Int. J. Anal. Appl. (2025), 23:293

International Journal of Analysis and Applications

A Study of Bi-Univalent Class in Leaf-Like Domains Using Quantum Calculus Through Subordination

Abdullah Alsoboh[1], Ala Amourah[2,3], Waggas Galib Atshan[4], Jamal Salah[1,*], Tala Sasa[5]

[1] *College of Applied and Health Sciences, A'Sharqiyah University, Post Box No. 42, Post Code No. 400, Ibra, Sultanate of Oman*

[2] *Mathematics Education Program, Faculty of Education and Arts, Sohar University, Sohar 311, Oman*

[3] *Jadara University Research Center, Jadara University, Jordan*

[4] *Department of Mathematics, College of Science, University of Al-Qadisiyah, Iraq*

[5] *Department of Mathematics, Faculty of Science, Applied Science Private University, Amman*

[*] *Corresponding author:* drjamals@asu.edu.om

Abstract. In this work, we introduce and study a new subclass of bi-univalent analytic functions that are associated with a leaf-shaped region, formulated through the framework of q-calculus and the principle of subordination. employing appropriate analytical techniques, we derive sharp coefficient estimates with a particular focus on the initial bounds for $|a_2|$ and $|a_3|$, which play a central role in geometric function theory. In addition, we establish Fekete–Szegö-type inequalities for functions belonging to the proposed class, thereby extending and complementing several existing results in the literature. To illustrate the theoretical findings, we provide explicit examples and graphical representations that highlight the behavior of functions in the class under consideration. The analysis not only broadens the applicability of q-calculus methods in the study of bi-univalent functions but also contributes to a deeper understanding of function classes connected with geometrically significant domains. Potential avenues for further research and connections to related subclasses are also discussed.

1. Introduction, Preliminaries and Definitions

Let f be a holomorphic function defined on the open unit disk Δ, where

$$\Delta = \{z \in \mathbb{C} : |z| < 1\}.$$

Received: Sep. 22, 2025.

2020 *Mathematics Subject Classification.* 30A36, 30C45, 81P68, 11B37.

Key words and phrases. analytic functions; univalent functions; starlike and convex functions; Fekete-Szegö; leaf-like domains; quantum calculus.

https://doi.org/10.28924/2291-8639-23-2025-293

ISSN: 2291-8639

A function belongs to the class $\mathcal{A}$ if it admits the Taylor–Maclaurin expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad z \in \Delta. \tag{1.1}$$

Within this family, the subclass $\mathcal{S}$ consists of functions that are univalent in Δ and satisfy the standard normalization conditions

$$f(0) = 0, \qquad f'(0) = 1. \tag{1.2}$$

An analytic function τ defined in Δ is called a *Schwarz function* if it satisfies

$$\tau(0) = 0 \quad \text{and} \quad |\tau(z)| < 1, \qquad z \in \Delta.$$

Given two analytic functions $f_1, f_2 \in \mathcal{A}$, we say that f_1 is *subordinate* to f_2, written as $f_1 \prec f_2$, if there exists a Schwarz function τ such that

$$f_1(z) = f_2(\tau(z)), \qquad z \in \Delta.$$

Another class that plays an important role is P, which is closely connected to the Carathéodory functions (see Miller [27]). These functions are analytic in Δ and satisfy the conditions

$$\Re\{\delta(z)\} > 0, \qquad z \in \Delta, \qquad \delta(0) = 1.$$

A Maclaurin series expansion provides a standard representation for a function $\delta(z)$ belonging to the class P, and may be written as

$$\delta(z) = 1 + \sum_{n=1}^{\infty} \delta_n z^n, \qquad z \in \Delta. \tag{1.3}$$

The coefficients δ_n are subject to the classical bound

$$|\delta_n| \le 2, \qquad n \ge 1, \tag{1.4}$$

which is a consequence of Carathéodory's lemma (see Duren [19]). Equivalently, a function δ belongs to the class P if and only if it satisfies the subordination relation

$$\delta(z) \prec \frac{1+z}{1-z}, \qquad z \in \Delta.$$

For every $f \in \mathcal{S}$, there exists an inverse function f^{-1} defined in a disk of radius $r_0(f) \ge \frac{1}{4}$. This inverse is characterized by the relations

$$z = f^{-1}(f(z)), \qquad \varsigma = f(f^{-1}(\varsigma)), \qquad |\varsigma| < r_0(f),\ z \in \Delta.$$

The expansion of f^{-1} about the origin takes the form

$$f^{-1}(\varsigma) = \chi(\varsigma) = \varsigma - a_2\varsigma^2 + (2a_2^2 - a_3)\varsigma^3 - (a_4 + 5a_2^3 - 5a_2a_3)\varsigma^4 + \cdots . \tag{1.5}$$

A function $f(z) \in \mathcal{S}$ is said to be *bi-univalent* if both f and its inverse f^{-1} are univalent in Δ. The family of such functions is denoted by Σ. Several illustrative examples of functions in Σ together with their corresponding inverses are listed in Table 1.

TABLE 1. Certain univalent functions along with their inverses.

$f_1(z) = \frac{z}{z+1}$	$f_1^{-1}(\varsigma) = \frac{\varsigma}{1-\varsigma}$
$f_2(z) = \frac{e^{2z}-1}{e^{2z}+1}$	$f_2^{-1}(\varsigma) = -\log(1-\varsigma)$
$f_3(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$	$f_3^{-1}(\varsigma) = \frac{e^{\varsigma}-1}{e^{\varsigma}}$

Ma and Minda [26] developed the class $K(\aleph)$ using subordination. They presented their method as follows:

$$K(\aleph) = \left\{ f \in \mathcal{A} : \quad 1 + \frac{zf''(z)}{f'(z)} \prec \aleph(z), \qquad (z \in \Delta) \right\}.$$

In this formula, $\aleph(z)$ denotes a function with a positive real component and is normalized according to criteria 1.2. The table below highlights the variety of approaches adopted by different authors in defining subclasses of functions, achieved by prescribing specific forms of $\aleph$.

TABLE 2. Representative subclasses of analytic functions defined via subordination relations.

$\aleph(z)$	**Authors**	**Reference**	**Year**
$\sqrt{1+z}$	Sokoł and Stankiewicz	[33]	1996
$\dfrac{z}{1-z}$	Piejko and Sokoł	[29]	2012
$z + \sqrt{1+z^2}$	Priya and Sharma	[30]	2018
$z + \sqrt[3]{1+z^3}$	Singh and Kaur	[32]	2021

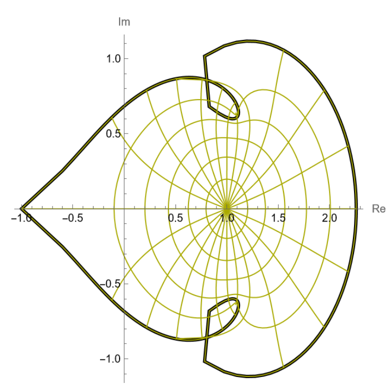


FIGURE 1. The image of $\aleph(\Delta)$ is displayed in a leaf-shaped region. Here, $\aleph(z) = z + \sqrt[3]{1+z^3}$

Quantum calculus, commonly known as q-calculus, provides a natural generalization of classical calculus by introducing a parameter $q \in (0,1)$, thereby extending the scope of standard analytical techniques. This framework has gained considerable attention because of its deep connections with physics, quantum mechanics, and Geometric Function Theory (GFT). A fundamental reference for q-difference calculus and its applications is the comprehensive treatise by Gasper and Rahman [24].

