

Gamma Like Matrix Operators With Some Applications

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Abstract. In this paper, we construct and investigate the space of weighted regular tribonacci matrix in Nakano sequence space of fuzzy functions. The idealization of the mappings has been achieved through the use of extended s -fuzzy functions and this sequence space of fuzzy functions. The topological and geometric properties of this new space, the multiplication maps stand-in on it, as well as the mappings' ideal that correspond to them, are discussed. We construct the existence of a fixed point of Kannan contraction mapping acting on this space and its associated pre-quasi ideal. It is interesting that several numerical experiments are presented to illustrate our results. Additionally, some successful applications to the existence of solutions of nonlinear difference equations of fuzzy functions are introduced.

1. INTRODUCTION

The mappings' ideal theory is well regarded in functional analysis. The closed mappings' ideals are certain to play an important function in the principle of Banach lattices. Fixed point theory, Banach space geometry, normal series theory, approximation theory, ideal transformations, etc. all use mappings' ideal. Using s -numbers is an essential technique. Pietsch [1–4] developed and studied the theory of s -numbers of linear bounded mappings between Banach spaces. He offered and explained some topological and geometric structures of the quasi ideals of ℓ_p type mappings. Then, Constantin [5], generalized the class of ℓ_p type mappings to the class of ces_p type mappings. Makarov and Faried [6], showed some inclusion relations of ℓ_p type mappings. As a generalization of ℓ_p type mappings, Stolz mappings and mappings' ideal were examined by Tita [7,8]. In [9], Maji and Srivastava studied the class $A_p^{(s)}$ of s -type ces_p mappings using s -number sequence and Cesàro sequence spaces and they introduced a new class $A_{p,q}^{(s)}$ of s -type $ces(p, q)$ mappings by weighted ces_p with $1 < p < \infty$. In [10], the class of s -type $Z(u, v; \ell_p)$ mappings was defined and some of their properties were explained. Pre-quasi mappings' ideals are more extensive than quasi mappings'

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ideals, according to Faried and Bakery [11]. Bakery and Abou Elmatty [12], investigated the necessary conditions on any s -type sequence space to form an operators ideal. They showed that the s -type Nakano generalized difference sequence space X fails to generate an operators ideal. They investigated the sufficient conditions on X to be premodular Banach special space of sequences and the constructed prequasi-operator ideal becomes a small, simple, and closed Banach space and has eigenvalues identical with its s -numbers. Finally, they introduced necessary and sufficient conditions on X explaining some topological and geometrical structures of the multiplication operator defined on X . Komal et al. [13], investigated the multiplication operators acting on Cesàro sequence spaces under the Luxemburg norm. The multiplication operators acting on Cesàro second order function spaces examined by İlkhani et al. [14]. The non-absolute type sequence spaces are a generalization of the equivalent absolute type. For that there exists a great interest to examine these sequence spaces. Newly, many authors in the literature have discussed a few non-absolute type sequence spaces and presented new interesting articles, for example, see Mursaleen and Noman [15], and Mursaleen and Başar [16]. The learning about of the variable exponent Lebesgue spaces $L_{(r)}$ obtained in addition impetus from the mathematical description of the hydrodynamics of non-Newtonian fluids (see [17, 18]). Applications of non-Newtonian fluids, known as electrorheological, vary from their use in army science to civil engineering and orthopedics. Guo and Zhu [19], investigated a class of stochastic Volterra-Levin equations with Poisson jumps. Mao et al. [20], concerned with neutral stochastic functional differential equations driven by pure jumps (NSFDEwPJs). They proved the existence and uniqueness of the solution to NSFDEwPJs whose coefficients satisfy the local Lipschitz condition and established the p -th exponential estimations and almost surely asymptotic estimations of the solution for NSFDEwJs. Yang and Zhu [21], concerned with a class of stochastic neutral functional differential equations of Sobolev-type with Poisson jumps. Since the booklet of the Banach fixed point theorem [22], many mathematicians have worked on many developments. Kannan [23] gave an example of a class of mappings with the same fixed point actions as contractions, though that fails to be continuous. The only attempt to describe Kannan operators in modular vector spaces was once made in Reference [24]. Bakery and Mohamed [25] explored the concept of the pre-quasi norm on Nakano sequence space such that its variable exponent in $(0, 1]$. They explained the sufficient conditions on it equipped with the definite pre-quasi norm to generate pre-quasi Banach and closed space, and examined the Fatou property of different pre-quasi norms on it. Moreover, they showed the existence of a fixed point of Kannan pre-quasi norm contraction maps on it and on the pre-quasi Banach operator ideal constructed by s -numbers which belong to this sequence space. After Zadeh [26] established the concept of fuzzy sets and fuzzy set operations, many researchers adopted the concept of fuzziness in cybernetics and artificial intelligence as well as in expert systems and fuzzy control. Javed et al. [27] investigated the Banach contraction in R -fuzzy b -metric spaces and discussed some related fixed point results to ensure a fixed point's existence and uniqueness. A nontrivial example is given to illustrate the feasibility of the proposed methods.

They offered an application to solve the first kind of Fredholm-type integral equation. In [28], Rehman and Aydi proved some common fixed point theorems for mappings involving generalized rational-type fuzzy cone-contraction conditions in fuzzy cone metric spaces. They gave a common solution of two definite Fredholm integral equations. The concept of orthogonal partial b -metric spaces was pioneered by Javed et al. [29]. They presented a unique fixed point for some orthogonal contractive mappings with some examples and an application. Humaira et al. [30], discussed the existence theorem for a unique solution to a coupled system of impulsive fractional differential equations in complex-valued fuzzy metric spaces and the fuzzy version of some fixed point results by using the definition and presented some properties of a complex-valued fuzzy metric space with some applications. In this study, Rome et al. [31] looked into the concept of extended fuzzy rectangular b -metric space. They explained that some fixed point results in the literature could be generalized by α -admittance in this space. They used this to show solutions for a group of integral equations. Many researchers in sequence spaces and summability theory studied fuzzy sequence spaces and their properties. Different classes of sequences of fuzzy real numbers have been discussed by Nanda [32], Nuray and Savas [33], Matloka [34], Altinok et al. [35], Colak et al. [36], Hazarika and Savas [37] and many others. In [33], the Nakano sequences of fuzzy integers were defined and analyzed. Tripathy and Baruah [38], introduced and examined some properties of a new type of difference sequence spaces of fuzzy real numbers.

We indicate the space of all bounded, finite rank linear mappings from an infinite dimensional Banach space $\mathcal{P}$ into an infinite dimensional Banach space $\mathcal{Q}$ by $\mathbb{B}(\mathcal{P}, \mathcal{Q})$, and $\mathbb{F}(\mathcal{P}, \mathcal{Q})$ and when $\mathcal{P} = \mathcal{Q}$, we inscribe $\mathbb{B}(\mathcal{P})$ and $\mathbb{F}(\mathcal{P})$.

Definition 1.1. [39] An s -number function is a mapping $s : \mathbb{B}(\mathcal{P}, \mathcal{Q}) \rightarrow \mathbb{R}^{+N}$ that gives all $V \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$ a $(s_d(V))_{d=0}^{\infty}$ holds the following conditions:

- (a): $\|V\| = s_0(V) \geq s_1(V) \geq s_2(V) \geq \dots \geq 0$, for every $V \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$,
- (b): $s_{l+d-1}(V_1 + V_2) \leq s_l(V_1) + s_d(V_2)$, for every $V_1, V_2 \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$ and $l, d \in \mathbb{N}$,
- (c): $s_d(VYW) \leq \|V\|s_d(Y) \|W\|$, for every $W \in \mathbb{B}(\Delta_0, \Delta)$, $Y \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$ and $V \in \mathbb{B}(\mathcal{Q}, \mathcal{Q}_0)$, where $\mathcal{P}_0$ and $\mathcal{Q}_0$ are arbitrary Banach spaces,
- (d): assume $V \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$ and $\gamma \in \mathbb{R}$, then $s_d(\gamma V) = |\gamma|s_d(V)$,
- (e): if $\text{rank}(V) \leq d$, then $s_d(V) = 0$, for all $V \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$,
- (f): $s_{l \geq a}(I_a) = 0$ or $s_{l < a}(I_a) = 1$, where I_a indicates the unit mapping on the a -dimensional Hilbert space ℓ_2^a .

We give here some examples of s -numbers:

- (1): The q -th Kolmogorov number, denoted by $d_q(X)$, is marked by

$$d_q(X) = \inf_{\dim J \leq q} \sup_{\|f\| \leq 1} \inf_{g \in J} \|Xf - g\|.$$
- (2): The q -th approximation number, indicated by $\alpha_q(X)$, is marked by

$$\alpha_q(X) = \inf \{ \|X - Y\| : Y \in \mathbb{B}(\mathcal{P}, \mathcal{Q}) \text{ and } \text{rank}(Y) \leq q \}.$$

The linear space of sequences of fuzzy functions is denoted by $\mathcal{E}^F$.

Notations 1.1. [40]

$$\begin{aligned}\overline{\mathbb{B}}^s_{\mathcal{E}^F} &:= \left\{ \overline{\mathbb{B}}^s_{\mathcal{E}^F}(\mathcal{P}, \mathcal{Q}) \right\}, \text{ where } \overline{\mathbb{B}}^s_{\mathcal{E}^F}(\mathcal{P}, \mathcal{Q}) := \left\{ V \in \mathbb{B}(\mathcal{P}, \mathcal{Q}) : ((s_j(V))_{j=0}^\infty) \in \mathcal{E}^F \right\}, \\ \overline{\mathbb{B}}^\alpha_{\mathcal{E}^F} &:= \left\{ \overline{\mathbb{B}}^\alpha_{\mathcal{E}^F}(\mathcal{P}, \mathcal{Q}) \right\}, \text{ where } \overline{\mathbb{B}}^\alpha_{\mathcal{E}^F}(\mathcal{P}, \mathcal{Q}) := \left\{ V \in \mathbb{B}(\mathcal{P}, \mathcal{Q}) : ((\alpha_j(V))_{j=0}^\infty) \in \mathcal{E}^F \right\}, \\ \overline{\mathbb{B}}^d_{\mathcal{E}^F} &:= \left\{ \overline{\mathbb{B}}^d_{\mathcal{E}^F}(\mathcal{P}, \mathcal{Q}) \right\}, \text{ where } \overline{\mathbb{B}}^d_{\mathcal{E}^F}(\mathcal{P}, \mathcal{Q}) := \left\{ V \in \mathbb{B}(\mathcal{P}, \mathcal{Q}) : ((d_j(V))_{j=0}^\infty) \in \mathcal{E}^F \right\},\end{aligned}$$

where

$$\overline{s_j(V)}(x) = \begin{cases} 1, & x = s_j(V) \\ 0, & x \neq s_j(V). \end{cases}$$

The goal of this paper is organized as follows: In Section 3, we give the definition and some inclusion relations of the space of weighted regular tribonacci matrix in Nakano sequence space of fuzzy functions, $(\tau_r^F(q, t))_v$, equipped with the function v . In Section 4, we explain the sufficient conditions on $(\tau_r^F(q, t))_v$ with known function v to construct private sequence space of fuzzy functions, or in short (pssf). This explains that $(\tau_r^F(q, t))_v$ is a pre-quasi Banach pssf and we examine some properties of s -type $(\tau_r^F(q, t))_v$ spaces. In Section 5, we act a multiplication operator on $(\tau_r^F(q, t))_v$, and investigate the necessity and enough setups on this sequence space so that the multiplication operator is bounded, approximable, invertible, Fredholm and closed range. In Section 6. First, we discuss the enough conditions (not necessary) on $(\tau_r^F(q, t))_v$, so that the closure of $\mathbb{F} = \overline{\mathbb{B}}^s_{(\tau_r^F(q, t))_v}$. This gives a negative answer of Rhoades [41] open problem about the linearity of s -type $(\tau_r^F(q, t))_v$ spaces. Second, we introduce the setups on $(\tau_r^F(q, t))_v$ such that the elements of $\overline{\mathbb{B}}^s_{(\tau_r^F(q, t))_v}$ are complete. Third, we offer the enough conditions on $(\tau_r^F(q, t))_v$ so that $\overline{\mathbb{B}}^\alpha_{(\tau_r^F(q, t))_v}$ is strictly contained for distinct weights and powers. We establish the setups for which the pre-quasi ideal $\overline{\mathbb{B}}^\alpha_{(\tau_r^F(q, t))_v}$ is minimum. Fourth, we introduce the conditions for which the Banach pre-quasi ideal $\overline{\mathbb{B}}^s_{(\tau_r^F(q, t))_v}$ is simple. Fifth, we give the enough conditions on $(\tau_r^F(q, t))_v$ so that the class $\mathbb{B}$ which sequence of eigenvalues in $(\tau_r^F(q, t))_v$ equals $\overline{\mathbb{B}}^s_{(\tau_r^F(q, t))_v}$. In Section 7, the existence of a fixed point of Kannan pre-quasi norm contraction operator on this sequence space and on its pre-quasi operator ideal constructed by $(\tau_r^F(q, t))_v$ and s -numbers are confirmed. Finally, in Section 8, we light our results by a few examples and applications to the existence of solutions of non-linear difference equations. We introduce our conclusion in Section 9.

2. DEFINITIONS AND PRELIMINARIES

Remember that Matloka [34], introduced bounded and convergent fuzzy numbers, investigated some of their properties, and demonstrated that any convergent fuzzy number sequence is bounded. Nanda [32], researched fuzzy number sequences and demonstrated that the set of all convergent fuzzy number sequences forms a complete metric space. Kumar et al. [42], presented the concept limit points and cluster points of sequences of fuzzy numbers. If Ω is the set of all

closed and bounded intervals on the real line $\mathfrak{R}$. Assume $f = [f_1, f_2]$ and $g = [g_1, g_2]$ in Ω , let

$$f \leq g \text{ if and only if } f_1 \leq g_1 \text{ and } f_2 \leq g_2.$$

Define a metric ρ on Ω by

$$\rho(f, g) = \max\{|f_1 - g_1|, |f_2 - g_2|\}.$$

Matloka [34] proved that ρ is a metric on Ω and (Ω, ρ) is a complete metric space. The relation $\leq$ is a partial order on Ω .

Definition 2.1. A fuzzy number f is a fuzzy subset of $\mathfrak{R}$ i.e., a mapping $f : \mathfrak{R} \rightarrow [0, 1]$ that verifies the four conditions:

- (a): f is fuzzy convex, i.e., for $x, y \in \mathfrak{R}$ and $\alpha \in [0, 1]$, $f(\alpha x + (1 - \alpha)y) \geq \min\{f(x), f(y)\}$;
- (b): f is normal, i.e., there is $y_0 \in \mathfrak{R}$ such that $f(y_0) = 1$;
- (c): f is an upper-semi continuous, i.e., for all $\alpha > 0$, $f^{-1}([0, \alpha])$ for all $x \in [0, 1]$ is open in the usual topology of $\mathfrak{R}$;
- (d): the closure of $f^0 := \{y \in \mathfrak{R} : f(y) > 0\}$ is compact.

The β -level set of a fuzzy real number f , $0 < \beta < 1$, denoted by f^β , is defined as

$$f^\beta = \{y \in \mathfrak{R} : f(y) \geq \beta\}.$$

The set of all upper semi-continuous, normal, convex fuzzy number, and f^β is compact, is marked by $\mathfrak{R}([0, 1])$. The set $\mathfrak{R}$ can be embedded in $\mathfrak{R}([0, 1])$, if we define $r \in \mathfrak{R}([0, 1])$ by

$$\bar{r}(t) = \begin{cases} 1, & t = r \\ 0, & t \neq r. \end{cases}$$

The additive identity and multiplicative identity in $\mathfrak{R}[0, 1]$ are denoted by $\bar{0}$ and $\bar{1}$, respectively. Assume $f, g \in \mathfrak{R}[0, 1]$ and the β -level sets are $[f]^\beta = [f_1^\beta, f_2^\beta]$, $[g]^\beta = [g_1^\beta, g_2^\beta]$, $\beta \in [0, 1]$. A partial ordering for any $f, g \in \mathfrak{R}[0, 1]$ as follows: $f \leq g$ if and only if $f^\beta \leq g^\beta$, for all $\beta \in [0, 1]$.

Assume $\bar{\rho} : \mathfrak{R}[0, 1] \times \mathfrak{R}[0, 1] \rightarrow \mathfrak{R}^+ \cup \{0\}$ is defined by $\bar{\rho}(f, g) = \sup_{0 \leq \beta \leq 1} \rho(f^\beta, g^\beta)$.

Recall that:

- (1) $(\mathfrak{R}[0, 1], \bar{\rho})$ is a complete metric space.
- (2) $\bar{\rho}(f + k, g + k) = \bar{\rho}(f, g)$ for all $f, g, k \in \mathfrak{R}[0, 1]$.
- (3) $\bar{\rho}(f + k, g + l) \leq \bar{\rho}(f, g) + \bar{\rho}(k, l)$.
- (4) $\bar{\rho}(\xi f, \xi g) = |\xi| \bar{\rho}(f, g)$, for all $\xi \in \mathfrak{R}$.

By c_0 , ℓ_∞ and ℓ_r , we denote the space of null, bounded and r -absolutely summable sequences of real numbers. The space of approximable and compact bounded linear mappings from $\mathcal{P}$ into $\mathcal{Q}$ will be denoted by $\mathcal{A}(\mathcal{P}, \mathcal{Q})$ and $\mathcal{K}(\mathcal{P}, \mathcal{Q})$, and if $\mathcal{P} = \mathcal{Q}$, we mark $\mathcal{A}(\mathcal{P})$ and $\mathcal{K}(\mathcal{P})$, respectively. The ideal of bounded, approximable and compact mappings between any arbitrary Banach spaces will be denoted by $\mathbb{B}$, $\mathcal{A}$ and $\mathcal{K}$, respectively.

Definition 2.2. [43] Let $\mathbb{B}$ be the class of all bounded linear operators within any two arbitrary Banach spaces. A sub class $\mathcal{U}$ of $\mathbb{B}$ is said to be a mappings' ideal, if every $\mathcal{U}(\mathcal{P}, \mathcal{Q}) = \mathcal{U} \cap \mathbb{B}(\mathcal{P}, \mathcal{Q})$ satisfies the following setups:

- (i): $I_\Gamma \in \mathcal{U}$, where Γ indicates Banach space of one dimension.
- (ii): The space $\mathcal{U}(\mathcal{P}, \mathcal{Q})$ is linear over $\mathbb{R}$.
- (iii): If $W \in \mathbb{B}(\mathcal{P}_0, \mathcal{P})$, $X \in \mathcal{U}(\mathcal{P}, \mathcal{Q})$ and $Y \in \mathbb{B}(\mathcal{Q}, \mathcal{Q}_0)$, then $YXW \in \mathcal{U}(\Delta_0, \mathcal{Q}_0)$.

Definition 2.3. [11] A function $H \in [0, \infty)^{\mathcal{U}}$ is said to be a pre-quasi norm on the ideal $\mathcal{U}$ if the following conditions hold:

- (1): Assume $V \in \mathcal{U}(\mathcal{P}, \mathcal{Q})$, $H(V) \geq 0$ and $H(V) = 0$, if and only if, $V = 0$,
- (2): one has $Q \geq 1$ with $H(\alpha V) \leq D|\alpha|H(V)$, for all $V \in \mathcal{U}(\mathcal{P}, \mathcal{Q})$ and $\alpha \in \mathbb{R}$,
- (3): there are $P \geq 1$ such that $H(V_1 + V_2) \leq P[H(V_1) + H(V_2)]$, for all $V_1, V_2 \in \mathcal{U}(\mathcal{P}, \mathcal{Q})$,
- (4): there are $\sigma \geq 1$ so that if $V \in \mathbb{B}(\Delta_0, \Delta)$, $X \in \mathcal{U}(\mathcal{P}, \mathcal{Q})$ and $Y \in \mathbb{B}(\mathcal{Q}, \mathcal{Q}_0)$ then $H(YXV) \leq \sigma \|Y\|H(X) \|V\|$.

Theorem 2.1. [11] H is a pre-quasi norm on the ideal $\mathcal{U}$, whenever H is a quasi norm on the ideal $\mathcal{U}$.

Lemma 2.2. [44] If $t_a > 0$ and $\lambda_a, \beta_a \in \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, for all $a \in \mathcal{N} := \{0, 1, 2, \dots\}$, and $\hbar = \max\{1, \sup_a t_a\}$, hence

$$|\lambda_a + \beta_a|^{t_a} \leq 2^{\hbar-1} (|\lambda_a|^{t_a} + |\beta_a|^{t_a}). \quad (2.1)$$

3. THE SEQUENCE SPACE $(\tau_r^F(q, t))_v$

In this section, we introduce the definition and some inclusion relations of the sequence space $(\tau_r^F(q, t))_v$ equipped with the function v .

The number sequence $(r_v)_{v=0}^\infty := (1, 1, 2, 4, 7, 13, 24, \dots)$ defined by the recurrence relation $r_v = r_{v-1} + r_{v-2} + r_{v-3}$, $v \geq 3$, with $r_0 = r_1 = 1$ and $r_2 = 2$, is called tribonacci sequence. Recall that, we name the sequence space $(\tau_r^F(q, t))_v$ as the domain of weighted regular tribonacci matrix in Nakano fuzzy sequence space since it is constructed by the domain of weighted regular tribonacci matrix defined in $\ell_{((t_l))}^F$, where the weighted regular tribonacci matrix, $\tau_r = (\lambda_{lz}(q))$, is defined as:

$$\lambda_{lz}(q) = \begin{cases} \frac{r_z q_z}{r_{l+2} + r_l - 1}, & 0 \leq z \leq l, \\ 0, & z > l, \end{cases}$$

where $q_z \in (0, \infty)$, for all $z \in \mathcal{N}$. Note that $\sum_{z=0}^l 2r_z = r_{l+2} + r_l - 1$. In [45], Yaying and Hazarika, studied some sequence spaces defined by the domain of a regular Tribonacci matrix.

