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Weaving Continuous Controlled K-G-Fusion Frames in Hilbert C^*-Modules

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Abstract. In this work, we introduce the framework of weaving continuous controlled K-g-fusion frames in Hilbert C^*-modules and provide several characterizations of this notion. Furthermore, we generalize some recent results on woven K-g-fusion frames and controlled K-g-fusion frames to the continuous controlled case. In addition, we establish a perturbation result for woven continuous controlled K-g-fusion frames. These advancements not only enhance our understanding of fusion frames but also open up new avenues for research in operator theory and signal processing. By exploring these concepts further, we hope to uncover additional properties and applications that could significantly impact the field.

1. Introduction and Preliminaries

The theory of frames has become a fundamental tool in functional analysis and its applications, providing stable and redundant representations of elements in Hilbert spaces. Over the last decades, various generalizations of frame theory have been investigated, including g-frames, fusion frames, and their controlled and K-operator versions. These extensions have been motivated not only by theoretical interest but also by their relevance in applied fields such as signal processing, sampling theory, image analysis, and data transmission.

In recent years, the concept of weaving frames has attracted considerable attention. Roughly speaking, two or more families of frames are said to be woven if any selection of their elements, when combined together, still forms a frame with universal bounds. This notion has been successfully extended to g-frames, fusion frames, and their controlled counterparts. On the other hand, the study of frames in Hilbert C^*-modules has emerged as a rich and powerful framework, where

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classical results from Hilbert spaces are generalized to the setting of operator algebras. For more detailed information on frames theory, readers are recommended to consult [3,5–8,10,14–17,19–32].

The aim of this paper is to introduce and investigate the notion of woven continuous controlled K-g-fusion frames in Hilbert C^*-modules. We first provide a rigorous definition and establish several characterizations of this new concept. Furthermore, we show that many recent results on woven K-g-fusion frames and controlled K-g-fusion frames naturally extend to the continuous case. Finally, we study the stability of these frames under perturbations and prove a perturbation theorem that ensures the robustness of the proposed framework.

This work not only unifies and extends several existing results in frame theory but also contributes to the development of the theory of frames in Hilbert C^*-modules, offering potential applications in both pure and applied mathematics.

Hilbert C^*-modules are generalizations of Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than in the field of complex numbers.

Let's now review the definition of a Hilbert C^*-module, the basic properties, and some facts concerning operators on Hilbert C^*-modules.

Definition 1.1. *[2] Let $\mathcal{A}$ be a unital C^*- algebra and $\mathcal{M}$ be a left $\mathcal{A}-$ module, such that the linear structures of $\mathcal{A}$ and $\mathcal{M}$ are compatible. $\mathcal{M}$ is a pre-Hilbert $\mathcal{A}$ module if $\mathcal{M}$ is equipped with an $\mathcal{A}-$valued inner product $\langle\cdot,\cdot\rangle_{\mathcal{A}} : \mathcal{M}\times\mathcal{M} \to \mathcal{A}$ such that is sesquilinear, positive definite and respects the module action. In the other words,*

1 - $\langle y, y\rangle_{\mathcal{A}} \geq 0$, $\forall y \in \mathcal{M}$ *and* $\langle y, y\rangle_{\mathcal{A}} = 0$ *if and only if* $y = 0$.

2 - $\langle az + y, x\rangle_{\mathcal{A}} = a\langle z, x\rangle_{\mathcal{A}} + \langle y, x\rangle_{\mathcal{A}}$ *for all* $a \in \mathcal{A}$ *and* $x, y, z \in \mathcal{M}$

3 - $\langle x, y\rangle_{\mathcal{A}} = \langle y, x\rangle^*_{\mathcal{A}}$ *for all* $x, y \in \mathcal{M}$.

For $y \in \mathcal{M}$, we define $\|y\| = \|\langle y, y\rangle_{\mathcal{A}}\|^{\frac{1}{2}}$. If $\mathcal{M}$ is complete with $\|.\|$, it is called a Hilbert $\mathcal{A}-$module or a Hilbert C^*-module over $\mathcal{A}$. For every x in C^*-algebra $\mathcal{A}$, we have $|x| = (x^*x)^{\frac{1}{2}}$ and the $\mathcal{A}-$valued norm on $\mathcal{M}$ is defined by $|x| = \langle x, x\rangle^{\frac{1}{2}}_{\mathcal{A}}$ for $x \in \mathcal{M}$.

Throughout this paper, $\mathcal{M}$ is considered to be a Hilbert C^*- modules over a C^*- algebra; we denote that $I_{\mathcal{M}}$ is the identity operator on $\mathcal{M}$. $\{H_w\}_{w\in\Omega}$ is a sequence of Hilbert C^*-submodules of $\mathcal{M}$, and $\{V_w\}_{w\in\Omega}$ is a sequence of Hilbert C^*-modules V.

We denote by $End^*_{\mathcal{A}}(\mathcal{M}, V_w)$ the set of all adjointable operators from $\mathcal{M}$ to V_w. In particular, $End^*_{\mathcal{A}}(\mathcal{M})$ denotes the set of all adjointable operators on $\mathcal{M}$. The range of an operator T is denoted by $R(T)$. We denote by $\mathcal{GL}^+(\mathcal{M})$ the set of all bounded, positive, and invertible linear operators on $\mathcal{M}$, i.e., those positive operators that possess a bounded inverse.

Definition 1.2. *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules. A map $T : \mathcal{H} \to \mathcal{K}$ is said to be adjointable if there exists a map $T^* : \mathcal{K} \to \mathcal{H}$ such that $\langle Tx, y\rangle_{\mathcal{A}} = \langle x, T^*y\rangle_{\mathcal{A}}$ for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$.*

Remark 1.1. *It follows from the definition that if T is adjointable, T^* is adjointable and $\langle T^*x, y\rangle_{\mathcal{A}} = \langle x, Ty\rangle_{\mathcal{A}}$. That is $(T^*)^* = T$*

Proposition 1.1. *[9] Let T be an adjointable map. Then T is a bounded linear module map.*

Proposition 1.2. *[9] For $T \in End^*_{\mathcal{A}}(\mathcal{H})$, we have $\langle Tx, Tx\rangle \leq \|T\|^2\langle x, x\rangle,\ \forall x \in \mathcal{H}$.*

The following proposition is given by Ljiljana Arambašić in [1].

Proposition 1.3. *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules over a C^*-algebra $\mathcal{A}$, and $T \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$. The following statements are equivalent:*

(1) *T is surjective.*
(2) *T^* is bounded below with respect to the norm, i.e., there is $m > 0$ such that*

$$\|T^*x\| \geq m\|x\|, \quad \forall x \in \mathcal{K}.$$

(3) *T^* is bounded below with respect to the inner product, i.e., there is $m' > 0$ such that*

$$\langle T^*x, T^*x\rangle \geq m'\langle x, x\rangle, \quad \forall x \in \mathcal{K}.$$

Lemma 1.1. *[1] Let $\mathcal{U}$ be a Hilbert $\mathcal{A}$-module over a C^*-algebra $\mathcal{A}$, and let $T \in End^*_{\mathcal{A}}(\mathcal{U})$ be such that $T^* = T$. The following statements are equivalent:*

(i) *T is surjective.*
(ii) *There exist constants $m, M > 0$ such that*

$$m\|x\| \leq \|Tx\| \leq M\|x\|, \quad \textit{for all } x \in U.$$

(iii) *There exist constants $m', M' > 0$ such that*

$$m'\langle x, x\rangle \leq \langle Tx, Tx\rangle \leq M'\langle x, x\rangle, \quad \textit{for all } x \in U.$$

Lemma 1.2. *[4] Let $\mathcal{E}$, $\mathcal{H}$, and $\mathcal{L}$ be Hilbert $\mathcal{A}$-modules, and let $T \in End^*_{\mathcal{A}}(\mathcal{E}, \mathcal{L})$ and $T' \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{L})$. Then the following statements are equivalent:*

(1) *$T'(T')^* \leq \lambda TT^*$ for some $\lambda > 0$.*
(2) *There exists $\mu > 0$ such that*

$$\|(T')^*z\| \leq \mu\|T^*z\|, \quad \textit{for all } z \in \mathcal{L}.$$

Lemma 1.3. *[11] Let $\{W_j\}_{j\in J}$ denote a sequence of orthogonally complemented closed submodules of H. Suppose $U \in \mathrm{End}^*_A(H)$ is invertible and satisfies*

$$U^*UW_j \subseteq W_j, \quad \forall j \in J.$$

Under these conditions, the sequence $\{UW_j\}_{j\in J}$ also forms a sequence of orthogonally complemented closed submodules, and the relation

$$\pi_{W_j}U^* = \pi_{W_j}U^*\pi_{UW_j}$$

holds for each $j \in J$.

Let X be a Banach space, (Ω, μ) be a measure space, and $f : \Omega \to X$ be a measurable function. Integral of the Banach-valued function f has been defined by Bochner and others. Most properties of this integral are similar to those of the integral of real-valued functions. Since every C^*-algebra and Hilbert C^*-module are Banach spaces, we can use this integral and its properties.

Let (Ω, μ) be a measure space, $\mathcal{M}$ and V be two Hilbert C^*-modules, and $\{V_k : k \in \Omega\}$ be a sequence of subspaces of V, and $End^*_{\mathcal{A}}(\mathcal{M}, V_w)$ is the collection of all adjointable $\mathcal{A}$-linear maps from U into V_k. We define

$$\bigoplus_{k\in\Omega} V_k = \left\{ G = \{G_k\}_{k\in\Omega} : G_k \in V_k, \left\| \int_\Omega |G_k|^2 d\mu(k) \right\| < \infty \right\}.$$

For any $F = \{F_k\}_{k\in\Omega}$ and $G = \{G_k\}_{k\in\Omega}$, the $\mathcal{A}$-valued inner product is defined by $\langle F, G\rangle = \int_\Omega \langle F_k, G_k\rangle d\mu(k)$ and the norm $\|G\| = \|\langle G, G\rangle\|^{\frac{1}{2}}$. In this case $\bigoplus_{k\in\Omega} V_k$ is a Hilbert C^*-module.

