International Journal of Analysis and Applications
ISSN 2291-8639
Volume 7, Number 2 (2015), 162-170
http://www.etamaths.com

ON THE ITERATED EXPONENT OF CONVERGENCE OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

ABDALLAH EL FARISSI

Abstract. In this paper, we investigate the relationship between solutions and their derivatives of the differential equation $f^{(k)}+A_{k-1}f^{(k-1)}+...+A_0f = 0$ for $k \geq 2$ and small functions, where A_j $(j = 0, 1, ..., k-1)$ are meromorphic functions of finite iterated p-order.

1. Introduction and main results

In this paper, a meromorphic function will mean meromorphic in the whole complexe plan. We shall use the standard notations in Nevanlinna value distribution of meromorphic functions [9, 11] such as $T(r,f), N(r,f), m(r,f)$. For the definition of the iterated order of a meromorphic function, we use the same definition as in [10], [2, p. 317], [9, p. 129]. For all $r \in \mathbb{R}$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp\left(\exp_p r\right)$, $p \in \mathbf{N}$. We also define for all r sufficiently large $\log_1 r := \log r$ and $\log_{p+1} r := \log\left(\log_p r\right)$, $p \in \mathbf{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$.

Definition 1.1: ([10], [11]) Let f be a meromorphic function. The iterated $p-$order $\rho_p(f)$ of f is defined by

$$\rho_p(f) = \overline{\lim_{r\to+\infty}} \frac{\log_p T(r,f)}{\log r} \qquad (p \geqslant 1 \text{ is an integer}), \tag{1.1}$$

For $p = 1$, this notation is called order, and for $p = 2$ hyper-order.

Definition 1.2: (*see* [10]) The finiteness degree of the order of a meromorphic function f is defined by

(1.2)
$$i(f) = \begin{cases} 0 & \text{for } f \text{ rational,} \\ \min\{n \in \mathbf{N} : \rho_j(f) < +\infty\} & \text{for } f \text{ transcendental for wich same } n \in \mathbb{N} \text{ with } \rho_n(f) < +\infty \text{ exists} \\ \infty & \text{for } f \text{ with } \rho_n(f) = +\infty \text{ for all } n \in \mathbb{N}. \end{cases}$$

2010 *Mathematics Subject Classification.* 34M10, 30D35.
Key words and phrases. linear differential equations; meromorphic functions; iterated $p-$order; iterated exponent of convergence of the sequence of distinct zeros.

Definition 1.3: (*see* [10]) Let f be a meromorphic function. The iterated convergence exponent of the sequence of zeros of $f(z)$ is defined by

$$\lambda_p(f) = \overline{\lim_{r\to+\infty}} \frac{\log_p N\left(r, \frac{1}{f}\right)}{\log r}\ ;\ p \geqslant 1 \text{ is an integer}, \tag{1.3}$$

where $N\left(r, \frac{1}{f}\right)$ is the counting function of zeros of $f(z)$ in $\{|z| < r\}$. Similary the iterated convergence exponent of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_p(f) = \overline{\lim_{r\to+\infty}} \frac{\log_p \overline{N}\left(r, \frac{1}{f}\right)}{\log r}\ ;\ p \geqslant 1 \text{ is an integer}, \tag{1.3}$$

where $\overline{N}\left(r, \frac{1}{f}\right)$ is the counting function of distinct zeros of $f(z)$ in $\{|z| < r\}$.

Definition 1.5 [10] The growth index of the iterated convergence exponent of the sequence of zero points of a meromorphic function f with iterated order is defined by

$$i_\lambda(f) = \begin{cases} 0 & \text{if } n\left(r, \frac{1}{f}\right) = O(\log r), \\ \min\{n \in \mathbb{N} : \lambda_n(f) < \infty\} & \text{if } \lambda_n(f) < \infty \text{ for some } n \in \mathbb{N}, \\ \infty & \text{if } \lambda_n(f) = \infty \text{ for all } n \in \mathbb{N}. \end{cases}$$

Similarly, we can define the growth index $i_{\overline{\lambda}}(f)$ of $\overline{\lambda}_p(f)$.