Definition 3.1. If $(t_l) \in \mathbb{R}^{+\mathcal{N}}$, where $\mathbb{R}^{+\mathcal{N}}$ is the space of all sequences of positive reals. The sequence space $(\tau_r^F(q, t))_v$ with the function v is defined by:

$$(\tau_r^F(q, t))_v = \left\{ \bar{f} = (\bar{f}_k) \in \omega(F) : v(\delta \bar{f}) < \infty, \text{ for some } \delta > 0 \right\}, \text{ where } v(\bar{f}) = \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \bar{f}_z, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l}.$$

Theorem 3.1. If $(t_l) \in \mathfrak{R}^{+\mathcal{N}} \cap \ell_\infty$, then

$$\left(\tau_r^F(q, t)\right)_v = \left\{ \bar{f} = (\bar{f}_k) \in \omega(F) : v(\delta \bar{f}) < \infty, \text{ for all } \delta > 0 \right\}.$$

Proof. Assume $(t_l) \in \mathfrak{R}^{+\mathcal{N}} \cap \ell_\infty$, we have

$$\begin{aligned} \left(\tau_r^F(q, t)\right)_v &= \left\{ \bar{f} = (\bar{f}_k) \in \omega(F) : v(\delta \bar{f}) < \infty, \text{ for some } \delta > 0 \right\} \\ &= \left\{ \bar{f} = (\bar{f}_k) \in \omega(F) : \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \delta \bar{f}_z, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} < \infty, \text{ for some } \delta > 0 \right\} \\ &= \left\{ \bar{f} = (\bar{f}_k) \in \omega(F) : \inf_l \delta^{t_l} \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \bar{f}_z, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} < \infty, \text{ for some } \delta > 0 \right\} \\ &= \left\{ \bar{f} = (\bar{f}_k) \in \omega(F) : \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \bar{f}_z, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} < \infty \right\} \\ &= \left\{ \bar{f} = (\bar{f}_k) \in \omega(F) : v(\delta \bar{f}) < \infty, \text{ for any } \delta > 0 \right\}. \end{aligned}$$

□

Theorem 3.2. If $(t_l) \in [1, \infty)^{\mathcal{N}} \cap \ell_\infty$, then $\left(\tau_r^F(q, t)\right)_v$ is a non-absolute type.

Proof. By choosing $\bar{f} = (\bar{1}, -\bar{1}, \bar{0}, \bar{0}, \bar{0}, \dots)$, then $|\bar{f}| = (\bar{1}, \bar{1}, \bar{0}, \bar{0}, \bar{0}, \dots)$. We have

$$\begin{aligned} v(\bar{f}) &= \left(\frac{q_0}{2}\right)^{t_0} + \left(\frac{|q_0 - q_1|}{4}\right)^{t_1} + \left(\frac{|q_0 - q_1|}{8}\right)^{t_2} + \dots \\ &\neq \left(\frac{q_0}{2}\right)^{t_0} + \left(\frac{|q_0 + q_1|}{4}\right)^{t_1} + \left(\frac{|q_0 + q_1|}{8}\right)^{t_2} + \dots = v(|\bar{f}|). \end{aligned}$$

Then, the sequence space $\left(\tau_r^F(q, t)\right)_v$ is non-absolute type.

□

Definition 3.2. Assume $(t_l) \in \mathfrak{R}^{+\mathcal{N}}$ and $t_l \geq 1$, for every $l \in \mathcal{N}$.

$$\left(|\tau_r^F|(q, t)\right)_\varphi := \left\{ \bar{f} = (\bar{f}_k) \in \omega(F) : \varphi(\delta \bar{f}) < \infty, \text{ for some } \delta > 0 \right\},$$

$$\text{where } \varphi(\bar{f}) = \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z |\bar{f}_z|, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l}.$$

Theorem 3.3. Suppose $(t_l) \in (1, \infty)^{\mathcal{N}} \cap \ell_\infty$ with $\left(\frac{l+1}{r_{l+2}+r_l-1}\right) \notin \ell_{(t_l)}$, hence $\left(|\tau_r^F|(q, t)\right)_\varphi \subsetneq \left(\tau_r^F(q, t)\right)_v$.

Proof. Let $\bar{f} \in \left(|\tau_r^F|(q, t)\right)_\varphi$, since

$$\sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \bar{f}_z, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \leq \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z |\bar{f}_z|, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} < \infty.$$

Hence, $\bar{f} \in \left(\tau_r^F(q, t)\right)_v$. If we choose $\bar{g} = \left(\frac{(-1)^z}{r_z q_z}\right)_{z \in \mathcal{N}}$, one gets $\bar{g} \in \left(\tau_r^F(q, t)\right)_v$ and $\bar{g} \notin \left(|\tau_r^F|(q, t)\right)_\varphi$. □

4. PRE-MODULAR PRIVATE SEQUENCE SPACE

We explain in this section the enough setup on $\tau_r^F(q, t)$ with definite function v to construct pre-modular $\mathfrak{pssf}$. Which investigates that $\tau_r^F(q, t)$ is a pre-quasi normed $\mathfrak{pssf}$.

Definition 4.1. The linear space of sequences $\mathcal{E}^F$ is named a private sequence space of fuzzy functions ($\mathfrak{pssf}$), if it satisfies the following conditions:

- (1): $\bar{e}_x \in \mathcal{E}^F$, with $x \in \mathcal{N}$, where $\bar{e}_x = (\bar{0}, \bar{0}, \dots, \bar{1}, \bar{0}, \bar{0}, \dots)$, while $\bar{1}$ displays at the x^{th} place,
- (2): $\mathcal{E}^F$ is solid i.e., for $\bar{h} = (\bar{h}_x) \in \omega(F)$, $|\bar{j}| = (|\bar{j}_x|) \in \mathcal{E}^F$ and $|\bar{h}_x| \leq |\bar{j}_x|$, with $x \in \mathcal{N}$, then $|\bar{h}| \in \mathcal{E}^F$,
- (3): $(|\bar{j}_{[\frac{i}{2}]}|)_{x=0}^\infty \in \mathcal{E}^F$, if $(|\bar{j}_x|)_{x=0}^\infty \in \mathcal{E}^F$, where $[a]$ denotes the integral part of real number a

Theorem 4.1. Assume the linear sequence space $\mathcal{E}^F$ is a $\mathfrak{pssf}$, then $\overline{\mathbb{B}}_{\mathcal{E}^F}^s$ is an operator ideal.

Proof. (i) Assume $V \in \mathbb{F}(\mathcal{P}, \mathcal{Q})$ and $\text{rank}(V) = n$ with $n \in \mathcal{N}$, as $\bar{e}_i \in \mathcal{E}^F$ for all $i \in \mathcal{N}$ and $\mathcal{E}^F$ is a linear space, one has

$(\overline{s_i(V)})_{i=0}^\infty = (\overline{s_0(V)}, \overline{s_1(V)}, \dots, \overline{s_{n-1}(V)}, \bar{0}, \bar{0}, \bar{0}, \dots) = \sum_{i=0}^{n-1} \overline{s_i(V)} \bar{e}_i \in \mathcal{E}^F$; for that $V \in \overline{\mathbb{B}}_{\mathcal{E}^F}^s(\mathcal{P}, \mathcal{Q})$ then $\mathbb{F}(\mathcal{P}, \mathcal{Q}) \subseteq \overline{\mathbb{B}}_{\mathcal{E}^F}^s(\mathcal{P}, \mathcal{Q})$.

(ii) Suppose $V_1, V_2 \in \overline{\mathbb{B}}_{\mathcal{E}^F}^s(\mathcal{P}, \mathcal{Q})$ and $\beta_1, \beta_2 \in \mathfrak{R}$ then by Definition 4.1 condition (3) one has $(\overline{s_{[\frac{i}{2}]}(V_1)})_{i=0}^\infty \in \mathcal{E}^F$ and $(\overline{s_{[\frac{i}{2}]}(V_2)})_{i=0}^\infty \in \mathcal{E}^F$, as $i \geq 2[\frac{i}{2}]$, by the definition of $\bar{s}$ -numbers and $\overline{s_i(P)}$ is a decreasing sequence, we have

$\overline{s_i(\beta_1 V_1 + \beta_2 V_2)} \leq \overline{s_{2[\frac{i}{2}]}(\beta_1 V_1 + \beta_2 V_2)} \leq \overline{s_{[\frac{i}{2}]}(\beta_1 V_1) + s_{[\frac{i}{2}]}(\beta_2 V_2)} = |\beta_1| \overline{s_{[\frac{i}{2}]}(V_1)} + |\beta_2| \overline{s_{[\frac{i}{2}]}(V_2)}$ for each $i \in \mathcal{N}$. In view of Definition 4.1 condition (2) and $\mathcal{E}^F$ is a linear space, one obtains $(\overline{s_i(\beta_1 V_1 + \beta_2 V_2)})_{i=0}^\infty \in \mathcal{E}^F$, hence $\beta_1 V_1 + \beta_2 V_2 \in \overline{\mathbb{B}}_{\mathcal{E}^F}^s(\mathcal{P}, \mathcal{Q})$.

(iii) Suppose $P \in \mathbb{B}(\mathcal{P}_0, \mathcal{P})$, $T \in \overline{\mathbb{B}}_{\mathcal{E}^F}^s(\mathcal{P}, \mathcal{Q})$ and $R \in \mathbb{B}(\mathcal{Q}, \mathcal{Q}_0)$, one has

$(\overline{s_i(T)})_{i=0}^\infty \in \mathcal{E}^F$ and as $\overline{s_i(RTP)} \leq \|R\| \overline{s_i(T)} \|P\|$, by Definition 4.1 conditions (1) and (2) one gets $(\overline{s_i(RTP)})_{i=0}^\infty \in \mathcal{E}^F$, then $RTP \in \overline{\mathbb{B}}_{\mathcal{E}^F}^s(\mathcal{P}_0, \mathcal{Q}_0)$. $\square$

Definition 4.2. A subspace of the $\mathfrak{pssf}$ is named a pre-modular $\mathfrak{pssf}$, if there is a function $v : \mathcal{E}^F \rightarrow [0, \infty)$ satisfies the following conditions:

- (i): For every $\bar{j} \in \mathcal{E}^F$, $\bar{j} = \bar{\theta} \iff v(|\bar{j}|) = 0$, and $v(\bar{j}) \geq 0$, with $\bar{\theta} = (\bar{0}, \bar{0}, \bar{0}, \dots)$,
- (ii): if $\bar{j} \in \mathcal{E}^F$ and $\delta \in \mathfrak{R}$, then there are $E_0 \geq 1$ with $v(\delta \bar{j}) \leq |\delta| E_0 v(\bar{j})$,
- (iii): $v(\bar{h} + \bar{j}) \leq G_0(v(\bar{h}) + v(\bar{j}))$ includes for some $G_0 \geq 1$, with $\bar{f}, \bar{g} \in \mathcal{E}^F$,
- (iv): assume $x \in \mathcal{N}$, $|\bar{h}_x| \leq |\bar{j}_x|$, we have $v(|\bar{h}_x|) \leq v(|\bar{j}_x|)$,
- (v): the inequality, $v(|\bar{j}_x|) \leq v(|\bar{j}_{[\frac{x}{2}]}|) \leq D_0 v(|\bar{j}_x|)$ verifies, for $D_0 \geq 1$,
- (vi): if $\mathcal{F}$ is the space of finite sequences of fuzzy numbers, then the closure of $\mathcal{F} = \mathcal{E}_v^F$,
- (vii): we have $\eta > 0$ such that $v(\bar{v}, \bar{0}, \bar{0}, \bar{0}, \dots) \geq \eta |v| v(\bar{1}, \bar{0}, \bar{0}, \bar{0}, \dots)$, where

$$\bar{v}(y) = \begin{cases} 1, & y = v \\ 0, & y \neq v. \end{cases}$$

Definition 4.3. The $\mathfrak{pssf} \mathcal{E}_v^F$ is named a pre-quasi normed $\mathfrak{pssf}$, if v confirms the setup (i)-(iii) of Definition 4.2. If $\mathcal{E}^F$ is complete equipped with v , then $\mathcal{E}_v^F$ is named a pre-quasi Banach $\mathfrak{pssf}$.

Theorem 4.2. Each pre-modular $\mathfrak{psf}_v^F$ is a pre-quasi normed $\mathfrak{psf}$.

By $\mathfrak{I}_{\nearrow}$ and $\mathfrak{I}_{\searrow}$, we denote the space of all monotonic increasing and decreasing sequences of positive reals, respectively.

Theorem 4.3. $\tau_r^F(q, t)$ is a $\mathfrak{psf}$, if the next setups are confirmed:

- (f1): $(t_l) \in \mathfrak{I}_{\nearrow} \cap \ell_\infty$ with $t_0 > 1$.
 (f2): $(r_z q_z)_{z=0}^\infty \in \mathfrak{I}_{\searrow}$ or, $(r_z q_z)_{z=0}^\infty \in \mathfrak{I}_{\nearrow} \cap \ell_\infty$ and there exists $C \geq 1$ such that $r_{2z+1} q_{2z+1} \leq C r_z q_z$.

Proof. (1-i) Assume $\bar{f}, \bar{g} \in \tau_r^F(q, t)$. One obtains

$$\sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z (\bar{f}_z + \bar{g}_z), \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \leq 2^{t_0-1} \left(\sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \bar{f}_z, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} + \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \bar{g}_z, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \right) < \infty,$$

hence, $\bar{f} + \bar{g} \in \tau_r^F(q, t)$.

(1-ii) Suppose $\delta \in \mathfrak{R}$, $\bar{f} \in \tau_r^F(q, t)$ and as $(t_l) \in \mathfrak{I}_{\nearrow} \cap \ell_\infty$, we get

$$\sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \delta \bar{f}_z, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \leq \sup_l |\delta|^{t_l} \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \bar{f}_z, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} < \infty.$$

So, $\delta \bar{f} \in \tau_r^F(q, t)$. In view of setup (1-i) and (1-ii), we have $\tau_r^F(q, t)$ is a linear space.

As $(t_l) \in \mathfrak{I}_{\nearrow} \cap \ell_\infty$ and $t_0 > 1$, one obtains

$$\sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z (\bar{e}_b)_z, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} = \sum_{l=b}^{\infty} \left(\frac{r_b q_b}{r_{l+2} + r_l - 1} \right)^{t_l} \leq \sup_{l=b} (r_b q_b)^{t_l} \sum_{l=b}^{\infty} \left(\frac{1}{r_{l+2} + r_l - 1} \right)^{t_l} < \infty.$$

Therefore, $\bar{e}_b \in \tau_r^F(q, t)$, for every $b \in \mathcal{N}$.

(2) Let $|\bar{f}_b| \leq |\bar{g}_b|$, with $b \in \mathcal{N}$ and $|\bar{g}| \in \tau_r^F(q, t)$. One has

$$\sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z |\bar{f}_z|, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \leq \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z |\bar{g}_z|, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} < \infty,$$

hence $|\bar{f}| \in \tau_r^F(q, t)$.

(3) Assume $(|\bar{f}_z|) \in \tau_r^F(q, t)$, with $(t_l) \in \mathfrak{I}_{\nearrow} \cap \ell_\infty$ and $(r_z q_z)_{z=0}^\infty \in \mathfrak{I}_{\searrow}$, we get

$$\sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z |\bar{f}_{[\frac{z}{2}]}|, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} = \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^{2l} r_z q_z |\bar{f}_{[\frac{z}{2}]}|, \bar{0} \right)}{r_{2l+2} + r_{2l} - 1} \right)^{t_{2l}} + \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^{2l+1} r_z q_z |\bar{f}_{[\frac{z}{2}]}|, \bar{0} \right)}{r_{2l+3} + r_{2l+1} - 1} \right)^{t_{2l+1}}$$

$$\begin{aligned}
&\leq \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^{2l} r_z q_z |\bar{f}_{[\frac{z}{2}]}|, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} + \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^{2l+1} r_z q_z |\bar{f}_{[\frac{z}{2}]}|, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \\
&\leq \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(r_{2l} q_{2l} |\bar{f}_l| + \sum_{z=0}^l (r_{2z} q_{2z} + r_{2z+1} q_{2z+1}) |\bar{f}_z|, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} + \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l (r_{2z} q_{2z} + r_{2z+1} q_{2z+1}) |\bar{f}_z|, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \\
&\leq 2^{h-1} \left(\sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z |\bar{f}_z|, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} + \sum_{l=0}^{\infty} \left(\frac{2\bar{\rho} \left(\sum_{z=0}^l r_z q_z |\bar{f}_z|, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \right) + \sum_{l=0}^{\infty} \left(\frac{2\bar{\rho} \left(\sum_{z=0}^l r_z q_z |\bar{f}_z|, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \\
&\leq (2^{2h-1} + 2^{h-1} + 2^h) \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z |\bar{f}_z|, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} < \infty,
\end{aligned}$$

hence $(|\bar{f}_{[\frac{z}{2}]}|) \in \tau_r^F(q, t)$. $\square$

In view of Theorem 4.1 and Theorem 4.3, we have the next Theorem.

Theorem 4.4. Suppose the setups of theorem 4.3 are settled, then $\overline{\mathbb{B}}_{\tau_r^F(q,t)}^s$ is an operator ideal.

Theorem 4.5. $(\tau_r^F(q, t))_v$ is a pre-modular pssf, if the setups of theorem 4.3 are settled.

Proof. (i) Definitely, $v(\bar{f}) \geq 0$ and $v(|\bar{f}|) = 0 \Leftrightarrow \bar{f} = \bar{0}$.

(ii) There are $E_0 = \max \left\{ 1, \sup_l |\rho|^{t_l-1} \right\} \geq 1$ with $v(\delta \bar{f}) \leq E_0 |\delta| v(\bar{f})$, for each $\bar{f} \in \tau_r^F(q, t)$ and $\delta \in \mathbb{R}$.

(iii) The inequality $v(\bar{f} + \bar{g}) \leq 2^{h-1} (v(\bar{f}) + v(\bar{g}))$ satisfies, with $\bar{f}, \bar{g} \in \tau_r^F(q, t)$.

(iv) Clearly, from the proof part (2) of Theorem 4.3.

(v) Obviously, the proof part (3) of Theorem 4.3, that $D_0 \geq (2^{2h-1} + 2^{h-1} + 2^h) \geq 1$.

(vi) Clearly, the closure of $\mathcal{F} = \tau_r^F(q, t)$.

(vii) One has $0 < \omega \leq \sup_l |\nu|^{t_l-1}$ with $v(\bar{\nu}, \bar{0}, \bar{0}, \bar{0}, \dots) \geq \omega |\nu| v(\bar{1}, \bar{0}, \bar{0}, \bar{0}, \dots)$, for all $\nu \neq 0$ and $\omega > 0$, if $\nu = 0$. $\square$

Theorem 4.6. If the setups of theorem 4.3 are established, then $(\tau_r^F(q, t))_v$ is a pre-quasi Banach pssf.

Proof. According to Theorem 4.5, the space $(\tau_r^F(q, t))_v$ is a pre-modular pssf. According to Theorem 4.2, the space $(\tau_r^F(q, t))_v$ is a pre-quasi normed pssf. To explain that $(\tau_r^F(q, t))_v$ is a pre-quasi Banach pssf, assume $\bar{f}^a = (\bar{f}_z^a)_{z=0}^{\infty}$ is a Cauchy sequence in $(\tau_r^F(q, t))_v$, then for all $\varepsilon \in (0, 1)$, there is $a_0 \in \mathcal{N}$ so that for all $a, b \geq a_0$, one gets

$$v(\bar{f}^a - \bar{f}^b) = \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z (\bar{f}_z^a - \bar{f}_z^b), \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} < \varepsilon^h.$$

Hence, for $a, b \geq a_0$ and $z \in \mathcal{N}$, we obtain $\bar{\rho} \left(\sum_{z=0}^l r_z q_z (\bar{f}_z^a - \bar{f}_z^b), \bar{0} \right) < \varepsilon$. Since $(\mathbb{R}[0, 1], \bar{\rho})$ is a complete metric space. So $(\bar{f}_z^b)$ is a Cauchy sequence in $\mathbb{R}[0, 1]$, for fixed $z \in \mathcal{N}$. This explains $\lim_{b \rightarrow \infty} \bar{f}_z^b = \bar{f}_z^0$, for fixed $z \in \mathcal{N}$. Hence $v(\bar{f}^a - \bar{f}^0) < \varepsilon^h$, for all $a \geq a_0$. Finally to investigate that $\bar{f}^0 \in (\tau_r^F(q, t))_v$, one has $v(\bar{f}^0) \leq 2^{h-1} (v(\bar{f}^a - \bar{f}^0) + v(\bar{f}^a)) < \infty$, then $\bar{f}^0 \in (\tau_r^F(q, t))_v$. This explains that $(\tau_r^F(q, t))_v$ is a pre-quasi Banach pssf. $\square$

Theorem 4.7. Suppose $s-$ type $\mathcal{E}_v^F := \left\{ \bar{h} = (\overline{s_x(H)}) \in \mathfrak{R}^N : H \in \mathbb{B}(\mathcal{P}, \mathcal{Q}) \text{ and } v(\bar{h}) < \infty \right\}$. If $\overline{\mathbb{B}^s}_{\mathcal{E}_v}$ is an operators ideal, then the following conditions are verified:

1. $\mathcal{F} \subset s-$ type $\mathcal{E}_v^F$.
2. Suppose $(\overline{s_x(H_1)})_{x=0}^\infty \in s-$ type $\mathcal{E}_v^F$ and $(\overline{s_x(H_2)})_{x=0}^\infty \in s-$ type $\mathcal{E}_v^F$, then $(\overline{s_x(H_1 + H_2)})_{x=0}^\infty \in s-$ type $\mathcal{E}_v^F$.
3. Assume $\lambda \in \mathfrak{R}$ and $(\overline{s_x(H)})_{x=0}^\infty \in s-$ type $\mathcal{E}_v^F$, then $|\lambda|(\overline{s_x(H)})_{x=0}^\infty \in s-$ type $\mathcal{E}_v^F$.
4. The sequence space $\mathcal{E}_v^F$ is solid. i.e., if $(\overline{s_x(J)})_{x=0}^\infty \in s-$ type $\mathcal{E}_v^F$ and $\overline{s_x(H)} \leq \overline{s_x(J)}$, for all $x \in \mathcal{N}$ and $H, J \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$, then $(\overline{s_x(H)})_{x=0}^\infty \in s-$ type $\mathcal{E}_v^F$.