At the core of analytic investigations in this setting lies the q-difference operator ∂_q, which serves as a central tool in studying the behavior of analytic functions. Notable developments include the work of Seoudy and Aouf [31], who extended the q-calculus methods to functions defined on the unit disk, thus enriching the analytic theory underlying GFT. For further perspectives, both classical and recent contributions can be found in a number of sources, including [1,3–9,14–18], and some applications can be found in [40,41,43,46,47].

More recently, Alsoboh and Oros [9] introduced and investigated a subclass of bi-univalent functions associated with a leaf-shaped domain within the framework of q-calculus. Their study was motivated by the following definition:

Definition 1.1 ([9]). *A function $f \in \mathcal{S}$ is said to belong to the class $\Sigma_q(\lambda, \aleph(z;q))$ if it has the form* (1.1) *and satisfies the subordination relation*

$$(1-\lambda)\,\frac{f(z)}{z} + \lambda\left(\partial_q f(z)\right) \prec \aleph(z;q), \qquad z \in \Delta, \tag{1.6}$$

where $\lambda \geq 1$, $\aleph(0;q) = 1$, and

$$\aleph(z;q) = \frac{\langle 2|q\rangle z}{2+(1-q)z} + \sqrt[3]{1+\left(\frac{\langle 2|q\rangle z}{2+(1-q)z}\right)^3}. \tag{1.7}$$

The operator ∂_q denotes the q-derivative of f, defined by

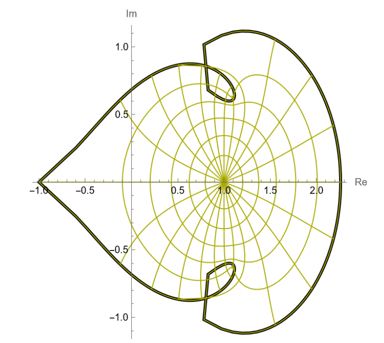
$$\partial_q f(z) = \begin{cases} \dfrac{f(z)-f(qz)}{z-qz}, & z \neq 0, \\[2ex] f'(z), & q \to 1^-,\ z \neq 0, \\[2ex] f'(0), & z = 0. \end{cases}$$

Here, $\langle n|q\rangle$ denotes the quantum number, given by

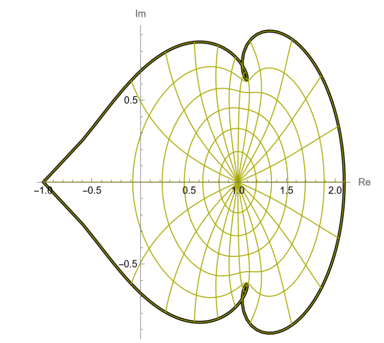
$$\langle n|q\rangle = \begin{cases} \dfrac{1-q^n}{1-q}, & 0<q<1,\ \ n \in \mathbb{C}\setminus\{0\}, \\[2ex] q^{n-1}+\cdots+q+1, & n \in \mathbb{N}, \\[2ex] 1, & q \to 0^+,\ \ n \in \mathbb{C}\setminus\{0\}, \\[2ex] n, & q \to 1^-,\ \ n \in \mathbb{C}\setminus\{0\}. \end{cases}$$

Remark 1.1. *The mapping* $\aleph(z;q)$ *transforms the unit disk* Δ *into a leaf-shaped region. The function* $\aleph(z;q)$ *is analytic and univalent in* Δ*, symmetric with respect to the real axis, and satisfies the normalization conditions*

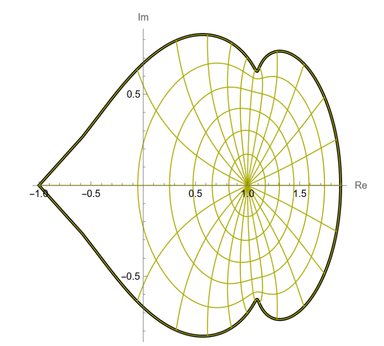
$$\aleph(0;q) = 1 \quad \textit{and} \quad \partial_q \aleph(0;q) = 1.$$



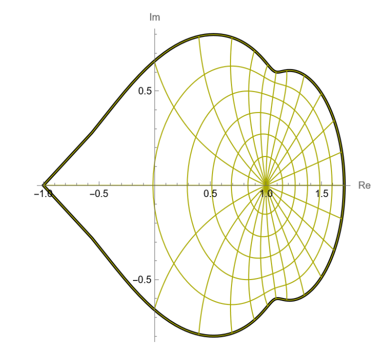
(**a**) The image of $\aleph(\Delta;\mathfrak{q})$ with $\mathfrak{q} \to 1^-$



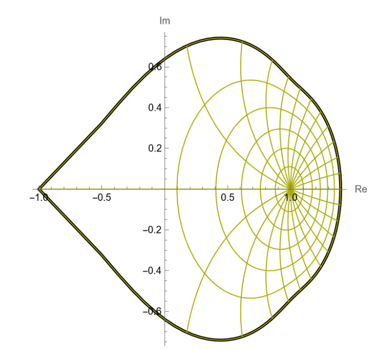
(**b**) The image of $\aleph(\Delta;0.89)$



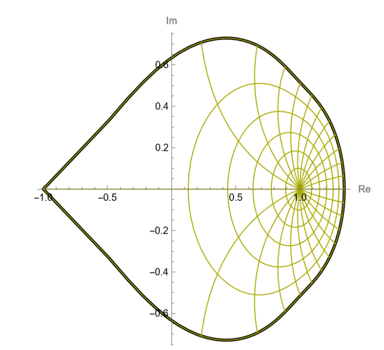
(c) The image of $\aleph(\Delta;0.73)$



(d) The image of $\aleph(\Delta;0.54)$



(e) The image of $\aleph(\Delta;0.1)$



(f) The image of $\aleph(\Delta)$ with $\mathfrak{q} \to 0^+$

Figure 2. Illustration of the leaf-shaped region $\aleph(\Delta;\mathfrak{q})$.

Recent work by Amourah et al. [10] introduced the starlike class $\mathcal{S}^*(q;\aleph(z;q))$, associated with the leaf-shaped domain in Δ. This class is defined as follows:

Definition 1.2. *A function $f \in \mathcal{S}$ is said to belong to the class $\mathcal{S}^*(q;\aleph(z;q))$ if it can be represented in the form* (1.1) *and satisfies the subordination condition*

$$\frac{z\,\partial_q f(z)}{f(z)} \prec \aleph(z;q), \qquad z \in \Delta, \tag{1.8}$$

where $\aleph(z;q)$ is given by (1.7).

