Int. J. Anal. Appl. (2025), 23:275

International Journal of Analysis and Applications

Construction of Proximal Fuzzy Iterated Function Systems

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Abstract. In this paper, we study the existence of best proximity point for set-valued non-self contractions, like p-cyclic contractions (p-CC) and p-cyclic ϕ-contractions (p-C-ϕ-C) in a given fuzzy metric space, which extends and improves some existing results in the literature. Furthermore, using the aforementioned contractions, we construct the Iterated Function Systems (IFSs).

1. Introduction

A key finding in fixed point theory is the Banach contraction principle. By varying the functions and the underlying spaces, several authors have established generalized versions of this result [5, 17, 18, 23, 24, 28]. The following result was obtained by Kirk et al. in [18] where they extended the Banach contraction principle to a novel class of mappings known as the p-cyclic contractions.

Theorem 1.1. *[18] Let $\Gamma_1, \Gamma_2, ..., \Gamma_p$, $(p \geq 2, p \in \mathbb{N})$ be non-empty closed subsets of a complete metric space (Ω, d). Let $g : \cup_{i=1}^{p}\Gamma_i \to \cup_{i=1}^{p}\Gamma_i$ satisfy the conditions below:*

(i) $g(\Gamma_i) \subseteq \Gamma_{i+1}$, $1 \leq i \leq p$, *where* $\Gamma_{p+1} = \Gamma_1$,

(ii) $\exists k \in (0, 1)$ *such that* $d(ga, ga^*) \leq kd(a, a^*)$, $a \in \Gamma_i$, $a^* \in \Gamma_{i+1}$.

Then g has a unique fixed point.

Suppose Γ is a non-empty subset of a metric space (Ω, d). A map $g : \Gamma \to \Gamma$ is said to have a fixed point if $gx = x$ for some $x \in \Gamma$. When the equation $gx = x$ does not have a solution, that is, $d(x, gx) > 0$ for all $x \in \Gamma$, we must look for an element $x \in \Gamma$ such that $d(x, \Gamma x)$ is minimal. That is,

Received: Sep. 19, 2025.

2020 *Mathematics Subject Classification.* 47H10, 54H25.

Key words and phrases. best attractor; iterated function systems; p-cyclic ϕ-contraction; p-cyclic contraction.

https://doi.org/10.28924/2291-8639-23-2025-275

ISSN: 2291-8639

let Γ_1, Γ_2 be two disjoint non-empty subsets of a metric space (Ω, d) and $g : \Gamma_1 \to \Gamma_2$ be a non-self map. Then g is said to have a best proximity point in Γ_1 if we can find an element $x \in \Gamma_1$ such that

$$d(x, gx) = d(\Gamma_1, \Gamma_2) = \inf\{d(x, y) : x \in \Gamma_1 \text{ and } y \in \Gamma_2\}.$$

The existence of the best proximity point for a p-cyclic contraction in a complete metric space was later proved by Karpagam and Sushama in [16], who extended the findings to the case where the intersection of Γ_i's is empty.

Definition 1.1. *[16] Let $\{\Gamma_i\}_{i=1}^{p}$ be non-empty subsets of a metric space (Ω, d) and let $g : \bigcup_{i=1}^{p} \Gamma_i \to \bigcup_{i=1}^{p} \Gamma_i$. Then g is a p-cyclic contraction, if*

(i) $g(\Gamma_i) \subseteq \Gamma_{i+1}$, $1 \le i \le p$, *where* $\Gamma_{p+i} = \Gamma_i$

(ii) $\exists k \in (0, 1)$ *such that*

$$d(ga, ga^*) \le kd(a, a^*) + (1 - k)d(\Gamma_i, \Gamma_{i+1}),\ a \in \Gamma_i, a^* \in \Gamma_{i+1}, 1 \le i \le p.$$

Then, a point $a \in \Gamma_i$ is known as a best proximity point of g, if $d(a, ga) = d(\Gamma_i, \Gamma_{i+1})$.

Additionally, Al-Thagafi and Shahzad [3] established the existence of a best proximity point for a cyclic ϕ-contraction map and introduced the idea of cyclic ϕ-contraction maps, a bigger class than cyclic contraction maps.

Furthermore, fuzzy logic enables us to analyze the world around us using only mathematics, despite the fact that it is full of uncertainty. Zadeh [32] introduced fuzzy sets and examined their properties using membership functions. After Zadeh's work, Kramosil and Michálek [19] established KM fuzzy metric space and extended fuzzy concepts to metric spaces. Later, George and Veeramani [11] generalized KM fuzzy metric spaces to GV fuzzy metric spaces. Many writers have provided various descriptions of the fuzzy metric space [8,10,33]. Since fuzzy metric spaces are an extension of metric spaces, many theorems from fixed point theory can be extended to fuzzy settings. See [1,9,13,15,25,31] for further details. Rodriguez and Romaguera [26] derived the fuzzy version of the Banach contraction principle for a single valued self map in a complete fuzzy metric space (FMS).

On the other hand, Mandelbrot [20] introduced the fractal theory. Later, Barnsley [7] used Iterated Function Systems (IFS) to create fractals. Additionally, he proved a fractal version of the Banach contraction principle, establishing the existence of an attractor of an IFS.

Definition 1.2. *[7] An IFS consists of a complete metric space (Ω, d) together with a finite set of contraction mappings $g_i : \Omega \to \Omega$ with respective contractivity factors k_i for $i = 1, 2, \cdots, s$. That is,*

$$d(g_i x, g_i y) \le k_i d(x, y),$$

for all $x, y \in \Omega$ and $n = 1, 2, \cdots, s$. The notation for the IFS is $\{\Omega, g_i,\ i = 1, 2, ..., s,\ s \in \mathbb{N}\}$ and its contractivity factor is $k = \max\{k_i : i = 1, 2, \cdots, s\}$.

Theorem 1.2. *[7] Let $\{\Omega, g_i,\ i = 1, 2, ..., s,\ s \in \mathbb{N}\}$ be an IFS with contractivity factor k and $\mathcal{P}_{cp}(\Omega)$ denotes the set of all non-empty compact subsets of Ω. Then, the mapping $G : \mathcal{P}_{cp}(\Omega) \to \mathcal{P}_{cp}(\Omega)$ defined by*

$$G(E) = \bigcup_{i=1}^{s} g_i(E),\ \forall\, E \in \mathcal{P}_{cp}(\Omega)$$

is also a contraction map on the complete metric space $(\mathcal{P}_{cp}(\Omega), h)$ with contractivity factor k, where h is the Hausdorff metric. That is

$$h(GE, GT) \leq k\, h(E, T)$$

for all $E, T \in \mathcal{P}_{cp}(\Omega)$. Its unique fixed point is given by, $E = \lim_{n\to\infty} G^{\circ n}(T)$, for any $T \in \mathcal{P}_{cp}(\Omega)$ and it obeys

$$E = G(E) = \bigcup_{i=1}^{s} g_i(E).$$

The attractor of the given IFS $\{\Omega, g_i, 1 \leq i \leq s\}$ is the fixed point of the function $G : \mathcal{P}_{cp}(\Omega) \to \mathcal{P}_{cp}(\Omega)$.

By varying the contraction and space, many researchers have established the existence of attractors of different IFS, refer to [2, 4, 6, 12, 14, 27]. In particular, Altun et al. [4] have extended the ideas of IFS and attractors to the proximal setting and thereby extending the Theorem 1.2 to non-self maps. Inspired by the work of Altun et al. [4], Unni and Pragadeeswarar [29] developed the proximal IFSs using cyclic Meir-Keeler contractions and constructed fractals using the new IFS. Analogously, in [30], they developed proximal fuzzy IFSs using a new class of contractions in the setting of fuzzy metric spaces and constructed fractals from the correesponding IFSs in their respective framework.