For $k \geqslant 2$, we consider the linear differential equation

$$f^{(k)} + A(z) f = 0, \tag{1.5}$$

where $A(z)$ is a transcendental meromorphic function of finite iterated order $\rho_p(A) = \rho > 0$. In [14], Wang and Lü have investigated the fixed points and hyper-order of solutions of second order linear differential equations with meromorphic coefficients and their derivatives. They have obtained the following result:

Theorem A [14] *Suppose that* $A(z)$ *is a transcendental meromorphic function satisfying* $\delta(\infty, A) = \underline{\lim}_{r\to+\infty} \frac{m(r,A)}{T(r,A)} = \delta > 0$, $\rho(A) = \rho < +\infty$. *Then every meromorphic solution* $f(z) \not\equiv 0$ *of the equation*

$$f'' + A(z) f = 0 \tag{1.6}$$

satisfies that f, f' *and* f'' *have infinitely many fixed points and*

$$\overline{\tau}(f) = \overline{\tau}\left(f'\right) = \overline{\tau}\left(f''\right) = \rho(f) = +\infty, \tag{1.7}$$

$$\overline{\tau}_2(f) = \overline{\tau}_2\left(f'\right) = \overline{\tau}_2\left(f''\right) = \rho_2(f) = \rho. \tag{1.8}$$

Theorem A has been generalized to higher order differential equations by Liu and Zhang as follows (see [12]):

Theorem B [12] *Suppose that* $k \geqslant 2$ *and* $A(z)$ *is a transcendental meromorphic function satisfying* $\delta(\infty, A) = \delta > 0$, $\rho(A) = \rho < +\infty$. *Then every meromorphic solution* $f(z) \not\equiv 0$ *of* (1.4), *satisfies that* f *and* $f', f'', ..., f^{(k)}$ *all have infinitely many fixed points and*

$$\overline{\tau}(f) = \overline{\tau}\left(f'\right) = \overline{\tau}\left(f''\right) = ... = \overline{\tau}\left(f^{(k)}\right) = \rho(f) = +\infty, \tag{1.9}$$

$$\overline{\tau}_2(f) = \overline{\tau}_2\left(f'\right) = \overline{\tau}_2\left(f''\right) = ... = \overline{\tau}_2\left(f^{(k)}\right) = \rho_2(f) = \rho. \tag{1.10}$$

Theorem A and B have been generalized by B. Belaïdi for iterated p-order (see [1]):

Theorem C [1] *Let* $k \geqslant 2$ *and* $A(z)$ *be transcendental meromorphic function of finite iterated order* $\rho_p(A) = \rho > 0$ *such that* $\delta(\infty, A) = \delta > 0$. *Suppose, moreover, that either:*

(i) *all poles of* f *are of uniformly multiplicity or that*

(ii) $\delta(\infty, f) > 0$.

If $\varphi \not\equiv 0$ *is a meromorphic function with finite* iterated $p-$order $\rho_p(\varphi) < +\infty$, *then every meromorphic solution* $f(z) \not\equiv 0$ *of* (1.5), *satisfies*

$$\overline{\lambda}_p(f - \varphi) = \overline{\lambda}_p\left(f' - \varphi\right) = ... = \overline{\lambda}_p\left(f^{(k)} - \varphi\right) = \rho_p(f) = +\infty, \tag{1.11}$$

and

$$\overline{\lambda}_{p+1}(f - \varphi) = \overline{\lambda}_{p+1}\left(f' - \varphi\right) = ... = \overline{\lambda}_{p+1}\left(f^{(k)} - \varphi\right) = \rho_{p+1}(f) = \rho. \tag{1.12}$$

For $k \geqslant 2$, we consider the linear differential equation

$$f^{(k)} + A_{k-1}f^{(k-1)} + ... + A_0 f = 0, k \geqslant 2, \tag{1.13}$$

Recently Bouabdelli and Belaïdi [3] investigate the relationship between small functions and derivative of solutions of equation (1.13) and obtain some theorems which extended the previous results given by Xu, Tu and Zheng (*see* [15]).