Firstly we give the definition of $K-g-$fusion frame in Hilbert C^*- Modules.

2. $K-g-$Fusion Frame in Hilbert C^*- Modules

Definition 2.1. *[11] Let $\mathcal{A}$ be a unital C^*-algebra, and let $\mathcal{H}$ be a countably generated Hilbert $\mathcal{A}-$ module. Let $(v_j)_{j\in J}$ be a family of weights in $\mathcal{A}$, i.e., each v_j is a positive invertible element from the center of $\mathcal{A}$. Let $(W_j)_{j\in J}$ be a collection of orthogonally complemented closed submodules of $\mathcal{H}$, and let $(\mathcal{K}_j)_{j\in J}$ be a sequence of closed submodules of another Hilbert $\mathcal{A}-$ module $\mathcal{K}$.*

For each $j \in J$, let $\Upsilon_j \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}_j)$ be an adjointable operator.

We say that

$$\Upsilon = \left(W_j, \Upsilon_j, v_j\right)_{j\in J}$$

is a g-fusion frame for $\mathcal{H}$ with respect to $(\mathcal{K}_j)_{j\in J}$ if there exist constants $0 < A \le B < \infty$ such that

$$A\langle x, x\rangle \le \sum_{j\in J} v_j^2 \langle \Upsilon_j P_{W_j} x, \Upsilon_j P_{W_j} x\rangle \le B\langle x, x\rangle, \quad \forall x \in \mathcal{H}.$$

The constants A and B are called the lower and upper frame bounds, respectively.

Definition 2.2. *[11] Let $\mathcal{A}$ be a unital C^*- algebra and $\mathcal{H}$ a countably generated Hilbert $\mathcal{A}-$ module. Let $(v_j)_{j\in J}$ be a family of weights in $\mathcal{A}$, i.e., each v_j is a positive invertible element from the center of $\mathcal{A}$. Let $(W_j)_{j\in J}$ be a collection of orthogonally complemented closed submodules of $\mathcal{H}$, and let $(\mathcal{K}_j)_{j\in J}$ be a sequence of closed submodules of a Hilbert $\mathcal{A}-$ module $\mathcal{K}$.*

For each $j \in J$, let $\Upsilon_j \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}_j)$ be an adjointable operator, and let $K \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$.

We say that

$$\Upsilon = \left(W_j, \Upsilon_j, v_j\right)_{j\in J}$$

is a K-g-fusion frame for $\mathcal{H}$ with respect to $(\mathcal{K}_j)_{j\in J}$ if there exist constants $0 < A \le B < \infty$ such that

$$A\langle K^* x, K^* x\rangle \le \sum_{j\in J} v_j^2 \langle \Upsilon_j P_{W_j} x, \Upsilon_j P_{W_j} x\rangle \le B\langle x, x\rangle, \quad \forall x \in \mathcal{H}.$$

The constants A and B are called the lower and upper bounds of the $K-g-$fusion frame in Hilbert C^*- Modules, respectively.

Definition 2.3. *[18] Let $\mathcal{H}$ be a countably generated Hilbert $\mathcal{A}-$ module. Let $C\ C' \in GL^+(\mathcal{H})$, and $K \in End^*_{\mathcal{A}}(\mathcal{H})$, and Let $(v_j)_{j\in J}$ be a family of weights in $\mathcal{A}$, i.e., each v_j is a positive invertible element from the center of $\mathcal{A}$. Let $(W_j)_{j\in J}$ be a collection of orthogonally complemented closed submodules of $\mathcal{H}$, and let $(\mathcal{K}_j)_{j\in J}$ be a sequence of closed submodules of another Hilbert $\mathcal{A}-$ module $\mathcal{K}$.*

For each $j \in J$, let $\Upsilon_j \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}_j)$. We say that

$$\Upsilon_{CC'} = \Big(W_j, \Upsilon_j, v_j\Big)_{j\in J}$$

is a C, C'-controlled - $K - g-$fusion frame for $\mathcal{H}$ if there exist constants $0 < A \leq B < \infty$ such that

$$A\langle K^*x, K^*x\rangle \leq \sum_{j\in J} v_j^2 \langle \Upsilon_j P_{W_j} Cx, \Upsilon_j P_{W_j} C'x\rangle \leq B\langle x, x\rangle, \quad \forall x \in \mathcal{H}.$$

The constants A and B are called the lower and upper frame bounds,of the C, C'-controlled $K - g-$fusion frame respectively.

2.1. **Continuous $K - g-$fusion frame in Hilbert C^*-modules.**

Definition 2.4. *[12] Let $K \in End^*_A(\mathcal{M})$. Let $\{H_w\}_{w\in\Omega}$ be a measurable family of closed submodules of $\mathcal{M}$, each orthogonally complemented. Let P_{H_w} be the orthogonal projection from U onto H_w. Let $\Upsilon_w \in End^*_A(\mathcal{M}, V_w)$ for all $w \in \Omega$, and let $\{v_w\}_{w\in\Omega}$ be a family of weights in A, i.e., each v_w is a positive, invertible element from the center of A.*

Then the family $\Upsilon = \{(H_w, \Upsilon_w, v_w)\}_{w\in\Omega}$ is called a continuous $K - g-$ fusion frame for $\mathcal{M}$ if the following conditions hold:

(1) *For each $x \in \mathcal{M}$, $\{P_{H_w}x\}_{w\in\Omega}$ is measurable;*
(2) *For each $x \in \mathcal{M}$, the function $\widehat{\Upsilon} : \Omega \to V_w$, defined by $\widehat{\Upsilon}(w) = \Upsilon_w x$, is measurable;*
(3) *There exist constants $0 < A \leq B < \infty$ such that for all $x \in \mathcal{M}$,*

$$A\langle K^*x, K^*x\rangle \leq \int_\Omega v_w^2 \langle \Upsilon_w P_{H_w} x, \Upsilon_w P_{H_w} x\rangle \, d\mu(w) \leq B\langle x, x\rangle. \tag{2.1}$$

We call A and B the lower and upper frame bounds of a continuous $K - g-$fusion frame, respectively.

If the left-hand inequality in (2.1) holds with equality, then $\{(H_w, \Upsilon_w, v_w)\}_{w\in\Omega}$ is called a tight continuous $K - g-$fusion frame.

If $A = B = 1$, then $\{(H_w, \Upsilon_w, v_w)\}_{w\in\Omega}$ is called a Parseval continuous $K - g-$fusion frame for $\mathcal{M}$.

If only the right-hand inequality in (2.1) holds, then $\{(H_w, \Upsilon_w, v_w)\}_{w\in\Omega}$ is called a continuous $K - g-$ fusion Bessel sequence with bound B for $\mathcal{M}$.

If $K = I_d$, then the family $\Upsilon = \{(H_w, \Upsilon_w, v_w)\}_{w\in\Omega}$ is called a continuous g-fusion frame for $\mathcal{M}$

2.2. **weaving frame.** Given two frames $\{f_k\}_{k\in I}$ and $\{g_k\}_{k\in I}$ for a Hilbert space H, they are said to be woven if there exist constants $0 < A \leq B < \infty$ such that, for every subset $\sigma \subset I$, the family

$$\{f_k\}_{k\in\sigma} \cup \{g_k\}_{k\in\sigma^c}$$

forms a frame for H.

3. Main Result

3.1. Continuous Controlled $K-g-$fusion frame in Hilbert C^*-modules.

Definition 3.1. *[13] Let $C, C' \in GL^+(\mathcal{M})$, and let $\{H_w\}_{w\in\Omega}$ be a family of closed submodules of $\mathcal{M}$, each orthogonally complemented in $\mathcal{M}$. Let P_{H_w} denote the orthogonal projection from $\mathcal{M}$ onto H_w , and $\Lambda_w \in End^*_{\mathcal{A}}(\mathcal{M}, V_w)$ for each $w \in \Omega$. Let $\{v_w\}_{w\in\Omega}$ be a family of weights in $\mathcal{A}$, where each v_w is a positive invertible element from the center of the C^*-algebra $\mathcal{A}$.*

We say that

$$\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$$

is a continuous (C, C')-controlled $K-g$-fusion frame for $\mathcal{M}$ if:

(1) *For each $x \in \mathcal{M}$, the family $\{P_{H_w}x\}_{w\in\Omega}$ is measurable;*
(2) *For each $x \in \mathcal{M}$, the function $\Lambda : \Omega \to V_w$ defined by $\Lambda(w) = \Lambda_w x$ is measurable;*
(3) *There exist constants $0 < A \le B < \infty$ such that*

$$A\langle K^*x, K^*x\rangle \le \int_\Omega v_w^2 \,\langle \Lambda_w P_{H_w} Cx,\ \Lambda_w P_{H_w} C'x\rangle \,d\mu(w) \le B\langle x, x\rangle, \quad \forall x \in \mathcal{M}. \tag{3.1}$$

The constants A and B are called the lower and upper continuous$(C, C')-$ controlled $K-g-$fusion frame bounds, respectively.

If only the right-hand inequality of (3.1) holds then , we call Λ a continuous $(C, C')-$ controlled $K-g-$ fusion Bessel sequence.

(1): *If $A = B$, Λ is called a tight continuous(C, C')-controlled $K-g-$ fusion frame.*
(2): *If $A = B = 1$, Λ is called a Parseval continuous (C, C')-controlled $K-g-$ fusion frame.*
(3): *If $C' = I_H$ then $\Lambda_{CC'}$ is called a continuous (C, I_H)-controlled $K-g-$fusion frame.*
(4): *If $C = C' = I_H$ then $\Lambda_{CC'}$ is called a continuous $K-g-$ fusion frame .*
(5): *If $K = I_H$ then $\Lambda_{CC'}$ is called a continuous $(C, C')-$controlled $g-$ fusion frame .*

Suppose that $\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ be a $(C, C')-$ controlled continuous $g-$ fusion Bessel sequence for $\mathcal{M}$. The bounded linear operator

$$T^{(C,C')} : \bigoplus_{w\in\Omega} V_w \to \mathcal{M}$$

define by

$$T^{(C,C')}\Big(\{x_w\}_{w\in\Omega}\Big) = \int_\Omega v_w (CC')^{\frac{1}{2}} P_{H_w} \Lambda_w^* x_w \,d\mu(w), \qquad \forall \{x_w\}_{w\in\Omega} \in \bigoplus_{w\in\Omega} V_w. \tag{2.2}$$

$T^{(C,C')}$ is called the synthesis operator for the continuous $(C, C')-$ controlled $g-$ fusion frame Λ. The adjoint operator $T^{(C,C')^*} : \mathcal{M} \to \bigoplus_{w\in\Omega} V_w$ given by

$$T^{(C,C')^*}(y) = \Big\{v_w \Lambda_w P_{H_w} (C'C)^{\frac{1}{2}} y\Big\}_{w\in\Omega}, \tag{2.3}$$

is called the analysis operator for the continuous $(C, C')-$ controlled $g-$ fusion frame Λ.