There are studies on the existence of an attractor for various cyclic contractions (see [21, 22] for further information). However, there is a research gap in the literature, on the existence of best attractor of the IFS consisting of cyclic maps in a fuzzy metric space.

In this work, we prove an extended fractal version of the Banach contraction principle for a p-cyclic contraction in a fuzzy metric space, inspired by the works of Karpagam [16] and Al-Thagafi [3]. Furthermore, we use p-cyclic ϕ-contractions to create an IFS.

The paper is organized as follows. We first prove the existence of the best attractor for a p-cyclic contraction. Then, we extend the results to a p-cyclic ϕ-contraction. In addition, we construct IFS using both contractions, and as a consequence, we have an application in fractals. Finally, we discuss the scope of future research in this direction.

2. Preliminaries

The following are a few definitions and results that are required in the sequel.

Definition 2.1. *[32] Let Ω be a space of points (with a generic element of Ω denoted by m). A fuzzy set M in Ω is characterized by a membership (characteristic) function*

$$\mu_M : \Omega \to [0, 1],$$

which associates with each point $m \in \Omega$ a real number in the interval $[0,1]$. The value $\mu_M(m)$ represents the grade of membership *of m in M*

Definition 2.2. *[11] A triplet $(\Omega, M, *)$ is said to be a fuzzy metric space (FMS), if Ω is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $\Omega^2 \times (0,\infty)$ satisfying the following conditions for all $z_1, z_2, z_3 \in \Omega$ and $t_1, t_2 > 0$:*

(1) $M(z_1, z_2, t_1) > 0$,
(2) $M(z_1, z_2, t_1) = 1 \iff z_1 = z_2$,
(3) $M(z_1, z_2, t_1) = M(z_2, z_1, t_1)$,
(4) $M(z_1, z_2, .) : (0,\infty) \to [0,1]$ *is continuous,*
(5) $M(z_1, z_3, t_1) * M(z_3, z_2, t_2) \leq M(z_1, z_2, t_1 + t_2)$.

Lemma 2.1. *[26] M is a continuous function on $\Omega \times \Omega \times (0,\infty)$.*

Lemma 2.2. *[13] $M(x_1, x_2, .)$ is non-decreasing for all x_1, x_2 in Ω.*

Here, we have some notions and they will be used frequently throughout the research article. Let Γ_1, Γ_2 be non-empty subsets of a FMS $(\Omega, M, *)$ and $\mathcal{P}_{cp}(\Omega)$ represents the family of all non-empty compact subsets of Ω.

Now, $M_H : \mathcal{P}_{cp}(\Omega) \times \mathcal{P}_{cp}(\Omega) \to [0,\infty)$ is defined as below.

$$M_H(\Gamma_1, \Gamma_2, t) = \min\{\inf_{v \in \Gamma_1} M(v, \Gamma_2, t),\ \inf_{w \in \Gamma_2} M(w, \Gamma_1, t)\},\ \forall\ t \geq 0.$$

Recall,

$$M(v, \Gamma_2, t) = \sup_{w \in \Gamma_2} M(v, w, t),\ \forall\ t \geq 0.$$

Here, M_H is known as the Hausdorff fuzzy metric.

Theorem 2.1. *[26] Let $(\Omega, M, *)$ be a FMS. Then, $(\mathcal{P}_{cp}(\Omega), M_H, *)$ is also a FMS.*

Now, we have some important Lemma's which support our results.

Lemma 2.3. *[30] $M_H(\Gamma_1, \Gamma_2, .)$ is non-decreasing for all $\Gamma_1, \Gamma_2 \in \mathcal{P}_{cp}(\Omega)$.*

Lemma 2.4. *[30] $M_H(\Gamma_1, \Gamma_2, t) = M(p_0, q_0, t)$ for some $p_0 \in \Gamma_1$ and $q_0 \in \Gamma_2$, where $\Gamma_1, \Gamma_2 \in \mathcal{P}_{cp}(\Omega)$.*

Lemma 2.5. *[30] If $\Gamma_1 \subset \Gamma_2$, then $M(v, \Gamma_2, t) \geq M(v, \Gamma_1, t)$, $\forall\, \Gamma_1, \Gamma_2 \in \mathcal{P}_{cp}(\Omega)$, $v \in \Omega$ and $t > 0$.*

Lemma 2.6. *[30] If $\Gamma_2 \subset \Gamma_3$, then $\inf_{w \in \Gamma_1} M(w, \Gamma_3, t) \geq \inf_{w \in \Gamma_1} M(w, \Gamma_2, t)$, for all $\Gamma_1, \Gamma_2, \Gamma_3 \in \mathcal{P}_{cp}(\Omega)$ and $t > 0$.*

Lemma 2.7. *[30] Let $(\Omega, M, *)$ be a FMS and $\Gamma_1, \Gamma_2, \Lambda_1, \Lambda_2 \in \mathcal{P}_{cp}(\Omega)$. Then, we have*

$$M_H(\Gamma_1 \cup \Gamma_2, \Lambda_1 \cup \Lambda_2, t)\ \geq\ \min\{M_H(\Gamma_1, \Lambda_1, t), M_H(\Gamma_2, \Lambda_2, t)\},\ \forall\, t > 0.$$

In general we have,

Lemma 2.8. *[30] Let $(\Omega, M, *)$ be a FMS and $\{X_i\}_{i=1}^s$, $\{Y_i\}_{i=1}^s$ be collections of subsets of $\mathcal{P}_{cp}(\Omega)$. Then we have,*

$$M_H(\bigcup_{i=1}^{s} X_i, \bigcup_{i=1}^{s} Y_i, t) \geq \min_{1 \leq i \leq s} M_H(X_i, Y_i, t), \quad \forall t > 0, s \in \mathbb{N}.$$

Now, we define the following notion,

$$\hat{M}(\mathcal{P}_{cp}(\Gamma_1), \mathcal{P}_{cp}(\Gamma_2), t) \quad = \quad \sup\{M_H(\Lambda_1, \Lambda_2, t) : \Lambda_1 \in \mathcal{P}_{cp}(\Gamma_1),\ \Lambda_2 \in \mathcal{P}_{cp}(\Gamma_2)\}, \forall t > 0,$$

where $\Gamma_1, \Gamma_2 \subseteq \Omega$.
Recall, $M(\Gamma_1, \Gamma_2, t) = \sup\{M(v, w, t) : v \in \Gamma_1 \text{ and } w \in \Gamma_2\}$.