Definition 1.3. *A function $f \in \mathcal{S}$ is said to belong to the class $C(q;\aleph(z;q))$ if it can be represented in the form* (1.1) *and satisfies the subordination condition*

$$1 + \frac{z\,\partial_q^2 f(z)}{\partial_q f(z)} \prec \aleph(z;q), \qquad z \in \Delta, \tag{1.9}$$

where $\aleph(z;q)$ is given by (1.7).

The primary objective of this work is to investigate the structural and analytic properties of bi-univalent functions associated with a leaf-shaped domain in Δ. In this section, we provide a rigorous definition of the class under consideration, supplemented by illustrative examples to clarify its geometric significance. In Section 3, we establish sharp coefficient bounds and examine the Fekete–Szegö functional for the proposed class. Furthermore, by applying the results developed in the earlier sections, we deduce several corollaries that address particular subclasses and special cases of interest.

2. The Bi-univalent Class $\mathsf{K}_\Sigma(\eta,q;\aleph(z;q))$

In this section, we make use of the q-difference operator in conjunction with the subordination principle for analytic functions, as outlined in the preliminaries, to provide a rigorous mathematical characterization of the newly introduced class $\mathsf{K}_\Sigma(\eta,q;\aleph(z;q))$ of bi-univalent functions associated with a leaf-shaped domain.

Definition 2.1. *Let $\eta \in [0,1]$. A bi-univalent function f of the form* (1.1) *is said to belong to the class $\mathsf{K}_\Sigma(\eta,q;\aleph(z;q))$ if and only if*

$$\Phi_1(z;q) = (1-\eta)\,\frac{z\,\partial_q f(z)}{f(z)} + \eta\,\frac{\partial_q\big(z\,\partial_q f(z)\big)}{\partial_q f(z)} \prec \aleph(z;q), \qquad z \in \Delta, \tag{2.1}$$

and

$$\Phi_2(\varsigma;q) = (1-\eta)\,\frac{\varsigma\,\partial_q \chi(\varsigma)}{\chi(\varsigma)} + \eta\,\frac{\partial_q\big(\varsigma\,\partial_q \chi(\varsigma)\big)}{\partial_q \chi(\varsigma)} \prec \aleph(\varsigma;q), \qquad \varsigma \in \Delta, \tag{2.2}$$

where $\chi = f^{-1}$ is given by (1.5), *and $\aleph(z;q)$ is of the form* (1.7).

Remark 2.1. *We emphasize that the class $\mathsf{K}_{\Sigma}(\eta, q; \aleph(z;q))$ is non-empty. To illustrate this, consider the function*

$$f_*(z) = \frac{z}{1-\vartheta z}, \qquad |\vartheta| < 0.55. \tag{2.3}$$

It is clear that $f_* \in \mathcal{S}$ and, moreover, $f_* \in \Sigma$ together with its inverse

$$f_*^{-1}(\varsigma) = \frac{\varsigma}{1+\vartheta\varsigma}, \qquad |\vartheta| < 0.55. \tag{2.4}$$

Using the notation introduced in (2.1) *and* (2.2)*, a straightforward computation yields*

$$\Phi_1(f_*(z);q) = \frac{1-\eta}{1-\vartheta q z} + \frac{\eta(1+\vartheta q z)}{1-\vartheta q^2 z}, \qquad |\vartheta| < 0.55,$$

and

$$\Phi_2(f_*^{-1}(\varsigma);q) = \frac{1-\eta}{1+\vartheta q \varsigma} + \frac{\eta(1-\vartheta q \varsigma)}{1+\vartheta q^2 \varsigma}, \qquad |\vartheta| < 0.55.$$

Furthermore, for all $z \in \Delta$, one observes that

$$\Phi_1(-\vartheta z;q) = \Phi_2(\vartheta\varsigma;q),$$

which implies that

$$\Phi_1(\Delta;q) = \Phi_2(\Delta;q).$$

GeoGebra Classic 6 is used to generate boundary visualizations $\partial\Delta$ using functions Φ_1, Φ_2 and $\aleph$, as illustrated in Table 3. This approach is particularly effective under the conditions $|\vartheta| \leq 0.55$ and $q \in (0,1)$.

It should be noted that $\aleph$ is univalent in Δ. Consequently, the subordinations

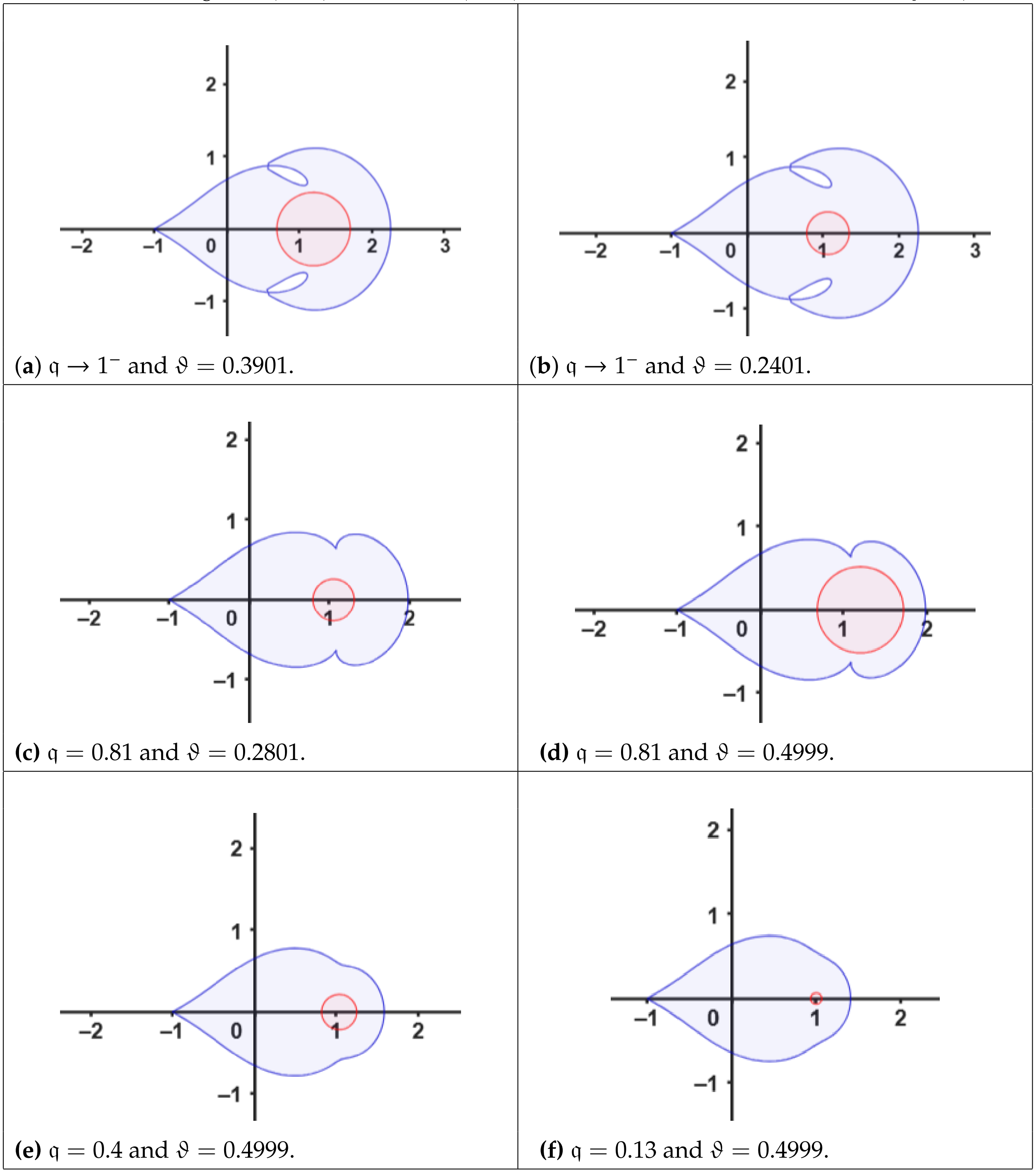
$$\Phi_1(z;q) \prec \aleph(z;q) \quad \text{and} \quad \Phi_2(\varsigma;q) \prec \aleph(\varsigma;q)$$