Theorem D [3] *Let* $k \geqslant 2$ *and* $(A_j)_{j=0,1,2,...k-1}$ *be entire functions of finite iterated order with* $i(A_0) = p$ $(0 < p < \infty)$ *and satisfy one of the following conditions:*

(*i*) $\max\{\rho_p(A_j), (j = 1, ..., k-1)\} < \rho_p(A_0) = \rho < +\infty$,

(*ii*) $\max\{\rho_p(A_j), (j = 1, ..., k-1)\} \leqslant \rho_p(A_0) = \rho$ $(0 < \rho < \infty)$ *and* $\max\{\sigma_p(A_j), \rho_p(A_j) = \rho_p(A_0)\} < \sigma_p(A_0) = \sigma$, $(0 < \sigma < \infty)$,

then for every solution $f \not\equiv 0$ *of* (1.13) *and for any entire function* $\varphi \not\equiv 0$ *satisfying* $\rho_{p+1}(\varphi) < \rho$, *we have*

$$\overline{\lambda}_{p+1}\left(f^{(i)} - \varphi\right) = \overline{\lambda}_{p+1}\left(f^{(i)} - \varphi\right) = \rho_{p+1}(f) = \rho, i \in \mathbb{N}.$$

Theorem E [3] *Let* $k \geqslant 2$ *and* $(A_j)_{j=0,1,2,...k-1}$ *be meromorphic functions of finite iterated order with* $i(A_0) = p$ $(0 < p < \infty)$ *satisfying*
$\max\{\rho_p(A_j), (j = 1, ..., k-1)\} < \rho_p(A_0) = \rho < +\infty$ *and* $\delta(\infty, A_0) > 0$. *Then for every meromorphic solution* $f \not\equiv 0$ *of* (1.13) *whose poles are of uniformly bounded multiplicity and for any meromorphic function* $\varphi \not\equiv 0$ *satisfying* $\rho_{p+1}(\varphi) < \rho$, *we have*

$$\overline{\lambda}_{p+1}\left(f^{(i)} - \varphi\right) = \overline{\lambda}_{p+1}\left(f^{(i)} - \varphi\right) = \rho_{p+1}(f) = \rho, i \in \mathbb{N}.$$

In all previous theorems we note that, the conditions on the coefficients gives us that any solution of the equation (1.13) is of infinite $p-$order and the same $(p+1)-$order. And there are several papers where the authors show that on certain conditions all solutions of the equation of infinite $p-$order and the same $(p+1)-$order (see $[2], [6], [7], [12]...$). The question that arises is: If any solution of the equation is of infinite $p-$order and the same $(p+1)-$order, is that we have the same results?.

In this paper we give an answer of above question and we prove the following theorems:

Theorem 1.1 *Let $k \geqslant 2$ and $(A_j)_{j=0,1,2,...k-1}$ be meromorphic functions of finite $p-$order. Suppose that all solution of the equation* (1.13) *of infinite $p-$order and $\rho_{p+1}(f) = \rho$. If $\varphi \not\equiv 0$ is a meromorphic function with $i(\varphi) < p+1$ or $\rho_{p+1}(\varphi) < \rho$, then every meromorphic solution $f \not\equiv 0$ of* (1.13) *satisfies*

$$i_{\overline{\lambda}}\left(f^{(i)} - \varphi\right) = i_{\lambda}\left(f^{(i)} - \varphi\right) = i(f) = p+1,\ i \in \mathbb{N} \tag{1.14}$$

and

$$\overline{\lambda}_{p+1}\left(f^{(i)} - \varphi\right) = \overline{\lambda}_{p+1}\left(f^{(i)} - \varphi\right) = \rho_{p+1}(f) = \rho,\ i \in \mathbb{N}. \tag{1.15}$$