When the C and C' commute with each other, and commute with the operator $P_{H_w}\Lambda_w^*\Lambda_w P_{H_w}$ for each $w \in \Omega$, then the continuous $(C,C')-$ controlled $g-$fusion frame operator $S^{(C,C')} : \mathcal{M} \to \mathcal{M}$ is defined as

$$S^{(C,C')}(x) = T^{(C,C')}T^{(C,C')^*}(x) = \int_\Omega v_w^2 C' P_{H_w}\Lambda_w^*\Lambda_w P_{H_w} Cx \, d\mu(w), \qquad \forall x \in \mathcal{M}. \tag{2.4}$$

Now, we present woven continuous controlled $K-g-$fusion frame for $\mathcal{M}$.

Definition 3.2. *A family of continuous $(C, C')-$ controlled $K-g-$ fusion frames $\Lambda = \left\{H_{jw}, \Lambda_{jw}, v_{jw}\right\}_{w\in\Omega,\ j\in[n]}$ for $\mathcal{M}$ is said to be woven continuous $(C, C')-$ controlled $K-g-$fusion frame if there exist universal positive constants $0 < A \le B < \infty$ such that for every partition $\left\{\sigma_j\right\}_{j\in[n]}$ of Ω, the family $\Lambda = \left\{H_{jw}, \Lambda_{jw}, v_{jw}\right\}_{w\in\sigma_j, j\in[n]}$ is a continuous $(C, C')-$ controlled $K-g-$ fusion frame for $\mathcal{M}$ with bounds A and B.*

Each family $\Lambda = \left\{H_{jw}, \Lambda_{jw}, v_{jw}\right\}_{w\in\sigma_j, j\in[n]}$ is called a weaving continuous (C, C')-controlled K-g-fusion frame.

Let : $\Gamma := \left\{H_{jw}, \Gamma_{jw}, v_{jw}\right\}_{w\in\Omega,\ j\in[n]}$ and $\Phi := \left\{F_{jw}, \Phi_{jw}, \mu_{jw}\right\}_{w\in\Omega,\ j\in[n]}$ two continuous $(C, C')-$ controlled K-g-fusion frame for $\mathcal{M}$

Definition 3.3. *Γ and Φ are said to be woven continuous $(C, C')-$ controlled $K-g-$fusion frame if there exist universal constants $0 < A \le B < \infty$ such that for every partition $\{\sigma_j\}_{j\in[n]}$ of Ω, the family $\Gamma \cup \Phi$ is a continuous $(C, C')-$ controlled $K-g-$ fusion frame for $\mathcal{M}$ with bounds A and B respectively, that is:*

$$\begin{aligned} A\langle x,x\rangle_{\mathcal{A}} \le &\int_\sigma v_{jw}^2 \langle \Gamma_{jw}P_{H_{jw}}Cx,\ \Gamma_{jw}P_{H_{jw}}C'x\rangle \, d\mu(w) \\ &+ \int_{\sigma^c} \mu_{jw}^2 \langle \Phi_{jw}P_{F_{jw}}Cx, \Phi_{jw}P_{F_{jw}}Cx\rangle_{\mathcal{A}} d\mu(w) \le B\langle x,x\rangle_{\mathcal{A}} \textit{ for all } x \in \mathcal{M}. \end{aligned}$$

Theorem 3.1. *Let $\Lambda = \left\{H_{jw}, \Lambda_{jw}, v_{jw}\right\}_{w\in\Omega}$ be a continuous $(C,C')-$ controlled $g-$fusion Bessel sequence for $\mathcal{M}$ for each $j \in [n]$.with bounds B_j . Then every weaving is a continuous $(C,C')-$controlled $g-$fusion Bessel sequence for $\mathcal{M}$ with bounds $\sum_{j\in[n]} B_j$.*

Proof. Let $\Lambda = \left\{H_{jw}, \Lambda_{jw}, v_{jw}\right\}_{w\in\Omega}$ be a $(C,C')-$ controlled continuous $g-$fusion Bessel sequence for $\mathcal{M}$ for each $j \in [n]$ So :

$$\int_\Omega v_{jw}^2 \langle \Lambda_{jw}P_{H_{jw}}Cx,\ \Lambda_{jw}P_{H_{jw}}C'x\rangle \, d\mu(w) \le B_j\langle x,x\rangle, \quad \forall x \in \mathcal{M}.$$

Since we have for any partition $\{\sigma_j\}_{j\in[n]}$ of Ω and $x \in \mathcal{M}$,:

$$\int_{\sigma_j} v_{jw}^2 \langle \Lambda_{jw}P_{H_{jw}}Cx,\ \Lambda_{jw}P_{H_{jw}}C'x\rangle \, d\mu(w) \le \int_\Omega v_{jw}^2 \langle \Lambda_{jw}P_{H_{jw}}Cx,\ \Lambda_{jw}P_{H_{jw}}C'x\rangle \, d\mu(w)$$

Hence

$$\sum_{j\in[n]}\int_{\sigma_j} v_{jw}^2 \langle \Lambda_{jw}P_{H_{jw}}Cx,\ \Lambda_{jw}P_{H_{jw}}C'x\rangle\, d\mu(w) \leq \sum_{j\in[n]}\int_{\Omega} v_{jw}^2 \langle \Lambda_{jw}P_{H_{jw}}Cx,\ \Lambda_{jw}P_{H_{jw}}C'x\rangle\, d\mu(w) \leq \sum_{j\in[n]} B_j\langle x,x\rangle_{\mathcal{A}}.$$

yielding the desired bound. □

Theorem 3.2. *Let $\Lambda = \left\{H_{jw}, \Lambda_{jw}, v_{jw}\right\}_{w\in\sigma_j\ ,j\in[n]}$ be a woven continuous $(C, C')-$ controlled $K-g-$ fusion frame for $\mathcal{M}$ with universal bounds A and B. If $V \in End^*_{\mathcal{A}}(\mathcal{M})$ be invertible operator such that V^* commutes with C, C' and V commutes with K and*

$$V^*VH_{jw} \subseteq H_{jw}, \quad \forall j \in J.$$

Then the family $\left\{VH_{jw}, \Lambda_{jw}P_{H_{jw}}V^*, v_{jw}\right\}_{w\in\sigma_j\ ,j\in[n]}$ is a woven continuous (C, C')-controlled K-g-fusion frame for $\mathcal{M}$.

Proof. Let $\Lambda = \left\{H_{jw}, \Lambda_{jw}, v_{jw}\right\}_{w\in\sigma_j\ ,j\in[n]}$ be a woven continuous $(C, C')-$ controlled $K-g-$-fusion frame for $\mathcal{M}$ with universal bounds A and B, and $V \in End^*_{\mathcal{A}}(\mathcal{M})$ be invertible operator such that V^* commutes with C, C' and V commutes with K and

$$V^*VH_{jw} \subseteq H_{jw}, \quad \forall j \in [n].$$

So by lemma (1.3) we have : $P_{H_{jw}}V^* = P_{H_{jw}}V^*P_{VH_{jw}}$ for all $w \in \sigma_j$ and $j \in [n]$, the mapping $w \to P_{VH_{jw}}$ is weakly measurable. For every $x \in \mathcal{M}$, we have

$$\begin{aligned}
&\sum_{j\in[n]}\int_{\Omega} v_{jw}^2 \langle \Lambda_{jw}P_{H_{jw}}V^*P_{VH_{jw}}Cx,\ \Lambda_{jw}P_{H_{jw}}V^*P_{VH_{jw}}C'x\rangle\, d\mu(w)\\
&= \sum_{j\in[n]}\int_{\Omega} v_{jw}^2 \langle \Lambda_{jw}P_{H_{jw}}V^*Cx,\ \Lambda_{jw}P_{H_{jw}}V^*C'x\rangle\, d\mu(w)\\
&= \sum_{j\in[n]}\int_{\Omega} v_{jw}^2 \langle \Lambda_{jw}P_{H_{jw}}CV^*x,\ \Lambda_{jw}P_{H_{jw}}C'V^*x\rangle\, d\mu(w)\\
&\leq B\langle V^*x\, V^*x\rangle\\
&\leq B\,\|V\|^2\,\langle x,x\rangle.
\end{aligned}$$

On the other hand, for every $x \in H$, we have:

$$A\langle K^*V^*x, K^*V^*x\rangle \leq \sum_{j\in[n]}\int_{\Omega} v_{jw}^2 \langle \Lambda_{jw}P_{H_{jw}}V^*P_{VH_{jw}}Cx,\ \Lambda_{jw}P_{H_{jw}}V^*P_{VH_{jw}}C'x\rangle\, d\mu(w)$$

And : $\langle K^*x, K^*x\rangle = \langle (V^{-1})^*)V^*K^*x, (V^{-1})^*)V^*K^*x\rangle. \leq \left\|V^{-1}\right\|^2 \langle V^*K^*x, V^*K^*x\rangle.$

Hence :

$$A\left\|V^{-1}\right\|^{-2}\langle K^*x, K^*x\rangle \leq \sum_{j\in[n]}\int_{\Omega} v_{jw}^2 \langle \Lambda_{jw}P_{H_{jw}}V^*P_{VH_{jw}}Cx,\ \Lambda_{jw}P_{H_{jw}}V^*P_{VH_{jw}}C'x\rangle\, d\mu(w)$$

Then the family $\left\{VH_{jw}, \Lambda_{jw}P_{H_{jw}}V^*, v_{jw}\right\}_{w\in\sigma_j\ , j\in[n]}$ is a woven continuous $(C, C')-$ controlled $K-g-$ fusion frame for $\mathcal{M}$ with bounds $A\left\|V^{-1}\right\|^{-2}$ and $B\,\|V\|^2$ □

Corollary 3.1. *Let $\Lambda = \left\{H_{jw}, \Lambda_{jw}, v_{jw}\right\}_{w\in\sigma_j\ , j\in[n]}$ be a woven continuous $(C, C')-$ controlled $K-g-$ fusion frame for $\mathcal{M}$ with universal bounds A and B. If $V \in End^*_{\mathcal{A}}(\mathcal{M})$ be invertible operator such that V^* commutes with C, C' and V commutes with K and*

$$V^*VH_{jw} \subseteq H_{jw}, \quad \forall j \in [n].$$

Then the family $\left\{VH_{jw}, \Lambda_{jw}P_{H_{jw}}V^*, v_{jw}\right\}_{w\in\sigma_j\ , j\in[n]}$ is a woven continuous $(C, C')-$ controlled VKV^*-g- fusion frame for $\mathcal{M}$.

