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Some Results on Modular Coloring Problems of Some Graph Operations

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Abstract. Let $k \geq 2$ be an integer. A modular k-coloring of a graph G is defined as a mapping from $V(G)$ to the set $\mathbb{Z}_k$ such that adjacent vertices may receive the same color and their color sums are distinct under modulo k. The least integer k that admits a valid modular coloring is referred to the modular chromatic number of G, represented by $\chi_{mc}(G)$. In this study, our focus is on the modular chromatic number for certain graph classes, particularly type-1 and type-2 trees and some operations of graphs and those are: cartesian product, rooted product and join graphs. Also, we provide a graph G satisfying $\chi_{mc}(G) = \chi(G) + 1$.

1. Introduction

In this paper, we consider only connected simple graphs G without isolated vertices. Notations and terminology used throughout this work are adopted from [1]. The *open neighborhood* of a vertex $u \in G$ is the set of all vertices adjacent to u and it is represented by $N_G(u)$ or $N(u)$. A [1] *Clique* is a maximal complete subgraph of graph G. Modular coloring is motivated by the checkerboard problem and it was first formulated by Okamoto, Salehi, and Zhang [4].

Let graph G be a simple connected graph. The vertex coloring is a function $g : V(G) \to \mathbb{Z}_k$ $(k \geq 2)$ defined as follows, $g(u) = g(w)$ and $\phi_g^+(u) = \sum_{w \in N(u)} g(w) \pmod{k}$, for all $u \in V(G)$. The function g is said to be modular k-coloring of G whenever $\phi_g^+(w) \neq \phi_g^+(u)$, $\forall\ uw \in E(G)$. The least integer $k \geq 2$ for which a modular k-coloring of G exists refers to the ([4]) *modular chromatic number* of $G(\chi_{mc}(G))$. Also, they proved that:

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Theorem 1.1. *[4] If G is simple connected graph. Then it holds that:* $\chi(G) \leq \chi_{mc}(G) \leq \Delta(G)(\Delta(G)+1)^{\chi(G)-2}+1$, *where* $\chi(G)$ *represent chromatic number and* $\Delta(G)$ *represent maximum degree of* G, *respectively.*

Okamoto, Salehi, and Zhang conjectured [4] that in an $m \times n$ checkerboard, coins are assigned to specific squares (ensuring no more than one coin per square) can satisfy: squares of the same color have neighbors with coin counts of equal parity, and squares of different colors have neighbors with coin counts of opposite parity.
Let G and H be simple connected graph. The *cartesian product* $G\square H$, as defined in [1], is the graph formed from G and H graphs with $V(G\square H) = V(G) \times V(H)$. The adjacent vertices $(u_1, w_1)(u_2, w_2) \in G\square H \Longleftrightarrow$ either$w_1 = w_2$, $u_1u_2 \in E(G)$ or $u_1 = u_2$, $w_1w_2 \in E(H)$.

A nontrivial tree T is termed as *type-1* precisely when $\chi_{mc}(T) = 2$ and as *type-2* when $\chi_{mc}(T) = 3$ ([6]). In particular, Okamoto et. al.([6]) demonstrates that $\chi_{mc}(T) = 2$ for any tree T with a diameter of atmost 6 and $\chi_{mc}(T) = 3$ for specific caterpillars T with large diameters(≥ 7). Also they gave the following problem.