Theorem 1.2 *Let $k \geqslant 2$ and $(A_j)_{j=0,1,2,...k-1}$ be meromorphic functions of finite $p-$order. Suppose that all solution of the equation* (1.13) *of infinite $p-$order. If $\varphi \not\equiv 0$ is an meromorphic function with $\rho_p(\varphi) < +\infty$, then every meromorphic solution $f \not\equiv 0$ of* (1.13) *satisfies*

$$\overline{\lambda}_p\left(f^{(i)} - \varphi\right) = \overline{\lambda}_p\left(f^{(i)} - \varphi\right) = \rho_p(f) = \infty,\ i \in \mathbb{N}. \tag{1.16}$$

Remark 1.2 The proof of Theorems 1.1, 1.2 are quite different from that in the proof of Theorem D and E (see [3]) we give a simple proof of theorems in the paper. The main ingredient in the proof is Lemma 2.5.

Corollary 1.1 *Under the assumptions of Theorem 1.1, if $\varphi(z) = z$, then for every meromorphic solution f of* (1.13)*, we have*

$$i_{\overline{\tau}}\left(f^{(i)}\right) = i_{\tau}\left(f^{(i)}\right) = i(f) = p+1,\ i \in \mathbb{N} \tag{1.17}$$

and

$$\overline{\tau}_{p+1}\left(f^{(i)}\right) = \tau_{p+1}\left(f^{(i)}\right) = \rho_{p+1}(f) = \rho_p(A_0) = \rho,\ i \in \mathbb{N}. \tag{1.18}$$

Corollary 1.2 *Suppose that $k \geqslant 2$ and $A(z)$ is a transcendental meromorphic function such that $0 < \rho_p(A) = \rho < +\infty$. If $\varphi \not\equiv 0$ is meromorphic function with $i(\varphi) < p+1$ or $\rho_{p+1}(\varphi) < \rho$, then every solution $f \not\equiv 0$ of* (1.5) *satisfies* (1.14) *and* (1.15).

Corollary 1.3 *Let $k \geqslant 2$ and $(A_j)_{j=0,1,2,...k-1}$ be entire functions of finite iterated p-order such that $i(A_0) = p; 0 < p < \infty$. Suppose that*
$\max\{i(A_j), (j = 1, ..., k-1)\} < i(A_0)$ *or*
$\max\{\rho_p(A_j), (j = 1, ..., k-1)\} < \rho_p(A_0) < +\infty$. *If $\varphi \not\equiv 0$ is an entire function with $i(\varphi) < p+1$ or $\rho_{p+1}(\varphi) < \rho_p(A_0)$, then every solution $f \not\equiv 0$ of* (1.13) *satisfies* (1.14) and (1.15).

For $p = 1$ in Theorem 1.1 we gat the following corollary (see [7])

Corollary 1.4 [7] *Let $k \geqslant 2$ and A_j $(j = 0, 1, ..., k-1)$ be meromorphic functions of finite order such that all solution of equation* (1.13) *satisfy $\rho(f) = +\infty$ and $\rho_2(f) = \rho$. Then if $\varphi \not\equiv 0$ is an meromorphic function with $\rho_2(\varphi) < \rho$, then every solution $f \not\equiv 0$ of* (1.13) *satisfies*

$$\overline{\lambda}\left(f^{(i)} - \varphi\right) = \lambda\left(f^{(i)} - \varphi\right) = \rho(f) = +\infty, \ i \in \mathbb{N} \tag{1.19}$$

and

$$\overline{\lambda}_2\left(f^{(i)} - \varphi\right) = \lambda_2\left(f^{(i)} - \varphi\right) = \rho_2(f) = \rho, \ i \in \mathbb{N}. \tag{1.20}$$

Remark 1.3: Theorem 1.1 is the improvement of theorems A, B, C and D and Theorem 1.2 is the improvement of theorem E.