Remark 2.1. *Let $(\Omega, M, *)$ be a FMS and $\Gamma_1, \Gamma_2 \subseteq \Omega$. Then we have, $\hat{M}(\mathcal{P}_{cp}(\Gamma_1), \mathcal{P}_{cp}(\Gamma_2), t) \leq M(\Gamma_1, \Gamma_2, t)$, $\forall t > 0$.*

Proof. For some $\Lambda_1 \subseteq \Gamma_1$, $\Lambda_2 \subseteq \Gamma_2$, we have,

$$M(\Gamma_1, \Gamma_2, t) \geq M(\Lambda_1, \Lambda_2, t) \geq M_H(\Lambda_1, \Lambda_2, t).$$

Therefore,

$$\begin{aligned} M(\Gamma_1, \Gamma_2, t) &\geq \sup\{M_H(\Lambda_1, \Lambda_2, t) : \Lambda_1 \in \mathcal{P}_{cp}(\Gamma_1),\ \Lambda_2 \in \mathcal{P}_{cp}(\Gamma_2)\}, \forall t > 0. \\ &= \hat{M}(\mathcal{P}_{cp}(\Gamma_1), \mathcal{P}_{cp}(\Gamma_2), t), \ \forall t > 0. \end{aligned}$$

□

Lemma 2.9. *Let $A, B \in \mathcal{P}_{cp}(\Omega)$. Then, for given $s_1 \in A$, we can find $s_2 \in B$ such that $M(s_1, s_2, t) \geq M_H(A, B, t)$.*

Proof. Let A and B be compact subsets of Ω. Hence for an arbitrary $s_1 \in A$, we can find $s_2 \in B$ such that

$$\begin{aligned} M(s_1, s_2, t) &= \sup_{s_2 \in B} M(s_1, B, t) \\ &\geq \inf_{s_1 \in A} M(A, B, t) \\ &\geq M_H(A, B, t). \end{aligned}$$

□

3. Main Results

Here, we prove the existence of the best proximity point for multivalued p-cyclic contractions and prove the same for multivalued p-cyclic ϕ-contractions. Throughout the paper, $\{\Gamma_i\}_{i=1}^p$ $(p \geq 2, p \in \mathbb{N})$ denotes a collection of non-empty subsets of a FMS $(\Omega, M, *)$.

3.1. **IFS consisting p-cyclic contraction.**

Definition 3.1. *Let $\{\Gamma_i\}_{i=1}^{p}$ be non-empty subsets of a metric space $(\Omega, M, *)$ and let $g : \bigcup_{i=1}^{p} \Gamma_i \to \bigcup_{i=1}^{p} \Gamma_i$; g is a p-cyclic contraction (p-CC), if it satisfies the following conditions below:*

(i) *$g(\Gamma_i) \subseteq \Gamma_{i+1}$, $i = 1, 2, \cdots p$, where $\Gamma_{p+i} = \Gamma_i$,*

(ii) *for some $0 < k < 1$,*

$$M(ga, ga^*, t) \geq kM(a, a^*, t) + (1-k)M(\Gamma_i, \Gamma_{i+1}, t),\ a \in \Gamma_i, a^* \in \Gamma_{i+1}, i = 1, 2, \cdots p,$$

$\forall t > 0$. Then, a point $a \in \Gamma_i$ is a best proximity point, if $M(a, ga, t) = M(\Gamma_i, \Gamma_{i+1}, t)$.

Theorem 3.1. *Let $\{\Gamma_i\}_{i=1}^{p}$ be a collection of non-empty subsets of a FMS $(\Omega, M, *)$ and let $g : \bigcup_{i=1}^{p} \Gamma_i \to \bigcup_{i=1}^{p} \Gamma_i$ be a continuous map. Suppose, g is a p-CC, then the following conditions hold:*

(i) *under g, the elements in $\bigcup_{i=1}^{p} \mathcal{P}_{cp}(\Gamma_i)$ are mapped to the elements in $\bigcup_{i=1}^{p} \mathcal{P}_{cp}(\Gamma_i)$,*

(ii) *for any $E \in \bigcup_{i=1}^{p} \mathcal{P}_{cp}(\Gamma_i)$, the mapping $g : \bigcup_{i=1}^{p} \mathcal{P}_{cp}(\Gamma_i) \to \bigcup_{i=1}^{p} \mathcal{P}_{cp}(\Gamma_i)$ defined as*

$$g(E) = \{g(s) : s \in E\}$$

is a p-CC with respect to the Hausdorff metric.

Proof. (i) Given g is continuous, hence the compact subsets of Γ_i are mapped to the compact subsets of Γ_{i+1}. For some $E \in \mathcal{P}_{cp}(\Gamma_i)$, we have $g(E) \in \mathcal{P}_{cp}(\Gamma_{i+1})$.

(ii) Let $P \in \mathcal{P}_{cp}(\Gamma_i)$ and $Q \in \mathcal{P}_{cp}(\Gamma_{i+1})$. Then, $gP \in \mathcal{P}_{cp}(\Gamma_{i+1})$ and $gQ \in \mathcal{P}_{cp}(\Gamma_{i+2})$. Thus for $s \in P$, we get

$$\begin{aligned} M(gs, gQ, t) &= \sup_{s^* \in Q} M(gs, gs^*, t) \\ &\geq \sup_{s^* \in Q} [kM(s, s^*, t) + (1-k)\, M(\Gamma_i, \Gamma_{i+1}, t)] \\ &\geq \sup_{s^* \in Q} [kM(s, s^*, t) + (1-k)\, \hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t)] \\ &= kM(s, Q, t) + (1-k)\, \hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t). \end{aligned}$$

Also, for $s^* \in Q$, we get

$$M(gs^*, gP, t) \geq kM(s^*, P, t) + (1-k)\, \hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t).$$

Now,

$$\begin{aligned} M_H(gP, gQ, t) &= \min\{\inf_{s \in P} M(gs, gQ, t), \inf_{s^* \in Q} M(gs^*, gP, t)\} \\ &\geq \min\{\inf_{s \in P} [kM(s, Q, t) + (1-k)\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t)], \\ &\qquad \inf_{s^* \in Q} [kM(s^*, P, t) + (1-k)\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t)]\} \\ &= k \min\{\inf_{s \in P} M(s, Q, t), \inf_{s^* \in Q} M(s^*, P, t)\} + (1-k)\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1})) \\ &= k\, M_H(P, Q, t) + (1-k)\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t). \end{aligned}$$

Hence,

$$M_H(gP, gQ, t) \geq k\, M_H(P, Q, t) + (1-k)\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t),$$

where $P \in \mathcal{P}_{cp}(\Gamma_i)$, $Q \in \mathcal{P}_{cp}(\Gamma_{i+1})$. Therefore, the result follows.

□

Proposition 3.1. *Let $\{\Gamma_i\}_{i=1}^{p}$ be a collection of non-empty subsets of a FMS $(\Omega, M, *)$. Given, continuous mappings $g_j : \bigcup_{i=1}^{p}\Gamma_i \to \bigcup_{i=1}^{p}\Gamma_i$, $j = 1, 2, ..., r\, (r \in \mathbb{N})$, and f_j's are p-CCs with contractivity factors $k_j \in (0,1), \forall 1 \leq j \leq r$,*

$$M(g_j s, g_j s^*, t) \geq k_j M(s, s^*, t) + (1-k_j)M(\Gamma_i, \Gamma_{i+1}, t),$$

for all $s \in \Gamma_i, s^* \in \Gamma_{i+1}$. Let $G : \bigcup_{i=1}^{p}\mathcal{P}_{cp}(\Gamma_i) \to \bigcup_{i=1}^{p}\mathcal{P}_{cp}(\Gamma_i)$ be defined as follows:

$$\begin{aligned} G(E) &= g_1(E) \cup g_2(E) \cup ... \cup g_r(E) \\ &= \bigcup_{j=1}^{r} g_j(E),\ \forall E \in \bigcup_{i=1}^{p}\mathcal{P}_{cp}(\Gamma_i). \end{aligned}$$

Then, G is also a p-CC with respect to the Hausdorff metric. That is,

$$M_H(GP, GQ, t) \geq k_* M_H(P, Q, t) + (1-k_*)\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t),$$

for all $P \in \mathcal{P}_{cp}(\Gamma_i), Q \in \mathcal{P}_{cp}(\Gamma_{i+1})$, where $k_* = \max_{1\leq j\leq r} k_j$.