Problem 1.1. *[6] Is there any characterization for type 1 (or type 2) trees?*

Towards the above problem, we have given some trees T with $\chi_{mc}(T) = 2$ and $\chi_{mc}(T) = 3$. Hence a question naturally arises that: $2 \leq \chi_{mc}(T_1\square T_2) \leq 3$ for any two trees T_1 and T_2. Okamoto et.al., in [4], proved that the grid graph $P_m\square P_n$ admits a modular 2-coloring, follows this results, we provide $\chi_{mc}(P_m\square S_n) = 2$ in section 4.
Paramaguru and Sampathkumar investigated modular colorings of cartesian product graphs, in [7, 8], except some special numeric value of m and n, $\chi_{mc}(C_m\square P_n) = \chi(C_m\square P_n)$ in [7]; and $\chi_{mc}(C_m\square C_n) = \chi(C_m\square C_n)$ in [8].
Subsequently, Rajarajachozhan and Sampathkumar, in [10], computed that: $\chi_{mc}(K_m\square G) = \max\{m, \chi(G)\}$, where $G \in \{P_n, C_n, K_m\}$ and $m, n \geq 4$; except, $\chi_{mc}(K_m\square P_n)$, when $m \equiv 0 \pmod 4$ and $n \equiv 2 \pmod 6$ or $m \equiv 1 \pmod 2$.
Moreover, Mahalakshmi and Rajasekaran, in [12], showed that: $\chi_{mc}(G\square H) = \max\{\chi(G), \chi(H)\}$, here $G \in \{P_m, C_m, K_m\}$ with H is regular bipartite graph. They also provided a partial solution to Problem 5.1 that was posed in [10].

A connected graph G is called *unicyclic* [1] iff it contains exactly one cycle.

A Graph G is *rooted graph* [3] means a single vertex is specifically designed as the root, thereby distinguishing it from all other vertices. Let H be a labelled graph with n vertices and $G_1, G_2, \ldots, G_n$ represents n isomorphic copies of a rooted graph of G. The *rooted product* [3] of H with G, represented as $H \circ G$, is the graph formed by merging the root vertex of each graph G_r to the r^{th}vertex of H, for each $r = \{1, 2, \ldots, n\}$.

Given any $n \in \mathbb{Z}^+$ and a subset $Q \subseteq \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$, a *circulant graph* $C_n(Q)$ [1] with $V(C_n(Q)) = \mathbb{Z}_n = \{v_0, v_1, \ldots, v_{n-1}\}$ and an edge $v_rv_{r+s \pmod n} \in E(C_n(Q))$ for each $s \in Q$ and $r \in \{0, 1, \ldots, n-1\}$ subscript taken addition modulo n. If n is even and the lengths $1, 3, \ldots, \lfloor \frac{n}{2} \rfloor$ are odd, then $C_n(1,3,\ldots,\lfloor\frac{n}{2}\rfloor) \cong K_{\frac{n}{2},\frac{n}{2}}$. If $Q = \left\{1,2,\ldots,\left\lfloor\frac{n}{2}\right\rfloor\right\}$, then $C_n(Q) \cong K_n$. Okamoto et al., in [4], proved that:$\chi_{mc}(K_n) = n$ and $\chi_{mc}(K_{\frac{n}{2},\frac{n}{2}}) = 2$.

Let $G \vee H$ denote a *join graph* G and H [1], specified as the super-graph of $G + H$ in which each vertex in G is adjacent to all vertices in H. Clearly, in [4], $\chi_{mc}(G \vee H) \geq \chi(G) + \chi(H)$. Paramaguru and Sampathkumar [9] established that $\chi_{mc}(G \vee H) = \chi(G) + \chi(H)$ for certain graphs G and H. Furthermore, they given that $\chi_{mc}(C_{4m+2} \vee C_{4n+2}) = \chi(C_{4m+2} \vee C_{4n+2}) + 1$.

Mahalakshmi and Rajasekaran, in [12], provided the partial solution for Problems 1 and 2 posed in [9]. Also they proved that: $\chi_{mc}(C_{2m+1} \vee H) = \chi(C_{2m+1} + H)$, where either $H \in \{K_q, C_{2q+1,K_{q,q}}\}$ or H is bipartite graph. To prove our results, we utilize the following theorems:

Theorem 1.2. *[4]*

$$\chi_{mc}(C_m) = \begin{cases} 2 & \text{if } m \equiv 0 \pmod 4, \\ 3 & \text{otherwise.} \end{cases}$$

Proposition 1.1. *[4] For $m \geq 3$, $\chi_{mc}(W_m) = \chi(W_m)$.*

In this article, we focus on finding the modular chromatic number for type-1 and type-2 trees, as well as specific operations such as the rooted product, cartesian product and join of graphs. For each class, some examples are provided and in it the notation $a(b)$ is consistently used throughout, where in a denotes the color assigned to a vertex and b denotes its corresponding color sum.