hold true. These relationships follow from the facts that

$$\Phi_1(0;q) = \Phi_2(0;q) = \aleph(0;q) = 1, \qquad \Phi_1(\Delta;q) \subset \aleph(\Delta;q), \qquad \Phi_2(\Delta;q) \subset \aleph(\Delta;q).$$

For further details and graphical evidence, see Figure 3.

FIGURE 3. Image of $\aleph(e^{i\theta};\mathfrak{q})$ (blue) and $\Phi(e^{i\theta};\mathfrak{q})$ (red) for various values of $\mathfrak{q}$, ϑ, and $\theta \in [0, 2\pi)$.



(a) $\mathfrak{q} \to 1^-$ and $\vartheta = 0.3901$.

(b) $\mathfrak{q} \to 1^-$ and $\vartheta = 0.2401$.

(c) $\mathfrak{q} = 0.81$ and $\vartheta = 0.2801$.

(d) $\mathfrak{q} = 0.81$ and $\vartheta = 0.4999$.

(e) $\mathfrak{q} = 0.4$ and $\vartheta = 0.4999$.

(f) $\mathfrak{q} = 0.13$ and $\vartheta = 0.4999$.

Example 2.1. *For* $\eta = 0$, *a bi-univalent function* f *of the form* (1.1) *belongs to the class* $\mathsf{K}_\Sigma(0, q; \aleph(z;q))$ *if and only if*

$$\frac{z\,\partial_q f(z)}{f(z)} \prec \aleph(z;q), \qquad z \in \Delta,$$

and

$$\frac{\varsigma\, \partial_q \chi(\varsigma)}{\chi(\varsigma)} \prec \aleph(\varsigma; q), \qquad \varsigma \in \Delta,$$

where $\chi = f^{-1}$ *is given by* (1.5), *and* $\aleph(z;q)$ *is defined in* (1.7).

Example 2.2. *For* $\eta = 1$, *a bi-univalent function* f *of the form* (1.1) *belongs to the class* $\mathsf{K}_\Sigma(1, q; \aleph(z;q))$ *if and only if*

$$\frac{\partial_q\Big(z\, \partial_q f(z)\Big)}{\partial_q f(z)} \prec \aleph(z; q), \qquad z \in \Delta,$$

and

$$\frac{\partial_q\Big(\varsigma\, \partial_q \chi(\varsigma)\Big)}{\partial_q \chi(\varsigma)} \prec \aleph(\varsigma; q), \qquad \varsigma \in \Delta,$$

where $\chi = f^{-1}$ *is given by* (1.5), *and* $\aleph(z;q)$ *is defined in* (1.7).

Example 2.3. *If* $q \to 1^-$, *then* $\Sigma_{\mathcal{K}}\left(\aleph(z;q)\right)$ *is reduced to* $\Sigma_{\mathcal{K}}\left(z + \sqrt[3]{1+z^3}\right)$ *defined by*

$$\frac{z\, f''(z)}{f'(z)} + 1 \prec z + \sqrt[3]{1+z^3},$$

and

$$\frac{\varsigma\left(f^{-1''}(\varsigma)\right)}{f'(\varsigma)} + 1 \prec \varsigma + \sqrt[3]{1+\varsigma^3},$$

for all z *and* ς *fall in the open unit desk* Δ.

3. The Bounds of the Coefficients and the Fekete-Szegö Functional within the Class $\mathsf{K}_\Sigma(\eta, q; \aleph(z;q))$

Theorem 3.1. *Let* $f \in \Sigma$ *be given by the expansion* (1.1) *and assume that* f *belongs to the class* $\mathsf{K}_\Sigma(\eta, q; \aleph(z;q))$. *Then the initial Taylor–Maclaurin coefficients of* f *satisfy the following estimates:*

$$|a_2| \;\leq\; \frac{\langle 2|q\rangle}{\sqrt{2q\left|\,\langle 2|q\rangle\Big(q + (1-q)\eta - (1-q)\eta\langle 2|q\rangle\Big) - q(3-q)(1+q\eta)^2\right|}},$$

and

$$|a_3| \;\leq\; \frac{1}{2q\left(1 + q\langle 2|q\rangle\eta\right)} + \frac{\langle 2|q\rangle^2}{4q^2(1+q\eta)^2}.$$

Proof. If $f \in \mathsf{K}_\Sigma(\eta, q; \aleph(z;q))$, then, according to Definition 2.1, there exist analytic functions τ_1 and τ_2 such that

$$\delta_1(z) = \frac{1+\tau_1(z)}{1-\tau_1(z)} = 1 + \omega_1 z + \omega_2 z^2 + \cdots, \qquad z \in \Delta,$$

and

$$\delta_2(\varsigma) = \frac{1+\tau_2(\varsigma)}{1-\tau_2(\varsigma)} = 1 + \imath_1 \varsigma + \imath_2 \varsigma^2 + \cdots, \qquad \varsigma \in \Delta.$$

Thus, $\delta_1, \delta_2 \in \mathrm{P}$.