Proof. We use Principle of Mathematical Induction for proving the result. For $r = 1$, the result is trivial. Now, we are going to verify it for $r = 2$.
Let $P \in \mathcal{P}_{cp}(\Gamma_i), Q \in \mathcal{P}_{cp}(\Gamma_{i+1})$. Then,

$$\begin{aligned} M_H(GP, GQ, t) &= M_H(\bigcup_{j=1}^{2} g_j(P), \bigcup_{j=1}^{2} g_j(Q), t) \\ &= M_H(g_1P \cup g_2P, g_1Q \cup g_2Q, t) \\ &\geq M_H(g_1P, g_1Q, t) \wedge M_H(g_2P, g_2Q, t) \\ &\geq \min\{k_1 M_H(P, Q, t) + (1-k_1)\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1})), \\ &\qquad k_2 M_H(P, Q, t) + (1-k_2)\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}))\} \\ &= \min\{k_1[M_H(P, Q, t) - \hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t)] + \hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t), \\ &\qquad k_2[M_H(P, Q, t) - \hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t)] + \hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t)\} \\ &\geq k_*[M_H(P, Q, t) - \hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t)] + \hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t), \end{aligned}$$

where $k_* = \max\{k_1, k_2\}$. That is,

$$\begin{aligned} M_H(GP, GQ, t) &\geq k_*[M_H(P, Q, t) - \hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t)] + \hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t) \\ &= k_* M_H(P, Q, t) + (1-k_*)\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t). \end{aligned}$$

Hence, the result can be easily proved using the Principle of Mathematical Induction. □

Proposition 3.2. *Let $\{\Gamma_i\}_{i=1}^{p}$ be a collection of non-empty closed subsets of a FMS $(\Omega, M, *)$. Given, continuous mappings $g_j : \bigcup_{i=1}^{p} \Gamma_i \to \bigcup_{i=1}^{p} \Gamma_i$, $j = 1, 2, ..., r$ $(r \in \mathbb{N})$, and g_j's are p-CCs with contractivity factors $k_j \in (0, 1), \forall\, 1 \le j \le r$. Consider $G : \bigcup_{i=1}^{p} \mathcal{P}_{cp}(\Gamma_i) \to \bigcup_{i=1}^{p} \mathcal{P}_{cp}(\Gamma_i)$ defined as in Proposition 3.1. Then, for every $P, Q \in \mathcal{P}_{cp}(\Gamma_i)$, for $1 \le i \le p$, we have*

(i) $\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t) = \hat{M}(\mathcal{P}_{cp}(\Gamma_{i+1}), \mathcal{P}_{cp}(\Gamma_{i+2}), t)$, $\forall\, 1 \le i \le p$, *where* $\Gamma_{p+i} = \Gamma_i$,

(ii) $M_H(G^{pn}P, G^{pn+1}Q, t) \to \hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t)$ *as* $n \to \infty$,

(iii) $M_H(G^{pn \mp p}P, G^{pn+1}Q, t) \to \hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t)$ *as* $n \to \infty$.

Proof. (i) By Proposition 3.1, we have G is also a p-CC with respect to the Hausdorff metric with contractivity factor $k = \max_{1 \le j \le r} k_j$.

Let $P \in \mathcal{P}_{cp}(\Gamma_i)$ and $Q \in \mathcal{P}_{cp}(\Gamma_{i+1})$. Then, we obtain

$$\begin{aligned}
\hat{M}(\mathcal{P}_{cp}(\Gamma_{i+1}), \mathcal{P}_{cp}(\Gamma_{i+2}), t) &\ge M_H(GP, GQ, t) \\
&\ge kM_H(P, Q, t) + (1-k)\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t) \\
&\ge kM_H(P, Q, t) + (1-k)M_H(P, Q, t) \\
&= M_H(P, Q, t).
\end{aligned}$$

Which implies

$$\hat{M}(\mathcal{P}_{cp}(\Gamma_{i+1}), \mathcal{P}_{cp}(\Gamma_{i+2}), t) \ge \hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t), \ \forall i = 1, 2, ..., p.$$

Therefore, we have

$$\begin{aligned}
\hat{M}(\mathcal{P}_{cp}(\Gamma_p), \mathcal{P}_{cp}(\Gamma_1), t) &\ge \hat{M}(\mathcal{P}_{cp}(\Gamma_{p-1}), \mathcal{P}_{cp}(\Gamma_p), t) \\
&\ge ... \ge \hat{M}(\mathcal{P}_{cp}(\Gamma_1), \mathcal{P}_{cp}(\Gamma_2), t) \ge \hat{M}(\mathcal{P}_{cp}(\Gamma_p), \mathcal{P}_{cp}(\Gamma_1), t).
\end{aligned}$$

Hence

$$\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t) = \hat{M}(\mathcal{P}_{cp}(\Gamma_{i+1}), \mathcal{P}_{cp}(\Gamma_{i+2}), t),$$

for all $i = 1, 2, ..., p$, where $\Gamma_{p+i} = \Gamma_i$.

(ii) For $P, Q \in \mathcal{P}_{cp}(\Gamma_i)$, we have

$$\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t) \ge M_H(G^{pn}P, G^{pn+1}Q, t).$$

Now, since G is a p-CC with contractivity factor $k = \max_{1 \le j \le r} k_j$, we get

$$\begin{aligned}
\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t) &\ge M_H(G^{pn}P, G^{pn+1}Q, t) \\
&\ge k\, M_H(G^{pn-1}P, G^{pn}Q, t) + (1-k)\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t) \\
&\ge k\, [k\, M_H(G^{pn-2}P, G^{pn-1}Q, t) + (1-k)\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t)] \\
&\qquad + (1-k)\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t) \\
&= k^2 M_H(G^{pn-2}P, G^{pn-1}Q, t) + (1-k^2)\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t).
\end{aligned}$$

Proceeding in this way, we have

$$M_H(G^{pn}P, G^{pn+1}Q, t) \geq k^{pn} M_H(P, GQ, t) + (1 - k^{pn})\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t).$$

Therefore, as $n \to \infty$, we get

$$M_H(G^{pn}P, G^{pn+1}Q, t) \to \hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t).$$

(iii) In the same way, we can prove that

$$M_H(G^{pn\pm p}P, G^{pn+1}Q, t) \to \hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t), \text{ as } n \to \infty.$$

□

Theorem 3.2. *Let $(\Omega, M, *)$ be a FMS and $\{\Gamma_i\}_{i=1}^{p}$ be a collection of non-empty closed subsets of Ω. If $g_j : \bigcup_{i=1}^{p} \Gamma_i \to \bigcup_{i=1}^{p} \Gamma_i$, $j = 1, 2, ..., r$ $(r \in \mathbb{N})$ be continuous mappings and each g_j is a p-CC. That is,*

$$M(g_j s, g_j s^*, t) \geq k_j M(s, s^*, t) + (1 - k_j) M(\Gamma_i, \Gamma_{i+1}, t),$$

for all $s \in \Gamma_i, s^* \in \Gamma_{i+1}$, where $k_j \in (0, 1), 1 \leq j \leq r$. Consider $G : \bigcup_{i=1}^{p} \mathcal{P}_{cp}(\Gamma_i) \to \bigcup_{i=1}^{p} \mathcal{P}_{cp}(\Gamma_i)$ defined as in Proposition 3.1. Let $B \in \mathcal{P}_{cp}(\Gamma_i)$ for some $i = 1, 2, ..., p$. Assume that $\{G^{pn}B\} \subseteq \mathcal{P}_{cp}(\Gamma_i)$ has a convergent subsequence in $\mathcal{P}_{cp}(\Gamma_i)$. Then, G has a best proximity point.