Proof. The universal upper bounds is $B\,\|V\|^2$. On the other hand, for each $x \in \mathcal{M}$, we have

$$\begin{aligned}
\frac{A}{\|V\|^2}\langle (VKV^*)^*x, (VKV^*)^*x\rangle &= \frac{A}{\|V\|^2}\langle (VKV^*)x, (VKV^*)x\rangle \\
&\le A\langle (KV^*)x, (KV^*)x\rangle \\
&\le \sum_{j\in[n]}\int_{\Omega} v_{jw}^2\,\langle \Lambda_{jw}P_{H_{jw}}CV^*x,\ \Lambda_{jw}P_{H_{jw}}C'V^*x\rangle\, d\mu(w) \\
&= \sum_{j\in[n]}\int_{\Omega} v_{jw}^2\,\langle \Xi_{jw}P_{VH_{jw}}Cx,\ \Xi_{jw}P_{VH_{jw}}C'x\rangle\, d\mu(w)
\end{aligned}$$

With $\Xi_{jw} = \Lambda_{jw}P_{H_{jw}}V^*$

Hence the family $\left\{VH_{jw}, \Lambda_{jw}P_{H_{jw}}V^*, v_{jw}\right\}_{w\in\sigma_j\ , j\in[n]}$ is a woven continuous $(C, C')-$ controlled VKV^*-g-fusion frame for $\mathcal{M}$. □

Theorem 3.3. *Let $V \in End^*_{\mathcal{A}}(\mathcal{M})$ be invertible operator such that V^* and $(V^{-1})^*$ commutes with C, C' and $\left\{VH_{jw}, \Lambda_{jw}P_{H_{jw}}V^*, v_{jw}\right\}_{w\in\sigma_j\ , j\in[n]}$ is a woven continuous (C, C')-controlled K-g-fusion frame for $\mathcal{M}$ with universal bounds A and B such that*

$$V^*VH_{jw} \subseteq H_{jw}, \quad \forall j \in [n].$$

So $\Lambda = \left\{H_{jw}, \Lambda_{jw}, v_{jw}\right\}_{w\in\sigma_j\ , j\in[n]}$ is a woven continuous $(C, C')-$ controlled $V^{-1}KV-g-$fusion frame for $\mathcal{M}$.

Proof. Let $x \in \mathcal{M}$, $w \in \sigma_j$, $j \in [n]$, by lemma (1.3) and getting $\Xi_{jw} = \Lambda_{jw}P_{H_{jw}}V^*$ we have:

$$\begin{aligned}
\frac{A}{\|V\|^2}\langle \left(V^{-1}KV\right)^*x, \left(V^{-1}KV\right)^*x\rangle &= \frac{A}{\|V\|^2}\langle V^*K^*(V^{-1})^*x, V^*K^*(V^{-1})^*x\rangle \\
&\le A\langle \left(K^*(V^{-1})^*\right)x, \left(K^*(V^{-1})^*\right)x\rangle \\
&\le \sum_{j\in[n]}\int_{\Omega} v_{jw}^2\,\langle \Xi_{jw}P_{VH_{jw}}C(V^{-1})^*x,\ \Xi_{jw}P_{VH_{jw}}C'(V^{-1})^*x\rangle\, d\mu(w) \\
&\le \sum_{j\in[n]}\int_{\Omega} v_{jw}^2\,\langle \Xi_{jw}C(V^{-1})^*x,\ \Xi_{jw}C'(V^{-1})^*x\rangle\, d\mu(w)
\end{aligned}$$

$$= \sum_{j\in[n]} \int_{\Omega} v_{jw}^2 \langle \Xi_{jw}(V^{-1})^* Cx, \Xi_{jw}(V^{-1})^* C' x\rangle\, d\mu(w)$$

$$= \sum_{j\in[n]} \int_{\Omega} v_{jw}^2 \langle \Lambda_{jw} P_{H_{jw}} Cx, \Lambda_{jw} P_{H_{jw}} C' x\rangle\, d\mu(w)$$

Then for each $x \in \mathcal{M}$ we have :

$$\frac{A}{\| V\|^2} \langle \left(V^{-1} K V\right)^* x, \left(V^{-1} K V^*\right)^* x\rangle \le \sum_{j\in[n]} \int_{\Omega} v_{jw}^2 \langle \Lambda_{jw} P_{H_{jw}} Cx, \Lambda_{jw} P_{H_{jw}} C' x\rangle\, d\mu(w)$$

For the upper bounds we put $L = \sum_{j\in[n]} \int_{\Omega} v_{jw}^2 \langle \Lambda_{jw} P_{H_{jw}} Cx, \Lambda_{jw} P_{H_{jw}} C' x\rangle\, d\mu(w)$:

$$\begin{aligned}
L &= \sum_{j\in[n]} \int_{\Omega} v_{jw}^2 \langle \Lambda_{jw} P_{H_{jw}} CV^*(V^{-1})^* x, \Lambda_{jw} P_{H_{jw}} C' V^*(V^{-1})^* x\rangle\, d\mu(w) \\
&\le \sum_{j\in[n]} \int_{\Omega} v_{jw}^2 \langle \Lambda_{jw} P_{H_{jw}} V^* P_{VH_{jw}}(V^{-1})^* Cx, \Lambda_{jw} P_{H_{jw}} V^* P_{VH_{jw}}(V^{-1})^* C' x\rangle\, d\mu(w) \\
&\le \sum_{j\in[n]} \int_{\Omega} v_{jw}^2 \langle \Lambda_{jw} P_{H_{jw}} V^* P_{VH_{jw}} C(V^{-1})^* x, \Lambda_{jw} P_{H_{jw}} V^* P_{VH_{jw}} C'(V^{-1})^* x\rangle\, d\mu(w) \\
&\le B\langle (V^{-1})^* x, (V^{-1})^* x\rangle \\
&\le B\|(V^{-1})\|^2 \langle x, x\rangle
\end{aligned}$$

Hence: $\Lambda = \left\{H_{jw}, \Lambda_{jw}, v_{jw}\right\}_{w\in\sigma_j, j\in[n]}$ is a woven continuous $(C, C')-$ controlled $V^{-1}KV-$ $g-$fusion frame for $\mathcal{M}$. □

Let $\Gamma := \left\{H_{jw}, \Gamma_{jw}, v_{jw}\right\}_{w\in\Omega,\, j\in[n]}$ and $\Phi := \left\{F_{jw}, \Phi_{jw}, \mu_{jw}\right\}_{w\in\Omega,\, j\in[n]}$ two continuous $(C, C')-$ controlled $K-g-$fusion frame for $\mathcal{M}$

Definition 3.4. *Γ and Φ are said to be woven continuous $(C, C')-$ controlled $K-g-$ fusion frame if there exist universal constants $0 < A \le B < \infty$ such that for every partition $\{\sigma_j\}_{j\in[n]}$ of Ω, the family $\Gamma \cup \Phi$ is a continuous $(C, C')-$ controlled $K-g-$fusion frame for $\mathcal{M}$ with bounds A and B respectively,that is:*

$$\begin{aligned}
A\langle x, x\rangle_{\mathcal{A}} \le & \int_{\sigma} v_{jw}^2 \langle \Gamma_{jw} P_{H_{jw}} Cx,\ \Gamma_{jw} P_{H_{jw}} C' x\rangle\, d\mu(w) \\
&+ \int_{\sigma^c} \mu_{jw}^2 \langle \Phi_{kj} P_{F_{jw}} Cx, \Phi_{jw} P_{F_{jw}} Cx\rangle_{\mathcal{A}} d\mu(w) \le B\langle x, x\rangle_{\mathcal{A}} \text{ for all } x \in \mathcal{M}.
\end{aligned}$$

Now, we will see that the intersection of components of a woven continuous $(C, C')-$ controlled $K-g-$ fusion frame, with a closed subspace is a be woven continuous $(C, C')-$controlled $K-$ $g-$fusion frame, for the smaller space.

Theorem 3.4. *Let $\{H_w, \Gamma_w, v_w\}\}_{w\in\Omega}$ and $\{F_w, \Phi_w, \mu_w\}_{w\in\Omega}$ be woven continuous $(C, C')-$ controlled $K-g-$ fusion frame for $\mathcal{M}$, and $\mathcal{W}$ be a closed subspace of $\mathcal{M}$. and $P_{F_w} P_{\mathcal{W}} = P_{\mathcal{W}} P_{F_w}$, $P_{H_w} P_{\mathcal{W}} = P_{\mathcal{W}} P_{H_w}$*

Then the families given by $\{H_w \cap \mathcal{W}, \Gamma_w, v_w\}\}_{w\in\Omega}$ *and* $\{F_w \cap \mathcal{W}, \Phi_w, \mu_w\}\}_{w\in\Omega}$ *are woven continuous* $(C, C')-$ *controlled* $K-g-$ *fusion frame for* $\mathcal{W}$.