Proof. If $\overline{\mathbb{B}^s}_{\mathcal{E}_v}$ is a mappings' ideal.

(i): We have $\mathbb{F}(\mathcal{P}, \mathcal{Q}) \subset \overline{\mathbb{B}^s}_{\mathcal{E}_v^F}(\mathcal{P}, \mathcal{Q})$. Hence for all $X \in \mathbb{F}(\mathcal{P}, \mathcal{Q})$, we have $(\overline{s_r(X)})_{r=0}^\infty \in \mathcal{F}$. This gives $(\overline{s_r(X)})_{r=0}^\infty \in s-$ type $\mathcal{E}_v^F$. Hence $\mathcal{F} \subset s-$ type $\mathcal{E}_v^F$.

(ii): The space $\overline{\mathbb{B}^s}_{\mathcal{E}_v^F}(\mathcal{P}, \mathcal{Q})$ is linear over $\mathfrak{R}$. Hence for each $\lambda \in \mathfrak{R}$ and $X_1, X_2 \in \overline{\mathbb{B}^s}_{\mathcal{E}_v^F}(\mathcal{P}, \mathcal{Q})$, we have $X_1 + X_2 \in \overline{\mathbb{B}^s}_{\mathcal{E}_v^F}(\mathcal{P}, \mathcal{Q})$ and $\lambda X_1 \in \overline{\mathbb{B}^s}_{\mathcal{E}_v^F}(\mathcal{P}, \mathcal{Q})$. This implies

$$(\overline{s_r(X_1)})_{r=0}^\infty \in s-\text{type } \mathcal{E}_v^F \text{ and } (\overline{s_r(X_2)})_{r=0}^\infty \in s-\text{type } \mathcal{E}_v^F \Rightarrow (\overline{s_r(X_1 + X_2)})_{r=0}^\infty \in s-\text{type } \mathcal{E}_v^F$$

and

$$\lambda \in \mathfrak{R} \text{ and } (\overline{s_r(X_1)})_{r=0}^\infty \in s-\text{type } \mathcal{E}_v^F \Rightarrow |\lambda|(\overline{s_r(X_1)})_{r=0}^\infty \in s-\text{type } \mathcal{E}_v^F.$$

(iii): If $A \in \mathbb{B}(\mathcal{P}_0, \mathcal{P})$, $B \in \overline{\mathbb{B}^s}_{\mathcal{E}_v^F}(\mathcal{P}, \mathcal{Q})$ and $D \in \mathbb{B}(\mathcal{Q}, \mathcal{Q}_0)$, then $DBA \in \overline{\mathbb{B}^s}_{\mathcal{E}_v^F}(\mathcal{P}_0, \mathcal{Q}_0)$, where $\mathcal{P}_0$ and $\mathcal{Q}_0$ are arbitrary Banach spaces. Therefore, since $(\overline{s_r(B)})_{r=0}^\infty \in s-$ type $\mathcal{E}_v^F$, then $(\overline{s_r(DBA)})_{r=0}^\infty \in s-$ type $\mathcal{E}_v^F$. Since $\overline{s_r(DBA)} \leq \|D\| \overline{s_r(B)} \|A\|$. By using condition 3, if $(\|D\| \|A\| \overline{s_r(B)})_{r=0}^\infty \in \mathcal{E}_v^F$, we have $(\overline{s_r(DBA)})_{r=0}^\infty \in s-$ type $\mathcal{E}_v^F$. This means $s-$ type $\mathcal{E}_v^F$ is solid.

□

In view of Theorem 4.7 and Theorem 4.4, we construct the next properties of the $s-$ type $(\tau_i^F(q, t))_v$.

Theorem 4.8. Let $s-$ type $(\tau_i^F(q, t))_v := \left\{ \bar{f} = (\overline{s_n(X)}) \in \mathfrak{R}^N : X \in \mathbb{B}(\mathcal{P}, \mathcal{Q}) \text{ and } v(\bar{f}) < \infty \right\}$. The next conditions are established:

1. One has $s-$ type $(\tau_i^F(q, t))_v \supset \mathcal{F}$.
2. Suppose $(\overline{s_n(X_1)})_{n=0}^\infty \in s-$ type $(\tau_i^F(q, t))_v$ and $(\overline{s_n(X_2)})_{n=0}^\infty \in s-$ type $(\tau_i^F(q, t))_v$, then $(\overline{s_n(X_1 + X_2)})_{n=0}^\infty \in s-$ type $(\tau_i^F(q, t))_v$.
3. Assume $\lambda \in \mathfrak{R}$ and $(\overline{s_n(X)})_{n=0}^\infty \in s-$ type $(\tau_i^F(q, t))_v$, hence $|\lambda|(\overline{s_n(X)})_{n=0}^\infty \in s-$ type $(\tau_i^F(q, t))_v$.
4. The $s-$ type $(\tau_i^F(q, t))_v$ is solid.

5. MULTIPLICATION OPERATORS ON $(\tau_r^F(q, t))_v$

We discuss here the necessity and enough setups on $(\tau_r^F(q, t))_v$ in order to the multiplication operator defined on it is bounded, invertible, approximable, Fredholm and closed range.

Definition 5.1. Suppose $\lambda = (\lambda_k) \in \mathfrak{R}^{\mathcal{N}}$ and $\mathcal{E}_v^F$ is a pre-quasi normed pssf. The operator $H_\lambda : \mathcal{E}_v^F \rightarrow \mathcal{E}_v^F$ is named a multiplication operator on $\mathcal{E}_v^F$, if $H_\lambda \bar{f} = (\lambda_b \bar{f}_b) \in \mathcal{E}_v^F$, with $f \in \mathcal{E}_v^F$. The multiplication operator is named created by λ , if $H_\lambda \in \mathbb{B}(\mathcal{E}_v^F)$.

Theorem 5.1. Suppose $\lambda \in \mathfrak{R}^{\mathcal{N}}$, the setups of theorem 4.3 are entrenched, hence

$$\lambda \in \ell_\infty \iff H_\lambda \in \mathbb{B}((\tau_r^F(q, t))_v).$$

Proof. Let $\lambda \in \ell_\infty$. Hence, there is $\nu > 0$ so that $|\lambda_b| \leq \nu$, for every $b \in \mathcal{N}$. Assume $\bar{f} \in (\tau_r^F(q, t))_v$, one has

$$v(H_\lambda \bar{f}) = v(\lambda \bar{f}) = \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l \lambda_z r_z q_z \bar{f}_z, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \leq \sup_l \nu^{t_l} \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \bar{f}_z, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} = \sup_l \nu^{t_l} v(\bar{f}).$$

Therefore, $H_\lambda \in \mathbb{B}((\tau_r^F(q, t))_v)$.

On the other hand, assume $H_\lambda \in \mathbb{B}((\tau_r^F(q, t))_v)$ and $\lambda \notin \ell_\infty$. Hence for all $b \in \mathcal{N}$, there are $x_b \in \mathcal{N}$ so that $\lambda_{x_b} > b$. We get

$$\begin{aligned} v(H_\lambda \bar{e}_{x_b}) &= v(\lambda \bar{e}_{x_b}) = \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l \lambda_z r_z q_z \overline{(e_{x_b})_z}, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \\ &= \sum_{l=x_b}^{\infty} \left(\frac{\lambda_{(x_b)} r_{(x_b)} q_{x_b}}{r_{l+2} + r_l - 1} \right)^{t_l} > \sum_{l=x_b}^{\infty} \left(\frac{b r_{(x_b)} q_{x_b}}{r_{l+2} + r_l - 1} \right)^{t_l} > b^{t_0} v(\bar{e}_{x_b}). \end{aligned}$$

Hence, $H_\lambda \notin \mathbb{B}((\tau_r^F(q, t))_v)$. So $\lambda \in \ell_\infty$. □

Theorem 5.2. Suppose $\lambda \in \mathfrak{R}^{\mathcal{N}}$ and $(\tau_r^F(q, t))_v$ is a pre-quasi normed pssf. Hence $|\lambda_b| = 1$, for every $b \in \mathcal{N}$, if and only if, H_λ is an isometry.

Proof. Let $|\lambda_b| = 1$, for every $b \in \mathcal{N}$. One obtains

$$v(H_\lambda \bar{f}) = v(\lambda \bar{f}) = \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \lambda_z \bar{f}_z, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} = \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \bar{f}_z, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} = v(\bar{f}),$$

for every $\bar{f} \in (\tau_r^F(q, t))_v$. Therefore, H_λ is an isometry.

Suppose the necessity setup is entrenched and $|\lambda_b| < 1$, for some $b = b_0$. We get

$$\begin{aligned} v(H_\lambda \bar{e}_{b_0}) &= v(\lambda \bar{e}_{b_0}) = \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \lambda_z \overline{(e_{b_0})_z}, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} = \sum_{l=b_0}^{\infty} \left(\frac{|\lambda_{b_0}| r_{b_0} q_{b_0}}{r_{l+2} + r_l - 1} \right)^{t_l} \\ &< \sum_{l=b_0}^{\infty} \left(\frac{r_{b_0} q_{b_0}}{r_{l+2} + r_l - 1} \right)^{t_l} = v(\bar{e}_{b_0}). \end{aligned}$$

Next if $|\lambda_{b_0}| > 1$, obviously $v(H_\lambda \overline{e_{b_0}}) > v(\overline{e_{b_0}})$. This explains a contradiction for the two cases. Therefore, $|\lambda_b| = 1$, for all $b \in \mathcal{N}$. $\square$

Theorem 5.3. Suppose $\lambda \in \mathfrak{R}^{\mathcal{N}}$, the setups of theorem 4.3 are entrenched. Hence

$$H_\lambda \in \mathcal{A}((\tau_r^F(q, t))_v) \iff (\lambda_b)_{b=0}^\infty \in c_0.$$

Proof. Let $H_\lambda \in \mathcal{A}((\tau_r^F(q, t))_v)$, then $H_\lambda \in \mathcal{K}((\tau_r^F(q, t))_v)$. Assume $\lim_{b \rightarrow \infty} \lambda_b \neq 0$. Therefore, we have $\varrho > 0$ such that the set $K_\varrho = \{b \in \mathcal{N} : |\lambda_b| \geq \varrho\} \not\subseteq \mathfrak{I}$, where $\mathfrak{I}$ is the space of all sets with finite number of elements. Assume $\{\alpha_b\}_{b \in \mathcal{N}} \subset K_\varrho$. Hence, $\{\overline{e_{\alpha_b}} : \alpha_b \in K_\varrho\} \in \ell_\infty^F$ is an infinite set in $(\tau_r^F(q, t))_v$, where ℓ_∞^F is the space of bounded sequences of fuzzy functions. Since

$$\begin{aligned} v(H_\lambda \overline{e_{\alpha_a}} - H_\lambda \overline{e_{\alpha_b}}) &= v(\lambda \overline{e_{\alpha_a}} - \lambda \overline{e_{\alpha_b}}) = \sum_{l=0}^\infty \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \lambda_z (\overline{(e_{\alpha_a})_z} - \overline{(e_{\alpha_b})_z}), \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \\ &\geq \sum_{l=0}^\infty \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \varrho (\overline{(e_{\alpha_a})_z} - \overline{(e_{\alpha_b})_z}), \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \geq \inf_l \varrho^{t_l} v(\overline{e_{\alpha_a}} - \overline{e_{\alpha_b}}), \end{aligned}$$

for every $\alpha_a, \alpha_b \in K_\varrho$. Then, $\{\overline{e_{\alpha_b}} : \alpha_b \in K_\varrho\} \in \ell_\infty^F$, which cannot have a convergent subsequence under H_λ . Hence $H_\lambda \notin \mathcal{K}((\tau_r^F(q, t))_v)$. This explains $H_\lambda \notin \mathcal{A}((\tau_r^F(q, t))_v)$, which indicates a contradiction. Hence, $\lim_{b \rightarrow \infty} \lambda_b = 0$. On the other hand, assume $\lim_{b \rightarrow \infty} \lambda_b = 0$. Therefore, for all $\varrho > 0$, one has $K_\varrho = \{b \in \mathcal{N} : |\lambda_b| \geq \varrho\} \subset \mathfrak{I}$. Hence, for every $\varrho > 0$, we have $\dim \left(\left((\tau_r^F(q, t))_v \right)_{K_\varrho} \right) = \dim(\mathfrak{R}^{K_\varrho}) < \infty$.

Therefore, $H_\lambda \in \mathbb{F} \left(\left((\tau_r^F(q, t))_v \right)_{K_\varrho} \right)$. Assume $\lambda_a \in \mathfrak{R}^{\mathcal{N}}$, for all $a \in \mathcal{N}$, where

$$(\lambda_a)_b = \begin{cases} \lambda_b, & b \in K_{\frac{1}{a+1}}, \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, $H_{\lambda_a} \in \mathbb{F} \left(\left((\tau_r^F(q, t))_v \right)_{B_{\frac{1}{a+1}}} \right)$, since $\dim \left(\left((\tau_r^F(q, t))_v \right)_{B_{\frac{1}{a+1}}} \right) < \infty$, for all $a \in \mathcal{N}$. According to $(t_l) \in \mathfrak{I} \nearrow \cap \ell_\infty$ with $t_0 > 1$, we have

$$\begin{aligned} v((H_\lambda - H_{\lambda_a})\bar{f}) &= v \left(\left((\lambda_b - (\lambda_a)_b) \bar{f}_b \right)_{b=0}^\infty \right) = \sum_{l=0}^\infty \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z (\lambda_z - (\lambda_a)_z) \bar{f}_z, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \\ &= \sum_{l=0, l \in K_{\frac{1}{a+1}}}^\infty \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z (\lambda_z - (\lambda_a)_z) \bar{f}_z, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} + \sum_{l=0, l \notin K_{\frac{1}{a+1}}}^\infty \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z (\lambda_z - (\lambda_a)_z) \bar{f}_z, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \\ &= \sum_{l=0, l \notin K_{\frac{1}{a+1}}}^\infty \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \lambda_z \bar{f}_z, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \leq \frac{1}{(a+1)^{t_0}} \sum_{l=0, l \notin K_{\frac{1}{a+1}}}^\infty \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \bar{f}_z, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \\ &< \frac{1}{(a+1)^{t_0}} \sum_{l=0}^\infty \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \bar{f}_z, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} = \frac{1}{(a+1)^{t_0}} v(\bar{f}). \end{aligned}$$

Hence, $\|H_\lambda - H_{\lambda_a}\| \leq \frac{1}{(a+1)^{t_0}}$. Which investigates that H_λ is a limit of finite rank maps. Therefore, $H_\lambda \in \mathcal{A}((\tau_r^F(q, t))_v)$. $\square$

Theorem 5.4. Assume $\lambda \in \mathfrak{R}^N$, the setups of theorem 4.3 are entrenched. Hence

$$H_\lambda \in \mathcal{K}((\tau_r^F(q, t))_v) \iff (\lambda_b)_{b=0}^\infty \in c_0.$$

Proof. Evidently, since $\mathcal{A}((\tau_r^F(q, t))_v) \subsetneq \mathcal{K}((\tau_r^F(q, t))_v)$. $\square$

Corollary 5.5. Suppose the setups of theorem 4.3 are proved, hence $\mathcal{K}((\tau_r^F(q, t))_v) \subsetneq \mathcal{B}((\tau_r^F(q, t))_v)$.

Proof. As $\lambda = (1, 1, \dots)$ creates the multiplication map I on $(\tau_r^F(q, t))_v$. Which explains $I \notin \mathcal{K}((\tau_r^F(q, t))_v)$ and $I \in \mathcal{B}((\tau_r^F(q, t))_v)$. $\square$

Theorem 5.6. If $(\tau_r^F(q, t))_v$ is a pre-quasi Banach pssf and $H_\lambda \in \mathcal{B}((\tau_r^F(q, t))_v)$. Hence there are $\alpha > 0$ and $\eta > 0$ such that $\alpha < |\lambda_b| < \eta$, with $b \in (\ker(\lambda))^c$, if and only if, $\text{Range}(H_\lambda)$ is closed.

Proof. Assume the enough conditions are proved. Hence, there is $\varrho > 0$ so that $|\lambda_b| \geq \varrho$, for all $b \in (\ker(\lambda))^c$. To explain that $\text{Range}(H_\lambda)$ is closed. Assume $\bar{g}$ is a limit point of $\text{Range}(H_\lambda)$. We obtain $H_\lambda \bar{f}_b \in (\tau_r^F(q, t))_v$, for every $b \in \mathcal{N}$ so that $\lim_{b \rightarrow \infty} H_\lambda \bar{f}_b = \bar{g}$. Evidently, the sequence $H_\lambda \bar{f}_b$ is a Cauchy sequence. As $(t_l) \in \mathfrak{I} \cap \ell_\infty$ with $t_0 > 1$, one gets

$$\begin{aligned} v(H_\lambda \bar{f}_a - H_\lambda \bar{f}_b) &= \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z (\lambda_z \overline{(f_a)_z} - \lambda_z \overline{(f_b)_z}), \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \\ &= \sum_{l=0, l \in (\ker(\lambda))^c}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z (\lambda_z \overline{(f_a)_z} - \lambda_z \overline{(f_b)_z}), \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} + \\ &\quad \sum_{l=0, l \notin (\ker(\lambda))^c}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z (\lambda_z \overline{(f_a)_z} - \lambda_z \overline{(f_b)_z}), \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \\ &\geq \sum_{l=0, l \in (\ker(\lambda))^c}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z (\lambda_z \overline{(f_a)_z} - \lambda_z \overline{(f_b)_z}), \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \\ &= \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z (\lambda_z \overline{(u_a)_z} - \lambda_z \overline{(u_b)_z}), \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \\ &> \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\varrho \sum_{z=0}^l r_z q_z (\overline{(u_a)_z} - \overline{(u_b)_z}), \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \geq \inf_l \varrho^{t_l} v(\overline{u_a} - \overline{u_b}), \end{aligned}$$

where

$$\overline{(u_a)_k} = \begin{cases} \overline{(f_a)_k}, & k \in (\ker(\lambda))^c, \\ 0, & k \notin (\ker(\lambda))^c. \end{cases}$$

Hence, $\{\overline{u_a}\}$ is a Cauchy sequence in $(\tau_r^F(q, t))_v$. As $(\tau_r^F(q, t))_v$ is complete. Therefore, there is $\bar{f} \in (\tau_r^F(q, t))_v$ so that $\lim_{b \rightarrow \infty} \overline{u_b} = \bar{f}$. Since $H_\lambda \in \mathcal{B}((\tau_r^F(q, t))_v)$, one has $\lim_{b \rightarrow \infty} H_\lambda \overline{u_b} = H_\lambda \bar{f}$. Since

$\lim_{b \rightarrow \infty} H_\lambda \overline{u_b} = \lim_{b \rightarrow \infty} H_\lambda \overline{f_b} = \overline{g}$. Therefore, $H_\lambda \overline{f} = \overline{g}$. Hence $\overline{g} \in \text{Range}(H_\lambda)$. So $\text{Range}(H_\lambda)$ is closed. Next, assume the necessity setup is confirmed. Hence, there is $\varrho > 0$ so that $v(H_\lambda \overline{f}) \geq \varrho v(\overline{f})$, with $\overline{f} \in \left((\tau_r^F(q, t))_v \right)_{(\ker(\lambda))^c}$. If $K = \left\{ b \in (\ker(\lambda))^c : |\lambda_b| < \varrho \right\} \neq \emptyset$, hence for $a_0 \in K$, one has

$$\begin{aligned} v(H_\lambda \overline{e_{a_0}}) &= v\left(\left(\lambda_b \overline{(e_{a_0})_b}\right)_{b=0}^\infty\right) = \sum_{l=0}^\infty \left(\frac{\overline{\rho} \left(\sum_{z=0}^l r_z q_z \overline{(e_{a_0})_z}, \overline{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \\ &< \sum_{l=0}^\infty \left(\frac{\overline{\rho} \left(\varrho \sum_{z=0}^l r_z q_z \overline{(e_{a_0})_z}, \overline{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \leq \sup_l \varrho^{t_l} v(\overline{e_{a_0}}), \end{aligned}$$

which introduces a contradiction. So $K = \emptyset$, we have $|\lambda_b| \geq \varrho$, with $b \in (\ker(\lambda))^c$. This proves the theorem. $\square$

Theorem 5.7. Suppose $\lambda \in \mathfrak{R}^N$ and $(\tau_r^F(q, t))_v$ is a pre-quasi Banach pssf. Hence, there are $\alpha > 0$ and $\eta > 0$ so that $\alpha < |\lambda_b| < \eta$, for every $b \in \mathcal{N}$, if and only if, $H_\lambda \in \mathbb{B}((\tau_r^F(q, t))_v)$ is invertible.

Proof. Assume the enough setup are proved. Suppose $\kappa \in \mathfrak{R}^N$ with $\kappa_b = \frac{1}{\lambda_b}$. In view of Theorem 5.1, the operators H_λ and H_κ are bounded linear. We get $H_\lambda H_\kappa = H_\kappa H_\lambda = I$. Hence $H_\kappa = H_\lambda^{-1}$. After, assume H_λ is invertible. Hence $\text{Range}(H_\lambda) = \left((\tau_r^F(q, t))_v \right)_\mathcal{N}$. So, $\text{Range}(H_\lambda)$ is closed. Therefore, by using Theorem 5.6, there is $\alpha > 0$ so that $|\lambda_b| \geq \alpha$, for every $b \in (\ker(\lambda))^c$. We have $\ker(\lambda) = \emptyset$, if $\lambda_{b_0} = 0$, with $b_0 \in \mathcal{N}$, which gives $e_{b_0} \in \ker(H_\lambda)$, this explains a contradiction, as $\ker(H_\lambda)$ is trivial. Therefore, $|\lambda_b| \geq \alpha$, for every $b \in \mathcal{N}$. Since $H_\lambda \in \ell_\infty$. By using Theorem 5.1, there is $\eta > 0$ so that $|\lambda_b| \leq \eta$, for every $b \in \mathcal{N}$. Therefore, we have $\alpha \leq |\lambda_b| \leq \eta$, with $b \in \mathcal{N}$. $\square$

Definition 5.2. [46] An operator $U \in \mathbb{B}(\mathcal{E})$ is named Fredholm if $\dim(\text{Range}(U))^c < \infty$, $\dim(\ker(U)) < \infty$ and $\text{Range}(U)$ is closed, where $(\text{Range}(U))^c$ marks the complement of $\text{Range}(U)$.