2. Auxiliary Lemmas

To prove our main results, we need the following lemmas.

Lemma 2.1 [5] *Suppose that $A_0, A_1, ..., A_{k-1}, F(\not\equiv 0)$ are meromorphic functions and let f be a meromorphic solution of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + ... + A_1 f' + A_0 f = F, \tag{2.1}$$

such that $i(f) = \rho + 1$ $(0 < p < \infty)$. If either
$\max\{i(F), i(A_j) \ \ j = 0, 1, ..., k-1\} < p+1$
or
$\max\{\rho_{p+1}(F), \rho_{p+1}(A_j) \ \ j = 0, 1, ..., k-1\} < \rho_{p+1}(f)$,
then we have $i_{\overline{\lambda}}(f) = i_\lambda(f) = i(f) = p+1$ and $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f)$.

Lemma 2.2 (see Remark 1.3 of [10]). *If f is a meromorphic function with $i(f) = p$, then $\rho_p\left(f'\right) = \rho_p(f)$.*

Lemma 2.3 [10] *Let $k \geqslant 2$ and A_j $(j = 0, 1, ..., k-1)$ be entire functions of finite iterated p-order such that $i(A_0) = p, (0 < p < \infty)$.*
Suppose that $\max\{i(A_j), (j = 1, ..., k-1)\} < i(A_0)$ *or*
$\max\{\rho_p(A_j), (j = 1, ..., k-1)\} < \rho_p(A_0) < +\infty$, *then every solution $f \not\equiv 0$ of* (1.13) *satisfies $i(f) = p+1$* and $\rho_{p+1}(f) = \rho_p(A_0)$.

Let A_j $(j = 0, 1, ..., k-1)$ be a functions. We define the following sequence of functions:

$$(2.2)\qquad \begin{cases} A_j^0 = A_j, & j = 0,1,...,k-1 \\ A_{k-1}^i = A_{k-1}^{i-1} - \dfrac{\left(A_0^{i-1}\right)'}{A_0^{i-1}}, & i \in \mathbb{N} \\ A_j^i = A_j^{i-1} + A_{j+1}^{i-1}\dfrac{\left(\Psi_{j+1}^{i-1}\right)'}{\Psi_{j+1}^{i-1}}, & j = 0,1,...,k-2,\ i \in \mathbb{N}, \end{cases}$$

where $\Psi_{j+1}^{i-1} = \dfrac{A_{j+1}^{i-1}}{A_0^{i-1}}$.

Remark 2.1: In the case where one of functions A_j^i $(j = 0,1,...,k-1)$ is equal to zero then $A_j^{i+1} = A_{j-1}^i\,(j = 0,1,...,k-1)$.

Lemma 2.4 *Suppose that f is a solution of (1.13). Then $g_i = f^{(i)}$ is a solution of the equation*

$$(2.3)\qquad g_i^{(k)} + A_{k-1}^i g_i^{(k-1)} + ... + A_0^i g_i = 0,$$

where A_j^i $(j = 0,1,...,k-1)$ are given by (2.2).