2. TREES

In this section, we find some class of trees which are type-1 and type-2.

2.1. Type-1.

Let $P_n := x_1x_2\ldots x_n$ denote a path on n vertices. A *triangular tree* T_n(introduced in [11]) is formed by associating each vertex x_r of P_n with the designated leaves (either only one pendent vertex or vertex of degree one) of a corresponding path P_r of order r. In this construction, the path P_n is *base* of the triangular tree T_n.

We next construct a generalized triangular tree T_n^m as follows: Let $T_n^1, T_n^2, \ldots, T_n^m$ be m-isomorphic copies of T_n and let $V(T_n^s) = \{x_r^{k,s} : 1 \leq r \leq n, 1 \leq k \leq r$ and $1 \leq s \leq m\}$ be a vertex set of T_n^j. The *generalized triangular tree* T_n^m is obtained by identify all first vertices $x_1^{1,s}$ of each copy $T_n^s(1 \leq s \leq n)$ into a single vertex v(say, rooted vertex). Hence the vertex set of T_n^m is $V(T_n^m) = \{v\} \cup \{x_r^{k,s} : 2 \leq r \leq n,\ 1 \leq k \leq r,\ 1 \leq s \leq m\}$.

Theorem 2.1. *For $m, n \geq 2$, $\chi_{mc}(T_n^m) = 2$.*

Proof. Clearly,by 1.1, we have $\chi_{mc}(T_n^m) \geq \chi(T_n^m) = 2$. Hence $\chi_{mc}(T_n^m) \geq 2$. Define $g : V(T_n^m) \to \mathbb{Z}_2$ as follows: for $s \in \{1,2,\ldots,m\}$,

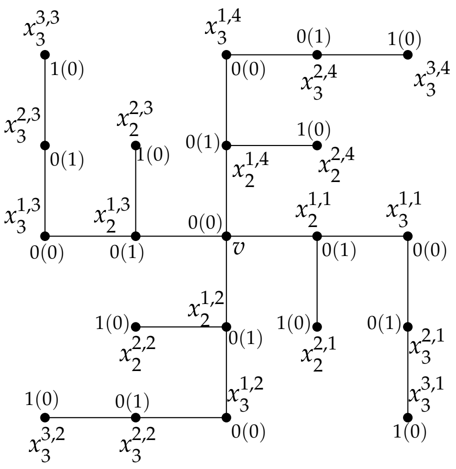

FIGURE 1. $\chi_{mc}(T_3^4) = 2$.

$$g(x_i^{k,j}) = \begin{cases} 1 & \text{if } (r,k) \equiv (0 \pmod 2, 2 \pmod 4); \\ & \quad \text{and } (r,k) \equiv (1 \pmod 2, 3 \pmod 4); \\ 0 & \text{otherwise.} \end{cases}$$

It clearly shows that adjacent vertices have different color sum (for illustrate, see Fig. 1). Hence $\chi_{mc}(T_n^m) = 2$. ☐

Let $n \in \mathbb{Z}^+$ and $t \geq 0$ be an integer. The graph $T(n,t)$ represents a *complete n-ary tree* [2] of length t such that each internal vertex has exactly n children and all leaf vertices are located at distance t from the root. If $t = 0$, $T(n,0)$ has only one vertex (i.e., root vertex of $T(n,t)$), say, v_0. Also if $t = 1$, $T(n,1) \cong K_{1,n}$.