Theorem 5.8. Suppose $(\tau_r^F(q, t))_v$ is a pre-quasi Banach pssf and $H_\lambda \in \mathbb{B}((\tau_r^F(q, t))_v)$. Hence H_λ is Fredholm operator, if and only if, (i) $\ker(\lambda) \subsetneq \mathcal{N} \cap \mathfrak{I}$ and (ii) $|\lambda_b| \geq \varrho$, with $b \in (\ker(\lambda))^c$.

Proof. Let the enough conditions be satisfied. Assume $\ker(\lambda) \subsetneq \mathcal{N}$ is an infinite, hence $\overline{e_b} \in \ker(H_\lambda)$, for every $b \in \ker(\lambda)$. Since $\overline{e_b}$'s are linearly independent, one obtains that $\dim(\ker(H_\lambda)) = \infty$, which explains a contradiction. Hence, $\ker(\lambda) \subsetneq \mathcal{N}$ must be finite. The setup (ii) follows from Theorem 5.6. Next, suppose the conditions (i) and (ii) are confirmed. In view of Theorem 5.6, the condition (ii) explains that $\text{Range}(H_\lambda)$ is closed. The setup (i) gives that $\dim(\ker(H_\lambda)) < \infty$ and $\dim((\text{Range}(H_\lambda))^c) < \infty$. Hence H_λ is Fredholm. $\square$

6. FEATURES OF PRE-QUASI IDEAL

In this section, we introduce the enough setup (not necessary) on $(\tau_r^F(q, t))_v$ such that the closure of $\mathbb{F} = \overline{\mathbb{B}^\alpha}_{(\tau_r^F(q, t))_v}$. This investigates a negative answer of Rhoades [41] open problem about the linearity of s -type $(\tau_r^F(q, t))_v$ spaces. Secondly, for which conditions on $(\tau_r^F(q, t))_v$, are $\overline{\mathbb{B}^s}_{(\tau_r^F(q, t))_v}$ closed and complete? Thirdly, we explain the enough setup on $(\tau_r^F(q, t))_v$ such

that $\overline{\mathbb{B}}^{\alpha}_{(\tau_r^F(q,t))_v}$ is strictly contained for different weights and powers. We offer the setup so that $\overline{\mathbb{B}}^{\alpha}_{(\tau_r^F(q,t))_v}$ is minimum. Fourthly, we introduce the conditions so that the Banach pre-quasi ideal $\overline{\mathbb{B}}^s_{(\tau_r^F(q,t))_v}$ is simple. Fifthly, we investigate the enough conditions on $(\tau_r^F(q,t))_v$ such that the space of all bounded linear operators which sequence of eigenvalues in $(\tau_r^F(q,t))_v$ equals $\overline{\mathbb{B}}^s_{(\tau_r^F(q,t))_v}$.

6.1. Finite rank pre-quasi ideal.

Theorem 6.1. $\overline{\mathbb{B}}^{\alpha}_{(\tau_r^F(q,t))_v}(\mathcal{P}, \mathcal{Q})$ = the closure of $\mathbb{F}(\mathcal{P}, \mathcal{Q})$, suppose the setups of theorem 4.3 are established. But the converse is not necessarily true.

Proof. To investigate that the closure of $\mathbb{F}(\mathcal{P}, \mathcal{Q}) \subseteq \overline{\mathbb{B}}^{\alpha}_{(\tau_r^F(q,t))_v}(\mathcal{P}, \mathcal{Q})$. As $\bar{e}_l \in (\tau_r^F(q,t))_v$, for every $l \in \mathcal{N}$ and $(\tau_r^F(q,t))_v$ is a linear space. Let $Z \in \mathbb{F}(\mathcal{P}, \mathcal{Q})$, one gets $(\alpha_l(Z))_{l=0}^{\infty} \in \mathcal{F}$. To explain that $\overline{\mathbb{B}}^{\alpha}_{(\tau_r^F(q,t))_v}(\mathcal{P}, \mathcal{Q}) \subseteq$ the closure of $\mathbb{F}(\mathcal{P}, \mathcal{Q})$. Assume $Z \in \overline{\mathbb{B}}^{\alpha}_{(\tau_r^F(q,t))_v}(\mathcal{P}, \mathcal{Q})$, we obtain $(\alpha_l(Z))_{l=0}^{\infty} \in (\tau_r^F(q,t))_v$. Since $v(\alpha_l(Z))_{l=0}^{\infty} < \infty$, let $\rho \in (0, 1)$, hence there is $l_0 \in \mathcal{N} - \{0\}$ with $v((\alpha_l(Z))_{l=l_0}^{\infty}) < \frac{\rho}{2^{h+3}\eta d}$, for some $d \geq 1$, where $\eta = \max \left\{ 1, \sum_{l=l_0}^{\infty} \left(\frac{1}{r_{l+2} + r_l - 1} \right)^{t_l} \right\}$. Since $\overline{\alpha_l(Z)} \in \mathfrak{I}_{\searrow}^F$, we get

$$\begin{aligned} \sum_{l=l_0+1}^{2l_0} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \overline{\alpha_{2l_0}(Z)}, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} &\leq \sum_{l=l_0+1}^{2l_0} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \overline{\alpha_z(Z)}, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \\ &\leq \sum_{l=l_0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \overline{\alpha_z(Z)}, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} < \frac{\rho}{2^{h+3}\eta d}. \end{aligned} \quad (6.1)$$

Hence there is $Y \in \mathbb{F}_{2l_0}(\mathcal{P}, \mathcal{Q})$ so that $\text{rank}(Y) \leq 2l_0$ and

$$\sum_{l=2l_0+1}^{3l_0} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \overline{\|Z - Y\|}, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \leq \sum_{l=l_0+1}^{2l_0} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \overline{\|Z - Y\|}, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} < \frac{\rho}{2^{h+3}\eta d'}, \quad (6.2)$$

since $(t_l) \in \mathfrak{I}_{\nearrow} \cap \ell_{\infty}$, we have

$$\sup_{l=l_0}^{\infty} \bar{\rho}^{t_l} \left(\sum_{z=0}^{l_0} r_z q_z \overline{\|Z - Y\|}, \bar{0} \right) < \frac{\rho}{2^{2h+2}\eta}. \quad (6.3)$$

Therefore, one has

$$\sum_{l=0}^{l_0} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \overline{\|Z - Y\|}, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} < \frac{\rho}{2^{h+3}\eta d}. \quad (6.4)$$

In view of inequalities (1)-(5), one gets

$$\begin{aligned} d(Z, Y) &= v(\overline{\alpha_l(Z - Y)})_{l=0}^{\infty} \\ &= \sum_{l=0}^{3l_0-1} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \overline{\alpha_z(Z - Y)}, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} + \sum_{l=3l_0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \overline{\alpha_z(Z - Y)}, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \\ &\leq \sum_{l=0}^{3l_0} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \overline{\|Z - Y\|}, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} + \sum_{l=l_0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^{l+2l_0} r_z q_z \overline{\alpha_z(Z - Y)}, \bar{0} \right)}{r_{l+2l_0+2} + r_{l+2l_0} - 1} \right)^{t_{l+2l_0}} \end{aligned}$$

$$\begin{aligned}
 &\leq \sum_{l=0}^{3l_0} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \|Z - Y\|, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} + \sum_{l=l_0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^{l+2l_0} r_z q_z \alpha_z(Z - Y), \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \\
 &\leq 3 \sum_{l=0}^{l_0} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \|Z - Y\|, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} + \sum_{l=l_0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^{2l_0-1} r_z q_z \alpha_z(Z - Y) + \sum_{z=2l_0}^{l+2l_0} r_z q_z \alpha_z(Z - Y), \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \\
 &\leq 3 \sum_{l=0}^{l_0} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \|Z - Y\|, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} + 2^{h-1} \sum_{l=l_0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^{2l_0-1} r_z q_z \alpha_z(Z - Y), \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} + \\
 &\quad 2^{h-1} \sum_{l=l_0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=2l_0}^{l+2l_0} r_z q_z \alpha_z(Z - Y), \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \\
 &\leq 3 \sum_{l=0}^{l_0} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \|Z - Y\|, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} + 2^{h-1} \sum_{l=l_0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^{2l_0-1} r_z q_z \alpha_z(Z - Y), \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} + \\
 &\quad 2^{h-1} \sum_{l=l_0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \|Z - Y\|, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \\
 &\leq 3 \sum_{l=0}^{l_0} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \|Z - Y\|, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} + 2^{2h-1} \sup_{l=l_0}^{\infty} \bar{\rho}^{t_l} \left(\sum_{z=0}^{l_0} r_z q_z \|Z - Y\|, \bar{0} \right) \sum_{l=l_0}^{\infty} \left(\frac{p_l}{r_{l+2} + r_l - 1} \right)^{t_l} + \\
 &\quad 2^{h-1} \sum_{l=l_0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \alpha_z(Z), \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} < \rho.
 \end{aligned}$$

On the other hand, one has a negative example as $I_2 \in \overline{\mathbb{B}}^{\alpha}_{(\tau_r^F(q,t))_v}(\mathcal{P}, \mathcal{Q})$, where $r_z q_z = 1$, for all $z \in \mathcal{N}$ and $t = (0, -1, 2, 2, 2, \dots)$, but $(t_l) \notin \mathfrak{I}_{\nearrow}$. This shows the proof. $\square$

6.2. Banach pre-quasi ideal.

Theorem 6.2. *If the setups of theorem 4.3 are established, then the function Ψ is a pre-quasi norm on $\overline{\mathbb{B}}^s_{(\tau_r^F(q,t))_v}$, with $\Psi(Z) = v(s_q(Z))_{q=0}^{\infty}$, for all $Z \in \overline{\mathbb{B}}^s_{(\tau_r^F(q,t))_v}(\mathcal{P}, \mathcal{Q})$.*

Proof. (1): When $X \in \overline{\mathbb{B}}^s_{(\tau_r^F(q,t))_v}(\mathcal{P}, \mathcal{Q})$, $\Psi(X) = v(s_q(X))_{q=0}^{\infty} \geq 0$ and $\Psi(X) = v(s_q(X))_{q=0}^{\infty} = 0$,

if and only if, $s_q(X) = \bar{0}$, for all $q \in \mathcal{N}$, if and only if, $X = 0$,

(2): there is $E_0 \geq 1$ with $\Psi(\alpha X) = v(s_q(\alpha X))_{q=0}^{\infty} \leq E_0 |\alpha| \Psi(X)$, for all $X \in \overline{\mathbb{B}}^s_{(\tau_r^F(q,t))_v}(\mathcal{P}, \mathcal{Q})$ and $\alpha \in \mathfrak{R}$,

(3): one has $D_0 G_0 \geq 1$ so that for $X_1, X_2 \in \overline{\mathbb{B}}^s_{(\tau_r^F(q,t))_v}(\mathcal{P}, \mathcal{Q})$, one can see

$$\begin{aligned}
 \Psi(X_1 + X_2) &= v(s_q(X_1 + X_2))_{q=0}^{\infty} \leq G_0 \left(v(s_{[\frac{q}{2}]}(X_1))_{q=0}^{\infty} + v(s_{[\frac{q}{2}]}(X_2))_{q=0}^{\infty} \right) \\
 &\leq D_0 G_0 \left(v(s_q(X_1))_{q=0}^{\infty} + v(s_q(X_2))_{q=0}^{\infty} \right),
 \end{aligned}$$

(4): we have $\varrho \geq 1$, if $X \in \mathbb{B}(\mathcal{P}_0, \mathcal{P})$, $Y \in \overline{\mathbb{B}}^s_{(\tau_r^F(q,t))_v}(\mathcal{P}, \mathcal{Q})$ and $Z \in \mathbb{B}(\mathcal{Q}, \mathcal{Q}_0)$, then $\Psi(ZYX) = v(s_q(ZYX))_{q=0}^{\infty} \leq v(\|X\| \|Z\| s_q(Y))_{q=0}^{\infty} \leq \varrho \|X\| \Psi(Y) \|Z\|$.

□

Theorem 6.3. *If the setups of theorem 4.3 are established, hence $(\overline{\mathbb{B}^s}_{(\tau_r^F(q,t))_v}, \Psi)$ is a pre-quasi Banach ideal, where $\Psi(X) = v\left(\overline{(s_l(X))}_{l=0}^\infty\right)$.*

Proof. As $(\tau_r^F(q,t))_v$ is a pre-modular pssf, hence from theorem 6.2, Ψ is a pre-quasi norm on $\overline{\mathbb{B}^s}_{(\tau_r^F(q,t))_v}$. Suppose $(X_b)_{b \in \mathcal{N}}$ is a Cauchy sequence in $\overline{\mathbb{B}^s}_{(\tau_r^F(q,t))_v}(\mathcal{P}, \mathcal{Q})$. As $\mathbb{B}(\mathcal{P}, \mathcal{Q}) \supseteq \overline{\mathbb{B}^s}_{(\tau_r^F(q,t))_v}(\mathcal{P}, \mathcal{Q})$, one obtains

$$\Psi(X_a - X_b) = \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \overline{s_z(X_a - X_b)}, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \geq \left(\frac{q_0}{2} \|X_a - X_b\| \right)^{t_0},$$

hence $(X_b)_{b \in \mathcal{N}}$ is a Cauchy sequence in $\mathbb{B}(\mathcal{P}, \mathcal{Q})$. Since $\mathbb{B}(\mathcal{P}, \mathcal{Q})$ is a Banach space, then there is $X \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$ with $\lim_{b \rightarrow \infty} \|X_b - X\| = 0$. Since $\overline{(s_l(X_b))}_{l=0}^\infty \in (\tau_r^F(q,t))_v$, for every $b \in \mathcal{N}$. According to Definition 4.2 setups (ii), (iii) and (v), one gets

$$\begin{aligned} \Psi(X) &= \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \overline{s_z(X)}, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \\ &\leq 2^{\hbar-1} \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \overline{s_{[\frac{z}{2}]}(X - X_b)}, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} + 2^{\hbar-1} \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \overline{s_{[\frac{z}{2}]}(X_b)}, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \\ &\leq 2^{\hbar-1} \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \|X - X_b\|, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} + 2^{\hbar-1} D_0 \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \overline{s_z(X_b)}, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} < \infty. \end{aligned}$$

Therefore, $\overline{(s_l(X))}_{l=0}^\infty \in (\tau_r^F(q,t))_v$, then $X \in \overline{\mathbb{B}^s}_{(\tau_r^F(q,t))_v}(\mathcal{P}, \mathcal{Q})$. □

6.3. Minimum pre-quasi ideal.

Theorem 6.4. *Suppose $\mathcal{P}$ and $\mathcal{Q}$ are Banach spaces with $\dim(\mathcal{P}) = \dim(\mathcal{Q}) = \infty$, and the setups of theorem 4.3 are confirmed with $1 < t_l^{(1)} < t_l^{(2)}$, and $0 < q_l^{(2)} \leq q_l^{(1)}$, for all $l \in \mathcal{N}$, hence*

$$\overline{\mathbb{B}^s}_{\left(\tau_r^F((q_l^{(1)}), (t_l^{(1)}))\right)_v}(\mathcal{P}, \mathcal{Q}) \subsetneq \overline{\mathbb{B}^s}_{\left(\tau_r^F((q_l^{(2)}), (t_l^{(2)}))\right)_v}(\mathcal{P}, \mathcal{Q}) \subsetneq \mathbb{B}(\mathcal{P}, \mathcal{Q}).$$

Proof. Let $Z \in \overline{\mathbb{B}^s}_{\left(\tau_r^F((q_l^{(1)}), (t_l^{(1)}))\right)_v}(\mathcal{P}, \mathcal{Q})$, then $\overline{(s_l(Z))} \in \left(\tau_r^F((q_l^{(1)}), (t_l^{(1)}))\right)_v$. One obtains

$$\sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z^{(2)} \overline{s_z(Z)}, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l^{(2)}} < \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z^{(1)} \overline{s_z(Z)}, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l^{(1)}} < \infty,$$

then $Z \in \overline{\mathbb{B}^s} \left(\tau_r^F((q_l^{(2)}), (t_l^{(2)})) \right)_v (\mathcal{P}, \mathcal{Q})$. Next, if we choose $(\overline{s_l(Z)})_{l=0}^\infty$ with $\bar{\rho} \left(\sum_{z=0}^l r_z q_z^{(1)} \overline{s_z(Z)}, \bar{0} \right) = \frac{r_{l+2} + r_l - 1}{t_l^{(1)} \sqrt{l+1}}$, one gets $Z \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$ such that

$$\sum_{l=0}^\infty \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z^{(1)} \overline{s_z(X)}, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l^{(1)}} = \sum_{l=0}^\infty \frac{1}{l+1} = \infty,$$

and

$$\sum_{l=0}^\infty \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z^{(2)} \overline{s_z(Z)}, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l^{(2)}} \leq \sum_{l=0}^\infty \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z^{(1)} \overline{s_z(Z)}, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l^{(2)}} = \sum_{l=0}^\infty \left(\frac{1}{l+1} \right)^{t_l^{(1)}} < \infty.$$

Therefore, $Z \notin \overline{\mathbb{B}^s} \left(\tau_r^F((q_l^{(1)}), (t_l^{(1)})) \right)_v (\mathcal{P}, \mathcal{Q})$ and $Z \in \overline{\mathbb{B}^s} \left(\tau_r^F((q_l^{(2)}), (t_l^{(2)})) \right)_v (\mathcal{P}, \mathcal{Q})$.

Clearly, $\overline{\mathbb{B}^s} \left(\tau_r^F((q_l^{(2)}), (t_l^{(2)})) \right)_v (\mathcal{P}, \mathcal{Q}) \subset \mathbb{B}(\mathcal{P}, \mathcal{Q})$. Next, if we put $(\overline{s_l(Z)})_{l=0}^\infty$ such that $\bar{\rho} \left(\sum_{z=0}^l r_z q_z^{(2)} \overline{s_z(Z)}, \bar{0} \right) = \frac{r_{l+2} + r_l - 1}{t_l^{(2)} \sqrt{l+1}}$. We have $Z \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$ such that $Z \notin \overline{\mathbb{B}^s} \left(\tau_r^F((q_l^{(2)}), (t_l^{(2)})) \right)_v (\mathcal{P}, \mathcal{Q})$. This explains the proof. $\square$

Theorem 6.5. Let $\mathcal{P}$ and $\mathcal{Q}$ be Banach spaces with $\dim(\mathcal{P}) = \dim(\mathcal{Q}) = \infty$, and the setups of theorem 4.3 are established with $\left(\frac{\sum_{z=0}^l r_z q_z}{r_{l+2} + r_l - 1} \right)_{l=0}^\infty \notin \ell_{((t_l))}$, hence $\overline{\mathbb{B}^\alpha}_{(\tau_r^F(q,t))_v}$ is minimum.

Proof. Suppose the enough setups are confirmed. Then $(\overline{\mathbb{B}^\alpha}_{\tau_r^F(q,t)}, \Psi)$, where

$\Psi(Z) = \sum_{l=0}^\infty \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \overline{\alpha_z(Z)}, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l}$, is a pre-quasi Banach ideal. Suppose $\overline{\mathbb{B}^\alpha}_{\tau_r^F(q,t)}(\mathcal{P}, \mathcal{Q}) = \mathbb{B}(\mathcal{P}, \mathcal{Q})$, hence there is $\eta > 0$ with $\Psi(Z) \leq \eta \|Z\|$, for every $Z \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$. According to Dvoretzky's theorem [47], for every $b \in \mathcal{N}$, one obtains quotient spaces $\mathcal{P}/Y_b$ and subspaces M_b of $\mathcal{Q}$ which can be mapped onto ℓ_2^b by isomorphisms V_b and X_b with $\|V_b\| \|V_b^{-1}\| \leq 2$ and $\|X_b\| \|X_b^{-1}\| \leq 2$. Let I_b be the identity operator on ℓ_2^b , T_b be the quotient operator from $\mathcal{P}$ onto $\mathcal{P}/Y_b$ and J_b is the natural embedding operator from M_b into $\mathcal{Q}$. Suppose m_z is the Bernstein numbers [2] then

$$\begin{aligned} 1 = m_z(I_b) &= m_z(X_b X_b^{-1} I_b V_b V_b^{-1}) \leq \|X_b\| m_z(X_b^{-1} I_b V_b) \|V_b^{-1}\| = \|X_b\| m_z(J_b X_b^{-1} I_b V_b) \|V_b^{-1}\| \\ &\leq \|X_b\| d_z(J_b X_b^{-1} I_b V_b) \|V_b^{-1}\| = \|X_b\| d_z(J_b X_b^{-1} I_b V_b T_b) \|V_b^{-1}\| \\ &\leq \|X_b\| \alpha_z(J_b X_b^{-1} I_b V_b T_b) \|V_b^{-1}\|, \end{aligned}$$

for $0 \leq l \leq b$. We have

$$\begin{aligned} \sum_{z=0}^l r_z q_z &\leq \bar{\rho} \left(\sum_{z=0}^l \|X_b\| r_z q_z \overline{\alpha_z(J_b X_b^{-1} I_b V_b T_b)} \|V_b^{-1}\|, \bar{0} \right) \Rightarrow \\ \left(\frac{\sum_{z=0}^l r_z q_z}{r_{l+2} + r_l - 1} \right)^{t_l} &\leq (\|X_b\| \|V_b^{-1}\|)^{t_l} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \overline{\alpha_z(J_b X_b^{-1} I_b V_b T_b)}, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l}. \end{aligned}$$

Hence, for some $\varrho \geq 1$, one gets

$$\begin{aligned} \sum_{l=0}^b \left(\frac{\sum_{z=0}^l r_z q_z}{r_{l+2} + r_l - 1} \right)^{t_l} &\leq \varrho \|X_b\| \|V_b^{-1}\| \sum_{l=0}^b \left(\frac{\overline{\rho \left(\sum_{z=0}^l r_z q_z \alpha_z (J_b X_b^{-1} I_b V_b T_b), \bar{0} \right)}}{r_{l+2} + r_l - 1} \right)^{t_l} \Rightarrow \\ \sum_{l=0}^b \left(\frac{\sum_{z=0}^l r_z q_z}{r_{l+2} + r_l - 1} \right)^{t_l} &\leq \varrho \|X_b\| \|V_b^{-1}\| \Psi(J_b X_b^{-1} I_b V_b T_b) \Rightarrow \\ \sum_{l=0}^b \left(\frac{\sum_{z=0}^l r_z q_z}{r_{l+2} + r_l - 1} \right)^{t_l} &\leq \varrho \eta \|X_b\| \|V_b^{-1}\| \|J_b X_b^{-1} I_b V_b T_b\| \Rightarrow \\ \sum_{l=0}^b \left(\frac{\sum_{z=0}^l r_z q_z}{r_{l+2} + r_l - 1} \right)^{t_l} &\leq \varrho \eta \|X_b\| \|V_b^{-1}\| \|J_b X_b^{-1}\| \|I_b\| \|V_b T_b\| = \varrho \eta \|X_b\| \|V_b^{-1}\| \|X_b^{-1}\| \|I_b\| \|V_b\| \leq 4\varrho \eta. \end{aligned}$$

Therefore, we have a contradiction, if $b \rightarrow \infty$. Then $\mathcal{P}$ and $\mathcal{Q}$ both cannot be infinite dimensional if $\overline{\mathbb{B}}_{\tau_{\mathcal{F}}^F(q,t)}^{\alpha}(\mathcal{P}, \mathcal{Q}) = \mathbb{B}(\mathcal{P}, \mathcal{Q})$. This shows the proof. $\square$

Theorem 6.6. Suppose $\mathcal{P}$ and $\mathcal{Q}$ are Banach spaces with $\dim(\mathcal{P}) = \dim(\mathcal{Q}) = \infty$, and the setups of theorem 4.3 are confirmed with $\left(\frac{\sum_{z=0}^l r_z q_z}{r_{l+2} + r_l - 1} \right)_{l \in \mathcal{N}} \notin \ell_{((t_l))}$, hence $\overline{\mathbb{B}}_{\tau_{\mathcal{F}}^F(q,t)}^d$ is minimum.