Proof: Assume that f is a solution of equation (1.13) and let $g_i = f^{(i)}$. We prove that g_i is an entire solution of the equation (2.3). Our proof is by induction: For $i = 1$, differentiating both sides of (1.13), we obtain

$$(2.4)\quad f^{(k+1)} + A_{k-1}f^{(k)} + \left(A_{k-1}' + A_{k-2}\right)f^{(k-1)} + ... + \left(A_1' + A_0\right)f' + A_0' f = 0,$$

and replacing f by

$$f = -\frac{(f^{(k)} + A_{k-1}f^{(k-1)} + ... + A_1 f')}{A_0},$$

we get

$$f^{(k+1)} + \left(A_{k-1} - \frac{A_0'}{A_0}\right)f^{(k)} + \left(A_{k-1}' + A_{k-2} - A_{k-1}\frac{A_0'}{A_0}\right)f^{(k-1)}...$$

$$+ \left(A_1' + A_0 - A_1\frac{A_0'}{A_0}\right)f' = 0.$$

That is

$$g_1^{(k)} + A_{k-1}^1 g_1^{(k-1)} + A_{k-2}^1 g_1^{(k-2)} ... + A_0^1 g_1 = 0.$$

Suppose that the assertion is true for the values which are strictly smaller than a certain i. We suppose g_{i-1} is a solution of the equation

$$(2.5)\qquad g_{i-1}^{(k)} + A_{k-1}^{i-1}g_{i-1}^{(k-1)} + A_{k-2}^{i-1}g_{i-1}^{(k-2)}... + A_0^{i-1}g_{i-1} = 0.$$

Differentiating both sides of (2.5), we can write

$$g_{i-1}^{(k+1)} + A_{k-1}^{i-1}g_{i-1}^{(k)} + \left(\left(A_{k-1}^{i-1}\right)' + A_{k-2}\right)g_{i-1}^{(k-1)} + ...$$

$$(2.6)\qquad + \left(\left(A_1^{i-1}\right)' + A_0^{i-1}\right)g_{i-1}' + A_0' g_{i-1} = 0.$$

In (2.6), replacing g_{i-1} by

$$g_{i-1}=-\frac{(g_{i-1}^{(k)}+A_{k-1}^{i-1}g_{i-1}^{(k-1)}+A_{k-2}^{i-1}g_{i-1}^{(k-2)}...+A\left(g_{i-1}\right)^{'})}{A_{0}^{i-1}},$$

yields

$$g_{i-1}^{(k+1)}+\left(A_{k-1}^{i-1}-\frac{\left(A_{0}^{i-1}\right)^{'}}{A_{0}^{i-1}}\right)g_{i-1}^{(k)}+\left(\left(A_{k-1}^{i-1}\right)^{'}+A_{k-2}-A_{k-1}^{i-1}\frac{\left(A_{0}^{i-1}\right)^{'}}{A_{0}^{i-1}}\right)g_{i-1}^{(k-1)}...+$$

$$+\left(\left(A_{1}^{i-1}\right)^{'}+A_{0}^{i-1}-A_{1}^{i-1}\frac{\left(A_{0}^{i-1}\right)^{'}}{A_{0}^{i-1}}\right)g_{i-1}^{'}=0. \tag{2.7}$$

That is

$$g_{i}^{(k)}+A_{k-1}^{i}g_{i}^{(k-1)}+A_{k-2}^{i}g_{i}^{(k-2)}...+A_{0}^{i}g_{i}=0.$$

Lemma 2.4 is thus proved.

Lemma 2.5 *Let* A_j $(j=0,1,...,k-1)$ *be meromorphic functions of finite order such that all solution of equation* (1.13) has infinit $p-$order and $\rho_{p+1}(f)=\rho$. *And let* $A_j^i, (j=0,1,...,k-1)$ *be defined as in* (2.2). *Then all nontrivial meromorphic solution* g *of the equation*

$$g^{(k)}+A_{k-1}^{i}g^{(k-1)}+...+A_{0}^{i}g=0,\ k\geqslant 2 \tag{2.8}$$

satisfies: $\rho_p(g)=+\infty$ *and* $\rho_{p+1}(g)=\rho$.