Theorem 2.2. *For $t, n \geq 2$, $\chi_{mc}(T(n,t)) = 2$.*

Proof. Let $\varphi \geq 0$ be levels of the tree $T(n,t)$. Clearly, by 1.1, we have $\chi_{mc}(T(n,t)) \geq 2$.
For n is even, the table shows the colorings and color sums of $T(n,t)$.

Level	φ_0	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9	φ_{10}	φ_{11}	φ_{12}	φ_{13}	φ_{14}	$\cdots$	φ_{t-1}	φ_t
Color	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	...	1	0
Color Sum	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	...	0	1

For n is even, the table shows the colorings and color sums of $T(n,t)$.

$t \equiv 1 (\text{mod } 4)$																		
Level	φ_0	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9	φ_{10}	φ_{11}	φ_{12}	φ_{13}	φ_{14}	$\cdots$	φ_{t-1}	φ_t
Color	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	...	0	1
Color Sum	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	...	1	0

$t \not\equiv 1 \pmod 4$																	
Level	φ_0	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9	φ_{10}	φ_{11}	φ_{12}	φ_{13}	φ_{14}	$\cdots$	φ_{t-1}	φ_t
Color	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	$\ldots$	0	1
Color Sum	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	$\ldots$	1	0

It clearly shows that adjacent vertices have different color sum and $\chi_{mc}(T(n,t)) \leq 2$. Hence $\chi_{mc}(T(n,t)) = 2$. $\square$

Let $T_1 = T(m,s)$ and $T_2 = T(n,t)$ be the complete m-ary and n-ary trees of length s and t respectively and root vertices u_0 and v_0. Then the tree $\mathcal{T}$ is constructed by a line joining between the root vertices u_0 and v_0 of T_1 and T_2, respectively. Hence $V(\mathcal{T}) = V(T_1) \cup V(T_2)$ and $E(\mathcal{T}) = E(T_1) \cup E(T_2) \cup \{u_0v_0\}$. Hence we have the following result.

Theorem 2.3. $\chi_{mc}(\mathcal{T}) = 2$.

Proof. Clearly, by 1.1, we have $\chi_{mc}(\mathcal{T}) \geq \chi(\mathcal{T}) = 2$. Hence $\chi_{mc}(\mathcal{T}) \geq 2$.
Let m and n be even, and let $\gamma, \mu \geq 0$ be the levels of $\mathcal{T}$. Then, for $s, t \geq 0$, the tables shows the colorings and color sums of $\mathcal{T}$.

Level	γ_0	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	$\cdots$	γ_{s-1}	γ_s
Color	1	0	1	0	1	0	1	$\ldots$	0	1
Color Sum	0	1	0	1	0	1	0	$\ldots$	1	0

Level	μ_0	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	$\cdots$	μ_{t-1}	μ_t
Color	0	1	0	1	0	1	0	$\ldots$	1	0
Color Sum	1	0	1	0	1	0	1	$\ldots$	0	1

Let $m$be even(odd) and n be odd (even), for $s \geq 0$, $t \equiv 2 \pmod 4$ and the tables shows the colorings and color sums of $\mathcal{T}$.

Level	γ_0	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	$\cdots$	γ_{s-1}	γ_s
Color	0	1	0	1	0	1	0	$\ldots$	0	1
Color Sum	1	0	1	0	1	0	1	$\ldots$	1	0

Level	μ_0	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	$\cdots$	μ_{t-1}	μ_t
Color	1	0	0	0	1	0	0	$\ldots$	1	0
Color Sum	0	1	0	1	0	1	0	$\ldots$	0	1

For $s \geq 0$ and $t \not\equiv 2 \pmod 4$, the tables shows the colorings and color sums of $\mathcal{T}$.