6.4. Simple Banach pre-quasi ideal.

Lemma 6.7. [3] If $M \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$ and $M \notin \mathcal{A}(\mathcal{P}, \mathcal{Q})$, then there are operators $Q \in \mathbb{B}(\mathcal{P})$ and $L \in \mathbb{B}(\mathcal{Q})$ so that $LMQe_x = e_x$, for all $x \in \mathcal{N}$.

Theorem 6.8. [3] Suppose $\mathcal{E}^F$ is a Banach space with $\dim(\mathcal{E}^F) = \infty$, then

$$\mathbb{F}(\mathcal{E}^F) \subsetneq \mathcal{A}(\mathcal{E}^F) \subsetneq \mathcal{K}(\mathcal{E}^F) \subsetneq \mathbb{B}(\mathcal{E}^F).$$

Theorem 6.9. Suppose $\mathcal{P}$ and $\mathcal{Q}$ are Banach spaces with $\dim(\mathcal{P}) = \dim(\mathcal{Q}) = \infty$, and the setups of theorem 4.3 are confirmed with $1 < t_l^{(1)} < t_l^{(2)}$, and $0 < q_l^{(2)} \leq q_l^{(1)}$, for all $l \in \mathcal{N}$, hence

$$\begin{aligned} &\mathbb{B}\left(\overline{\mathbb{B}}^s_{\left(\tau_{\mathcal{F}}^F((q_l^{(2)}), (t_l^{(2)}))\right)}_v\right)(\mathcal{P}, \mathcal{Q}), \overline{\mathbb{B}}^s_{\left(\tau_{\mathcal{F}}^F((q_l^{(1)}), (t_l^{(1)}))\right)}_v(\mathcal{P}, \mathcal{Q}) \\ &= \mathcal{A}\left(\overline{\mathbb{B}}^s_{\left(\tau_{\mathcal{F}}^F((q_l^{(2)}), (t_l^{(2)}))\right)}_v\right)(\mathcal{P}, \mathcal{Q}), \overline{\mathbb{B}}^s_{\left(\tau_{\mathcal{F}}^F((q_l^{(1)}), (t_l^{(1)}))\right)}_v(\mathcal{P}, \mathcal{Q}) \right). \end{aligned}$$

Proof. Let $X \in \mathbb{B}\left(\overline{\mathbb{B}}^s_{\left(\tau_{\mathcal{F}}^F((q_l^{(2)}), (t_l^{(2)}))\right)}_v\right)(\mathcal{P}, \mathcal{Q}), \overline{\mathbb{B}}^s_{\left(\tau_{\mathcal{F}}^F((q_l^{(1)}), (t_l^{(1)}))\right)}_v(\mathcal{P}, \mathcal{Q})\right)$ and

$X \notin \mathcal{A}\left(\overline{\mathbb{B}}^s_{\left(\tau_{\mathcal{F}}^F((q_l^{(2)}), (t_l^{(2)}))\right)}_v\right)(\mathcal{P}, \mathcal{Q}), \overline{\mathbb{B}}^s_{\left(\tau_{\mathcal{F}}^F((q_l^{(1)}), (t_l^{(1)}))\right)}_v(\mathcal{P}, \mathcal{Q})\right)$. In view of Lemma 6.7, there are $Y \in \mathbb{B}\left(\overline{\mathbb{B}}^s_{\left(\tau_{\mathcal{F}}^F((q_l^{(2)}), (t_l^{(2)}))\right)}_v\right)(\mathcal{P}, \mathcal{Q})\right)$ and $Z \in \mathbb{B}\left(\overline{\mathbb{B}}^s_{\left(\tau_{\mathcal{F}}^F((q_l^{(1)}), (t_l^{(1)}))\right)}_v\right)(\mathcal{P}, \mathcal{Q})\right)$ with $ZXYI_b = I_b$. Therefore, for

every $b \in \mathcal{N}$, we get

$$\begin{aligned} \|I_b\|_{\overline{\mathbb{B}^s}_{(\tau_r^F((q_l^{(1)}), (t_l^{(1)}))_v)}(\mathcal{P}, \mathcal{Q})} &= \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z^{(1)} \overline{s_z(I_b)}, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l^{(1)}} \\ &\leq \|ZXY\| \|I_b\|_{\overline{\mathbb{B}^s}_{(\tau_r^F((q_l^{(2)}), (t_l^{(2)}))_v)}(\mathcal{P}, \mathcal{Q})} \leq \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z^{(2)} \overline{s_z(I_b)}, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l^{(2)}}. \end{aligned}$$

This contradicts Theorem 6.4. Then $X \in \mathcal{A}(\overline{\mathbb{B}^s}_{(\tau_r^F((q_l^{(2)}), (t_l^{(2)}))_v)}(\mathcal{P}, \mathcal{Q}), \overline{\mathbb{B}^s}_{(\tau_r^F((q_l^{(1)}), (t_l^{(1)}))_v)}(\mathcal{P}, \mathcal{Q}))$, which finishes the proof. $\square$

Corollary 6.10. Assume $\mathcal{P}$ and $\mathcal{Q}$ are Banach spaces with $\dim(\mathcal{P}) = \dim(\mathcal{Q}) = \infty$, and the setups of theorem 4.3 are established with $1 < t_l^{(1)} < t_l^{(2)}$, and $0 < q_l^{(2)} \leq q_l^{(1)}$, for all $l \in \mathcal{N}$, hence

$$\begin{aligned} &\mathbb{B}(\overline{\mathbb{B}^s}_{(\tau_r^F((q_l^{(2)}), (t_l^{(2)}))_v)}(\mathcal{P}, \mathcal{Q}), \overline{\mathbb{B}^s}_{(\tau_r^F((q_l^{(1)}), (t_l^{(1)}))_v)}(\mathcal{P}, \mathcal{Q})) \\ &= \mathcal{K}(\overline{\mathbb{B}^s}_{(\tau_r^F((q_l^{(2)}), (t_l^{(2)}))_v)}(\mathcal{P}, \mathcal{Q}), \overline{\mathbb{B}^s}_{(\tau_r^F((q_l^{(1)}), (t_l^{(1)}))_v)}(\mathcal{P}, \mathcal{Q})). \end{aligned}$$

Proof. Evidently, as $\mathcal{A} \subset \mathcal{K}$. $\square$

Definition 6.1. [3] A Banach space $\mathcal{E}^F$ is named simple if the algebra $\mathbb{B}(\mathcal{E}^F)$ includes one and only one non-trivial closed ideal.

Theorem 6.11. Let $\mathcal{P}$ and $\mathcal{Q}$ be Banach spaces with $\dim(\mathcal{P}) = \dim(\mathcal{Q}) = \infty$, and the setups of theorem 4.3 are satisfied, hence $\overline{\mathbb{B}^s}_{(\tau_r^F(q, t))_v}$ is simple.

Proof. Assume the closed ideal $\mathcal{K}(\overline{\mathbb{B}^s}_{(\tau_r^F(q, t))_v}(\mathcal{P}, \mathcal{Q}))$ includes an operator $X \notin \mathcal{A}(\overline{\mathbb{B}^s}_{(\tau_r^F(q, t))_v}(\mathcal{P}, \mathcal{Q}))$. In view of Lemma 6.7, we have $Y, Z \in \mathbb{B}(\overline{\mathbb{B}^s}_{(\tau_r^F(q, t))_v}(\mathcal{P}, \mathcal{Q}))$ with $ZXYI_b = I_b$. This gives that $I_{\overline{\mathbb{B}^s}_{(\tau_r^F(q, t))_v}(\mathcal{P}, \mathcal{Q})} \in \mathcal{K}(\overline{\mathbb{B}^s}_{(\tau_r^F(q, t))_v}(\mathcal{P}, \mathcal{Q}))$. Then $\mathbb{B}(\overline{\mathbb{B}^s}_{(\tau_r^F(q, t))_v}(\mathcal{P}, \mathcal{Q})) = \mathcal{K}(\overline{\mathbb{B}^s}_{(\tau_r^F(q, t))_v}(\mathcal{P}, \mathcal{Q}))$. Hence, $\overline{\mathbb{B}^s}_{(\tau_r^F(q, t))_v}$ is simple Banach space. $\square$

6.5. Eigenvalues of s-type operators.

Notations 6.12.

$$\begin{aligned} (\overline{\mathbb{B}^s}_{\mathcal{E}^F})^\lambda &:= \left\{ (\overline{\mathbb{B}^s}_{\mathcal{E}^F})^\lambda(\mathcal{P}, \mathcal{Q}); \mathcal{P} \text{ and } \mathcal{Q} \text{ are Banach Spaces} \right\}, \text{ where} \\ (\overline{\mathbb{B}^s}_{\mathcal{E}^F})^\lambda(\mathcal{P}, \mathcal{Q}) &:= \left\{ X \in \mathbb{B}(\mathcal{P}, \mathcal{Q}) : ((\overline{\lambda_l(X)})_{l=0}^\infty \in \mathcal{E}^F \text{ and } \|X - \bar{\rho}(\overline{\lambda_l(X)}, \bar{0})I\| \text{ is not invertible, for all } l \in \mathcal{N}) \right\}. \end{aligned}$$

Theorem 6.13. Let $\mathcal{P}$ and $\mathcal{Q}$ be Banach spaces with $\dim(\mathcal{P}) = \dim(\mathcal{Q}) = \infty$, and the setups of theorem 4.3 are established with $\inf_l \left(\frac{\sum_{z=0}^l r_z q_z}{r_{l+2} + r_l - 1} \right)^{t_l} > 0$, hence

$$(\overline{\mathbb{B}^s}_{(\tau_r^F(q, t))_v})^\lambda(\mathcal{P}, \mathcal{Q}) = \overline{\mathbb{B}^s}_{(\tau_r^F(q, t))_v}(\mathcal{P}, \mathcal{Q}).$$

Proof. Suppose $X \in \left(\overline{\mathbb{B}^s}_{(\tau_r^F(q,t))_v}\right)^\lambda(\mathcal{P}, \mathcal{Q})$, hence $(\overline{s_l(X)})_{l=0}^\infty \in (\tau_r^F(q,t))_v$ and $\|X - \overline{\rho}(\overline{s_l(X)}, \overline{0})I\| = 0$, for all $l \in \mathcal{N}$. We have $X = \overline{\rho}(\overline{s_l(X)}, \overline{0})I$, for every $l \in \mathcal{N}$, so

$$\overline{\rho}(\overline{s_l(X)}, \overline{0}) = \overline{\rho}(\overline{s_l(\overline{\rho}(\overline{s_l(X)}, \overline{0})I)}, \overline{0}) = \overline{\rho}(\overline{s_l(X)}, \overline{0}),$$

for every $l \in \mathcal{N}$. Therefore, $(\overline{s_l(X)})_{l=0}^\infty \in (\tau_r^F(q,t))_v$, then $X \in \overline{\mathbb{B}^s}_{(\tau_r^F(q,t))_v}(\mathcal{P}, \mathcal{Q})$.

Secondly, suppose $X \in \overline{\mathbb{B}^s}_{(\tau_r^F(q,t))_v}(\mathcal{P}, \mathcal{Q})$. Then $(\overline{s_l(X)})_{l=0}^\infty \in (\tau_r^F(q,t))_v$. Hence, we have

$$\sum_{l=0}^\infty \left(\frac{\overline{\rho}(\sum_{z=0}^l r_z q_z \overline{s_z(X)}, \overline{0})}{r_{l+2} + r_l - 1} \right)^{t_l} \geq \inf_l \left(\frac{\sum_{z=0}^l r_z q_z}{r_{l+2} + r_l - 1} \right)^{t_l} \sum_{l=0}^\infty [\overline{\rho}(\overline{s_z(X)}, \overline{0})]^{t_l}.$$

Therefore, $\lim_{l \rightarrow \infty} \overline{s_l(X)} = \overline{0}$. Assume $\|X - \overline{\rho}(\overline{s_l(X)}, \overline{0})I\|^{-1}$ exists, for every $l \in \mathcal{N}$. Hence $\|X - \overline{\rho}(\overline{s_l(X)}, \overline{0})I\|^{-1}$ exists and bounded, for every $l \in \mathcal{N}$. Then, $\lim_{l \rightarrow \infty} \|X - \overline{\rho}(\overline{s_l(X)}, \overline{0})I\|^{-1} = \|X\|^{-1}$ exists and bounded. As $(\overline{\mathbb{B}^s}_{(\tau_r^F(q,t))_v}, \Psi)$ is a pre-quasi operator ideal, we get

$$I = XX^{-1} \in \overline{\mathbb{B}^s}_{(\tau_r^F(q,t))_v}(\mathcal{P}, \mathcal{Q}) \Rightarrow (\overline{s_l(I)})_{l=0}^\infty \in \tau_r^F(q,t) \Rightarrow \lim_{l \rightarrow \infty} \overline{s_l(I)} = \overline{0}.$$

So we have a contradiction, since $\lim_{l \rightarrow \infty} \overline{s_l(I)} = \overline{1}$. Hence $\|X - \overline{\rho}(\overline{s_l(X)}, \overline{0})I\| = 0$, for every $l \in \mathcal{N}$. Hence $\|X - \overline{\rho}(\overline{s_l(X)}, \overline{0})I\| = 0$, for every $l \in \mathcal{N}$. This gives $X \in \left(\overline{\mathbb{B}^s}_{(\tau_r^F(q,t))_v}\right)^\lambda(\mathcal{P}, \mathcal{Q})$. This shows the proof. $\square$

7. KANNAN CONTRACTION OPERATOR

Definition 7.1. A pre-quasi normed pss† v on $\mathcal{E}^F$ confirms the Fatou property, if for every sequence $\{\overline{t^a}\} \subseteq \mathcal{E}_v^F$ with $\lim_{a \rightarrow \infty} v(\overline{t^a} - \overline{t}) = \overline{0}$ and each $\overline{z} \in \mathcal{E}_v^F$ then $v(\overline{z} - \overline{t}) \leq \sup_j \inf_{a \geq j} v(\overline{z} - \overline{t^a})$.

Theorem 7.1. The function $v(\overline{f}) = \left[\sum_{l=0}^\infty \left(\frac{\overline{\rho}(\sum_{z=0}^l r_z q_z \overline{f_z}, \overline{0})}{r_{l+2} + r_l - 1} \right)^{t_l} \right]^{\frac{1}{h}}$ establishes the Fatou property, for all $\overline{f} \in \tau_r^F(q,t)$, assume the setups of theorem 4.3 are confirmed.

Proof. Suppose $\{\overline{g^b}\} \subseteq (\tau_r^F(q,t))_v$ with $\lim_{b \rightarrow \infty} v(\overline{g^b} - \overline{g}) = 0$. As the space $(\tau_r^F(q,t))_v$ is a pre-quasi closed space, then $\overline{g} \in (\tau_r^F(q,t))_v$. Hence, for all $\overline{f} \in (\tau_r^F(q,t))_v$, we have

$$\begin{aligned} v(\overline{f} - \overline{g}) &= \left[\sum_{l=0}^\infty \left(\frac{\overline{\rho}(\sum_{z=0}^l r_z q_z (\overline{f_z} - \overline{g_z}), \overline{0})}{r_{l+2} + r_l - 1} \right)^{t_l} \right]^{\frac{1}{h}} \\ &\leq \left[\sum_{l=0}^\infty \left(\frac{\overline{\rho}(\sum_{z=0}^l r_z q_z (\overline{f_z} - \overline{g_z^b}), \overline{0})}{r_{l+2} + r_l - 1} \right)^{t_l} \right]^{\frac{1}{h}} + \left[\sum_{l=0}^\infty \left(\frac{\overline{\rho}(\sum_{z=0}^l r_z q_z (\overline{g_z^b} - \overline{g_z}), \overline{0})}{r_{l+2} + r_l - 1} \right)^{t_l} \right]^{\frac{1}{h}} \\ &\leq \sup_j \inf_{b \geq j} v(\overline{f} - \overline{g^b}). \end{aligned}$$

$\square$

Theorem 7.2. The function $v(\bar{f}) = \sum_{l=0}^{\infty} \left(\frac{\bar{\rho}(\sum_{z=0}^l r_z q_z \bar{f}_z, \bar{0})}{r_{l+2} + r_l - 1} \right)^{t_l}$ does not establish the Fatou property, for all $\bar{f} \in \tau_r^F(q, t)$, suppose the setups of theorem 4.3 are satisfied.

Proof. Suppose $\{\bar{g}^b\} \subseteq (\tau_r^F(q, t))_v$ with $\lim_{b \rightarrow \infty} v(\bar{g}^b - \bar{g}) = 0$. As the space $(\tau_r^F(q, t))_v$ is a pre-quasi closed space, then $\bar{g} \in (\tau_r^F(q, t))_v$. Hence, for all $\bar{f} \in (\tau_r^F(q, t))_v$, we obtains

$$\begin{aligned} v(\bar{f} - \bar{g}) &= \sum_{l=0}^{\infty} \left(\frac{\bar{\rho}(\sum_{z=0}^l r_z q_z (\bar{f}_z - \bar{g}_z), \bar{0})}{r_{l+2} + r_l - 1} \right)^{t_l} \\ &\leq 2^{h-1} \left[\sum_{l=0}^{\infty} \left(\frac{\bar{\rho}(\sum_{z=0}^l r_z q_z (\bar{f}_z - \bar{g}_z^b), \bar{0})}{r_{l+2} + r_l - 1} \right)^{t_l} + \sum_{l=0}^{\infty} \left(\frac{\bar{\rho}(\sum_{z=0}^l r_z q_z (\bar{g}_z^b - \bar{g}_z), \bar{0})}{r_{l+2} + r_l - 1} \right)^{t_l} \right] \\ &\leq 2^{h-1} \sup_j \inf_{b \geq j} v(\bar{f} - \bar{g}^b). \end{aligned}$$

Therefore, v does not establish the Fatou property. $\square$

Now, we investigate the enough setups on $(\tau_r^F(q, t))_v$ under definite pre-quasi norm so that there is an unique fixed point of Kannan contraction operator.

Definition 7.2. [25] An operator $W : \mathcal{E}_v^F \rightarrow \mathcal{E}_v^F$ is named a Kannan v -contraction, if there is $\lambda \in [0, \frac{1}{2})$, such that $v(W\bar{z} - W\bar{t}) \leq \lambda(v(W\bar{z} - \bar{z}) + v(W\bar{t} - \bar{t}))$, for every $\bar{z}, \bar{t} \in \mathcal{E}_v^F$.

A vector $\bar{z} \in \mathcal{E}_v^F$ is named a fixed point of W , if $W(\bar{z}) = \bar{z}$.

Theorem 7.3. Suppose the setups of theorem 4.3 are established, and $W : (\tau_r^F(q, t))_v \rightarrow (\tau_r^F(q, t))_v$ is Kannan v -contraction operator, where $v(\bar{f}) = \left[\sum_{l=0}^{\infty} \left(\frac{\bar{\rho}(\sum_{z=0}^l r_z q_z \bar{f}_z, \bar{0})}{r_{l+2} + r_l - 1} \right)^{t_l} \right]^{\frac{1}{h}}$, for every $\bar{f} \in \tau_r^F(q, t)$, then W has a unique fixed point.