Proof. The orthogonal projections of $\mathcal{M}$ onto $F_w \cap \mathcal{W}$ is the operator defined by $P_{F_w\cap\mathcal{W}} = P_{F_w}P_{\mathcal{W}}$ and The orthogonal projections of $\mathcal{M}$ onto $H_w \cap \mathcal{W}$ is the operator defined by $P_{H_w\cap\mathcal{W}} = P_{H_w}P_{\mathcal{W}}$. Let σ be a measurable subset of Ω. Then for every $x \in \mathcal{W}$, we have:

$$\begin{aligned} I &= \int_{\sigma} v_w^2\langle \Gamma_w P_{H_w} Cx, \Gamma_w P_{H_w} C'x\rangle \, d\mu(w) + \int_{\sigma^c} \mu_w^2\langle \Phi_w P_{F_w} Cx, \Phi_w P_{F_w} C'x\rangle \, d\mu(w) \\ &= \int_{\sigma} v_w^2\langle \Gamma_w P_{H_w} P_{\mathcal{W}} Cx, \Gamma_w P_{H_w} P_{\mathcal{W}} C'x\rangle \, d\mu(w) + \int_{\sigma^c} \mu_w^2\langle \Phi_w P_{F_w} P_{\mathcal{W}} Cx, \Phi_w P_{F_w} P_{\mathcal{W}} C'x\rangle \, d\mu(w) \\ &= \int_{\sigma} v_w^2\langle \Gamma_w P_{H_w\cap\mathcal{W}} Cx, \Gamma_w P_{H_w\cap\mathcal{W}} C'x\rangle \, d\mu(w) + \int_{\sigma^c} \mu_w^2\langle \Phi_w P_{F_w\cap\mathcal{W}} Cx, \Phi_w P_{F_w\cap\mathcal{W}} C'x\rangle \, d\mu(w). \end{aligned}$$

Then the families given by $\{H_w \cap \mathcal{W}, \Gamma_w, v_w\}_{w\in\Omega}$ and $\{F_w \cap \mathcal{W}, \Phi_w, \mu_w\}_{w\in\Omega}$ are woven continuous $(C, C')-$ controlled $K-g-$fusion frame for $\mathcal{W}$. □

Theorem 3.5. *Let* $\Lambda = \left\{H_{jw}, \Lambda_{jw}, v_{jw}\right\}_{w\in\sigma_j, j\in[n]}$ *be a woven continuous* $(C, C')-$*controlled* $K-g-$*fusion frame for* $\mathcal{M}$ *with universal bounds* A *and* B, *and* $V \in End^*_{\mathcal{A}}(\mathcal{M})$ *be invertible operator such that* V^* *commutes with* C, C' *we suppose that:*

$$V^*VH_{jw} \subseteq H_{jw}, \quad \forall j \in [n].$$

and K *have closed range, Then the family* $\left\{VH_{jw}, \Lambda_{jw}P_{H_{jw}}V^*, v_{jw}\right\}_{w\in\sigma_j, j\in[n]}$ *is a woven continuous* $(C, C')-$*controlled* $K-g-$*fusion frame for* $\mathcal{M}$, *if and only if there exists a* $\beta > 0$ *such that for each* $x \in \mathcal{M}$, *we have* $\langle V^*x, V^*x\rangle \geq \beta\langle K^*x, K^*x\rangle$.

Proof. Let $\left\{VH_{jw}, \Lambda_{jw}P_{H_{jw}}V^*, v_{jw}\right\}_{w\in\sigma_j, j\in[n]}$ is a woven continuous (C, C')-controlled K-g-fusion frame for $\mathcal{M}$,with bounds A' and B'.Then for each $x \in \mathcal{M}$, $w \in \sigma_j$, $j \in [n]$, by lemma (1.3) and getting $\Xi_{jw} = \Lambda_{jw}P_{H_{jw}}V^*$ we have:

$$\begin{aligned} A'\langle K^*x, K^*x\rangle &\leq \sum_{j\in[n]} \int_{\Omega} v_{jw}^2 \langle \Xi_{jw} P_{VH_{jw}} Cx, \Xi_{jw} P_{VH_{jw}} C'x\rangle \, d\mu(w) \\ &= \sum_{j\in[n]} \int_{\Omega} v_{jw}^2 \langle \Lambda_{jw} P_{H_{jw}} V^* Cx, \Lambda_{jw} P_{H_{jw}} V^* C'x\rangle \, d\mu(w) \\ &= \sum_{j\in[n]} \int_{\Omega} v_{jw}^2 \langle \Lambda_{jw} P_{H_{jw}} CV^* x, \Lambda_{jw} P_{H_{jw}} C'V^* x\rangle \, d\mu(w) \\ &\leq B\langle V^* x\, V^* x\rangle \end{aligned}$$

So for $x \in \mathcal{M}$ we have $\langle V^*x, V^*x\rangle \geq \beta\langle K^*x, K^*x\rangle$, with $\beta = \sqrt{\frac{A'}{B}}$.
Now we suppose $\langle V^*x, V^*x\rangle \geq \beta\langle K^*x, K^*x\rangle$. for all $x \in \mathcal{M}$, Since K have a closed range we have :

$$\begin{aligned}\langle V^*x, V^*x\rangle &= \langle (K^\dagger)^*K^*V^*x, (K^\dagger)^*K^*V^*x\rangle \\ &\leq \| K^\dagger \|^2\langle K^*V^*x, K^*V^*x\rangle\end{aligned}$$

for all $x \in \mathcal{M}$ we have:

$$\begin{aligned}&= \sum_{j\in[n]} \int_\Omega v_{jw}^2 \langle \Lambda_{jw}P_{H_{jw}}V^*P_{VH_{jw}}Cx, \Lambda_{jw}P_{H_{jw}}V^*P_{VH_{jw}}C'x\rangle \, d\mu(w) \\ &= \sum_{j\in[n]} \int_\Omega v_{jw}^2 \langle \Lambda_{jw}P_{H_{jw}}CV^*x, \Lambda_{jw}P_{H_{jw}}C'V^*x\rangle \, d\mu(w) \\ &\geq A\langle K^*V^*x, K^*V^*x\rangle \\ &\geq A\| K^\dagger \|^{-2}\langle V^*x, V^*x\rangle \\ &\geq A\| K^\dagger \|^{-2}\beta^2\langle K^*x, K^*x\rangle.\end{aligned}$$

Then the family $\left\{VH_{jw}, \Lambda_{jw}P_{H_{jw}}V^*, v_{jw}\right\}_{w\in\sigma_j, j\in[n]}$ is a woven continuous $(C, C')-$ controlled $K-g-$fusion frame for $\mathcal{M}$, if and only if there exists a $\beta > 0$ such that for each $x \in \mathcal{M}$, we have : $\langle V^*x, V^*x\rangle \geq \beta\langle K^*x, K^*x\rangle$. □

The following theorem shows that it suffices to verify continuous weaving controlled $K-g-$fusion frames on a smaller measurable space than the original one.

Theorem 3.6. *For each $j \in [n]$ let $\Lambda = \left\{H_{jw}, \Lambda_{jw}, v_{jw}\right\}_{w\in\Omega}$ be a woven continuous $(C, C')-$ controlled $K-g-$fusion frame for $\mathcal{M}$ with universal bounds A'_j and B'_j . If there exists a measurable subset $Z \subset \Omega$ such that the family $\Lambda = \left\{H_{jw}, \Lambda_{jw}, v_{jw}\right\}_{j\in[n], w\in Z}$ is a woven continuous $(C, C')-$ controlled $K-g-$fusion frame for $\mathcal{M}$ with universal bounds A' and B' . Then $\Lambda = \left\{H_{jw}, \Lambda_{jw}, v_{jw}\right\}_{j\in[n], w\in\Omega}$ is a woven continuous $(C, C')-$controlled $K-g-$fusion frame for $\mathcal{M}$.*

Proof. For the Upper bound.
Let $\{\gamma_j\}_{j\in[n]}$ an arbitrary measurable partition of Ω , for each $x \in \mathcal{M}$, we have :

$$\begin{aligned}&\sum_{j\in[n]} \int_{\gamma_j} v_{jw}^2 \langle \Lambda_{jw}P_{H_{jw}}Cx, \Lambda_{jw}P_{H_{jw}}C'x\rangle \, d\mu(w) \\ &\leq \sum_{j\in[n]} \int_\Omega v_{jw}^2 \langle \Lambda_{jw}P_{H_{jw}}Cx, \Lambda_{jw}P_{H_{jw}}C'x\rangle \, d\mu(w) \\ &\leq (\sum_{j\in[n]} B_j)\langle x, x\rangle\end{aligned}$$

For the Lower bound.
We have $\left\{\gamma_j \cap Z\right\}_{i\in[n]}$ is a partitions of Z. then, the family $\left\{H_{jw}, \Lambda_{jw}, v_{jw}\right\}_{j\in[n], w\in\gamma_j\cap Z}$ is a continuous

$(C, C')-$controlled$K-g-$fusion frame for $\mathcal{M}$ with lower bound A'. Thus

$$\begin{aligned}
&\sum_{j\in[n]}\int_{\gamma_j} v_{jw}^2 \langle \Lambda_{jw}P_{H_{jw}}Cx, \Lambda_{jw}P_{H_{jw}}C'x\rangle\, d\mu(w)\\
&\geq \sum_{j\in[n]}\int_{\gamma_j\cap Z} v_{jw}^2 \langle \Lambda_{jw}P_{H_{jw}}Cx, \Lambda_{jw}P_{H_{jw}}C'x\rangle\, d\mu(w)\\
&\geq A'\langle K^*x, K^*x\rangle.
\end{aligned}$$

This establishes the desired result □

Theorem 3.7. *Let $\left\{H_{jw}, \Lambda_{jw}, v_{jw}\right\}_{j\in[n], w\in\Omega}$ be a woven continuous $(C, C')-$ controlled $K-g-$fusion frame for $\mathcal{M}$. with bounds $A'$and B' If there exists a measurable subset $Z \subset \Omega$ and $0 < \Delta < A'$ such that for $x \in \mathcal{M}$*

$$\sum_{j\in[n]}\int_{Z} v_{jw}^2 \langle \Lambda_{jw}P_{H_{jw}}Cx, \Lambda_{jw}P_{H_{jw}}C'x\rangle\, d\mu(w) \leq \Delta\langle K^*x, K^*x\rangle.$$