Level	γ_0	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	$\cdots$	γ_{s-1}	γ_s
Color	1	0	1	0	1	0	1	$\ldots$	0	1
Color Sum	0	1	0	1	0	1	0	$\ldots$	1	0

Level	μ_0	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	$\cdots$	μ_{t-1}	μ_t
Color	0	1	0	1	0	1	0	$\ldots$	1	0
Color Sum	1	0	1	0	1	0	1	$\ldots$	0	1

Let m and n be odd, for $s \not\equiv 0 \pmod 4$,$t \not\equiv 3 \pmod 4$ and the tables shows the colorings and color sums of $\mathcal{T}$.

Level	γ_0	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	$\cdots$	γ_{s-1}	γ_s
Color	1	0	0	0	1	0	0	$\ldots$	0	1
Color Sum	0	1	0	1	0	1	0	$\ldots$	1	0

Level	μ_0	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	$\cdots$	μ_{t-1}	μ_t
Color	0	0	0	1	0	0	0	$\ldots$	0	1
Color Sum	1	0	1	0	1	0	1	$\ldots$	1	0

For $s \equiv 0 (\bmod 4)$, $t \equiv 3 (\bmod 4)$ and the tables shows the colorings and color sums of $\mathcal{T}$.

Level	γ_0	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	$\ldots$	γ_{s-1}	γ_s
Color	0	1	0	0	0	1	0	$\ldots$	0	1
Color Sum	1	0	1	0	1	0	1	$\ldots$	1	0

Level	μ_0	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	$\ldots$	μ_{t-1}	μ_t
Color	0	0	1	0	0	0	1	$\ldots$	1	0
Color Sum	0	1	0	1	0	1	0	$\ldots$	0	1

It clearly shows that adjacent vertices have different color sum and $\chi_{mc}(\mathcal{T}) \leq 2$. Hence $\chi_{mc}(\mathcal{T}) = 2$. □

2.2. **Type-2.** Consider a tree T^* as shown in Fig. 2. We prove that $\chi_{mc}(T^*) = 3$. Assume $\chi_{mc}(T^*) \leq 2$.

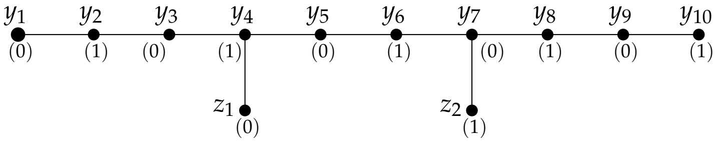

FIGURE 2. $\chi_{mc}(T^*) = 3$.

Thus, a modular 2-coloring g exists for T^*. By symmetry, the color sum of the vertices of T^* are as depicted in Fig. 2. $g(y_9) = g(y_7) = 1$ and hence $\phi_g^+(y_8) = 0$ in $\mathbb{Z}_2$, this yields a contradiction to the condition $\phi_g^+(y_8) = 1$. Hence $\chi_{mc}(T^*) = 3$.
From the above observation, we have: Let T be any nontrivial tree that contains an induced subtree T^* (as shown in Figure. 2). Then, $\chi_{mc}(T) = 3$.

3. CARTESIAN PRODUCT GRAPHS

In this section, we compute the modular coloring of cartesian product graphs that are: $P_m \square S_n$,$S_m \square S_n$ and $K_n \square G$, where $V(K_n) = \{u_1, u_2, \ldots, u_n\}$. A complete bipartite graph of form $K_{1,m}$ is a star [1]. I.e., $S_m \cong K_{1,m}$ with $V(S_m) = \{v_0, v_1, v_2, \ldots, v_m\}$and $E(S_m) = \{v_0 v_t : 1 \leq t \leq m\}$.
Observation 3.1. Let S_m and S_n denote star graphs with m and n vertices, respectively. Consider the Cartesian product $S_m \square S_n$ and it contains only vertices of even degree. Assign the color 1 to a single vertex (the vertex at positionv_{00}) and 0 elsewhere. With this assignment, the color sum of the vertices of $v_{0t} (t \in \{0, 1, 2, 3, \ldots, n\})$ and $v_{s0} (s \in \{0, 1, 2, 3, \ldots, m\})$ is 1, except the vertex v_{00}, while all other vertices have the color sum zero. Therefore, $\chi_{mc}(S_m \square S_n) = 2$.