Proof. Suppose $\bar{f} \in \tau_r^F(q, t)$, then $W^m \bar{f} \in \tau_r^F(q, t)$. Since W is a Kannan v -contraction operator, we have

$$\begin{aligned} v(W^{m+1} \bar{f} - W^m \bar{f}) &\leq \lambda (v(W^{m+1} \bar{f} - W^m \bar{f}) + v(W^m \bar{f} - W^{m-1} \bar{f})) \Rightarrow \\ v(W^{m+1} \bar{f} - W^m \bar{f}) &\leq \frac{\lambda}{1-\lambda} v(W^m \bar{f} - W^{m-1} \bar{f}) \leq \left(\frac{\lambda}{1-\lambda} \right)^2 v(W^{m-1} \bar{f} - W^{m-2} \bar{f}) \leq \dots \\ &\leq \left(\frac{\lambda}{1-\lambda} \right)^m v(W \bar{f} - \bar{f}). \end{aligned}$$

Therefore, for every $m, n \in \mathcal{N}$ with $n > m$, we have

$$\begin{aligned} v(W^m \bar{f} - W^n \bar{f}) &\leq \lambda (v(W^m \bar{f} - W^{m-1} \bar{f}) + v(W^n \bar{f} - W^{n-1} \bar{f})) \\ &\leq \lambda \left(\left(\frac{\lambda}{1-\lambda} \right)^{m-1} + \left(\frac{\lambda}{1-\lambda} \right)^{n-1} \right) v(W \bar{f} - \bar{f}). \end{aligned}$$

Hence, $\{W^m \bar{f}\}$ is a Cauchy sequence in $(\tau_r^F(q, t))_v$. Since the space $(\tau_r^F(q, t))_v$ is pre-quasi Banach space. Then, there exists $\bar{g} \in (\tau_r^F(q, t))_v$ so that $\lim_{m \rightarrow \infty} W^m \bar{f} = \bar{g}$. To explain that $W\bar{g} = \bar{g}$. As v has the Fatou property, we have

$$v(W\bar{g} - \bar{g}) \leq \sup_i \inf_{m \geq i} v(W^{m+1} \bar{f} - W^m \bar{f}) \leq \sup_i \inf_{m \geq i} \left(\frac{\lambda}{1-\lambda} \right)^m v(W\bar{f} - \bar{f}) = 0,$$

hence $W\bar{g} = \bar{g}$. So, $\bar{g}$ is a fixed point of W . To investigate that the fixed point is unique. Assume we have two distinct fixed points $\bar{b}, \bar{g} \in (\tau_r^F(q, t))_v$ of W . Then, one obtains

$$v(\bar{b} - \bar{g}) \leq v(W\bar{b} - W\bar{g}) \leq \lambda (v(W\bar{b} - \bar{b}) + v(W\bar{g} - \bar{g})) = 0.$$

Hence, $\bar{b} = \bar{g}$. □

Corollary 7.4. Assume the setups of theorem 4.3 are confirmed, and $W : (\tau_r^F(q, t))_v \rightarrow (\tau_r^F(q, t))_v$ is

Kannan v -contraction operator, where $v(\bar{f}) = \left[\sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \bar{f}_z, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \right]^{\frac{1}{h}}$, for all $\bar{f} \in \tau_r^F(q, t)$, hence W has an unique fixed point $\bar{b}$ with $v(W^m \bar{f} - \bar{b}) \leq \lambda \left(\frac{\lambda}{1-\lambda} \right)^{m-1} v(W\bar{f} - \bar{f})$.

Proof. According to Theorem 7.3, there is an unique fixed point $\bar{b}$ of W . Therefore, one obtains

$$v(W^m \bar{f} - \bar{b}) = v(W^m \bar{f} - W\bar{b}) \leq \lambda (v(W^m \bar{f} - W^{m-1} \bar{f}) + v(W\bar{b} - \bar{b})) = \lambda \left(\frac{\lambda}{1-\lambda} \right)^{m-1} v(W\bar{f} - \bar{f}).$$

□

Definition 7.3. Assume $\mathcal{E}_v^F$ is a pre-quasi normed pssf, $W : \mathcal{E}_v^F \rightarrow \mathcal{E}_v^F$ and $\bar{b} \in \mathcal{E}_v^F$. The operator W is named v -sequentially continuous at $\bar{b}$, if and only if, if $\lim_{a \rightarrow \infty} v(\bar{t}_a - \bar{b}) = 0$, then $\lim_{a \rightarrow \infty} v(W\bar{t}_a - W\bar{b}) = 0$.

Theorem 7.5. Suppose the setups of theorem 4.3 are established, and $W : (\tau_r^F(q, t))_v \rightarrow (\tau_r^F(q, t))_v$, where

$v(\bar{f}) = \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \bar{f}_z, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l}$, for all $\bar{f} \in \tau_r^F(q, t)$. The vector $\bar{g} \in (\tau_r^F(q, t))_v$ is the only fixed point of W , if the next setup are verified:

- (a): W is Kannan v -contraction operator,
- (b): W is v -sequentially continuous at $\bar{g} \in (\tau_r^F(q, t))_v$,
- (c): there is $\bar{v} \in (\tau_r^F(q, t))_v$ so that the sequence of iterates $\{W^m \bar{v}\}$ has a subsequence $\{W^{m_i} \bar{v}\}$ converges to $\bar{g}$.

Proof. Suppose the enough setups are established. Let $\bar{g}$ be not a fixed point of W , then $W\bar{g} \neq \bar{g}$. In view of the setups (b) and (c), one obtains

$$\lim_{m_i \rightarrow \infty} v(W^{m_i} \bar{f} - \bar{g}) = 0 \text{ and } \lim_{m_i \rightarrow \infty} v(W^{m_i+1} \bar{f} - W\bar{g}) = 0.$$

Since the operator W is Kannan v -contraction, we have

$$\begin{aligned} 0 < v(W\bar{g} - \bar{g}) &= v\left((W\bar{g} - W^{m_i+1}\bar{f}) + (W^{m_i}\bar{f} - \bar{g}) + (W^{m_i+1}\bar{f} - W^{m_i}\bar{f})\right) \\ &\leq 2^{2h-2}v\left(W^{m_i+1}\bar{v} - W\bar{g}\right) + 2^{2h-2}v\left(W^{m_i}\bar{v} - \bar{g}\right) + 2^{h-1}\lambda\left(\frac{\lambda}{1-\lambda}\right)^{m_i-1}v(W\bar{f} - \bar{f}). \end{aligned}$$

let $m_i \rightarrow \infty$, we get a contradiction. Hence, $\bar{g}$ is a fixed point of W . To show that the fixed point $\bar{g}$ is one. Assume we have two distinct fixed points $\bar{g}, \bar{b} \in \left(\tau_r^F\left(q, t\right)\right)_v$ of W . Hence, one gets

$$v(\bar{g} - \bar{b}) \leq v(W\bar{g} - W\bar{b}) \leq \lambda\left(v(W\bar{g} - \bar{g}) + v(W\bar{b} - \bar{b})\right) = 0.$$

Therefore, $\bar{g} = \bar{b}$. □

Example 7.6. If $T : \left(\tau_r^F\left(\left(\frac{1}{(l+5)r_l}\right)_{l=0}^\infty, \left(\frac{2l+3}{l+2}\right)_{l=0}^\infty\right)\right)_v \rightarrow \left(\tau_r^F\left(\left(\frac{1}{(l+5)r_l}\right)_{l=0}^\infty, \left(\frac{2l+3}{l+2}\right)_{l=0}^\infty\right)\right)_v$, where

$$v(\bar{f}) = \sqrt{\sum_{l=0}^\infty \left(\frac{\bar{\rho}\left(\sum_{z=0}^l \frac{\bar{f}_z}{z+5}, \bar{0}\right)}{r_{l+2} + r_l - 1}\right)^{\frac{2l+3}{l+2}}}, \text{ with } \bar{f} \in \left(\tau_r^F\left(\left(\frac{1}{(l+5)r_l}\right)_{l=0}^\infty, \left(\frac{2l+3}{l+2}\right)_{l=0}^\infty\right)\right)_v \text{ and}$$

$$T(\bar{f}) = \begin{cases} \frac{\bar{f}}{4}, & v(\bar{f}) \in [0, 1), \\ \frac{\bar{f}}{5}, & v(\bar{f}) \in [1, \infty). \end{cases}$$

Since for all $\bar{f}, \bar{g} \in \left(\tau_r^F\left(\left(\frac{1}{(l+5)r_l}\right)_{l=0}^\infty, \left(\frac{2l+3}{l+2}\right)_{l=0}^\infty\right)\right)_v$ with $v(\bar{f}), v(\bar{g}) \in [0, 1)$, we have

$$v(T\bar{f} - T\bar{g}) = v\left(\frac{\bar{f}}{4} - \frac{\bar{g}}{4}\right) \leq \frac{1}{\sqrt[4]{27}}\left(v\left(\frac{3\bar{f}}{4}\right) + v\left(\frac{3\bar{g}}{4}\right)\right) = \frac{1}{\sqrt[4]{27}}\left(v(T\bar{f} - \bar{f}) + v(T\bar{g} - \bar{g})\right).$$

For all $\bar{f}, \bar{g} \in \left(\tau_r^F\left(\left(\frac{1}{(l+5)r_l}\right)_{l=0}^\infty, \left(\frac{2l+3}{l+2}\right)_{l=0}^\infty\right)\right)_v$ with $v(\bar{f}), v(\bar{g}) \in [1, \infty)$, one has

$$v(T\bar{f} - T\bar{g}) = v\left(\frac{\bar{f}}{5} - \frac{\bar{g}}{5}\right) \leq \frac{1}{\sqrt[4]{64}}\left(v\left(\frac{4\bar{f}}{5}\right) + v\left(\frac{4\bar{g}}{5}\right)\right) = \frac{1}{\sqrt[4]{64}}\left(v(T\bar{f} - \bar{f}) + v(T\bar{g} - \bar{g})\right).$$

For every $\bar{f}, \bar{g} \in \left(\tau_r^F\left(\left(\frac{1}{(l+5)r_l}\right)_{l=0}^\infty, \left(\frac{2l+3}{l+2}\right)_{l=0}^\infty\right)\right)_v$ with $v(\bar{f}) \in [0, 1)$ and $v(\bar{g}) \in [1, \infty)$, we obtain

$$\begin{aligned} v(T\bar{f} - T\bar{g}) &= v\left(\frac{\bar{f}}{4} - \frac{\bar{g}}{5}\right) \leq \frac{1}{\sqrt[4]{27}}v\left(\frac{3\bar{f}}{4}\right) + \frac{1}{\sqrt[4]{64}}v\left(\frac{4\bar{g}}{5}\right) \leq \frac{1}{\sqrt[4]{27}}\left(v\left(\frac{3\bar{f}}{4}\right) + v\left(\frac{4\bar{g}}{5}\right)\right) \\ &= \frac{1}{\sqrt[4]{27}}\left(v(T\bar{f} - \bar{f}) + v(T\bar{g} - \bar{g})\right). \end{aligned}$$

Therefore, the operator T is Kannan v -contraction. Since v confirms the Fatou property. In view of Theorem 7.3, the operator T has a unique fixed point $\bar{\theta} \in \left(\tau_r^F\left(\left(\frac{1}{(l+5)r_l}\right)_{l=0}^\infty, \left(\frac{2l+3}{l+2}\right)_{l=0}^\infty\right)\right)_v$.

Suppose $\{\bar{f}^{(a)}\} \subseteq \left(\tau_r^F\left(\left(\frac{1}{(l+5)r_l}\right)_{l=0}^\infty, \left(\frac{2l+3}{l+2}\right)_{l=0}^\infty\right)\right)_v$ with $\lim_{a \rightarrow \infty} v(\bar{f}^{(a)} - \bar{f}^{(0)}) = 0$, where

$\overline{f^{(0)}} \in \left(\tau_r^F \left(\left(\frac{1}{(l+5)r_l} \right)_{l=0}^\infty, \left(\frac{2l+3}{l+2} \right)_{l=0}^\infty \right) \right)_v$ with $v(\overline{f^{(0)}}) = 1$. Since the pre-quasi norm v is continuous, we have

$$\lim_{a \rightarrow \infty} v(T\overline{f^{(a)}} - T\overline{f^{(0)}}) = \lim_{a \rightarrow \infty} v\left(\frac{\overline{f^{(a)}}}{4} - \frac{\overline{f^{(0)}}}{5}\right) = v\left(\frac{\overline{f^{(0)}}}{20}\right) > 0.$$

Hence, T is not v -sequentially continuous at $\overline{f^{(0)}}$. Therefore, the operator T is not continuous at $\overline{f^{(0)}}$.

$$\text{Let } v(\bar{f}) = \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l \frac{\bar{f}_z}{z+5}, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{\frac{2l+3}{l+2}}, \text{ with } \bar{f} \in \left(\tau_r^F \left(\left(\frac{1}{(l+5)r_l} \right)_{l=0}^\infty, \left(\frac{2l+3}{l+2} \right)_{l=0}^\infty \right) \right)_v.$$

Since for all $\bar{f}, \bar{g} \in \left(\tau_r^F \left(\left(\frac{1}{(l+5)r_l} \right)_{l=0}^\infty, \left(\frac{2l+3}{l+2} \right)_{l=0}^\infty \right) \right)_v$ with $v(\bar{f}), v(\bar{g}) \in [0, 1)$, we have

$$v(T\bar{f} - T\bar{g}) = v\left(\frac{\bar{f}}{4} - \frac{\bar{g}}{4}\right) \leq \frac{2}{\sqrt{27}} \left(v\left(\frac{3\bar{f}}{4}\right) + v\left(\frac{3\bar{g}}{4}\right) \right) = \frac{2}{\sqrt{27}} \left(v(T\bar{f} - \bar{f}) + v(T\bar{g} - \bar{g}) \right).$$

Let $\bar{f}, \bar{g} \in \left(\tau_r^F \left(\left(\frac{1}{(l+5)r_l} \right)_{l=0}^\infty, \left(\frac{2l+3}{l+2} \right)_{l=0}^\infty \right) \right)_v$ with $v(\bar{f}), v(\bar{g}) \in [1, \infty)$, we have

$$v(T\bar{f} - T\bar{g}) = v\left(\frac{\bar{f}}{5} - \frac{\bar{g}}{5}\right) \leq \frac{1}{4} \left(v\left(\frac{4\bar{f}}{5}\right) + v\left(\frac{4\bar{g}}{5}\right) \right) = \frac{1}{4} \left(v(T\bar{f} - \bar{f}) + v(T\bar{g} - \bar{g}) \right).$$

For every $\bar{f}, \bar{g} \in \left(\tau_r^F \left(\left(\frac{1}{(l+5)r_l} \right)_{l=0}^\infty, \left(\frac{2l+3}{l+2} \right)_{l=0}^\infty \right) \right)_v$ with $v(\bar{f}) \in [0, 1)$ and $v(\bar{g}) \in [1, \infty)$, we obtain

$$\begin{aligned} v(T\bar{f} - T\bar{g}) &= v\left(\frac{\bar{f}}{4} - \frac{\bar{g}}{5}\right) \leq \frac{2}{\sqrt{27}} v\left(\frac{3\bar{f}}{4}\right) + \frac{1}{4} v\left(\frac{4\bar{g}}{5}\right) \leq \frac{2}{\sqrt{27}} \left(v\left(\frac{3\bar{f}}{4}\right) + v\left(\frac{4\bar{g}}{5}\right) \right) \\ &= \frac{2}{\sqrt{27}} \left(v(T\bar{f} - \bar{f}) + v(T\bar{g} - \bar{g}) \right). \end{aligned}$$

Therefore, the operator T is Kannan v -contraction and $T^m(\bar{f}) = \begin{cases} \frac{\bar{f}}{4^m}, & v(\bar{f}) \in [0, 1), \\ \frac{\bar{f}}{5^m}, & v(\bar{f}) \in [1, \infty). \end{cases}$

Evidently, T is v -sequentially continuous at $\bar{\theta} \in \left(\tau_r^F \left(\left(\frac{1}{(l+5)r_l} \right)_{l=0}^\infty, \left(\frac{2l+3}{l+2} \right)_{l=0}^\infty \right) \right)_v$ and $\{T^m \bar{f}\}$ has a subsequence

$\{T^{m_j} \bar{f}\}$ converges to $\bar{\theta}$. According Theorem 7.5, the element

$\bar{\theta} \in \left(\tau_r^F \left(\left(\frac{1}{(l+5)r_l} \right)_{l=0}^\infty, \left(\frac{2l+3}{l+2} \right)_{l=0}^\infty \right) \right)_v$ is the only fixed point of T .

Example 7.7. Let

$$T : \left(\tau_r^F \left(\left(\frac{1}{(l+5)r_l} \right)_{l=0}^\infty, \left(\frac{2l+3}{l+2} \right)_{l=0}^\infty \right) \right)_v \rightarrow \left(\tau_r^F \left(\left(\frac{1}{(l+5)r_l} \right)_{l=0}^\infty, \left(\frac{2l+3}{l+2} \right)_{l=0}^\infty \right) \right)_v, \text{ with}$$

$$v(\bar{f}) = \sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l \frac{\bar{f}_z}{z+5}, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{\frac{2l+3}{l+2}}, \text{ with } \bar{f} \in \left(\tau_r^F \left(\left(\frac{1}{(l+5)r_l} \right)_{l=0}^\infty, \left(\frac{2l+3}{l+2} \right)_{l=0}^\infty \right) \right)_v \text{ and}$$

$$T(\bar{f}) = \begin{cases} \frac{1}{4}(\bar{e}_1 + \bar{f}), & \bar{f}_0(t) \in [0, \frac{1}{3}), \\ \frac{1}{3}\bar{e}_1, & \bar{f}_0(t) = \frac{1}{3}, \\ \frac{1}{4}\bar{e}_1, & \bar{f}_0(t) \in (\frac{1}{3}, 1]. \end{cases}$$

Since for all $\bar{f}, \bar{g} \in \left(\tau_r^F \left(\left(\frac{1}{(l+5)r_l} \right)_{l=0}^\infty, \left(\frac{2l+3}{l+2} \right)_{l=0}^\infty \right) \right)_v$ with $\bar{f}_0, \bar{g}_0 \in [0, \frac{1}{3})$, we get

$$\begin{aligned} v(T\bar{f} - T\bar{g}) &= v\left(\frac{1}{4}(\bar{f}_0 - \bar{g}_0, \bar{f}_1 - \bar{g}_1, \bar{f}_2 - \bar{g}_2, \dots)\right) \leq \frac{2}{\sqrt{27}} \left(v\left(\frac{3\bar{f}}{4}\right) + v\left(\frac{3\bar{g}}{4}\right) \right) \\ &\leq \frac{2}{\sqrt{27}} \left(v(T\bar{f} - \bar{f}) + v(T\bar{g} - \bar{g}) \right). \end{aligned}$$

For all $\bar{f}, \bar{g} \in \left(\tau_r^F \left(\left(\frac{1}{(l+5)r_l} \right)_{l=0}^\infty, \left(\frac{2l+3}{l+2} \right)_{l=0}^\infty \right) \right)_v$ with $\bar{f}_0, \bar{g}_0 \in (\frac{1}{3}, 1]$, hence for all $\varepsilon > 0$, we have

$$v(T\bar{f} - T\bar{g}) = 0 \leq \varepsilon \left(v(T\bar{f} - \bar{f}) + v(T\bar{g} - \bar{g}) \right).$$

For all $\bar{f}, \bar{g} \in \left(\tau_r^F \left(\left(\frac{1}{(l+5)r_l} \right)_{l=0}^\infty, \left(\frac{2l+3}{l+2} \right)_{l=0}^\infty \right) \right)_v$ with $\bar{f}_0 \in [0, \frac{1}{3})$ and $\bar{g}_0 \in (\frac{1}{3}, 1]$, one has

$$v(T\bar{f} - T\bar{g}) = v\left(\frac{\bar{f}}{4}\right) \leq \frac{1}{\sqrt{27}} v\left(\frac{3\bar{f}}{4}\right) = \frac{1}{\sqrt{27}} v(T\bar{f} - \bar{f}) \leq \frac{1}{\sqrt{27}} \left(v(T\bar{f} - \bar{f}) + v(T\bar{g} - \bar{g}) \right).$$

Therefore, the operator T is Kannan v -contraction. Obviously, T is v -sequentially continuous at $\frac{1}{3}\bar{e}_1 \in \left(\tau_r^F \left(\left(\frac{1}{(l+5)r_l} \right)_{l=0}^\infty, \left(\frac{2l+3}{l+2} \right)_{l=0}^\infty \right) \right)_v$ and there is $\bar{f} \in \left(\tau_r^F \left(\left(\frac{1}{(l+5)r_l} \right)_{l=0}^\infty, \left(\frac{2l+3}{l+2} \right)_{l=0}^\infty \right) \right)_v$ with $\bar{f}_0 \in [0, \frac{1}{3})$ such that the sequence of iterates $\{T^m \bar{f}\} = \left\{ \sum_{a=1}^m \frac{1}{4^a} \bar{e}_1 + \frac{1}{4^m} \bar{f} \right\}$ includes a subsequence $\{T^{m_j} \bar{f}\} = \left\{ \sum_{a=1}^{m_j} \frac{1}{4^a} \bar{e}_1 + \frac{1}{4^{m_j}} \bar{f} \right\}$ converges to $\frac{1}{3}\bar{e}_1$. In view of Theorem 7.5, the operator T has one fixed point $\frac{1}{3}\bar{e}_1 \in \left(\tau_r^F \left(\left(\frac{1}{(l+5)r_l} \right)_{l=0}^\infty, \left(\frac{2l+3}{l+2} \right)_{l=0}^\infty \right) \right)_v$. Note that T is not continuous at $\frac{1}{3}\bar{e}_1 \in \left(\tau_r^F \left(\left(\frac{1}{(l+5)r_l} \right)_{l=0}^\infty, \left(\frac{2l+3}{l+2} \right)_{l=0}^\infty \right) \right)_v$.