From these representations, it follows that

$$\tau_1(z) = \frac{\delta_1(z) - 1}{\delta_1(z) + 1}, \qquad z \in \Delta,$$

and

$$\tau_2(\varsigma) = \frac{\delta_2(\varsigma) - 1}{\delta_2(\varsigma) + 1}, \qquad \varsigma \in \Delta.$$

Now, combining these relations with (2.1) and (2.2), we obtain

$$\Phi_1(z;q) = (1-\eta)\,\frac{z\,\partial_q f(z)}{f(z)} + \eta\,\frac{\partial_q\Big(z\,\partial_q f(z)\Big)}{\partial_q f(z)} = \aleph(\delta_1(z);q),$$

and

$$\Phi_2(\varsigma;q) = (1-\eta)\,\frac{\varsigma\,\partial_q \chi(\varsigma)}{\chi(\varsigma)} + \eta\,\frac{\partial_q\Big(\varsigma\,\partial_q \chi(\varsigma)\Big)}{\partial_q \chi(\varsigma)} = \aleph(\delta_2(\varsigma);q)\,.$$

$$\begin{aligned}\aleph(\tau_1(z);q) &= \tfrac{\langle 2|q\rangle(\delta_1(z)-1)}{1+3\delta_1(z)+q(1-\delta_1(z))} + \sqrt[3]{1+\left(\tfrac{\langle 2|q\rangle(\delta_1(z)-1)}{1+3\delta_1(z)+q(1-\delta_1(z))}\right)^3}\\ &= 1 + \tfrac{\langle 2|q\rangle}{4}\omega_1 z + \tfrac{\langle 2|q\rangle}{4}\left(\omega_2 - \tfrac{(3-q)}{4}\omega_1^2\right)z^2 + \cdots, \qquad (z \in \Delta)\,,\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\aleph(\tau_2(\varsigma);q) &= \tfrac{\langle 2|q\rangle(\delta_2(\varsigma)-1)}{1+3\delta_2(\varsigma)+q(1-\delta_2(\varsigma))} + \sqrt[3]{1+\left(\tfrac{\langle 2|q\rangle(\delta_2(\varsigma)-1)}{1+3\delta_2(\varsigma)+q(1-\delta_2(\varsigma))}\right)^3}\\ &= 1 + \tfrac{\langle 2|q\rangle}{4}\imath_1 \varsigma + \tfrac{\langle 2|q\rangle}{4}\left(\imath_2 - \tfrac{(3-q)}{4}\imath_1^2\right)\varsigma^2 + \cdots, \qquad (\varsigma \in \Delta)\,.\end{aligned} \tag{3.2}$$

Since

$$(1-\eta)\tfrac{z\partial_q f(z)}{f(z)} + \eta\tfrac{\partial_q\left(z\partial_q f(z)\right)}{\partial_q f(z)} = q(1+q\eta)a_2 z + q\langle 2|q\rangle(1+q\langle 2|q\rangle\eta)a_3 - q\left(1+\eta(\langle 2|q\rangle^2-1)\right)a_2^2 z^2 + \cdots, \tag{3.3}$$

and

$$\begin{aligned}(1-\eta)\tfrac{\varsigma\partial_q \chi(\varsigma)}{\chi(\varsigma)} + \eta\tfrac{\partial_q\left(\varsigma\partial_q \chi(\varsigma)\right)}{\partial_q \chi(\varsigma)} = -q(1+q\eta)a_2 z + \Big(&2q\langle 2|q\rangle(1+q\langle 2|q\rangle\eta) - q\left(1+\eta(\langle 2|q\rangle^2-1)\right)a_2^2\\ &-q\langle 2|q\rangle(1+q\langle 2|q\rangle\eta)a_3\Big)z^2 + \cdots.\end{aligned} \tag{3.4}$$

By comparing (3.1) and (3.3), yields

$$\begin{gathered}q(1+q\eta)a_2 z +\\ q\langle 2|q\rangle(1+q\langle 2|q\rangle\eta)a_3 - q\left(1+\eta(\langle 2|q\rangle^2-1)\right)a_2^2 z^2 + \cdots\\ = 1 + \tfrac{1+q}{4}\omega_1 z + \tfrac{1+q}{4}\left(\omega_2 - \tfrac{(3-q)}{4}\omega_1^2\right)z^2 + \cdots.\end{gathered} \tag{3.5}$$

In addition, comparing (3.2) and (3.4), yields

$$\begin{gathered}-q(1+q\eta)a_2\varsigma + \Big(2q\langle 2|q\rangle(1+q\langle 2|q\rangle\eta) - q\left(1+\eta(\langle 2|q\rangle^2-1)\right)a_2^2\\ -q\langle 2|q\rangle(1+q\langle 2|q\rangle\eta)a_3\Big)\varsigma^2 + \cdots = 1 + \tfrac{\langle 2|q\rangle}{4}\imath_1\varsigma + \tfrac{\langle 2|q\rangle}{4}\left(\imath_2 - \tfrac{(3-q)}{4}\imath_1^2\right)\varsigma^2 + \cdots,\end{gathered} \tag{3.6}$$

By equating the corresponding coefficients in equations (3.5) and (3.6), and using the series expansions of δ_1 and δ_2, we obtain the following relations.

$$q(1+q\eta)\,a_2 = \frac{\langle 2|q\rangle}{4}\,\omega_1, \tag{3.7}$$

$$-q(1+q\eta)\,a_2 = \frac{\langle 2|q\rangle}{4}\,\imath_1, \tag{3.8}$$

$$q\langle 2|q\rangle(1+q\langle 2|q\rangle\eta)\,a_3 - q\Big(1+\eta(\langle 2|q\rangle^2-1)\Big)a_2^2 = \frac{\langle 2|q\rangle}{4}\left(\omega_2 - \frac{3-q}{4}\,\omega_1^2\right). \tag{3.9}$$

and

$$\Big(2q\langle 2|q\rangle(1+q\langle 2|q\rangle\eta) - q\left(1+\eta(\langle 2|q\rangle^2-1)\right)\Big)a_2^2 - q\langle 2|q\rangle(1+q\langle 2|q\rangle\eta)a_3 = \tfrac{\langle 2|q\rangle}{4}\left(\imath_2 - \tfrac{(3-q)}{4}\imath_1^2\right) \tag{3.10}$$

It follows directly from (3.7) and (3.8) that

$$\omega_1 = -\imath_1, \qquad \omega_1^2 = \imath_1^2. \tag{3.11}$$

$$2q^2\left(1+q\eta\right)^2 a_2^2 = \tfrac{\langle 2|q\rangle^2}{16}\left(\omega_1^2+\imath_1^2\right) \Longleftrightarrow \quad \omega_1^2+\imath_1^2 = \tfrac{32q^2(1+q\eta)^2}{\langle 2|q\rangle^2}a_2^2. \tag{3.12}$$

By adding equations (3.9) and (3.10), and performing straightforward simplifications, we obtain the following

$$\Big(q+\big(1-\langle 2|q\rangle(1-q)\big)\eta\Big)a_2^2 = \frac{\langle 2|q\rangle}{8q}\left[(\omega_2+\imath_2) + \frac{(3-q)}{4}\left(\omega_1^2+\imath_1^2\right)\right].$$

Substituting the value of $\left(\omega_1^2+\imath_1^2\right)$ from (3.12) into the relation, we obtain

$$a_2^2 = \frac{\langle 2|q\rangle^2\,(\omega_2+\imath_2)}{8q\left[\langle 2|q\rangle\Big(q+\big(1-\langle 2|q\rangle(1-q)\big)\eta\Big) - q(3-q)(1+q\eta)^2\right]}. \tag{3.13}$$

Applying (1.4) to the coefficients ω_2 and $\imath_2$, we obtain

$$|a_2| \leq \frac{\langle 2|q\rangle}{\sqrt{2q\left|\langle 2|q\rangle\Big(q+\big(1-\langle 2|q\rangle(1-q)\big)\eta\Big) - q(3-q)\,(1+q\eta)^2\right|}},$$