Theorem 3.1. *For $n, m \geq 2$, $\chi_{mc}(P_m \square S_n) = 2$.*

Proof. Let $V(P_m \square S_n) = \{v_{st} : 1 \leq s \leq m, 0 \leq t \leq n\}$. Clearly,by 1.1, we have $\chi_{mc}(P_m \square S_n) \geq \chi(P_m \square S_n) = 2$. Hence $\chi_{mc}(P_m \square S_n) \geq 2$. Next to prove that $\chi_{mc}(P_m \square S_n) \leq 2$. Define $g : V(P_m \square S_n) \to \mathbb{Z}_2$ as follows:
Let $m = 2$. Then, $g(v_{st}) = 1$ if $s = 1, t = 0$ and 0 otherwise.
Let $m \geq 3$. The evidence is then based on the cases listed below.
Case 1. n is odd.
For $m \equiv 0 \ (\bmod\ 3)$, $g(v_{st}) = 1$ if $s \equiv 2 (\bmod 6), t = 0$ and $s \equiv 5 (\bmod 6), t \neq 0$;

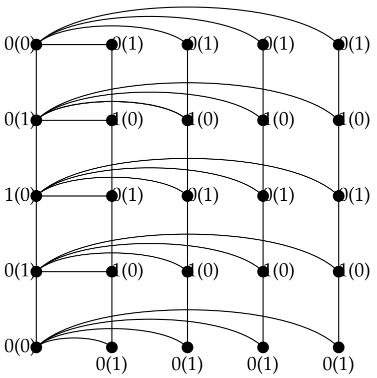

Figure 3. $\chi_{mc}(P_5\square S_5) = 2$.

for $m \not\equiv 0(\text{mod}\,3)$, $g(v_{st}) = 1$ if $s \equiv 1(\text{mod}\,6), t = 0$ and $s \equiv 4(\text{mod}\,6), t \neq 0$, and the remaining vertices are assigned color 0.

Case 2. n is even.

For $m \equiv 1,2 \pmod 8$, $g(v_{st}) = 1$ if $s \equiv 1(\text{mod}\,4), t = 0$ and $s \equiv 4,6 \pmod 8, t \neq 0$; for $m \equiv 0,5 \pmod 8$, $g(v_{st}) = 1$ if $s \equiv 3(\text{mod}\,4), t = 0$ and $s \equiv 2,4 \pmod 8, t \neq 0$; for $m \equiv 3,4,6,7 \pmod 8$, $g(v_{st}) = 1$ if $s \equiv 2(\text{mod}\,4), t = 0$ and $s \equiv 1,3 \pmod 8, t \neq 0$ and the remaining vertices are assigned color 0. It clearly shows that adjacent vertices have different color sum (For example, see Fig. 3). It follows that $\chi_{mc}(P_m\square S_n) \leq 2$. Hence $\chi_{mc}(P_m\square S_n) = 2$. □

Theorem 3.2. *If G is 2k-regular graph with* $\chi(G) = 3$. *Then,*

(i) For $m \geq 4$ *with* $m \not\equiv 0 \pmod 3$, $\chi_{mc}(K_m\square G) = m$.

(ii) For $m \equiv 0 \pmod 3$, $\chi_{mc}(K_m\square G) \leq m+1$.