Proof. Let $\{G^{pn_m}B\}$ be the convergent subsequence in $\mathcal{P}_{cp}(\Gamma_i)$ and it converges to some $B_0 \in \mathcal{P}_{cp}(\Gamma_i)$. Then,

$$\begin{aligned} \hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t) &= \hat{M}(\mathcal{P}_{cp}(\Gamma_{i-1}), \mathcal{P}_{cp}(\Gamma_i), t) \\ &\geq M_H(G^{pn_m-1}B, B_0, t) \\ &= M_H(B_0, G^{pn_m-1}B, t). \end{aligned}$$

Now,

$$\begin{aligned} \hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t) &\geq M_H(B_0, G^{pn_m-1}B, t) \\ &\geq M_H(B_0, G^{pn_m}B, \epsilon) * M_H(G^{pn_m}B, G^{pn_m-1}B, t - \epsilon), \ \forall \epsilon > 0. \end{aligned}$$

Hence as $m \to \infty$, by Proposition 3.2, we have

$$\begin{aligned} \hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t) &\geq \lim_{m\to\infty} M_H(B_0, G^{pn_m-1}B, t) \\ &\geq \hat{M}(\mathcal{P}_{cp}(\Gamma_{i-1}), \mathcal{P}_{cp}(\Gamma_i), t - \epsilon) \\ &= \hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t - \epsilon) \ \forall t > 0. \end{aligned}$$

Therefore, we obtain $M_H(B_0, G^{pn_m-1}B, t) \to \hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t)$.
Now,

$$\begin{aligned} \hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t) &\geq M_H(G^{pn_m}B, GB_0, t) \\ &\geq kM_H(G^{pn_m-1}B, B_0, t) + (1 - k)\,\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t). \end{aligned}$$

As $m \to \infty$, we obtain

$$M_H(B_0, GB_0, t) = \hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t).$$

$\square$

Remark 3.1. *In general, Theorem 3.2 does not guarantee the uniqueness of the best proximity point.*

Here, we are going to define the IFS.

Definition 3.2. *Let $(\Omega, M, *)$ be a FMS and $\{\Gamma_i\}_{i=1}^{p}$ be a collection of non-empty subsets of Ω. If $g_j : \bigcup_{i=1}^{p} \Gamma_i \to \bigcup_{i=1}^{p} \Gamma_i$, $j = 1, 2, ..., r$ $(r \in \mathbb{N})$ be continuous mappings and each g_j is a p-CC. That is,*

$$M(g_j s, g_j s^*, t) \geq k_j M(s, s^*, t) + (1 - k_j) M(\Gamma_i, \Gamma_{i+1}, t),$$

for all $s \in \Gamma_i, s^* \in \Gamma_{i+1}$, where $k_j \in (0, 1), 1 \leq j \leq r$. Then, $\{(\Omega, \Gamma_1, \Gamma_2, ..., \Gamma_p); g_j : j = 1, 2, \cdots r\}$ is called proximal fuzzy p-CC IFS.

Remark 3.2. *Theorem 3.2 ensures the existence of best proximity point for a multivalued map defined on the FMS $(\mathcal{P}_{cp}(\Omega), M_H, t)$. On the other hand, the uniqueness of the Best attractor for the p-CC IFS defined in Definition 3.2 cannot be guaranteed using the assumptions in Theorem 3.2. As uniqueness is an important for the construction of fractals from the IFS, finding the sufficient conditions on the space and function that ensures the uniqueness of the Best attractor for the proximal fuzzy p-CC IFS is a promising direction for future research.*

3.2. **IFS consisting p-cyclic ϕ-contraction.**

Definition 3.3. *[3] Let Γ_1 and Γ_2 be non-empty subsets of a FMS $(\Omega, M, *)$ and let $g : \Gamma_1 \cup \Gamma_2 \to \Gamma_1 \cup \Gamma_2$ such that $g(\Gamma_1) \subseteq \Gamma_2$ and $g(\Gamma_2) \subseteq \Gamma_1$. The map g is said to be a cyclic ϕ-contraction if $\phi : [0, \infty) \to [0, \infty)$ is a strictly increasing map and*

$$M(gs, gs^*, t) \geq M(s, s^*, t) - \phi(M(s, s^*, t)) + \phi(M(\Gamma_1, \Gamma_2, t)), \forall s \in \Gamma_1, s^* \in \Gamma_2.$$

Lemma 3.1. *Let Γ_1 and Γ_2 be non-empty subsets of a FMS $(\Omega, M, *)$ and let $g : \Gamma_1 \cup \Gamma_2 \to \Gamma_1 \cup \Gamma_2$ be a cyclic ϕ-contraction map. For $s_0 \in \Gamma_1 \cup \Gamma_2$, define $s_{n+1} = gs_n$ for each $n \geq 0$. Then,*

(i) $-\phi(M(s, s^*, t)) + \phi(M(\Gamma_1, \Gamma_2, t)) \geq 0$, $\forall s \in \Gamma_1, s^* \in \Gamma_2$,

(ii) $M(gs, gs^*, t) \geq M(s, s^*, t)$, $\forall s \in \Gamma_1, s^* \in \Gamma_2$,

(iii) $M(s_{n+2}, s_{n+1}, t) = M(gs_{n+1}, gs_n, t) \geq M(s_{n+1}, s_n, t), \forall n \geq 0$.

Proof. The proof is straightforward and hence omitted. $\square$

Definition 3.4. *Let $\{\Gamma_i\}_{i=1}^{p}$ be non-empty subsets of a fuzzy metric space $(\Omega, M, *)$ and let $g : \bigcup_{i=1}^{p} \Gamma_i \to \bigcup_{i=1}^{p} \Gamma_i$; g is called p-cyclic ϕ-contraction (p-C-ϕ-C), if*

(i) $g(\Gamma_i) \subseteq \Gamma_{i+1}$, $i = 1, 2, \cdots p$, *where* $\Gamma_{p+i} = \Gamma_i$,

(ii) *Let* $\phi : [0, \infty) \to [0, \infty)$ *is a strictly increasing map, and*

$$M(gs, gs^*, t) \geq M(s, s^*, t) - \phi(M(s, s^*, t)) + \phi(M(\Gamma_i, \Gamma_{i+1}, t)),$$

for all $s \in \Gamma_i, s^* \in \Gamma_{i+1}, 1 \leq i \leq p$.

Theorem 3.3. *Let $\{\Gamma_i\}_{i=1}^{p}$ be a collection of non-empty subsets of a FMS $(\Omega, M, *)$ and let $g : \bigcup_{i=1}^{p} \Gamma_i \to \bigcup_{i=1}^{p} \Gamma_i$ be a continuous map. Suppose, g is a p-C-ϕ-C, then the following conditions hold:*

(i) *under g, the elements in $\bigcup_{i=1}^{p} \mathcal{P}_{cp}(\Gamma_i)$ are mapped to elements in $\bigcup_{i=1}^{p} \mathcal{P}_{cp}(\Gamma_i)$,*

(ii) *for any $E \in \bigcup_{i=1}^{p} \mathcal{P}_{cp}(\Gamma_i)$, the mapping $g : \bigcup_{i=1}^{p} \mathcal{P}_{cp}(\Gamma_i) \to \bigcup_{i=1}^{p} \mathcal{P}_{cp}(\Gamma_i)$ defined as*

$$g(E) = \{g(s) : s \in E\}$$

is a p-C-ϕ-C with respect to the Hausdorff metric.