By subtracting (3.10) from (3.9), and using $\omega_1^2 = \imath_1^2$, we get

$$2q\langle 2|q\rangle(1+q\langle 2|q\rangle\eta)\left(a_3 - a_2^2\right) = \frac{\langle 2|q\rangle}{4}\,(\omega_2 - \imath_2)\,. \tag{3.14}$$

Then, in view of (3.12), (3.14) becomes

$$a_3 = \frac{\langle 2|q\rangle^2}{32q^2\left(1+q\eta\right)^2}\left(\omega_1^2+\imath_1^2\right) + \frac{\langle 2|q\rangle}{8q\langle 2|q\rangle(1+q\langle 2|q\rangle\eta)}\,(\omega_2-\imath_2)$$

Thus applying (1.4), we conclude that

$$|a_3| \leq \frac{1}{2\,q(1+q\langle 2|q\rangle\eta)} + \frac{\langle 2|q\rangle^2}{4\,q^2\left(1+q\eta\right)^2}.$$

□

Theorem 3.2. *Let $\mu \in \mathbb{R}$ and let f be given by* (1.1)*, belonging to class* $\mathsf{K}_{\Sigma}(\eta, q; \aleph(z;q))$*. Then the Fekete–Szegö functional satisfies the estimate*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \dfrac{\langle 2|q\rangle}{4q\langle 2|q\rangle(1+q\langle 2|q\rangle\eta)}, & \text{if } |1-\mu| \leq \left|\dfrac{\langle 2|q\rangle\Big(q+(1-q)\eta-(1-q)\eta\langle 2|q\rangle\Big)-q(3-q)(1+q\eta)^2}{\langle 2|q\rangle^2(1+q\langle 2|q\rangle\eta)}\right|, \\ \dfrac{\langle 2|q\rangle}{4q}\left|\mathcal{Y}(\mu)\right|, & \text{if } |1-\mu| \geq \left|\dfrac{\langle 2|q\rangle\Big(q+(1-q)\eta-(1-q)\eta\langle 2|q\rangle\Big)-q(3-q)(1+q\eta)^2}{\langle 2|q\rangle^2(1+q\langle 2|q\rangle\eta)}\right|, \end{cases}$$

where

$$\mathcal{Y}(\mu) = \frac{(1-\mu)\,\langle 2|q\rangle}{\langle 2|q\rangle\Big(q+(1-q)\eta-(1-q)\eta\langle 2|q\rangle\Big)-q(3-q)(1+q\eta)^2}.$$

Proof. If $f \in \mathsf{K}_{\Sigma}(\eta, q; \aleph(z;q))$ is represented in the form (1.1), then from (3.13) and (3.14) we deduce that

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{(1-\mu)\,\langle 2|q\rangle^2(\omega_2+\imath_2)}{8q\left[\langle 2|q\rangle\Big(q+(1-q)\eta-(1-q)\eta\langle 2|q\rangle\Big)-q(3-q)(1+q\eta)^2\right]} \\ &\quad + \frac{(1+q)(\omega_2-\imath_2)}{8q\langle 2|q\rangle(1+q\langle 2|q\rangle\eta)} \\ &= \frac{1+q}{8q}\left[\left(\mathcal{Y}(\mu)+\frac{1}{\langle 2|q\rangle(1+q\langle 2|q\rangle\eta)}\right)\omega_2+\left(\mathcal{Y}(\mu)-\frac{1}{\langle 2|q\rangle(1+q\langle 2|q\rangle\eta)}\right)\imath_2\right], \end{aligned}$$

where

$$\mathcal{Y}(\mu) = \frac{(1-\mu)\langle 2|q\rangle}{\left[\langle 2|q\rangle\left(q+\Big(1-\langle 2|q\rangle(1-q)\Big)\eta\right)-q(3-q)\,(1+q\eta)^2\right]}.$$

Then we conclude that

$$\left|a_3 - \mu\, a_2^2\right| \leq \begin{cases} \frac{(\langle 2|q\rangle)}{4q\langle 2|q\rangle(1+q\langle 2|q\rangle\eta)}, & \mathcal{Y}(\mu) \leq \frac{1}{\langle 2|q\rangle(1+q\langle 2|q\rangle\eta)} \\ \frac{\langle 2|q\rangle}{4q}\left|\mathcal{Y}(\mu)\right|, & \mathcal{Y}(\mu) \geq \frac{1}{\langle 2|q\rangle(1+q\langle 2|q\rangle\eta)} \end{cases}$$

This completes the proof of Theorem 3.2. □

Theorems 3.1 and 3.2 yield the following corollary, which is typically associated with Example 2.1, 2.2 and 2.3 .

Corollary 3.1. *Let $f \in \Sigma$ of the form (1.1) be included in the class* $\mathsf{K}_{\Sigma}(0, q; \aleph(z;q))$*. Then*

$$|a_2| \leq \frac{\langle 2|q\rangle}{\sqrt{2q\left|q\langle 2|q\rangle - q(3-q)\right|}}, \qquad |a_3| \leq \frac{1}{2\,q} + \frac{\langle 2|q\rangle^2}{4\,q^2},$$

and

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{1}{4q\langle 2|q\rangle}, & |1-\mu| \leq \frac{|2q(2-q)|}{\langle 2]_q^3}, \\ \\ \frac{\langle 2|q\rangle}{4q}\left|\mathcal{Y}(\mu)\right|, & |1-\mu| \leq \frac{|2q(2-q)|}{\langle 2]_q^3}, \end{cases}$$

where

$$\mathcal{Y}(\mu)=\frac{(1-\mu)\langle 2|q\rangle}{[2q(1-q)]}.$$

Corollary 3.2. *Let $f \in \Sigma$ of the form (1.1) be included in the class $\mathsf{K}_\Sigma(1,q;\aleph(z;q))$. Then*

$$|a_2| \leq \frac{\langle 2|q\rangle}{\sqrt{2q\left|\langle 2|q\rangle\Big(q+\big(1-\langle 2|q\rangle(1-q)\big)\Big)-q(3-q)\,(1+q)^2\right|}}, \qquad |a_3| \leq \frac{1}{2\,q(1+q\langle 2|q\rangle)}+\frac{1}{4\,q^2},$$

and

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{(\langle 2|q\rangle)}{4q\langle 2|q\rangle(1+q\langle 2|q\rangle)}, \ |1-\mu| \leq \left|\frac{1-\langle 2|q\rangle(1-q)-q(3-q)}{\langle 2|q\rangle(1+q\langle 2|q\rangle)}\right| \\ \\ \frac{(\langle 2|q\rangle)}{4q}\left|\mathcal{Y}(\mu)\right|, \ |1-\mu| \leq \left|\frac{1-\langle 2|q\rangle(1-q)-q(3-q)}{\langle 2|q\rangle(1+q\langle 2|q\rangle)}\right| \end{cases}$$

where

$$\mathcal{Y}(\mu)=\frac{(1-\mu)}{\Big(q+\big(1-\langle 2|q\rangle(1-q)\big)\Big)-q(3-q)\,(1+q)}.$$

Corollary 3.3. *Consider f defined as in equation (1.1), which belongs to class $\Sigma_{\mathcal{K}}\left(\aleph(z;q)\right)$. Then, we can state the following*

$$|a_2| \leq \frac{1}{2\sqrt{2}},\ |a_3| \leq \frac{7}{48},\ \textit{and}\ \left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{1}{12},,\ |1-\mu| \leq \frac{2}{3} \\ \\ \frac{1}{8}\left|1-\mu\right|,\ |1-\mu| \geq \frac{2}{3} \end{cases}$$

4. Conclusion

In this paper, we studied a new subclass of bi-univalent functions on the unit disk Δ, introduced in Definition 2.1. This subclass, denoted by $\mathsf{K}_\Sigma(\eta,q;\aleph(z;q))$, was examined in terms of its analytic and geometric behavior. We obtained sharp bounds for the first two Taylor–Maclaurin coefficients, $|a_2|$ and $|a_3|$, and also derived estimates for the Fekete–Szegö functional. These results highlight important structural features of the class and contribute to the ongoing study of coefficient inequalities in geometric function theory.