Hence $\left\{H_{jw}, \Lambda_{jw}, v_{jw}\right\}_{j\in[n], w\in(\Omega\setminus Z)}$ is a woven continuous $(C, C')-$ controlled $K-g-$fusion frame for $\mathcal{M}$. with bounds $A'-\Delta$ and B'

Proof. Let $\left\{H_{jw}, \Lambda_{jw}, v_{jw}\right\}_{j\in[n], w\in\Omega}$ be a woven continuous $(C, C')-$controlled $K-g-$fusion frame for $\mathcal{M}$. with bounds $A'$and B' Thus for each $x \in \mathcal{M}$

$$A'\langle K^*x, K^*x\rangle \leq \sum_{j\in[n]}\int_{\Omega} v_{jw}^2 \langle \Lambda_{jw}P_{H_{jw}}Cx,\ \Lambda_{jw}P_{H_{jw}}C'x\rangle\, d\mu(w) \leq B'\langle x, x\rangle_{\mathcal{A}}.$$

So:

$$\begin{aligned}
L &= \sum_{j\in[n]}\int_{(\Omega\setminus Z)} v_{jw}^2 \langle \Lambda_{jw}P_{H_{jw}}Cx,\ \Lambda_{jw}P_{H_{jw}}C'x\rangle\, d\mu(w)\\
&= \sum_{j\in[n]}\int_{\Omega} v_{jw}^2 \langle \Lambda_{jw}P_{H_{jw}}Cx,\ \Lambda_{jw}P_{H_{jw}}C'x\rangle\, d\mu(w) - \sum_{j\in[n]}\int_{Z} v_{jw}^2 \langle \Lambda_{jw}P_{H_{jw}}Cx, \Lambda_{jw}P_{H_{jw}}C'x\rangle\, d\mu(w)\\
&\geq (A'-\Delta)\langle K^*x, K^*x\rangle.
\end{aligned}$$

For the upper bound for all $x \in \mathcal{M}$ we have :

$$\begin{aligned}
L &= \sum_{j\in[n]}\int_{\Omega\setminus Z} v_{jw}^2 \langle \Lambda_{jw}P_{H_{jw}}Cx,\ \Lambda_{jw}P_{H_{jw}}C'x\rangle\, d\mu(w)\\
&\leq \sum_{j\in[n]}\int_{\Omega} v_{jw}^2 \langle \Lambda_{jw}P_{H_{jw}}Cx,\ \Lambda_{jw}P_{H_{jw}}C'x\rangle\, d\mu(w)\\
&\leq B'\langle x, x\rangle_{\mathcal{A}}.
\end{aligned}$$

Hence :

$$(A'-\Delta)\langle K^*x, K^*x\rangle \leq \sum_{j\in[n]}\int_{\Omega\setminus Z} v_{jw}^2 \langle \Lambda_{jw}P_{H_{jw}}Cx,\ \Lambda_{jw}P_{H_{jw}}C'x\rangle\, d\mu(w) \leq B'\langle x, x\rangle_{\mathcal{A}}$$

Then, $\left\{H_{jw}, \Lambda_{jw}, v_{jw}\right\}_{j\in[n],\, w\in(\Omega\setminus Z)}$ is a woven continuous $(C, C')-$ controlled $K-g-$fusion frame for $\mathcal{M}$. with bounds $A'-\Delta$ and B' □

Proposition 3.1. *Let $\{H_w, \Gamma_w, v_w\}_{w\in\Omega}$ be a continuous $(C, C')-$controlled $K-g-$fusion frame for $\mathcal{M}$ with bounds A and B. If $V \in End^*_{\mathcal{A}}(\mathcal{M})$ be invertible operator and K be a closed range operator such that $\|I_{\mathcal{M}} - V\|^2 \left\|K^{\dagger}\right\|^2 \leq \frac{A}{B}\left\|K^{\dagger}\right\|^{-2}$, V commutes with C, C' and*

$$V^*VH_{jw} \subseteq H_{jw}, \quad \forall j \in [n].$$

Then : $\left\{V^{-1}H_w, \Gamma_w V, v_w\right\}_{w\in\Omega}$ and $\{VH_w, \Gamma_w, v_w\}_{w\in\Omega}$ are woven continuous $(C, C')-$-controlled $K-g-$fusion frame for $\mathcal{R}_{\mathcal{K}}$.

Proof. Let γ a partition of Ω and $x \in \mathcal{R}_{\mathcal{K}}$ and by lemma (1.3) we have:
$\langle x, x\rangle = \langle (K^{\dagger}K)^* x, (K^{\dagger}K)^* x\rangle \leq \left\|K^{\dagger}\right\|^2 \langle K^* x, K^* x\rangle$. We put :

$$E = \int_{\gamma} v_w^2 \langle \Gamma_w P_{H_w} Cx, \Gamma_w P_{H_w} C'x\rangle\, d\mu(w) + \int_{\gamma^c} v_w^2 \langle \Gamma_w V P_{V^{-1}H_w} Cx, \Gamma_w V P_{V^{-1}H_w} C'x\rangle\, d\mu(w)$$

We have :

$$\begin{aligned}
E &= \int_{\gamma} v_w^2 \langle \Gamma_w P_{H_w} Cx, \Gamma_w P_{H_w} C'x\rangle\, d\mu(w) + \int_{\gamma^c} v_w^2 \langle \Gamma_w V P_{V^{-1}H_w} Cx, \Gamma_w V P_{V^{-1}H_w} C'x\rangle\, d\mu(w)\\
&= \int_{\gamma} v_w^2 \langle \Gamma_w P_{H_w} Cx, \Gamma_w P_{H_w} C'x\rangle\, d\mu(w) + \int_{\gamma^c} v_w^2 \langle \Gamma_w P_{H_w} CVx, \Gamma_w P_{H_w} C'Vx\rangle\, d\mu(w)\\
&\geq \int_{\gamma} v_w^2 \langle \Gamma_w P_{H_w} Cx, \Gamma_w P_{H_w} C'x\rangle\, d\mu(w) - \int_{\gamma^c} v_w^2 \langle \Gamma_w P_{H_w} C(I_{\mathcal{M}} - V)x, \Gamma_w P_{H_w} C'(I_{\mathcal{M}} - V)x\rangle d\mu(w)\\
&\geq A\langle K^*x, K^*x\rangle - B\|I_{\mathcal{M}} - V\|^2 \langle x, x\rangle.\\
&\geq A\langle K^*x, K^*x\rangle - B\|I_{\mathcal{M}} - V\|^2 \left\|K^{\dagger}\right\|^2 \langle K^* x, K^* x\rangle.\\
&= \left(A - B\|I_{\mathcal{M}} - V\|^2 \left\|K^{\dagger}\right\|^2\right) \langle K^* x, K^* x\rangle.
\end{aligned}$$

So: $\left(A - B\|I_{\mathcal{M}} - V\|^2)\left\|K^{\dagger}\right\|^2\right) \langle K^* x, K^* x\rangle \leq E$
For the upper bound we have:

$$\begin{aligned}
E &= \int_{\gamma} v_w^2 \langle \Gamma_w P_{H_w} Cx, \Gamma_w P_{H_w} C'x\rangle\, d\mu(w) + \int_{\gamma^c} v_w^2 \langle \Gamma_w V P_{V^{-1}H_w} Cx, \Gamma_w V P_{V^{-1}H_w} C'x\rangle\, d\mu(w)\\
&= \int_{\gamma} v_w^2 \langle \Gamma_w P_{H_w} Cx, \Gamma_w P_{H_w} C'x\rangle\, d\mu(w) + \int_{\gamma^c} v_w^2 \langle \Gamma_w P_{H_w} CVx, \Gamma_w P_{H_w} C'Vx\rangle\, d\mu(w)\\
&\leq \int_{\Omega} v_w^2 \langle \Gamma_w P_{H_w} Cx, \Gamma_w P_{H_w} C'x\rangle\, d\mu(w) + \int_{\Omega} v_w^2 \langle \Gamma_w P_{H_w} CVx, \Gamma_w P_{H_w} C'Vx\rangle\, d\mu(w)\\
&\leq \left(B + \|V\|^2\right)\langle x, x\rangle.
\end{aligned}$$

Then: $\left\{V^{-1}H_w, \Gamma_w V, v_w\right\}_{w\in\Omega}$ and $\{VH_w, \Gamma_w, v_w\}_{w\in\Omega}$ are woven continuous $(C, C')-$ controlled $K-g-$fusion frame for $\mathcal{R}_{\mathcal{K}}$. □

The following theorem establishes a sufficient condition for the existence of a weaving continuous controlled $K-g-$fusion frame , formulated in terms of the positive operators corresponding to the underlying continuous controlled $K-g-$ fusion frames.

Theorem 3.8. *Let* $\Gamma = \left\{H_w, \phi_w, v_w\right\}_{w\in\Omega}$ *and* $\Psi = \{F_w, \psi_w, v_w\}_{w\in\Omega}$ *be a continuous* $(C, C')-$*controlled* $K-g-$*fusion frame for* $\mathcal{M}$ *with* A' *,*B' *and* C' *,*D' *the frame bounds of* Γ *and* Ψ *,respectively. ,Suppose for each* $w \in \Omega$*, the operator* $C'_w : \mathcal{M} \to \mathcal{M}$ *defined by*

$$\langle C'_w(x), y\rangle = \int_{\Omega} v_w^2 \langle C^*\Theta_w C' x, y\rangle\, d\mu_w, \quad x, y \in \mathcal{M}.$$

where $\Theta_w = P_{F_w}\psi_w^*\psi_w P_{F_w} - P_{H_w}\phi_w^*\phi_w P_{H_w}$*, is a positive operator. Hence* Ψ *and* Γ *are Woven continuous* $(C, C')-$*controlled* $K-g-$*fusion frame for* $\mathcal{M}$.