Proof of (i). Since G is $2k$-regular graph and 3-chromatic, then $\exists$ a proper vertex coloring f on G and the color classes, say V_0, V_1, V_2 corresponding to the colors $0,1,2$ respectively. Therefore $V(K_m\square G) = \{(u_r, v) : 0 \leq r \leq m-1, v \in V_c, c = 0,1,2\}$. Clearly, by 1.1, we have $\chi_{mc}(K_m\square G) \geq \chi(K_m\square G) = max\{\chi(K_m), \chi(G)\} = max\{m, 3\} = m$. Hence $\chi_{mc}(K_m\square G) \geq m$. Define $g : V(K_m\square G) \to \mathbb{Z}_m$ as follows: $g((u_t, v)) = (t+c)(\text{mod}\,m)$ for $0 \leq r \leq m-1, v \in V_c, c = 0,1,2$.

Then, the color sum corresponding to each vertex $(u_t, v) \in V_0$ are:

$$\phi_g^+((u_t, v)) = \left(\frac{m(m-1)}{2} + k(t+1) + k(t+2) - t\right) (\text{mod}\,m).$$

Then the sequence of the color sums are:

$$\phi_g^+((u_t, v)) = \left\{\left(\frac{m}{2} + t(2k-1) + 3k\right)(\text{mod}\,m) \text{ if } m \equiv 0(\text{mod}\,2)\right\},$$
$$\cup \{(t(2k-1) + 3k) \ (\text{mod}\,m) \text{ if } m \equiv 1(\text{mod}\,2)\}.$$

The color sum of the vertices $(u_t, v) \in V_1$ are:

$$\phi_g^+((u_t, v)) = \left(\frac{m(m-1)}{2} + kt + k(t+2) - (t+1)\right)(\text{mod}\,m).$$

Then the sequence of the color sums are:

$$\phi_g^+((u_t,v)) = \left\{\left(\frac{m}{2} + (t+1)(2k-1)\right)(\bmod m) \text{ if } m \equiv 0 \ (\bmod 2)\right\} \cup \{(t+1)(2k-1) \ (\bmod m) \text{ if } m \equiv 1(\bmod 2)\}.$$

The color sum of the vertices $(u_t, v) \in V_2$ are:

$$\phi_g^+((u_t,v)) = \left(\frac{m(m-1)}{2} + k(t+1) + kt - (t+2)\right)(\bmod m).$$

Then the sequence of the color sums are:

$$\phi_g^+((u_t,v)) = \left\{\left(\frac{m}{2} + t(2k-1) + k - 2\right)(\bmod m) \text{ if } m \equiv 0(\bmod 2)\right\} \cup \{(t(2k-1) + k - 2)(\bmod m) \text{ if } m \equiv 1(\bmod 2)\}.$$

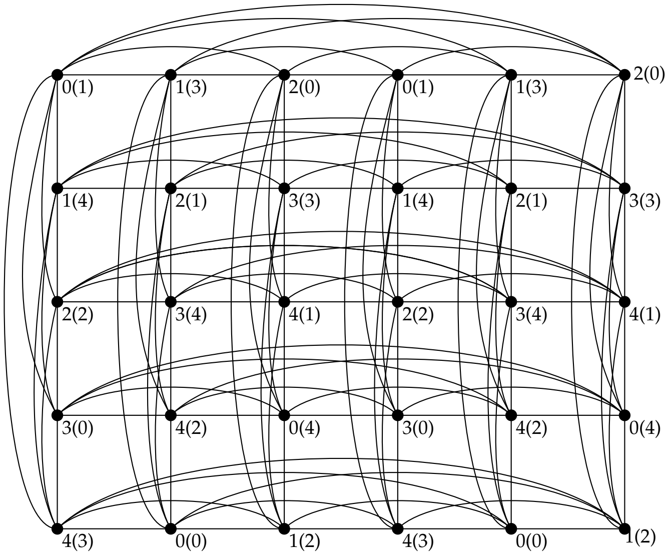

FIGURE 4. $\chi_{mc}(K_5 \square C_6(1,2)) = 5$.

It clearly shows that adjacent vertices have different color sum (For example, see Fig. 4). It follows that $\chi_{mc}(K_m \square G) \le m$. Hence $\chi_{mc}(K_m \square G) = m$.