The work presented here also points to several natural directions for further research. One promising avenue is to study sharp bounds related to the Zalcman conjecture in the setting of bi-univalent functions. Another is to investigate Hankel determinants for subclasses such as bi-convex and bi-close-to-convex functions, which may shed more light on the interplay between coefficient problems and geometric properties. It would also be interesting to extend the current approach to other q-analogue operators, connecting q-calculus with the modern theory of bi-univalent functions. Finally, potential applications of these results could be explored in related areas such as approximation theory and special functions.

In summary, this study adds new results for the class $\mathsf{K}_{\Sigma}(\eta, q; \aleph(z;q))$, while also opening several pathways for future exploration. We believe that pursuing these directions will not only deepen the understanding of bi-univalent functions but also enrich geometric function theory more broadly.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] O. Ahuja, N. Bohra, A. Cetinkaya, S. Kumar, Univalent Functions Associated With the Symmetric Points and Cardioid-Shaped Domain Involving (p,q)-Calculus, Kyungpook Math. J. 61 (2021), 75–98.

[2] R.P. Agarwal, Certain Fractional q-Integrals and q-Derivatives, Math. Proc. Camb. Philos. Soc. 66 (1969), 365–370. https://doi.org/10.1017/S0305004100045060.

[3] A. Alsoboh, A. Amourah, K. Al Mashrafi, T. Sasa, Bi-Starlike and Bi-Convex Function Classes Connected to Shell-Like Curves and the q-Analogue of Fibonacci Numbers, Int. J. Anal. Appl. 23 (2025), 201. https://doi.org/10.28924/2291-8639-23-2025-201.

[4] M. Ahmed, A. Alsoboh, A. Amourah, J. Salah, On the Fractional q-Differintegral Operator for Subclasses of Bi-Univalent Functions Subordinate to q-Ultraspherical Polynomials, Eur. J. Pure Appl. Math. 18 (2025), 6586. https://doi.org/10.29020/nybg.ejpam.v18i3.6586.

[5] A. Alsoboh, M. Darus, New Subclass of Analytic Functions Defined by q-Differential Operator With Respect to k-Symmetric Points, Int. J. Math. Comput. Sci. 14 (2019), 761–773.

[6] A. Alsoboh, A. Amourah, O. Alnajar, M. Ahmed, T.M. Seoudy, Exploring q-Fibonacci Numbers in Geometric Function Theory: Univalence and Shell-Like Starlike Curves, Mathematics 13 (2025), 1294. https://doi.org/10.3390/math13081294.

[7] A. Alsoboh, A. Amourah, M. Darus, C.A. Rudder, Investigating New Subclasses of Bi-Univalent Functions Associated with q-Pascal Distribution Series Using the Subordination Principle, Symmetry 15 (2023), 1109. https://doi.org/10.3390/sym15051109.

[8] A. Alsoboh, A. Amourah, F.M. Sakar, O. Ogilat, G.M. Gharib, et al., Coefficient Estimation Utilizing the Faber Polynomial for a Subfamily of Bi-Univalent Functions, Axioms 12 (2023), 512. https://doi.org/10.3390/axioms12060512.

[9] A. Alsoboh, G.I. Oros, A Class of Bi-Univalent Functions in a Leaf-Like Domain Defined Through Subordination via q-Calculus, Mathematics 12 (2024), 1594. https://doi.org/10.3390/math12101594.

[10] A. Amourah, A. Alsoboh, D. Breaz, S.M. El-Deeb, A Bi-Starlike Class in a Leaf-Like Domain Defined Through Subordination via q-Calculus, Mathematics 12 (2024), 1735. https://doi.org/10.3390/math12111735.

[11] A. Amourah, A. Alsoboh, J. Salah, K. Al Kalbani, Bounds on Initial Coefficients for Bi-Univalent Functions Linked to q-Analog of le Roy-Type Mittag-Leffler Function, WSEAS Trans. Math. 23 (2024), 714–722. https://doi.org/10.37394/23206.2024.23.73.

[12] T. Al-Hawary, A. Amourah, A. Alsoboh, O. Ogilat, I. Harny, and M. Darus, Applications of q-Ultraspherical polynomials to bi-univalent functions defined by q-Saigo's fractional integral operators, AIMS Math., 9, 2024, 17063–17075.

[13] T. Al-Hawary, A. Amourah, A. Alsoboh, A. M. Freihat, O. Ogilat, I. Harny, M. Darus, Subclasses of Yamakawa-Type Bi-Starlike Functions Subordinate to Gegenbauer Polynomials Associated With Quantum Calculus, Results Nonlinear Anal. 7 (2024), 75–83.

[14] O. Al-Refai, A. Amourah, T. Al-Hawary, B.A. Frasin, A New Method for Estimating General Coefficients to Classes of Bi-Univalent Functions, J. Funct. Spaces 2024 (2024), 9889253. https://doi.org/10.1155/2024/9889253.

[15] A. Amourah, O. Alnajar, M. Darus, A. Shdouh, O. Ogilat, Estimates for the Coefficients of Subclasses Defined by the Bell Distribution of Bi-Univalent Functions Subordinate to Gegenbauer Polynomials, Mathematics 11 (2023), 1799. https://doi.org/10.3390/math11081799.

[16] F. Yousef, A.A. Amourah, M. Darus, Differential Sandwich Theorems for p-Valent Functions Associated With a Certain Generalized Differential Operator and Integral Operator, Ital. J. Pure Appl. Math. 36 (2016), 543–556.

[17] A.A. Amourah, F. Yousef, Some Properties of a Class of Analytic Functions Involving a New Generalized Differential Operator, Bol. Soc. Parana. Mat. 38 (2019), 33–42. https://doi.org/10.5269/bspm.v38i6.40530.

[18] L. Andrei, V. Caus, Subordinations Results on a q-Derivative Differential Operator, Mathematics 12 (2024), 208. https://doi.org/10.3390/math12020208.

[19] P.L. Duren, Univalent Functions, Grundlehren der Mathematischen Wissenschaften, Vol. 259, Springer, 1983.