Proof. Let τ a partition of Ω so for every $x \in \mathcal{M}$ we have :

$$\begin{aligned}
A'\langle K^*x, K^*x\rangle &\le \int_{\Omega} v_w^2\langle \phi_w P_{H_w} Cx, \phi_w P_{H_w} C'x\rangle\, d\mu(w)\\
&= \int_{\tau} v_w^2\langle \phi_w P_{H_w} Cx, \phi_w P_{H_w} C'x\rangle\, d\mu(w) + \int_{\tau^c} v_w^2\langle \phi_w P_{H_w} Cx, \phi_w P_{H_w} C'x\rangle\, d\mu(w)\\
&= \int_{\tau} v_w^2\langle \phi_w P_{H_w} Cx, \phi_w P_{H_w} C'x\rangle\, d\mu(w) + \int_{\tau^c} v_w^2\langle C'^* P_{H_w}\phi_w^*\phi_w P_{H_w} Cx, x\rangle\, d\mu(w)\\
&= \int_{\tau} v_w^2\langle \phi_w P_{H_w} Cx, \phi_w P_{H_w} C'x\rangle\, d\mu(w) + \int_{\tau^c} v_w^2\langle C'^* P_{F_w}\psi_w^*\psi_w P_{F_w} Cx, x\rangle\, d\mu(w) - \int_{\tau^c} v_w^2\langle C'^*\Theta_w C'x, x\rangle\, d\mu(w)\\
&\le \int_{\tau} v_w^2\langle \phi_w P_{H_w} Cx, \phi_w P_{H_w} C'x\rangle\, d\mu(w) + \int_{\tau^c} v_w^2\langle \psi_w P_{F_w} Cx, \psi_w P_{F_w} C'x\rangle\, d\mu(w)\\
&\le (B' + D')\langle x, x\rangle.
\end{aligned}$$

Hence for all $x \in \mathcal{M}$

$$\begin{aligned}
A'\langle K^*x, K^*x\rangle \le &\int_{\tau} v_w^2\langle \phi_w P_{H_w} Cx, \phi_w P_{H_w} C'x\rangle\, d\mu(w)\\
&+ \int_{\tau^c} v_w^2\langle \psi_w P_{F_w} Cx, \psi_w P_{F_w} C'x\rangle\, d\mu(w) \le (B' + D')\langle x, x\rangle.
\end{aligned}$$

Thus Γ and Ψ are woven continuous $(C, C')-$controlled $K-g-$fusion frame for $\mathcal{M}$ with bounds A' and $B' + D'$ □

Theorem 3.9. *Let* $\Gamma_j = \{H_{jw}, \Lambda_{jw}, v_{jw}\}_{k\in\Omega}$ *be a family of continuous* $(C, C')-$ *controlled* $K-g-$*fusion frame for* $\mathcal{M}$*. with bounds* A_j *and* B_j *For each* $j \in [n]$*. Suppose there exists D such that*

$$0 \leq \int\limits_{Z} \Big\langle \Psi_{j,k}\, C\, x\,,\, \Psi_{j,k}\, C'\, x \Big\rangle\, d\mu_w$$

$$\leq\ D \min\left\{ \int\limits_{\Omega} v_{jw}^2 \,\langle \Lambda_{jw} P_{H_{jw}} C\, x\,, \Lambda_{jw} P_{H_{jw}} C' x\rangle\, d\mu(w),\ \int\limits_{\Omega} v_{kw}^2 \,\langle \Lambda_{kw} P_{H_{kw}} C\, x\,,\ \Lambda_{kw} P_{H_{kw}} C' x\rangle\, d\mu(w) \right\}.$$

for all $x \in \mathcal{M}$, $j \neq k \in [n]$ and $Z \subset \Omega$.
And $\Psi_{j,k} = v_{jw}^2 \Lambda_{jw} P_{H_{jw}} - v_{kw}^2 \Lambda_{kw} P_{H_{kw}}$
Then, the family $\Gamma = \{H_{jw}, \Lambda_{jw}, v_{jw}\}_{k\in\Omega,\, j\in[n]}$ is woven continuous $(C, C')-$ controlled $K-g-$fusion frame for $\mathcal{M}$ with universal bounds $\dfrac{\sum_{j\in[n]} A_j}{(n-1)(D+1)+1}$ and $\sum_{j\in[n]} B_j$.

Proof. Let $\left\{\sigma_j\right\}_{j\in[n]}$ be a partition of Ω. So for $x \in \mathcal{M}$, we have:

$$\sum_{j\in[n]} A_j \,\langle K^* x, K^* x\rangle \leq \sum_{j\in[n]} \int\limits_{\Omega} v_{jw}^2 \,\langle \Lambda_{jw} P_{H_{jw}} Cx,\ \Lambda_{jw} P_{H_{jw}} C' x\rangle\, d\mu(w)$$

$$= \sum_{j\in[n]} \sum_{k\in[n]} \int\limits_{\sigma_k} v_{jw}^2 \,\langle \Lambda_{jw} P_{H_{jw}} Cx,\ \Lambda_{jw} P_{H_{jw}} C' x\rangle\, d\mu(w)$$

$$\leq \sum_{j\in[n]} \Bigg[\int\limits_{\sigma_j} v_{jw}^2 \,\langle \Lambda_{jw} P_{H_{jw}} Cx,\ \Lambda_{jw} P_{H_{jw}} C' x\rangle\, d\mu(w) +$$

$$\sum_{k\in[n],\, k\neq j} \int\limits_{\sigma_k} \langle\, \Psi_{j,k}\, Cx,\ \Psi_{j,k}\, C' x\,\rangle\, d\mu_w +$$

$$\sum_{k\in[n],\, k\neq j} \int\limits_{\sigma_k} v_{kw}^2 \,\langle \Lambda_{kw} P_{H_{kw}} Cx,\ \Lambda_{kw} P_{H_{kw}} C' x\rangle\, d\mu(w) \Bigg],$$

$$\leq \sum_{j\in[n]} \Bigg[v_{jw}^2 \,\langle \Lambda_{jw} P_{H_{jw}} Cx,\ \Lambda_{jw} P_{H_{jw}} C' x\rangle\, d\mu(w) +$$

$$\sum_{k\in[n],\, k\neq j} (\,D\, +\, 1\,) \int\limits_{\sigma_k} v_{kw}^2 \,\langle \Lambda_{kw} P_{H_{kw}} Cx,\ \Lambda_{kw} P_{H_{kw}} C' x\rangle\, d\mu(w) \Bigg],$$

$$= \{\, (\,n\, -\, 1\,)\,(\,D\, +\, 1\,)\, +\, 1\,\} \sum_{j\in[n]} \int\limits_{\sigma_j} v_{jw}^2 \,\langle \Lambda_{jw} P_{H_{jw}} Cx,\ \Lambda_{jw} P_{H_{jw}} C' x\rangle\, d\mu(w).$$

So, for each $x \in \mathcal{M}$, we have

$$\frac{\sum_{j\in[n]} A_j}{(n-1)(D+1)+1} \,\langle K^* x, K^* x\rangle \leq \sum_{j\in[n]} \int\limits_{\sigma_j} v_{jw}^2 \,\langle \Lambda_{jw} P_{H_{jw}} Cx,\ \Lambda_{jw} P_{H_{jw}} C' x\rangle\, d\mu(w)\ \leq\ B' \,\langle\, x, x\,\rangle.$$

Then, the family $\Gamma = \{H_{jw}, \Lambda_{jw}, v_{jw}\}_{k\in\Omega,\, j\in[n]}$ is woven continuous $(C, C')-$ controlled $K-g-$fusion frame for $\mathcal{M}$ with universal bounds $\dfrac{\sum_{j\in[n]} A_j}{(n-1)(D+1)+1}$ and $\sum_{j\in[n]} B_j$. □

4. Perturbation of Woven Continuous Controlled g-Fusion Frame

In frame theory, a central issue is the stability of frames under perturbations. In this section, we show that under certain small perturbations, continuous controlled K-g-fusion frames remain woven continuous ,$(C, C')-$ controlled $K-g-$fusion frame.

Theorem 4.1. *Let $\Gamma = \left\{H_w, \phi_w, v_w\right\}_{w\in\Omega}$ and $\Psi = \{F_w, \psi_w, v_w\}_{w\in\Omega}$ be a continuous $(C, C')--$ controlled $K-g-$fusion frame for $\mathcal{M}$ with A' , B' and C' , D' the frame bounds of Γ and Ψ , respectively. Assume that there are constants non- negative γ , δ , and β with $0 < \gamma < 1$, $\delta < (1-\gamma)A' - \beta B'$ such that for each $x \in \mathcal{M}$, we put : $\Theta_w = P_{F_w}\psi_w^*\psi_w P_{F_w} - P_{H_w}\phi_w^*\phi_w P_{H_w}$ and we have :*

$$\begin{aligned}
0 &\le \int_\Omega v_w^2 \langle C^* \Theta_w C' x, x\rangle\, d\mu_w \\
&\le \gamma \int_\Omega v_w^2 \langle \psi_w P_{F_w} Cx, \psi_w P_{F_w} C'x\rangle\, d\mu(w) + \beta \int_\Omega v_w^2 \langle \phi_w P_{H_w} Cx, \phi_w P_{H_w} C'x\rangle\, d\mu(w) \\
&+ \delta \langle K^* x, K^* x\rangle.
\end{aligned}$$

Hence Ψ and Γ are Woven continuous $(C, C')-$ controlled $K-g-$ fusion frame for $\mathcal{M}$

