, Graph Theory, Addison-Wesley, 1972.