Proof: Let $\{f_1,f_2,...,f_k\}$ be a fundamental system of solutions of (1.13). We show that $\left\{f_1^{(i)},f_2^{(i)},...,f_k^{(i)}\right\}$ is a fundamental system of solutions of (2.8). By Lemma 2.4, it follows that $f_1^{(i)},f_2^{(i)},...,f_k^{(i)}$ is solutions of (2.8). Let $\alpha_1,\alpha_2,...,\alpha_k$ be constants such that

$$\alpha_1f_1^{(i)}+\alpha_2f_2^{(i)}+...+\alpha_kf_k^{(i)}=0.$$

Then, we have

$$\alpha_1f_1+\alpha_2f_2+...+\alpha_kf_k=P(z),$$

where $P(z)$ is a polynomial of degree less than i. Since $\alpha_1f_1+\alpha_2f_2+...+\alpha_kf_k$ is a solution of (1.13), then P is a solution of (1.13), and by the conditions of the Lemma 2.5, we conclude that P is an infinite solution of (1.13); this leads to a contradiction. Therefore, P is a trivial solution. We deduce that $\alpha_1f_1+\alpha_2f_2+...+\alpha_kf_k=0$. Using the fact that $\{f_1,f_2,...,f_k\}$ is a fundamental solution of (1.13), we get $\alpha_1=\alpha_2=...=\alpha_k=0$. Now, let g be a non trivial solution of (2.8). Then, using the fact that $\left\{f_1^{(i)},f_2^{(i)},...,f_k^{(i)}\right\}$ is a fundamental solution of (2.8), we claim that there exist constants $\alpha_1,\alpha_2,...,\alpha_k$ not all equal to zero, such that $g=\alpha_1f_1^{(i)}+\alpha_2f_2^{(i)}+...+\alpha_kf_k^{(i)}$. Let $h=\alpha_1f_1+\alpha_2f_2+...+\alpha_kf_k$, h be a solution of (1.13) and $h^{(i)}=g$. Hence, by Lemma 2.2, we have $\rho_{p+1}(h)=\rho_{p+1}(g)$, and by the conditions of the Lemma 2.5, we have $\rho_p(h)=\rho_p(g)=+\infty$ and $\rho_{p+1}(h)=\rho_{p+1}(g)=\rho$.

3. **Proof of Theorems and corollary 1.3**

Firstily we proof the theorem 1.1

Proof of Theorem 1.1

Assume that f is a solution of equation (1.13). By the conditions of theorem 1.1, we can write $i(f) = p + 1$, $\rho_{p+1}(f) = \rho$. Taking $g_i = f^{(i)}$, then, using Lemma 2.2, we have $i(g_i) = p + 1$, $\rho_{p+1}(g_i) = \rho$. Now, let $w(z) = g_i(z) - \varphi(z)$, where φ is a meromorphic function with $\rho_{p+1}(\varphi) < \rho_p(A_0)$.
Then $i(w) = i(g_i) = p + 1$, and $\rho_{p+1}(w) = \rho_{p+1}(g_i) = \rho_{p+1}(f) = \rho(A_0)$.
In order to prove $i_{\overline{\lambda}}(g_i - \varphi) = i_\lambda(g_i - \varphi) = p+1$ and $\overline{\lambda}_{p+1}(g_i - \varphi) = \lambda_{p+1}(g_i - \varphi) = \rho(A_0)$, we need to prove only $i_{\overline{\lambda}}(w) = i_\lambda(w) = p + 1$ and $\overline{\lambda}_{p+1}(w) = \rho(A_0)$. Using the fact that $g_i = w + \varphi$, and by Lemma 2.4 we get

$$(3.1)\quad w^{(k)} + A_{k-1}^i w^{(k-1)} + ... + A_0^i w = -\left(\varphi^{(k)} + A_{k-1}^i \varphi^{(k-1)} + ... + A_0^i \varphi\right) = F.$$

By $\rho_p\left(A_j^i\right) < \infty$, $\rho_{p+1}(\varphi) < \rho$ and Lemma 2.5, we get $F \not\equiv 0$ and $\rho_{p+1}(F) < \infty$. By Lemma 2.1 $i_{\overline{\lambda}}(w) = i_\lambda(w) = p + 1$ and $\overline{\lambda}_{p+1}(w) = \lambda_{p+1}(w) = \rho_{p+1}(w) = \rho(A_0)$. the proof of theorem 1.1 is complete.