[20] S.H. Hadi, T.G. Shaba, Z.S. Madhi, M. Darus, A.A. Lupaş, et al., Boundary Values of Hankel and Toeplitz Determinants for q-Convex Functions, MethodsX 13 (2024), 102842. https://doi.org/10.1016/j.mex.2024.102842.

[21] S.H. Hadi, M. Darus, B. Alamri, Ş. Altınkaya, A. Alatawi, On Classes of z-Uniformly q-Analogue of Analytic Functions with Some Subordination Results, Appl. Math. Sci. Eng. 32 (2024), 2312803. https://doi.org/10.1080/27690911.2024.2312803.

[22] A. Alatawi, M. Darus, The Fekete–Szegö Inequality for a Subfamily of q-Analogue Analytic Functions Associated with the Modified q-Opoola Operator, Asian-Eur. J. Math. 17 (2024), 2450027. https://doi.org/10.1142/s179355712450027x.

[23] M. Fekete, G. Szegö, Eine Bemerkung Über Ungerade Schlichte Funktionen, J. Lond. Math. Soc. s1-8 (1933), 85–89. https://doi.org/10.1112/jlms/s1-8.2.85.

[24] G. Gasper, M. Rahman, Basic Hypergeometric Series, 2nd ed., Cambridge University Press, 2004. https://doi.org/10.1017/CBO9780511526251.

[25] M.F. Khan, M. AbaOud, New Applications of Fractional q-Calculus Operator for a New Subclass of q-Starlike Functions Related with the Cardioid Domain, Fractal Fract. 8 (2024), 71. https://doi.org/10.3390/fractalfract8010071.

[26] W. Ma, A Unified Treatment of Some Special Classes of Univalent Functions, in: Proceedings of the Conference on Complex Analysis, Tianjin, China, pp. 157–169, 1992.

[27] S.S. Miller, Differential Inequalities and Carathéodory Functions, Bull. Am. Math. Soc. 81 (1975), 79–81.

[28] B. Khan, H.M. Srivastava, N. Khan, M. Darus, M. Tahir, et al., Coefficient Estimates for a Subclass of Analytic Functions Associated with a Certain Leaf-Like Domain, Mathematics 8 (2020), 1334. https://doi.org/10.3390/math8081334.

[29] K. Piejko, J. Sokół, On the Convolution and Subordination of Convex Functions, Appl. Math. Lett. 25 (2012), 448–453. https://doi.org/10.1016/j.aml.2011.09.034.

[30] M. Hari Priya, R. Sharma, On a Class of Bounded Turning Functions Subordinate to a Leaf-Like Domain, J. Phys.: Conf. Ser. 1000 (2018), 012056. https://doi.org/10.1088/1742-6596/1000/1/012056.

[31] T.M. Seoudy, M.K. Aouf, Coefficient Estimates of New Classes of q-Starlike and q-Convex Functions of Complex Order, J. Math. Inequal. 10 (2016), 135–145. https://doi.org/10.7153/jmi-10-11.

[32] G. Singh, C. Kaur, Starlike and Convex Functions Subordinate to Leaf-Like Domain, Turk. J. Comput. Math. Educ. 12 (2021), 6098–6102.

[33] J. Sokół, J. Stankiewicz, Radius of Convexity of Some Subclasses of Strongly Starlike Functions, Zesz. Nauk. Politech. Rzeszowskiej Mat. 19 (1996), 101–105.

[34] Q.A. Shakir, A.S. Tayyah, D. Breaz, L. Cotîrlă, E. Rapeanu, et al., Upper Bounds of the Third Hankel Determinant for Bi-Univalent Functions in Crescent-Shaped Domains, Symmetry 16 (2024), 1281. https://doi.org/10.3390/sym16101281.

[35] A.S. Tayyah, W.G. Atshan, Starlikeness and Bi-Starlikeness Associated with a New Carathéodory Function, J. Math. Sci. 290 (2025), 232–256. https://doi.org/10.1007/s10958-025-07604-8.

[36] A.S. Tayyah, W.G. Atshan, G.I. Oros, Third-Order Differential Subordination Results for Meromorphic Functions Associated with the Inverse of the Legendre Chi Function via the Mittag-Leffler Identity, Mathematics 13 (2025), 2089. https://doi.org/10.3390/math13132089.

[37] S.A. AL-Ameedee, W.G. Atshan, F.A. AL-Maamori, Second Hankel Determinant for Certain Subclasses of Bi-Univalent Functions, J. Phys.: Conf. Ser. 1664 (2020), 012044. https://doi.org/10.1088/1742-6596/1664/1/012044.

[38] S.A. Al-Ameedee, W. Galib Atshan, F. Ali Al-Maamori, Coefficients Estimates of Bi-Univalent Functions Defined by New Subclass Function, J. Phys.: Conf. Ser. 1530 (2020), 012105. https://doi.org/10.1088/1742-6596/1530/1/012105.

[39] W. Galib Atshan, E. Ibraham Badawi, Results on Coefficient Estimates for Subclasses of Analytic and Bi-Univalent Functions, J. Phys.: Conf. Ser. 1294 (2019), 032025. https://doi.org/10.1088/1742-6596/1294/3/032025.

[40] M.H. Darassi, O. Yasin, M. Ahmed, A Semi-Analytical Method to Solve the Fitzhugh-Nagumo Equation, J. Interdiscip. Math. 28 (2025), 1489–1504. https://doi.org/10.47974/jim-2064.

[41] M. Ahmed, Universal Covariant Representations and Positive Elements, Azerbaijan J. Math. 15 (2025), 44–52. https://doi.org/10.59849/2218-6816.2025.1.44.

[42] M. Ahmed, F. Moh'd, The Graded Annihilating Submodule Graph, AKCE Int. J. Graphs Comb. (2025). https://doi.org/10.1080/09728600.2025.2492075.

[43] M. Ahmed, Amenable Quase-Lattice Ordered Groups and True Representations, Bol. Soc. Parana. Mat. 41 (2022), 1–10. https://doi.org/10.5269/bspm.62552.

[44] F. Moh'd, M. Ahmed, M. Refai, Flexible Modules and Graded Rings, Bol. Soc. Parana. Mat. 41 (2022), 1–14. https://doi.org/10.5269/bspm.62550.

[45] Y.A. Al-Qudah, F.A. Al-Sharqi, M. Mishlish, M.M. Rasheed, Hybrid Integrated Decision-Making Algorithm Based on AO of Possibility Interval-Valued Neutrosophic Soft Settings, Int. J. Neutrosophic Sci. 22 (2023), 84–98. https://doi.org/10.54216/ijns.220306.

[46] A. Yousef, K. Alhazaymeh, N. Hassan, H. Qoqazeh, M. Almousa, et al., Transitive Closure of Vague Soft Set Relations and Its Operators, Int. J. Fuzzy Log. Intell. Syst. 22 (2022), 59–68. https://doi.org/10.5391/ijfis.2022.22.1.59.

[47] H. Qoqazeh, Y. Al-Qudah, M. Almousa, A. Jaradat, on D-Compact Topological Spaces, J. Appl. Math. Inform. 39 (2021), 883–894. https://doi.org/10.14317/JAMI.2021.883.