Proof. (i) Given g is continuous, hence the compact subsets of Γ_i are mapped to compact the subsets of Γ_{i+1}. For some $E \in \mathcal{P}_{cp}(\Gamma_i)$, we have $g(E) \in \mathcal{P}_{cp}(\Gamma_{i+1})$.

(ii) Let $P \in \mathcal{P}_{cp}(\Gamma_i)$ and $Q \in \mathcal{P}_{cp}(\Gamma_{i+1})$. Then, $gP \in \mathcal{P}_{cp}(\Gamma_{i+1})$ and $gQ \in \mathcal{P}_{cp}(\Gamma_{i+2})$. By Lemma 2.9, for each $s \in P$, we can find $s^* \in Q$ such that $M(s, s^*, t) \geq M_H(P, Q, t)$.

Now, consider

$$\begin{aligned}
M(gs, gQ, t) &\geq M(gs, gs^*, t) \\
&\geq M(s, s^*, t) - \phi(M(s, s^*, t)) + \phi(M(\Gamma_i, \Gamma_{i+1}, t)) \\
&\geq M(s, s^*, t) - \phi(M(s, s^*, t)) + \phi(\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t)) \\
&\geq M_H(P, Q, t) - \phi(M(s, s^*, t)) + \phi(\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t)) \\
&\geq M_H(P, Q, t) - \phi(M(s, Q, t)) + \phi(\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t)).
\end{aligned}$$

Also, for $s^* \in Q$, we get

$$M(gs^*, gP, t) \geq M_H(P, Q, t) - \phi(M(s^*, P, t)) + \phi(\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t)).$$

Consider

$$\begin{aligned}
M_H(gP, gQ, t) &= min\,\{\inf_{s\in P} M(gs, gQ, t), \inf_{s^*\in Q} M(gs^*, gP, t)\} \\
&\geq min\,\{\inf_{s\in P} [M_H(P, Q, t) - \phi(M(s, Q, t)) + \phi(\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t))], \\
&\qquad \inf_{s^*\in Q} [M_H(P, Q, t) - \phi(M(s^*, P, t)) + \phi(\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t))]\} \\
&= min\,\{M_H(P, Q, t) - \phi(\inf_{s\in P} M(s, Q, t)) + \phi(\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t)), \\
&\qquad M_H(P, Q, t) - \phi(\inf_{s^*\in Q} M(s^*, P, t)) + \phi(\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t))\} \\
&= M_H(P, Q, t) - \phi(min\,\{\inf_{s\in P} M(s, Q, t), \inf_{s^*\in Q} M(s^*, P, t)\}) + \phi(\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}))) \\
&= M_H(P, Q, t) - \phi(M_H(P, Q, t)) + \phi(\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t)).
\end{aligned}$$

Hence,

$$M_H(gP, gQ, t) \geq M_H(P, Q, t) - \phi(M_H(P, Q, t)) + \phi(\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t)),$$

where $P \in \mathcal{P}_{cp}(\Gamma_i)$, $Q \in \mathcal{P}_{cp}(\Gamma_{i+1})$. Therefore, the result follows.

□

Proposition 3.3. *Let $\{\Gamma_i\}_{i=1}^p$ be a collection of non-empty subsets of a FMS $(\Omega, M, *)$. Given, continuous mappings $g_j : \bigcup_{i=1}^p \Gamma_i \to \bigcup_{i=1}^p \Gamma_i$, $j = 1, 2, ..., r$ $(r \in \mathbb{N})$ and g_j's are p-cyclic ϕ_j-contractions. That is,*

$$M(g_j s, g_j s^*, t) \geq M(s, s^*, t) - \phi_j(M(s, s^*, t)) + \phi_j(M(\Gamma_i, \Gamma_{i+1}), t),$$

for all $s \in \Gamma_i, s^* \in \Gamma_{i+1}$. Let $G : \bigcup_{i=1}^p \mathcal{P}_{cp}(\Gamma_i) \to \bigcup_{i=1}^p \mathcal{P}_{cp}(\Gamma_i)$ be defined as follows:

$$\begin{aligned} G(E) &= g_1(E) \cup g_2(E) \cup ... \cup g_r(E) \\ &= \bigcup_{j=1}^r g_j(E), \ \forall E \in \bigcup_{i=1}^p \mathcal{P}_{cp}(\Gamma_i). \end{aligned}$$

Then, G is also a p-C-ϕ-C with respect to the Hausdorff metric. That is,

$$M_H(GC, GD, t) \geq M_H(C, D, t) - \phi_*(M_H(C, D, t)) + \phi_*(\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t)),$$

for all $C \in \mathcal{P}_{cp}(\Gamma_i), D \in \mathcal{P}_{cp}(\Gamma_{i+1})$, where $\phi_*(t) = \max_{1 \leq j \leq r} \phi(t)$.

Proof. We use Principle of Mathematical Induction for proving the result. For $r = 1$, the result is trivial. Now, we are going to verify it for $r = 2$.
Let $C \in \mathcal{P}_{cp}(\Gamma_i), D \in \mathcal{P}_{cp}(\Gamma_{i+1})$. Then,

$$\begin{aligned} M_H(GC, GD, t) &= M_H(\bigcup_{j=1}^2 g_j(C), \bigcup_{j=1}^2 g_j(D), t) \\ &= M_H(g_1 C \cup g_2 C, g_1 D \cup g_2 D, t) \\ &\geq min\{M_H(g_1 C, g_1 D, t), M_H(g_2 C, g_2 D, t)\} \\ &\geq min\{M_H(C, D, t) - \phi_1(M_H(C, D, t)) + \phi_1(\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t)), \\ &\qquad M_H(C, D, t) - \phi_2(M_H(C, D, t)) + \phi_2(\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t))\} \\ &= min\{M_H(C, D, t) + \phi_1(\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t)) - \phi_1(M_H(C, D, t)), \\ &\qquad M_H(C, D, t) + \phi_2(\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t)) - \phi_2(M_H(C, D, t))\}. \end{aligned}$$

Given, ϕ_1 and ϕ_2 are strictly increasing functions and $M_H(C, D, t) \leq \hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t)$, $\forall C \in \mathcal{P}_{cp}(\Gamma_i), D \in \mathcal{P}_{cp}(\Gamma_{i+1})$. For instance, if $\phi_1(t) = \max\{\phi_1(t), \phi_2(t)\}$, then

$$\phi_1(\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t)) - \phi_1(M_H(C, D, t)) \geq \phi_2(\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t)) - \phi_2(M_H(C, D, t)).$$

Hence we obtain

$$\begin{aligned} M_H(GC, GD, t) &\geq M_H(C, D, t) + \phi_*(\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t)) - \phi_*(M_H(C, D, t)) \\ &= M_H(C, D, t) - \phi_*(M_H(C, D, t)) + \phi_*(\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t)), \end{aligned}$$

where $\phi_*(t) = \max\{\phi_1(t), \phi_2(t)\}$. Hence, the result can be easily proved using the Principle of Mathematical Induction. □

Lemma 3.2. *Let $\{\Gamma_i\}_{i=1}^p$ be non-empty subsets of a FMS $(\Omega, M, *)$ and $G : \bigcup_{i=1}^p \mathcal{P}_{cp}(\Gamma_i) \to \bigcup_{i=1}^p \mathcal{P}_{cp}(\Gamma_i)$ be a p-C-ϕ-C. Then, the following conditions hold:*

(i) $-\phi(M_H(C,D,t)) + \phi(\hat{M}(\mathcal{P}_{cp}(\Gamma_i),\mathcal{P}_{cp}(\Gamma_{i+1}),t)) \geq 0$, $\forall C \in \mathcal{P}_{cp}(\Gamma_i), D \in \mathcal{P}_{cp}(\Gamma_{i+1})$,

(ii) $M_H(GC,GD,t) \geq M_H(C,D,t)$, $\forall C \in \mathcal{P}_{cp}(\Gamma_i), D \in \mathcal{P}_{cp}(\Gamma_{i+1})$,

(iii) $M_H(G^{pn}C, G^{pn+1}D,t) \geq M_H(G^{pn-1}C, G^{pn}D,t)$, $\forall n \geq 0$, *where* $C,D \in \mathcal{P}_{cp}(\Gamma_i)$ *and* $1 \leq i \leq p$.