Proof of Theorem 1.2

By the same reasoning as before we can prove Theorem 1.2.

Proof of corollary 1.3

By Lemma 2.3 we get $i(f) = p + 1$ and $\rho_{p+1}(f) = \rho_p(A_0)$ appliying theorem 1.2 we can easily get the conclusions of corollary 1.3

References

[1] B. Belaïdi, *Oscillation of fixed points of solutions of some linear differential equations*, Acta. Math. Univ. Comenianae, 77 (2008), 263-269.

[2] L. G. Bernal, *On growth k-order of solutions of a complex homogeneous linear differential equation*, Proceedings of the American Mathematical Society, 101 (1987), 317–322.

[3] R. Bouabdelli and B. Belaïdi, *On the iterated exponent of convergence of solutions of linear differential equations with entire and meromorphic coefficients*, Journal of Mathematics, 2013 (2013), Article ID 429083.

[4] Z. X. Chen, *The fixed points and hyper-order of solutions of second order complex differential equations,* Acta Mathematica Scientia, 20 (2000), 425-432. (in Chinese).

[5] T. B. Cao, Z. X. Chen, X. M. Zheng, and J. Tu, *On the iterated order of meromorphic solutions of higher order linear differential equations*, Annals of Differential Equations, 21 (2005), 111–122.

[6] A. El Farissi, and M. Benbachir*, Oscillation of fixed points of solutions to complex linear differential equations,* Electron. J. Diff. Equ., 2013 (2013), Article ID 41.

[7] A. El Farissi, *Value Distribution of Meromorphic Solutions and their Derivatives of Complex Differential Equations*, Hindawi publishing corporation, ISRN Mathematical Analysis, 2013 (2013), Article ID 497921.

[8] G. G. Gundersen, *Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates*, J. London Math. Soc., 2 (1988), 88-104.

[9] W. K. Hayman, *Meromorphic functions*, Clarendon Press, Oxford, 1964

[10] L. Kinnunen, *Linear differential equations with solutions of finite iterated order*, Southeast Asian Bull. Math., 22 (1998), 385-405.

[11] I. Laine, *Nevanlinna Theory and Complex Differential Equations,* Walter de Gruyter, Berlin, New York, 1993.

[12] M. S. Liu and X. M. Zhang, *Fixed points of meromorphic solutions of higher order Linear differential equations,* Ann. Acad. Sci. Fenn. Ser. A. I. Math., 31 (2006), 191-211.

[13] R. Nevanlinna, *Eindeutige analytische Funktionen*, Zweite Auflage. Reprint. Die Grundlehren der mathematischen Wissenschaften, Band 46. Springer-Verlag, Berlin-New York, 1974.

[14] J. Wang and W. R. Lü, *The fixed points and hyper-order of solutions of second order linear differential equations with meromorphic coefficients*, Acta Math. Appl. Sin. 27 (2004), 72-80. (in Chinese).

[15] H. Y. Xu, J. Tu, X. M. Zheng, *On the hyper exponent of convergence of zeros of* $f^{(j)} - \varphi$ *of higher order linear differential equations*, Advances in Difference Equations, 2012 (2012), article 114.

[16] H. X. Yi and C. C. Yang, *The Uniqueness Theory of Meromorphic Functions*, Science Press, Beijing, 1995 (in Chinese).

[17] Q. T. Zhang and C. C. Yang, *The Fixed Points and Resolution Theory of Meromorphic Functions*, Beijing University Press, Beijing, 1988 (in Chinese).

Department of Mathematics and Informatics, Faculty of Exact Sciences, University of Bechar, Algeria