Proof. Let τ a partition of Ω. So for each $x \in \mathcal{M}$, we have:

$$\begin{aligned}
L &= \int_\tau v_w^2 \langle \psi_w P_{F_w} Cx, \psi_w P_{F_w} C'x\rangle d\mu(w) + \int_{\tau^c} v_w^2 \langle \phi_w P_{H_w} Cx, \phi_w P_{H_w} C'x\rangle\, d\mu(w) \\
&\ge \int_\tau v_w^2 \langle \psi_w P_{F_w} Cx, \psi_w P_{F_w} C'x\rangle d\mu(w) \\
&- \int_{\tau^c} v_w^2 \Big\langle C^* \left(P_{F_w}\psi_w^*\psi_w P_{F_w} - P_{H_w}\phi_w^*\phi_w P_{H_w}\right) C' x, x\Big\rangle\, d\mu_w + \int_{\tau^c} v_w^2 \langle \psi_w P_{F_w} Cx, \psi_w P_{F_w} C'x\rangle d\mu(w) \\
&\ge \int_\Omega v_w^2 \langle \psi_w P_{F_w} Cx, \psi_w P_{F_w} C'x\rangle d\mu(w) - \int_\Omega v_w^2 \langle C^* \Theta_w C' x, x\rangle\, d\mu_w. \\
&\ge \int_\Omega v_w^2 \langle \psi_w P_{F_w} Cx, \psi_w P_{F_w} C'x\rangle d\mu(w) \\
&- \left(\gamma \int_\Omega v_w^2 \langle \psi_w P_{F_w} Cx, \psi_w P_{F_w} C'x\rangle\, d\mu(w) + \beta \int_\Omega v_w^2 \langle \phi_w P_{H_w} Cx, \phi_w P_{H_w} C'x\rangle\, d\mu(w) + \delta \langle K^* x, K^* x\rangle\right) \\
&\ge (1-\gamma) \int_\Omega v_w^2 \langle \psi_w P_{F_w} Cx, \psi_w P_{F_w} C'x\rangle d\mu(w) - \beta \int_\Omega v_w^2 \langle \phi_w P_{H_w} Cx, \phi_w P_{H_w} C'x\rangle\, d\mu(w) - \delta \langle K^* x, K^* x\rangle \\
&\ge \Big((1-\gamma)A' - \beta B' - \delta\Big) \langle K^* x, K^* x\rangle.
\end{aligned}$$

For the upper bounds we have:

$$L = \int_\tau v_w^2 \langle \psi_w P_{F_w} Cx, \psi_w P_{F_w} C'x\rangle d\mu(w) + \int_{\tau^c} v_w^2 \langle \phi_w P_{H_w} Cx, \phi_w P_{H_w} C'x\rangle\, d\mu(w)$$

$$\leq \int\limits_{\Omega} v_w^2 \langle \psi_w P_{F_w} Cx, \psi_w P_{F_w} C'x\rangle d\mu(w) + \int\limits_{\Omega} v_w^2 \langle \phi_w P_{H_w} Cx, \phi_w P_{H_w} C'x\rangle d\mu(w)$$
$$\leq (B' + D') \langle x, x\rangle$$

Hence:

$$\Big((1-\gamma) A' - \beta B' - \delta \Big) \langle K^* x, K^* x\rangle \leq L \leq (B' + D') \langle x, x\rangle$$

Thus Ψ and Γ are Woven continuous $(C, C')-$ controlled $K-g-$ fusion frame for $\mathcal{M}$. □

Conclusion

In this work, we have explored the concept of a woven continuous controlled $K-g-$ fusion frame in Hilbert C^*-modules and provided several propositions. We also present new results related to controlled generalized frames and discuss several properties of woven continuous controlled $K-g-$ fusion frames in Hilbert C^*-modules. Finally, we have explored the perturbation theory related to woven continuous controlled $K-g-$ fusion frames.

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References

[1] L. Arambašić, On Frames for Countably Generated Hilbert C^*-Modules, Proc. Am. Math. Soc. 135 (2006), 469–478. https://doi.org/10.1090/s0002-9939-06-08498-x.
[2] I. Kaplansky, Modules Over Operator Algebras, Am. J. Math. 75 (1953), 839–858. https://doi.org/10.2307/2372552.
[3] R. El Jazzar, R. Mohamed, On Frames in Hilbert Modules Over Locally C^*-Algebras, Int. J. Anal. Appl. 21 (2023), 130. https://doi.org/10.28924/2291-8639-21-2023-130.
[4] X. Fang, M.S. Moslehian, Q. Xu, On Majorization and Range Inclusion of Operators on Hilbert C^*-Modules, Linear Multilinear Algebr. 66 (2018), 2493–2500. https://doi.org/10.1080/03081087.2017.1402859.
[5] A. Karara, M. Rossafi, A. Touri, K-Biframes in Hilbert Spaces, J. Anal. 33 (2024), 235–251. https://doi.org/10.1007/s41478-024-00831-3.
[6] A. Karara, M. Rossafi, M. Klilou, S. Kabbaj, Construction of continuous K-g-Frames in Hilbert C^*-Modules, Palest. J. Math. 13 (2024), 198–209.
[7] A. Lfounoune, R. El Jazzar, K-Frames in Super Hilbert C^*-Modules, Int. J. Anal. Appl. 23 (2025), 19. https://doi.org/10.28924/2291-8639-23-2025-19.
[8] A. Lfounoune, H. Massit, A. Karara, M. Rossafi, Sum of G-Frames in Hilbert C^*-Modules, Int. J. Anal. Appl. 23 (2025), 64. https://doi.org/10.28924/2291-8639-23-2025-64.
[9] V. Manuilov, E. Troitsky, Hilbert C^*-Modules, American Mathematical Society, Providence, 2005. https://doi.org/10.1090/mmono/226.
[10] H. Massit, M. Rossafi, C. Park, Some Relations Between Continuous Generalized Frames, Afr. Mat. 35 (2023), 12. https://doi.org/10.1007/s13370-023-01157-2.
[11] F. Nhari, R. Echarghaoui, M. Rossafi, K-g-Fusion Frames in Hilbert C^*-Modules, Int. J. Anal. Appl. 19 (2021), 836–857. https://doi.org/10.28924/2291-8639-19-2021-836.
[12] F. Nhari, C. Park, M. Rossafi, Continuous K-g-Fusion Frames in Hilbert C^*-Modules, J. Mat. Stat. Komput. 19 (2023), 240–265. https://doi.org/10.20956/j.v19i2.23961.

[13] F.D. Nhari, M. Rossafi, Controlled Continuous K-g-Fusion Frame in Hilbert C^*-Modules, Trans. A. Razmadze Math. Inst. 178 (2024), 259–274.

[14] E.H. Ouahidi, M. Rossafi, Woven Continuous Generalized Frames in Hilbert C^*-Modules, Int. J. Anal. Appl. 23 (2025), 84. https://doi.org/10.28924/2291-8639-23-2025-84.

[15] M. Rossafi, A. Karara, R. El Jazzar, Biframes in Hilbert C^*-Modules, Montes Taurus J. Pure Appl. Math. 7 (2025), 69–80.

[16] M. Rossafi, M. Ghiati, M. Mouniane, F. Chouchene, A. Touri, S. Kabbaj, Continuous Frame in Hilbert C^*-Modules, J. Anal. 31 (2023), 2531–2561. https://doi.org/10.1007/s41478-023-00581-8.

[17] M. Rossafi, F. Nhari, C. Park, S. Kabbaj, Continuous g-Frames with C^*-Valued Bounds and Their Properties, Complex Anal. Oper. Theory 16 (2022), 44. https://doi.org/10.1007/s11785-022-01229-4.

[18] M. Rossafi, F. Nhari, Controlled K-g-fusion Frames in Hilbert C^*-Modules, Int. J. Anal. Appl. 20 (2022), 1. https://doi.org/10.28924/2291-8639-20-2022-1.

[19] M. Rossafi, F.D. Nhari, K-g-Duals in Hilbert C^*-Modules, Int. J. Anal. Appl. 20 (2022), 24. https://doi.org/10.28924/2291-8639-20-2022-24.

[20] M. Rossafi, S. Kabbaj, Generalized Frames for $B(H,K)$, Iran. J. Math. Sci. Inform. 17 (2022), 1–9. https://doi.org/10.52547/ijmsi.17.1.1.

[21] M. Rossafi, F. Nhari, A. Touri, Continuous Generalized Atomic Subspaces for Operators in Hilbert Spaces, J. Anal. 33 (2024), 927–947. https://doi.org/10.1007/s41478-024-00869-3.

[22] M. Rossafi, H. Massit, C. Park, Weaving Continuous Generalized Frames for Operators, Montes Taurus J. Pure Appl. Math. 6 (2024), 64–73.

[23] M. Rossafi, S. Kabbaj, Some Generalizations of Frames in Hilbert Modules, Int. J. Math. Math. Sci. 2021 (2021), 5522671. https://doi.org/10.1155/2021/5522671.

[24] M. Rossafi, S. Kabbaj, $*$-K-Operator Frame for $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$, Asian-Eur. J. Math. 13 (2020), 2050060. https://doi.org/10.1142/S1793557120500606.

[25] M. Rossafi, S. Kabbaj, Operator Frame for $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$, J. Linear Topol. Algebra 8 (2019), 85–95.

[26] M. Rossafi, S. Kabbaj, $*$-K-g-Frames in Hilbert $\mathcal{A}$-modules, J. Linear Topol. Algebra 7 (2018), 63–71.

[27] M. Rossafi, S. Kabbaj, $*$-g-Frames in Tensor Products of Hilbert C^*-Modules, Ann. Univ. Paedagog. Crac. Stud. Math. 17 (2018), 17–25.

[28] M. Rossafi, K. Mabrouk, M. Ghiati, M. Mouniane, Weaving Operator Frames for $B(H)$, Methods Funct. Anal. Topol. 29 (2023), 111–124.

[29] S. Touaiher, R. El Jazzar, M. Rossafi, Properties and Characterizations of Controlled K-G-Fusion Frames Within Hilbert C^*-Modules, Int. J. Anal. Appl. 23 (2025), 111. https://doi.org/10.28924/2291-8639-23-2025-111.

[30] S. Touaiher, M. Rossafi, Construction of New Continuous K-G-Frames Within Hilbert C^*-Modules, Int. J. Anal. Appl. 23 (2025), 232. https://doi.org/10.28924/2291-8639-23-2025-232.

[31] A. Touri, H. Labrigui, M. Rossafi, S. Kabbaj, Perturbation and Stability of Continuous Operator Frames in Hilbert C^*-Modules, J. Math. 2021 (2021), 5576534. https://doi.org/10.1155/2021/5576534.

[32] A. Touri, H. Labrigui, M. Rossafi, New Properties of Dual Continuous K-g-Frames in Hilbert Spaces, Int. J. Math. Math. Sci. 2021 (2021), 6671600. https://doi.org/10.1155/2021/6671600.