Proof. The results can be easily derived from the definition of p-C-ϕ-C and the characteristics of the function ϕ. □

Proposition 3.4. *Let* $\{\Gamma_i\}_{i=1}^p$ *be non-empty closed subsets of a FMS* $(\Omega, M, *)$ *and* $G : \bigcup_{i=1}^p \mathcal{P}_{cp}(\Gamma_i) \to \bigcup_{i=1}^p \mathcal{P}_{cp}(\Gamma_i)$ *be a p-C-ϕ-C. Then, for every* $C,D \in \mathcal{P}_{cp}(\Gamma_i)$, *for* $1 \leq i \leq p$, *we have*

(i) $\hat{M}(\mathcal{P}_{cp}(\Gamma_i),\mathcal{P}_{cp}(\Gamma_{i+1}),t) = \hat{M}(\mathcal{P}_{cp}(\Gamma_{i+1}),\mathcal{P}_{cp}(\Gamma_{i+2}),t)$, $\forall\, 1 \leq i \leq p$, *where* $\Gamma_{p+i} = \Gamma_i$,

(ii) $M_H(G^{pn}C, G^{pn+1}D,t) \to \hat{M}(\mathcal{P}_{cp}(\Gamma_i),\mathcal{P}_{cp}(\Gamma_{i+1}),t)$ *as* $n \to \infty$,

(iii) $M_H(G^{pn\mp p}C, G^{pn+1}D,t) \to \hat{M}(\mathcal{P}_{cp}(\Gamma_i),\mathcal{P}_{cp}(\Gamma_{i+1}),t)$ *as* $n \to \infty$.

Proof. (i) Let $C \in \mathcal{P}_{cp}(\Gamma_i)$ and $D \in \mathcal{P}_{cp}(\Gamma_{i+1})$. Then,

$$\begin{aligned}
\hat{M}(\mathcal{P}_{cp}(\Gamma_{i+1}),\mathcal{P}_{cp}(\Gamma_{i+2}),t) &\geq M_H(GC,GD,t)\\
&\geq M_H(C,D,t) - \phi(M_H(C,D,t)) + \phi(\hat{M}(\mathcal{P}_{cp}(\Gamma_i),\mathcal{P}_{cp}(\Gamma_{i+1}),t))\\
&\geq M_H(C,D,t) - \phi(M_H(C,D,t)) + \phi(M_H(C,D,t))\\
&= M_H(C,D,t).
\end{aligned}$$

Which implies

$$\hat{M}(\mathcal{P}_{cp}(\Gamma_{i+1}),\mathcal{P}_{cp}(\Gamma_{i+2}),t) \geq \hat{M}(\mathcal{P}_{cp}(\Gamma_i),\mathcal{P}_{cp}(\Gamma_{i+1}),t),\ \forall i = 1,2,...,p.$$

Therefore

$$\begin{aligned}
\hat{M}(\mathcal{P}_{cp}(\Gamma_p),\mathcal{P}_{cp}(\Gamma_1),t) &\geq \hat{M}(\mathcal{P}_{cp}(\Gamma_{p-1}),\mathcal{P}_{cp}(\Gamma_p),t)\\
&\geq ... \geq \hat{M}(\mathcal{P}_{cp}(\Gamma_1),\mathcal{P}_{cp}(\Gamma_2),t) \geq \hat{M}(\mathcal{P}_{cp}(\Gamma_p),\mathcal{P}_{cp}(\Gamma_1),t).
\end{aligned}$$

Hence

$$\hat{M}(\mathcal{P}_{cp}(\Gamma_i),\mathcal{P}_{cp}(\Gamma_{i+1}),t) = \hat{M}(\mathcal{P}_{cp}(\Gamma_{i+1}),\mathcal{P}_{cp}(\Gamma_{i+2}),t),$$

for all $i = 1,2,...,p$, where $\Gamma_{p+i} = \Gamma_i$.

(ii) Let $C,D \in \mathcal{P}_{cp}(\Gamma_i)$ and $n \in \mathbb{N}$. Assume $h_n = M_H(G^{pn}C, G^{pn+1}D,t)$, $\forall n \geq 1$ and clearly $0 < h_n < M(\Gamma_{i+1},\Gamma_{i+2},t)$, $\forall n \geq 1$. By Lemma 3.2, we have $\{h_n\}$ is increasing and bounded above. Therefore, $\{h_n\}$ converges to some l, as $n \to \infty$.
Since G is a p-C-ϕ-C, we have

$$\begin{aligned}
&M_H(G^{pn}C, G^{pn+1}D,t)\\
&\quad\geq M_H(G^{pn-1}C, G^{pn}D,t) - \phi(M_H(G^{pn-1}C, G^{pn}D,t)) + \phi(\hat{M}(\mathcal{P}_{cp}(\Gamma_i),\mathcal{P}_{cp}(\Gamma_{i+1}),t)).
\end{aligned}$$

That is, we can write

$$h_n \geq h_{n-1} - \phi(h_{n-1}) + \phi(\hat{M}(\mathcal{P}_{cp}(\Gamma_i),\mathcal{P}_{cp}(\Gamma_{i+1}),t)), \tag{3.1}$$

and by part (i) in Proposition 3.4, we get

$$\phi(\hat{M}(\mathcal{P}_{cp}(\Gamma_i),\mathcal{P}_{cp}(\Gamma_{i+1}),t)) = \phi(\hat{M}(\mathcal{P}_{cp}(\Gamma_{i-1}),\mathcal{P}_{cp}(\Gamma_i),t)) \geq \phi(M_H(G^{pn-1}C,G^{pn}D,t)) = \phi(h_{n-1}). \tag{3.2}$$

Thus, from inequality 3.1 and inequality 3.2, $\forall n \geq 1$, we obtain

$$\phi(\hat{M}(\mathcal{P}_{cp}(\Gamma_i),\mathcal{P}_{cp}(\Gamma_{i+1}),t)) \geq \phi(h_{n-1}) \geq h_{n-1} - h_n + \phi(\hat{M}(\mathcal{P}_{cp}(\Gamma_i),\mathcal{P}_{cp}(\Gamma_{i+1}),t)).$$

As $n \to \infty$, we have

$$\phi(\hat{M}(\mathcal{P}_{cp}(\Gamma_i),\mathcal{P}_{cp}(\Gamma_{i+1}),t)) \geq \lim_{n\to\infty} \phi(h_{n-1}) \geq \phi(\hat{M}(\mathcal{P}_{cp}(\Gamma_i),\mathcal{P}_{cp}(\Gamma_{i+1}),t)). \tag{3.3}$$