Let $v(\bar{f}) = \sqrt{\sum_{l=0}^\infty \left(\frac{\bar{\rho} \left(\sum_{z=0}^l \frac{\bar{f}_z}{z+5}, \bar{0} \right) \right)^{\frac{2l+3}{l+2}}}{r_{l+2} + r_l - 1}}$, for all $\bar{f} \in \left(\tau_r^F \left(\left(\frac{1}{(l+5)r_l} \right)_{l=0}^\infty, \left(\frac{2l+3}{l+2} \right)_{l=0}^\infty \right) \right)_v$. Since for all $\bar{f}, \bar{g} \in \left(\tau_r^F \left(\left(\frac{1}{(l+5)r_l} \right)_{l=0}^\infty, \left(\frac{2l+3}{l+2} \right)_{l=0}^\infty \right) \right)_v$ with $\bar{f}_0, \bar{g}_0 \in [0, \frac{1}{3})$, we have

$$\begin{aligned} v(T\bar{f} - T\bar{g}) &= v\left(\frac{1}{4}(\bar{f}_0 - \bar{g}_0, \bar{f}_1 - \bar{g}_1, \bar{f}_2 - \bar{g}_2, \dots)\right) \leq \frac{1}{\sqrt[4]{27}} \left(v\left(\frac{3\bar{f}}{4}\right) + v\left(\frac{3\bar{g}}{4}\right) \right) \\ &\leq \frac{1}{\sqrt[4]{27}} \left(v(T\bar{f} - \bar{f}) + v(T\bar{g} - \bar{g}) \right). \end{aligned}$$

For all $\bar{f}, \bar{g} \in \left(\tau_r^F \left(\left(\frac{1}{(l+5)r_l} \right)_{l=0}^\infty, \left(\frac{2l+3}{l+2} \right)_{l=0}^\infty \right) \right)_v$ with $\bar{f}_0, \bar{g}_0 \in (\frac{1}{3}, 1]$, hence for all $\varepsilon > 0$, one has

$$v(T\bar{f} - T\bar{g}) = 0 \leq \varepsilon \left(v(T\bar{f} - \bar{f}) + v(T\bar{g} - \bar{g}) \right).$$

For all $\bar{f}, \bar{g} \in \left(\tau_r^F \left(\left(\frac{1}{(l+5)r_l} \right)_{l=0}^\infty, \left(\frac{2l+3}{l+2} \right)_{l=0}^\infty \right) \right)_v$ with $\bar{f}_0 \in [0, \frac{1}{3})$ and $\bar{g}_0 \in (\frac{1}{3}, 1]$, we have

$$v(T\bar{f} - T\bar{g}) = v\left(\frac{\bar{f}}{4}\right) \leq \frac{1}{\sqrt[4]{27}} v\left(\frac{3\bar{f}}{4}\right) = \frac{1}{\sqrt[4]{27}} v(T\bar{f} - \bar{f}) \leq \frac{1}{\sqrt[4]{27}} \left(v(T\bar{f} - \bar{f}) + v(T\bar{g} - \bar{g}) \right).$$

Therefore, the operator T is Kannan ν -contraction. Since ν confirms the Fatou property. According to Theorem 7.3, the operator T has an unique fixed point $\frac{1}{3}e_1 \in \left(\tau_r^F \left(\left(\frac{1}{(l+5)r_l} \right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2} \right)_{l=0}^{\infty} \right) \right)_v$.

We offer the existence of a fixed point of Kannan contraction operator in the pre-quasi Banach operator ideal generated by $(\tau_r^F(q, t))_v$ and s -numbers.

Definition 7.4. A pre-quasi norm Ψ on the ideal $\overline{\mathbb{B}}^s_{\mathcal{E}^F}$, where $\Psi(W) = \nu \left((\overline{s_a(W)})_{a=0}^{\infty} \right)$, confirms the Fatou property if for every sequence $\{W_a\}_{a \in \mathcal{N}} \subseteq \overline{\mathbb{B}}^s_{\mathcal{E}^F}(Z, M)$ with $\lim_{a \rightarrow \infty} \Psi(W_a - W) = 0$ and each $V \in \overline{\mathbb{B}}^s_{\mathcal{E}^F}(Z, M)$, then $\Psi(V - W) \leq \sup_a \inf_{i \geq a} \Psi(V - W_i)$.

Theorem 7.8. The pre-quasi norm $\Psi(W) = \left[\sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z s_z(W), \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \right]^{\frac{1}{h}}$ does not establish the Fatou property, for every $W \in \overline{\mathbb{B}}^s_{(\tau_r^F(q, t))_v}(\mathcal{P}, \mathcal{Q})$, when the setups of theorem 4.3 are satisfied.

Proof. Let the conditions be confirmed and $\{W_m\}_{m \in \mathcal{N}} \subseteq \overline{\mathbb{B}}^s_{(\tau_r^F(q, t))_v}(\mathcal{P}, \mathcal{Q})$ with $\lim_{m \rightarrow \infty} \Psi(W_m - W) = 0$. As the space $\overline{\mathbb{B}}^s_{(\tau_r^F(q, t))_v}$ is a pre-quasi closed ideal. Hence, $W \in \overline{\mathbb{B}}^s_{(\tau_r^F(q, t))_v}(\mathcal{P}, \mathcal{Q})$. Then, for all $V \in \overline{\mathbb{B}}^s_{(\tau_r^F(q, t))_v}(\mathcal{P}, \mathcal{Q})$, one has

$$\begin{aligned} \Psi(V - W) &= \left[\sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z s_z(V - W), \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \right]^{\frac{1}{h}} \\ &\leq \left[\sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z s_z(V - W_i), \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \right]^{\frac{1}{h}} + \left[\sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z s_z(W - W_i), \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \right]^{\frac{1}{h}} \\ &\leq (2^{2h-1} + 2^{h-1} + 2^h)^{\frac{1}{h}} \sup_m \inf_{i \geq m} \left[\sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z s_z(V - W_i), \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \right]^{\frac{1}{h}}. \end{aligned}$$

Hence, Ψ does not verify the Fatou property. $\square$

Definition 7.5. [25] An operator $W : \overline{\mathbb{B}}^s_{\mathcal{E}^F}(Z, M) \rightarrow \overline{\mathbb{B}}^s_{\mathcal{E}^F}(Z, M)$ is called a Kannan Ψ -contraction, if there is $\lambda \in [0, \frac{1}{2})$, so that $\Psi(WV - WT) \leq \lambda(\Psi(WV - V) + \Psi(WT - T))$, for each $V, T \in \overline{\mathbb{B}}^s_{\mathcal{E}^F}(Z, M)$.

Definition 7.6. For the pre-quasi norm Ψ on the ideal $\overline{\mathbb{B}}^s_{\mathcal{E}^F}$, where $\Psi(W) = \nu \left((\overline{s_a(W)})_{a=0}^{\infty} \right)$, $G : \overline{\mathbb{B}}^s_{\mathcal{E}^F}(Z, M) \rightarrow \overline{\mathbb{B}}^s_{\mathcal{E}^F}(Z, M)$ and $B \in \overline{\mathbb{B}}^s_{\mathcal{E}^F}(Z, M)$. The operator G is named Ψ -sequentially continuous at B , if and only if, if $\lim_{m \rightarrow \infty} \Psi(W_m - B) = 0$, then $\lim_{m \rightarrow \infty} \Psi(GW_m - GB) = 0$.

Theorem 7.9. Assume the setups of theorem 4.3 are established and $G : \overline{\mathbb{B}}^s_{(\tau_r^F(q, t))_v}(\mathcal{P}, \mathcal{Q}) \rightarrow$

$\overline{\mathbb{B}}^s_{(\tau_r^F(q, t))_v}(\mathcal{P}, \mathcal{Q})$, where $\Psi(W) = \left[\sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z s_z(W), \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \right]^{\frac{1}{h}}$, for every $W \in \overline{\mathbb{B}}^s_{(\tau_r^F(q, t))_v}(\mathcal{P}, \mathcal{Q})$.

The vector $A \in \overline{\mathbb{B}}^s_{(\tau_r^F(q, t))_v}(\mathcal{P}, \mathcal{Q})$ is the unique fixed point of G , if the next setups are fulfilled:

- (a): G is Kannan Ψ -contraction mapping,
- (b): G is Ψ -sequentially continuous at a point $A \in \overline{\mathbb{B}^s}_{(\tau_r^F(q,t))_v}(\mathcal{P}, \mathcal{Q})$,
- (c): there is $B \in \overline{\mathbb{B}^s}_{(\tau_r^F(q,t))_v}(\mathcal{P}, \mathcal{Q})$ such that the sequence of iterates $\{G^{m_i}B\}$ has a subsequence $\{G^{m_i}B\}$ converges to A .

Proof. Let the enough setups be satisfied. Assume A is not a fixed point of G , then $GA \neq A$. In view of the conditions (b) and (c), one has

$$\lim_{m_i \rightarrow \infty} \Psi(G^{m_i}B - A) = 0 \text{ and } \lim_{m_i \rightarrow \infty} \Psi(G^{m_i+1}B - GA) = 0.$$

As G is Kannan Ψ -contraction operator, we get

$$\begin{aligned} 0 < \Psi(GA - A) &= \Psi((GA - G^{m_i+1}B) + (G^{m_i}B - A) + (G^{m_i+1}B - G^{m_i}B)) \\ &\leq (2^{2h-1} + 2^{h-1} + 2^h)^{\frac{1}{h}} \Psi(G^{m_i+1}B - GA) + (2^{2h-1} + 2^{h-1} + 2^h)^{\frac{2}{h}} \Psi(G^{m_i}B - A) \\ &\quad + (2^{2h-1} + 2^{h-1} + 2^h)^{\frac{2}{h}} \lambda \left(\frac{\lambda}{1-\lambda} \right)^{m_i-1} \Psi(GB - B). \end{aligned}$$

For $m_i \rightarrow \infty$, one obtains a contradiction. Hence, A is a fixed point of G . To prove that the fixed point A is unique. Assume we have two distinct fixed points $A, D \in \overline{\mathbb{B}^s}_{(\tau_r^F(q,t))_v}(\mathcal{P}, \mathcal{Q})$ of G . Therefore, one gets

$$\Psi(A - D) \leq \Psi(GA - GD) \leq \lambda(\Psi(GA - A) + \Psi(GD - D)) = 0.$$

So, $A = D$. □

Example 7.10. Suppose

$$\begin{aligned} M : S_{\left(\tau_r^F\left(\left(\frac{1}{(l+4)r_l}\right)_{l=0}^\infty, \left(\frac{2l+3}{l+2}\right)_{l=0}^\infty\right)\right)_v}(\mathcal{P}, \mathcal{Q}) &\rightarrow S_{\left(\tau_r^F\left(\left(\frac{1}{(l+4)r_l}\right)_{l=0}^\infty, \left(\frac{2l+3}{l+2}\right)_{l=0}^\infty\right)\right)_v}(\mathcal{P}, \mathcal{Q}), \text{ where} \\ \Psi(H) &= \sqrt{\sum_{l=0}^\infty \left(\frac{\bar{\rho} \left(\sum_{z=0}^l \frac{s_z(H)}{z+4}, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{\frac{2l+3}{l+2}}}, \text{ for each } H \in S_{\left(\tau_r^F\left(\left(\frac{1}{(l+4)r_l}\right)_{l=0}^\infty, \left(\frac{2l+3}{l+2}\right)_{l=0}^\infty\right)\right)_v}(\mathcal{P}, \mathcal{Q}) \text{ and} \end{aligned}$$

$$M(H) = \begin{cases} \frac{H}{6}, & \Psi(H) \in [0, 1), \\ \frac{H}{7}, & \Psi(H) \in [1, \infty). \end{cases}$$

Since for all $H_1, H_2 \in S_{\left(\tau_r^F\left(\left(\frac{1}{(l+4)r_l}\right)_{l=0}^\infty, \left(\frac{2l+3}{l+2}\right)_{l=0}^\infty\right)\right)_v}$ with $\Psi(H_1), \Psi(H_2) \in [0, 1)$, we have

$$\begin{aligned} \Psi(MH_1 - MH_2) &= \Psi\left(\frac{H_1}{6} - \frac{H_2}{6}\right) \leq \frac{\sqrt{2}}{\sqrt[4]{125}} \left(\Psi\left(\frac{5H_1}{6}\right) + \Psi\left(\frac{5H_2}{6}\right) \right) \\ &= \frac{\sqrt{2}}{\sqrt[4]{125}} \left(\Psi(MH_1 - H_1) + \Psi(MH_2 - H_2) \right). \end{aligned}$$

For all $H_1, H_2 \in S_{\left(\tau_r^F\left(\left(\frac{1}{(l+4)r_l}\right)_{l=0}^\infty, \left(\frac{2l+3}{l+2}\right)_{l=0}^\infty\right)\right)_v}$ with $\Psi(H_1), \Psi(H_2) \in [1, \infty)$, one has

$$\begin{aligned}\Psi(MH_1 - MH_2) &= \Psi\left(\frac{H_1}{7} - \frac{H_2}{7}\right) \leq \frac{\sqrt{2}}{\sqrt[4]{216}}\left(\Psi\left(\frac{6H_1}{7}\right) + \Psi\left(\frac{6H_2}{7}\right)\right) \\ &= \frac{\sqrt{2}}{\sqrt[4]{216}}\left(\Psi(MH_1 - H_1) + \Psi(MH_2 - H_2)\right).\end{aligned}$$

For all $H_1, H_2 \in S_{\left(\tau_r^F\left(\left(\frac{1}{(l+4)r_l}\right)_{l=0}^\infty, \left(\frac{2l+3}{l+2}\right)_{l=0}^\infty\right)\right)_v}$ with $\Psi(H_1) \in [0, 1)$ and $\Psi(H_2) \in [1, \infty)$, one gets

$$\begin{aligned}\Psi(MH_1 - MH_2) &= \Psi\left(\frac{H_1}{6} - \frac{H_2}{7}\right) \leq \frac{\sqrt{2}}{\sqrt[4]{125}}\Psi\left(\frac{5H_1}{6}\right) + \frac{\sqrt{2}}{\sqrt[4]{216}}\Psi\left(\frac{6H_2}{7}\right) \\ &\leq \frac{\sqrt{2}}{\sqrt[4]{125}}\left(\Psi(MH_1 - H_1) + \Psi(MH_2 - H_2)\right).\end{aligned}$$

Therefore, the operator M is Kannan Ψ -contraction and $M^m(H) = \begin{cases} \frac{H}{6^m}, & \Psi(H) \in [0, 1), \\ \frac{H}{7^m}, & \Psi(H) \in [1, \infty). \end{cases}$

Evidently, M is Ψ -sequentially continuous at the zero operator $\Theta \in S_{\left(\tau_r^F\left(\left(\frac{1}{(l+4)r_l}\right)_{l=0}^\infty, \left(\frac{2l+3}{l+2}\right)_{l=0}^\infty\right)\right)_v}$ and $\{M^m H\}$ has a subsequence $\{M^{m_i} H\}$ converges to Θ . According to Theorem 7.9, the zero operator $\Theta \in S_{\left(\tau_r^F\left(\left(\frac{1}{(l+4)r_l}\right)_{l=0}^\infty, \left(\frac{2l+3}{l+2}\right)_{l=0}^\infty\right)\right)_v}$ is the only fixed point of M .

Assume $\{H^{(a)}\} \subseteq S_{\left(\tau_r^F\left(\left(\frac{1}{(l+4)r_l}\right)_{l=0}^\infty, \left(\frac{2l+3}{l+2}\right)_{l=0}^\infty\right)\right)_v}$ is such that $\lim_{a \rightarrow \infty} \Psi(H^{(a)} - H^{(0)}) = 0$, where

$H^{(0)} \in S_{\left(\tau_r^F\left(\left(\frac{1}{(l+4)r_l}\right)_{l=0}^\infty, \left(\frac{2l+3}{l+2}\right)_{l=0}^\infty\right)\right)_v}$ with $\Psi(H^{(0)}) = 1$. Since the pre-quasi norm Ψ is continuous, we have

$$\lim_{a \rightarrow \infty} \Psi(MH^{(a)} - MH^{(0)}) = \lim_{a \rightarrow \infty} \Psi\left(\frac{H^{(0)}}{6} - \frac{H^{(0)}}{7}\right) = \Psi\left(\frac{H^{(0)}}{42}\right) > 0.$$

Hence, M is not Ψ -sequentially continuous at $H^{(0)}$. Therefore, the operator M is not continuous at $H^{(0)}$.

8. EXISTENCE OF SOLUTIONS OF NON-LINEAR DIFFERENCE EQUATIONS

In this section, we explore a solution in $\left(\tau_r^F(q, t)\right)_v$ to summable equations say (8.1), defined

in [48], where the setups of theorem 4.3 are established and $v(\bar{f}) = \left[\sum_{l=0}^\infty \left(\frac{\bar{\rho}\left(\sum_{z=0}^l r_z q_z \bar{f}_z, \bar{0}\right)}{r_{l+2} + r_l - 1}\right)^{t_l}\right]^{\frac{1}{h}}$, for all $\bar{f} \in \tau_r^F(q, t)$.

Examine the summable equations:

$$\bar{f}_z = \bar{y}_z + \sum_{m=0}^\infty A(z, m)g(m, \bar{f}_m), \quad (8.1)$$

and assume $W : \left(\tau_r^F(q, t)\right)_v \rightarrow \left(\tau_r^F(q, t)\right)_v$ is constructed by

$$W(\bar{f}_z)_{z \in \mathcal{N}} = \left(\bar{y}_z + \sum_{m=0}^\infty A(z, m)g(m, \bar{f}_m)\right)_{z \in \mathcal{N}}. \quad (8.2)$$

Theorem 8.1. *The summable equation (8.1) holds an unique solution in $(\tau_r^F(q, t))_v$, when $A : \mathcal{N}^2 \rightarrow \mathfrak{R}$, $g : \mathcal{N} \times \mathfrak{R}[0, 1] \rightarrow \mathfrak{R}[0, 1]$, $\bar{f} : \mathcal{N} \rightarrow \mathfrak{R}[0, 1]$, $\bar{y} : \mathcal{N} \rightarrow \mathfrak{R}[0, 1]$, $\bar{\eta} : \mathcal{N} \rightarrow \mathfrak{R}[0, 1]$, assume there is $\lambda \in \mathfrak{R}$ so that $\sup_l |\lambda|^{\frac{t_l}{h}} \in [0, \frac{1}{2})$ and for all $l \in \mathcal{N}$, we have*

$$\left| \sum_{z=0}^l \left(\sum_{m \in \mathcal{N}} A(z, m) [g(m, \bar{f}_m) - g(m, \bar{\eta}_m)] \right) r_z q_z \right| \leq \left| \sum_{z=0}^l \left(\bar{y}_z - \bar{f}_z + \sum_{m=0}^{\infty} A(z, m) g(m, \bar{f}_m) \right) r_z q_z \right| + |\lambda| \left| \sum_{z=0}^l \left(\bar{y}_z - \bar{\eta}_z + \sum_{m=0}^{\infty} A(z, m) g(m, \bar{\eta}_m) \right) r_z q_z \right|.$$

Proof. Let the conditions be established. Assume the mapping $W : (\tau_r^F(q, t))_v \rightarrow (\tau_r^F(q, t))_v$ is defined by equation (8.2). Hence

$$\begin{aligned} v(W\bar{f} - W\bar{\eta}) &= \left[\sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z (W\bar{f}_z - W\bar{\eta}_z), \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \right]^{\frac{1}{h}} \\ &= \left[\sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l \left(\sum_{m \in \mathcal{N}} A(z, m) [g(m, \bar{f}_m) - g(m, \bar{\eta}_m)] \right) r_z q_z, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \right]^{\frac{1}{h}} \\ &\leq \sup_l |\lambda|^{\frac{t_l}{h}} \left[\sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l (\bar{y}_z - \bar{f}_z + \sum_{m=0}^{\infty} A(z, m) g(m, \bar{f}_m)) r_z q_z, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \right]^{\frac{1}{h}} + \\ &\sup_l |\lambda|^{\frac{t_l}{h}} \left[\sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l (\bar{y}_z - \bar{\eta}_z + \sum_{m=0}^{\infty} A(z, m) g(m, \bar{\eta}_m)) r_z q_z, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \right]^{\frac{1}{h}} \\ &= \sup_l |\lambda|^{\frac{t_l}{h}} (v(W\bar{f} - \bar{f}) + v(W\bar{\eta} - \bar{\eta})). \end{aligned}$$

□

In view of Theorem 7.3, we obtain an unique solution of equation (8.1) in $(\tau_r^F(q, t))_v$.

Example 8.2. *Suppose the sequence space $\left(\tau_r^F \left(\left(\frac{1}{(l+1)r_l} \right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2} \right)_{l=0}^{\infty} \right) \right)_v$, where*

$$v(\bar{f}) = \sqrt{\sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l \frac{\bar{f}_z}{z+1}, \bar{0} \right)^{\frac{2l+3}{l+2}}}{r_{l+2} + r_l - 1} \right)}, \text{ for all } \bar{f} \in \left(\tau_r^F \left(\left(\frac{1}{(l+1)r_l} \right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2} \right)_{l=0}^{\infty} \right) \right)_v.$$

Assume the non-linear difference equations:

$$\bar{f}_z = \overline{e^{-(3z+6)}} + \sum_{m=0}^{\infty} (-1)^{z+m} \frac{\overline{f_{z-2}^b}}{\overline{f_{z-1}^d} + \overline{m^2 + 1}}, \quad (8.3)$$

with $b, d, \bar{f}_{-2}(t), \bar{f}_{-1}(t) > 0$, for all $t \in \mathfrak{R}$ and suppose

$$W : \left(\tau_r^F \left(\left(\frac{1}{(l+1)r_l} \right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2} \right)_{l=0}^{\infty} \right) \right)_v \rightarrow \left(\tau_r^F \left(\left(\frac{1}{(l+1)r_l} \right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2} \right)_{l=0}^{\infty} \right) \right)_v,$$

is defined by

$$W(\bar{f}_z)_{z=0}^\infty = \left(\overline{e^{-(3z+6)}} + \sum_{m=0}^\infty (-1)^{z+m} \frac{\overline{f_{z-2}^b}}{\overline{f_{z-1}^d} + \overline{m^2 + 1}} \right)_{z=0}^\infty. \quad (8.4)$$

Evidently, there is $\lambda \in \mathfrak{R}$ such that $\sup_l |\lambda|^{\frac{2l+3}{2l+4}} \in [0, \frac{1}{2})$ and for all $l \in \mathcal{N}$, one has

$$\begin{aligned} & \left| \sum_{z=0}^l \left(\sum_{m=0}^\infty (-1)^z \frac{\overline{f_{z-2}^b}}{\overline{f_{z-1}^d} + \overline{m^2 + 1}} \left((-1)^m - (-1)^m \right) \right) r_z q_z \right| \\ & \leq |\lambda| \left| \sum_{z=0}^l \left(\overline{e^{-(3z+6)}} - \bar{f}_z + \sum_{m=0}^\infty (-1)^{z+m} \frac{\overline{f_{z-2}^b}}{\overline{f_{z-1}^d} + \overline{m^2 + 1}} \right) r_z q_z \right| + \\ & |\lambda| \left| \sum_{z=0}^l \left(\overline{e^{-(3z+6)}} - \bar{\eta}_z + \sum_{m=0}^\infty (-1)^{z+m} \frac{\overline{\eta_{z-2}^b}}{\overline{\eta_{z-1}^d} + \overline{m^2 + 1}} \right) r_z q_z \right|. \end{aligned}$$

According to Theorem 8.1, the non-linear difference equations (8.3) contain an unique solution in $\left(\tau_r^F \left(\left(\frac{1}{(l+1)r_l} \right)_{l=0}^\infty, \left(\frac{2l+3}{l+2} \right)_{l=0}^\infty \right) \right)_v$.