Thus $\lim_{n\to\infty} \phi(h_{n-1}) = \phi(\hat{M}(\mathcal{P}_{cp}(\Gamma_i),\mathcal{P}_{cp}(\Gamma_{i+1}),t))$.
For all $n \in \mathbb{N}$, we have $h_n \leq l$ and $\hat{M}(\mathcal{P}_{cp}(\Gamma_i),\mathcal{P}_{cp}(\Gamma_{i+1}),t) \geq h_n$. Hence as $n \to \infty$, we get $\hat{M}(\mathcal{P}_{cp}(\Gamma_i),\mathcal{P}_{cp}(\Gamma_{i+1}),t) \geq l$.
Therefore, inequality 3.3 can be modified as below.

$$\lim_{n\to\infty} \phi(h_{n-1}) = \phi(\hat{M}(\mathcal{P}_{cp}(\Gamma_i),\mathcal{P}_{cp}(\Gamma_{i+1}),t)) \geq \phi(l). \tag{3.4}$$

Now, since ϕ is a strictly increasing function and $\hat{M}(\mathcal{P}_{cp}(\Gamma_i),\mathcal{P}_{cp}(\Gamma_{i+1}),t) \geq l \geq h_n,\ \forall n \in \mathbb{N}$, we get

$$\phi(\hat{M}(\mathcal{P}_{cp}(\Gamma_i),\mathcal{P}_{cp}(\Gamma_{i+1}),t)) \geq \phi(l) \geq \phi(h_n).$$

As $n \to \infty$, we have

$$\phi(\hat{M}(\mathcal{P}_{cp}(\Gamma_i),\mathcal{P}_{cp}(\Gamma_{i+1}),t)) \geq \phi(l) \geq \lim_{n\to\infty} \phi(h_n). \tag{3.5}$$

Thus by inequality 3.4 and inequality 3.5, we obtain $\lim_{n\to\infty} \phi(h_n) = \phi(\hat{M}(\mathcal{P}_{cp}(\Gamma_i),\mathcal{P}_{cp}(\Gamma_{i+1}),t)) = \phi(l)$. Here, ϕ is a striclty increasing function, hence we obtain $l = \hat{M}(\mathcal{P}_{cp}(\Gamma_i),\mathcal{P}_{cp}(\Gamma_{i+1}),t)$.

(iii) Similar manner, we can prove that

$$M_H(G^{pn\pm p}C, G^{pn+1}D,t) \to \hat{M}(\mathcal{P}_{cp}(\Gamma_i),\mathcal{P}_{cp}(\Gamma_{i+1}),t) \text{ as } n \to \infty.$$

□

Theorem 3.4. *Let $(\Omega, M, *)$ be a FMS and $\{\Gamma_i\}_{i=1}^p$ be a collection of non-empty closed subsets of Ω. If $g_j : \bigcup_{i=1}^p \Gamma_i \to \bigcup_{i=1}^p \Gamma_i$, $j = 1,2,...,r$ ($r \in \mathbb{N}$) be continuous mappings and each g_j is a p-C-ϕ-C. That is,*

$$M(g_j s, g_j s^*, t) \geq M(s,s^*,t) - \phi_j(M(s,s^*,t)) + \phi_j(M(\Gamma_i,\Gamma_{i+1},t)),$$

for all $s \in \Gamma_i, s^* \in \Gamma_{i+1}$. Consider $G : \bigcup_{i=1}^p \mathcal{P}_{cp}(\Gamma_i) \to \bigcup_{i=1}^p \mathcal{P}_{cp}(\Gamma_i)$ defined as in Proposition 3.1. Let $B \in \mathcal{P}_{cp}(\Gamma_i)$ for some $i = 1,2,...,p$. Assume that $\{G^{pn}B\} \subseteq \mathcal{P}_{cp}(\Gamma_i)$ has a convergent subsequence in $\mathcal{P}_{cp}(\Gamma_i)$. Then, G has a best proximity point.

Proof. Let $\{G^{pn_m}B\}$ be the convergent subsequence in $\mathcal{P}_{cp}(\Gamma_i)$ and it converges to some $B_0 \in \mathcal{P}_{cp}(\Gamma_i)$. Then,

$$\begin{aligned}\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t) &= \hat{M}(\mathcal{P}_{cp}(\Gamma_{i-1}), \mathcal{P}_{cp}(\Gamma_i), t) \\ &\geq M_H(G^{pn_m - 1}B, B_0, t) \\ &= M_H(B_0, G^{pn_m - 1}B, t).\end{aligned}$$

Now

$$\begin{aligned}\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t) &\geq M_H(B_0, G^{pn_m - 1}B, t) \\ &\geq M_H(B_0, G^{pn_m}B, \epsilon) * M_H(G^{pn_m}B, G^{pn_m - 1}B, t - \epsilon),\ \forall \epsilon > 0.\end{aligned}$$

Hence as $m \to \infty$, by Proposition 3.4 we have

$$\begin{aligned}\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t) &\geq \lim_{m\to\infty} M_H(B_0, G^{pn_m - 1}B, t - \epsilon) \\ &\geq \hat{M}(\mathcal{P}_{cp}(\Gamma_{i-1}), \mathcal{P}_{cp}(\Gamma_i), t) \\ &= \hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t).\end{aligned}$$

Therefore we get $M_H(B_0, G^{pn_m - 1}B, t) \to \hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t)$.
Now,

$$\begin{aligned}\hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t) &\geq M_H(G^{pn_m}B, GB_0, t) \\ &\geq M_H(G^{pn_m - 1}B, B_0, t).\end{aligned}$$

Therefore, as $m \to \infty$, we obtain

$$M_H(B_0, GB_0, t) = \hat{M}(\mathcal{P}_{cp}(\Gamma_i), \mathcal{P}_{cp}(\Gamma_{i+1}), t).$$

□

Remark 3.3. *When $\phi(t) = kt$, where $0 \leq k < 1$, p-C-ϕ-C reduces to p-CC.*

Remark 3.4. *In general, Theorem 3.4 does not guarantee the uniqueness of the best proximity point.*

Here, we are going to define the IFS.

Definition 3.5. *Let $(\Omega, M, *)$ be a FMS and $\{\Gamma_i\}_{i=1}^p$ be a collection of non-empty subsets of Ω. If $g_j : \bigcup_{i=1}^p \Gamma_i \to \bigcup_{i=1}^p \Gamma_i$, $j = 1, 2, ..., r$ $(r \in \mathbb{N})$ be continuous mappings and each g_j is a p-C-ϕ-C. That is,*

$$M(g_j s, g_j s^*, t) \geq M(s, s^*, t) - \phi_j(M(s, s^*, t)) + \phi_j(M(\Gamma_i, \Gamma_{i+1}, t)),$$

for all $s \in \Gamma_i, s^* \in \Gamma_{i+1}$. Then, $\{(\Omega, \Gamma_1, \Gamma_2, ..., \Gamma_p); g_j : j = 1, 2, \cdots r\}$ is called proximal p-C-ϕ-C IFS.

Remark 3.5. *Theorem 3.4 ensures the existence of best proximity point for a multivalued map defined on the FMS $(\mathcal{P}_{cp}(\Omega), M_H, t)$. On the other hand, the uniqueness of the Best attractor for the p-CC IFS defined in Definition 3.5 cannot be guaranteed using the assumptions in Theorem 3.4. As uniqueness is an important for the construction of fractals from the IFS, finding the sufficient conditions on the space and function that* *ensures the uniqueness of the Best attractor for the proximal fuzzy p-CC IFS is a promising direction for future research.*

4. Conclusions

In this work, we have proved the existence of best proximity points for set valued p-CC and p-C-ϕ-C in a FMS. Our results extend those of Al-Thagafi [3] and Karpagam [16].

Acknowledgment: This work was funded by the Basque Government through Grant IT1555-22.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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