Example 8.3. Suppose the sequence space $\left(\tau_r^F \left(\left(\frac{1}{(l+1)r_l} \right)_{l=0}^\infty, \left(\frac{2l+3}{l+2} \right)_{l=0}^\infty \right) \right)_v$, where

$$v(\bar{f}) = \sqrt{\sum_{l=0}^\infty \left(\frac{\bar{\rho} \left(\sum_{z=0}^l \frac{\bar{f}_z}{r_{l+2} + r_l - 1}, \bar{0} \right)^{\frac{2l+3}{l+2}}}{r_{l+2} + r_l - 1} \right)}, \text{ for all } \bar{f} \in \left(\tau_r^F \left(\left(\frac{1}{(l+1)r_l} \right)_{l=0}^\infty, \left(\frac{2l+3}{l+2} \right)_{l=0}^\infty \right) \right)_v.$$

Assume the non-linear difference equations:

$$\bar{f}_z = \bar{y}_z + \sum_{m=0}^\infty e^{z+m} \frac{\overline{f_{z-2}^b}}{\overline{f_{z-1}^d} + \overline{f_{z-1}^b} + \bar{2}}, \quad (8.5)$$

with $b, d, \overline{f_{-2}}(t), \overline{f_{-1}}(t) > 0$, for all $t \in \mathfrak{R}$ and suppose

$$W : \left(\tau_r^F \left(\left(\frac{1}{(l+1)r_l} \right)_{l=0}^\infty, \left(\frac{2l+3}{l+2} \right)_{l=0}^\infty \right) \right)_v \rightarrow \left(\tau_r^F \left(\left(\frac{1}{(l+1)r_l} \right)_{l=0}^\infty, \left(\frac{2l+3}{l+2} \right)_{l=0}^\infty \right) \right)_v,$$

is defined by

$$W(\bar{f}_z)_{z=0}^\infty = \left(\bar{y}_z + \sum_{m=0}^\infty e^{z+m} \frac{\overline{f_{z-2}^b}}{\overline{f_{z-1}^d} + \overline{f_{z-1}^b} + \bar{2}} \right)_{z=0}^\infty. \quad (8.6)$$

Evidently, there is $\lambda \in \mathfrak{R}$ such that $\sup_l |\lambda|^{\frac{2l+3}{2l+4}} \in [0, \frac{1}{2})$ and for all $l \in \mathcal{N}$, one has

$$\begin{aligned} & \left| \sum_{z=0}^l \left(\sum_{m=0}^\infty e^z \frac{\overline{f_{z-2}^b}}{\overline{f_{z-1}^d} + \overline{f_{z-1}^b} + \bar{2}} (e^m - e^m) \right) r_z q_z \right| \leq |\lambda| \left| \sum_{z=0}^l \left(\bar{y}_z - \bar{f}_z + \sum_{m=0}^\infty e^{z+m} \frac{\overline{f_{z-2}^b}}{\overline{f_{z-1}^d} + \overline{f_{z-1}^b} + \bar{2}} \right) r_z q_z \right| + \\ & |\lambda| \left| \sum_{z=0}^l \left(\bar{y}_z - \bar{\eta}_z + \sum_{m=0}^\infty e^{z+m} \frac{\overline{\eta_{z-2}^b}}{\overline{\eta_{z-1}^d} + \overline{\eta_{z-1}^b} + \bar{2}} \right) r_z q_z \right|. \end{aligned}$$

According to Theorem 8.1, the non-linear difference equations (8.5) contain an unique solution in $\left(\tau_r^F\left(\left(\frac{1}{(l+1)r_l}\right)_{l=0}^\infty, \left(\frac{2l+3}{l+2}\right)_{l=0}^\infty\right)\right)_v$.

Theorem 8.4. Consider the summable equations (8.1), and assume $W : \left(\tau_r^F(q, t)\right)_v \rightarrow \left(\tau_r^F(q, t)\right)_v$ is defined by (8.2), where the setups of theorem 4.3 are established and

$v(\bar{f}) = \sum_{l=0}^\infty \left(\frac{\bar{\rho}\left(\sum_{z=0}^l r_z q_z \bar{f}_z, \bar{0}\right)}{r_{l+2} + r_l - 1} \right)^{t_l}$, for all $\bar{f} \in \tau_r^F(q, t)$. The summable equation (8.1) has an unique solution $\bar{Z} \in \left(\tau_r^F(q, t)\right)_v$, if the following conditions are satisfied:

- (1): If $A : \mathcal{N}^2 \rightarrow \mathfrak{R}$, $g : \mathcal{N} \times \mathfrak{R}[0, 1] \rightarrow \mathfrak{R}[0, 1]$, $\bar{f} : \mathcal{N} \rightarrow \mathfrak{R}[0, 1]$, $\bar{y} : \mathcal{N} \rightarrow \mathfrak{R}[0, 1]$, $\bar{\eta} : \mathcal{N} \rightarrow \mathfrak{R}[0, 1]$, assume there is $\lambda \in \mathfrak{R}$ so that $2^{h-1} \sup_l |\lambda|^{t_l} \in [0, \frac{1}{2})$ and for all $l \in \mathcal{N}$, we have

$$\left| \sum_{z=0}^l \left(\sum_{m \in \mathcal{N}} A(z, m) [g(m, \bar{f}_m) - g(m, \bar{\eta}_m)] \right) r_z q_z \right| \leq |\lambda| \left| \sum_{z=0}^l \left(\bar{y}_z - \bar{f}_z + \sum_{m=0}^\infty A(z, m) g(m, \bar{f}_m) \right) r_z q_z \right| +$$

$$|\lambda| \left| \sum_{z=0}^l \left(\bar{y}_z - \bar{\eta}_z + \sum_{m=0}^\infty A(z, m) g(m, \bar{\eta}_m) \right) r_z q_z \right|.$$

- (2): W is v -sequentially continuous at $\bar{Z} \in \left(\tau_r^F(q, t)\right)_v$,

- (3): there is $\bar{Y} \in \left(\tau_r^F(q, t)\right)_v$ with $\{W^m \bar{Y}\}$ has $\{W^{m_j} \bar{Y}\}$ converging to $\bar{Z}$.

Proof. One has

$$\begin{aligned} v(W\bar{f} - W\bar{\eta}) &= \sum_{l=0}^\infty \left(\frac{\bar{\rho}\left(\sum_{z=0}^l r_z q_z (W\bar{f}_z - W\bar{\eta}_z), \bar{0}\right)}{r_{l+2} + r_l - 1} \right)^{t_l} \\ &= \sum_{l=0}^\infty \left(\frac{\bar{\rho}\left(\sum_{z=0}^l \left(\sum_{m \in \mathcal{N}} A(z, m) [g(m, \bar{f}_m) - g(m, \bar{\eta}_m)] \right) r_z q_z, \bar{0}\right)}{r_{l+2} + r_l - 1} \right)^{t_l} \\ &\leq 2^{h-1} \sup_l |\lambda|^{t_l} \sum_{l=0}^\infty \left(\frac{\bar{\rho}\left(\sum_{z=0}^l (\bar{y}_z - \bar{f}_z + \sum_{m=0}^\infty A(z, m) g(m, \bar{f}_m)) r_z q_z, \bar{0}\right)}{r_{l+2} + r_l - 1} \right)^{t_l} + \\ &2^{h-1} \sup_l |\lambda|^{t_l} \sum_{l=0}^\infty \left(\frac{\bar{\rho}\left(\sum_{z=0}^l (\bar{y}_z - \bar{\eta}_z + \sum_{m=0}^\infty A(z, m) g(m, \bar{\eta}_m)) r_z q_z, \bar{0}\right)}{r_{l+2} + r_l - 1} \right)^{t_l} \\ &= 2^{h-1} \sup_l |\lambda|^{t_l} (v(W\bar{f} - \bar{f}) + v(W\bar{\eta} - \bar{\eta})). \end{aligned}$$

By Theorem 7.5, one gets an unique solution $\bar{Z} \in \left(\tau_r^F(q, t)\right)_v$ of equation(8.1). □

Example 8.5. Suppose the sequence space $\left(\tau_r^F\left(\left(\frac{1}{(l+1)r_l}\right)_{l=0}^\infty, \left(\frac{2l+3}{l+2}\right)_{l=0}^\infty\right)\right)_v$, where

$$v(\bar{f}) = \sum_{l=0}^\infty \left(\frac{\bar{\rho}\left(\sum_{z=0}^l \frac{\bar{f}_z}{z+1}, \bar{0}\right)}{r_{l+2} + r_l - 1} \right)^{\frac{2l+3}{l+2}}, \text{ for all } \bar{f} \in \left(\tau_r^F\left(\left(\frac{1}{(l+1)r_l}\right)_{l=0}^\infty, \left(\frac{2l+3}{l+2}\right)_{l=0}^\infty\right)\right)_v.$$

Consider the summable equations (8.3).

Let $W : \left(\tau_r^F \left(\left(\frac{1}{(l+1)r_l} \right)_{l=0}^\infty, \left(\frac{2l+3}{l+2} \right)_{l=0}^\infty \right) \right)_v \rightarrow \left(\tau_r^F \left(\left(\frac{1}{(l+1)r_l} \right)_{l=0}^\infty, \left(\frac{2l+3}{l+2} \right)_{l=0}^\infty \right) \right)_v$ defined by (8.4). Assume W is v -sequentially continuous at $\bar{Z} \in \left(\tau_r^F \left(\left(\frac{1}{(l+1)r_l} \right)_{l=0}^\infty, \left(\frac{2l+3}{l+2} \right)_{l=0}^\infty \right) \right)_v$, and there is $\bar{Y} \in \left(\tau_r^F \left(\left(\frac{1}{(l+1)r_l} \right)_{l=0}^\infty, \left(\frac{2l+3}{l+2} \right)_{l=0}^\infty \right) \right)_v$ with $\{W^m \bar{Y}\}$ has $\{W^m \bar{Y}\}$ converging to $\bar{Z}$. Evidently, there is $\lambda \in \mathfrak{R}$ such that $2^{h-1} \sup_l |\lambda|^{\frac{2l+3}{l+2}} \in [0, \frac{1}{2})$ and for all $l \in \mathcal{N}$, one has

$$\begin{aligned} & \left| \sum_{z=0}^l \left(\sum_{m=0}^\infty (-1)^z \frac{\overline{f_{z-2}^b}}{f_{z-1}^d + m^2 + 1} \left((-1)^m - (-1)^m \right) \right) r_z q_z \right| \\ & \leq |\lambda| \left| \sum_{z=0}^l \left(\overline{e^{-(3z+6)}} - f_z + \sum_{m=0}^\infty (-1)^{z+m} \frac{\overline{f_{z-2}^b}}{f_{z-1}^d + m^2 + 1} \right) r_z q_z \right| + \\ & |\lambda| \left| \sum_{z=0}^l \left(\overline{e^{-(3z+6)}} - \bar{\eta}_z + \sum_{m=0}^\infty (-1)^{z+m} \frac{\overline{\eta_{z-2}^b}}{\eta_{z-1}^d + m^2 + 1} \right) r_z q_z \right|. \end{aligned}$$

By Theorem 8.6, the summable equations (8.3) have an unique solution

$$\bar{Z} \in \left(\tau_r^F \left(\left(\frac{1}{(l+1)r_l} \right)_{l=0}^\infty, \left(\frac{2l+3}{l+2} \right)_{l=0}^\infty \right) \right)_v.$$

In this part, we search for a solution to nonlinear matrix equations (8.7) at $D \in \overline{\mathbb{B}^s}_{(\tau_r^F(q,t))_v}(\mathcal{P}, \mathcal{Q})$,

the conditions of theorem 4.3 are satisfied, and $\Psi(G) = \left[\sum_{l=0}^\infty \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \overline{s_z(G)}, \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \right]^{\frac{1}{h}}$, for every $G \in \overline{\mathbb{B}^s}_{(\tau_r^F(q,t))_v}(\mathcal{P}, \mathcal{Q})$. Consider the summable equations

$$\overline{s_z(G)} = \overline{s_z(P)} + \sum_{m=0}^\infty A(z, m) f(m, \overline{s_m(G)}), \quad (8.7)$$

and suppose $W : \overline{\mathbb{B}^s}_{(\tau_r^F(q,t))_v}(\mathcal{P}, \mathcal{Q}) \rightarrow \overline{\mathbb{B}^s}_{(\tau_r^F(q,t))_v}(\mathcal{P}, \mathcal{Q})$ is defined by

$$W(G) = \left(\overline{s_z(P)} + \sum_{m=0}^\infty A(z, m) f(m, \overline{s_m(G)}) \right) I. \quad (8.8)$$

Theorem 8.6. The summable equations (8.7) have one solution $D \in \overline{\mathbb{B}^s}_{(\tau_r^F(q,t))_v}(\mathcal{P}, \mathcal{Q})$, if the following conditions are satisfied:

- (a): $A : \mathcal{N}^2 \rightarrow \mathfrak{R}$, $f : \mathcal{N} \times \mathfrak{R}[0, 1] \rightarrow \mathfrak{R}[0, 1]$, $P \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$, $T \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$, and for every $z \in \mathcal{N}$, there is κ so that $\sup_z \kappa^{\frac{t_z}{h}} \in [0, 0.5)$, with

$$\begin{aligned} & \left| \sum_{m \in \mathcal{N}} A(z, m) \left(f(m, \overline{s_m(G)}) - f(m, \overline{s_m(T)}) \right) \right| \\ & \leq \kappa \left[\left| \overline{s_z(P)} - \overline{s_z(G)} + \sum_{m \in \mathcal{N}} A(a, m) f(m, \overline{s_m(G)}) \right| + \left| \overline{s_z(P)} - \overline{s_z(T)} + \sum_{m \in \mathcal{N}} A(z, m) f(m, \overline{s_m(T)}) \right| \right], \end{aligned}$$

- (b): W is Ψ -sequentially continuous at a point $D \in \overline{\mathbb{B}^s}_{(\tau_r^F(q,t))_v}(\mathcal{P}, \mathcal{Q})$,

(c): there is $B \in \overline{\mathbb{B}}^s_{(\tau_r^F(q,t))_v}(\mathcal{P}, \mathcal{Q})$ so that the sequence of iterates $\{W^a B\}$ has a subsequence $\{W^{a_i} B\}$ converging to D .

Proof: Suppose the settings are verified. Consider the mapping $W : \overline{\mathbb{B}}^s_{(\tau_r^F(q,t))_v}(\mathcal{P}, \mathcal{Q}) \rightarrow \overline{\mathbb{B}}^s_{(\tau_r^F(q,t))_v}(\mathcal{P}, \mathcal{Q})$ defined by (8.8). We have

$$\begin{aligned} \Psi(WG - WT) &= \left[\sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z (\overline{s_z(G)} - \overline{s_z(T)}), \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \right]^{\frac{1}{h}} \\ &= \left[\sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z \sum_{m \in \mathcal{N}} A(a, m) (f(m, \overline{s_m(G)}) - f(m, \overline{s_m(T)})), \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \right]^{\frac{1}{h}} \\ &\leq \sup_z \kappa^{\frac{t_z}{h}} \left[\sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z (\overline{s_z(P)} - \overline{s_z(G)} + \sum_{m \in \mathcal{N}} A(z, m) f(m, \overline{s_m(G)})), \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \right]^{\frac{1}{h}} + \\ &\sup_z \kappa^{\frac{t_z}{h}} \left[\sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l r_z q_z (\overline{s_z(T)} - \overline{s_z(G)} + \sum_{m \in \mathcal{N}} A(z, m) f(m, \overline{s_m(T)})), \bar{0} \right)}{r_{l+2} + r_l - 1} \right)^{t_l} \right]^{\frac{1}{h}} \\ &= \sup_z \kappa^{\frac{t_z}{h}} (\Psi(WG - G) + \Psi(WT - T)). \end{aligned}$$

In view of Theorem 7.9, one obtains an unique solution of equation (8.7) at $D \in \overline{\mathbb{B}}^s_{(\tau_r^F(q,t))_v}(\mathcal{P}, \mathcal{Q})$.

Example 8.7. Assume the class $\overline{\mathbb{B}}^s_{(\tau_r^F(q,t)((\frac{1}{l!r_l}), (\frac{2l+3}{l+2})))_v}(\mathcal{P}, \mathcal{Q})$, where

$$\Psi(G) = \sqrt{\sum_{l=0}^{\infty} \left(\frac{\bar{\rho} \left(\sum_{z=0}^l \frac{\overline{s_z(G)}}{z!}, \bar{0} \right)^{\frac{2l+3}{l+2}}}{r_{l+2} + r_l - 1} \right)}, \text{ for all } G \in \overline{\mathbb{B}}^s_{(\tau_r^F(q,t)((\frac{1}{l!r_l}), (\frac{2l+3}{l+2})))_v}(\mathcal{P}, \mathcal{Q}).$$

Consider the non-linear difference equations:

$$\overline{s_z(G)} = \overline{e^{-(2z+3)}} + \sum_{m=0}^{\infty} \frac{\tan(2m+1) \cosh(3m-z) \cos^b |\overline{s_{z-2}(G)}|}{\sinh^d |\overline{s_{z-1}(G)}| + \overline{\sin mz} + 1}, \quad (8.9)$$

where $z \geq 2$ and $b, d > 0$ and let $W : \overline{\mathbb{B}}^s_{(\tau_r^F(q,t)((\frac{1}{l!r_l}), (\frac{2l+3}{l+2})))_v}(\mathcal{P}, \mathcal{Q}) \rightarrow \overline{\mathbb{B}}^s_{(\tau_r^F(q,t)((\frac{1}{l!r_l}), (\frac{2l+3}{l+2})))_v}(\mathcal{P}, \mathcal{Q})$ be defined as

$$W(G) = \left(\overline{e^{-(2z+3)}} + \sum_{m=0}^{\infty} \frac{\tan(2m+1) \cosh(3m-z) \cos^b |\overline{s_{z-2}(G)}|}{\sinh^d |\overline{s_{z-1}(G)}| + \overline{\sin mz} + 1} \right) I. \quad (8.10)$$

Suppose W is Ψ -sequentially continuous at a point $D \in \overline{\mathbb{B}}^s_{(\tau_r^F(q,t)((\frac{1}{l!r_l}), (\frac{2l+3}{l+2})))_v}(\mathcal{P}, \mathcal{Q})$, and there is $B \in \overline{\mathbb{B}}^s_{(\tau_r^F(q,t)((\frac{1}{l!r_l}), (\frac{2l+3}{l+2})))_v}(\mathcal{P}, \mathcal{Q})$ so that the sequence of iterates $\{W^a B\}$ has a subsequence $\{W^{a_i} B\}$ converging to

D. It is easy to see that

$$\begin{aligned} & \left| \sum_{m=0}^{\infty} \frac{\cosh(3m-z) \cos^b |\overline{s_{z-2}(G)}|}{\sinh^d |\overline{s_{z-1}(G)}| + \overline{\sin mz} + \overline{1}} \left(\tan(2m+1) - \tan(2m+1) \right) \right| \\ & \leq \frac{1}{25} \left| \overline{e^{-(2z+3)}} - \overline{s_z(G)} + \sum_{m=0}^{\infty} \frac{\tan(2m+1) \cosh(3m-z) \cos^b |\overline{s_{z-2}(G)}|}{\sinh^d |\overline{s_{z-1}(G)}| + \overline{\sin mz} + \overline{1}} \right| \\ & + \frac{1}{25} \left| \overline{e^{-(2z+3)}} - \overline{s_z(T)} + \sum_{m=0}^{\infty} \frac{\tan(2m+1) \cosh(3m-z) \cos^b |\overline{s_{z-2}(T)}|}{\sinh^d |\overline{s_{z-1}(T)}| + \overline{\sin mz} + \overline{1}} \right|. \end{aligned}$$

By Theorem 8.6, the non-linear difference equations (8.9) have one solution $D \in \overline{\mathbb{B}^s} \left(\tau_r^F(q, t) \left(\left(\frac{1}{n_l} \right), \left(\frac{2l+3}{l+2} \right) \right) \right)_v (\mathcal{P}, \mathcal{Q})$.

9. CONCLUSION

In this article, we offer some topological and geometric properties of $(\tau_r^F(q, t))_v$, of the multiplication maps acting on $(\tau_r^F(q, t))_v$, of the class $\overline{\mathbb{B}^s}(\tau_r^F(q, t))_v$, and of the class $\left(\overline{\mathbb{B}^s}(\tau_r^F(q, t))_v \right)^\lambda$. We investigate the existence of a fixed point of Kannan contraction map acting on these spaces. Some several numerical experiments are introduced to light our results. Furthermore, some successful applications to the existence of solutions of non-linear difference equations are discussed. This article has a number of advantages for researchers such as studying the fixed points of any contraction maps on this pre-quasi normed sequence space which is a generalization of the quasi normed sequence spaces, a new general space of solutions for many difference equations, the spectrum of any bounded linear operators between any two Banach spaces with s -numbers in this sequence space and note that the closed operator ideals are certain to play an important function in the principle of Banach lattices.

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