Proof of (ii). Define $g : V(K_m \square G) \to \mathbb{Z}_{m+1}$ as follows: $g((u_t, v)) = (t+c)(\bmod(m+1))$ for $0 \le t \le m-1, v \in V_c, c = 0,1,2$.

Then, the color sum corresponding to each vertex $(u_t, v) \in V_0$ are:

$$\phi_g^+((u_t,v)) = \left(\frac{m(m-1)}{2} + k(t+1) + k(t+2) - t\right)(\bmod(m+1)).$$

Then the sequence of the color sums are:

$$\phi_g^+((u_t,v)) = \{(t(2k-1) + 3k + 1)(\bmod(m+1)) \text{ for } m \equiv 0 \ (\bmod 2)\} \cup \left\{\left(\frac{m+3}{2} + t(2k-1) + 3k\right) \ (\bmod \ (m+1)) \text{ for } m \equiv 1 \ (\bmod 2)\right\}.$$

The color sum of the vertices $(u_t, v) \in V_1$ are:

$$\phi_g^+((u_t, v)) = \left(\frac{m(m+1)}{2} + kt + k(t+2) - (t+1)\right)(\bmod(m+1)).$$

Then the sequence of the color sums are:

$$\begin{aligned}\phi_g^+((u_t, v)) = &\{((t+1)(2k-1))(\bmod(m+1)) \text{ if } m \equiv 0(\bmod 2)\}\\ &\cup \left\{\left(\frac{m+1}{2} + (t+1)(2k-1)\right)(\bmod(m+1)) \text{ if } m \equiv 1(\bmod 2)\right\}.\end{aligned}$$

The color sum corresponding to each vertex $(u_t, v) \in V_2$ are:

$$\phi_g^+((u_t, v)) = \left(m + \frac{m(m+1)}{2} + k(t+1) + kt - (t+2)\right)(\bmod(m+1)).$$

Then the sequence of the color sums are:

$$\begin{aligned}\phi_g^+((u_t, v)) = &\{(t(2k-1) + k - 2)\ (\bmod(m+1)) \text{ if } m \equiv 0(\bmod 2)\}\\ &\cup \left\{\left(\frac{3m+1}{2} + t(2k-1) + k - 2\right)(\bmod(m+1)) \text{ if } m \equiv 1(\bmod 2)\right\}.\end{aligned}$$

It clearly shows that adjacent vertices have different color sum. Hence $\chi_{mc}(K_m \square G) \leq m+1$. □

Corollary 3.1. *Let $G = C_n(a_1, a_2, \ldots, a_{\lfloor \frac{n}{2} \rfloor})$ be a $2k$-regular with $\chi(G) = 3$. Then,*
(i)For $m \not\equiv 0 \ (\bmod 3)$, $m \geq n$, $\chi_{mc}(K_m \square G) = m$.
(ii)For $m \equiv 0 \ (\bmod 3)$, $m \geq n$, $\chi_{mc}(K_m \square G) \leq m+1$.

Corollary 3.2. *If G is $2k$-regular bipartite graph, $\chi(G \square C_n) = 3$, then,*
(i) For $m \not\equiv 0 \ (\bmod 3), n \geq 3$ is odd, $\chi_{mc}(K_m \square G \square C_n) = m$.
(ii) For $m \equiv 0 \ (\bmod 3), n \geq 3$ is odd, $\chi_{mc}(K_m \square G \square C_n) \leq m+1$.

Theorem 3.3. *Let $G = C_n(a, b)$ be a 4-regular circulant graph with order $n \not\equiv 0 \ (\bmod 3)$ and $\chi(G) = 4$.*
(i) For $m \geq 4$ with $m \not\equiv 0 \ (\bmod 3)$, $\chi_{mc}(K_m \square G) = m$.
(iii) For $m \equiv 0 \ (\bmod 3)$ and $m \neq 3$, $\chi_{mc}(K_m \square G) \leq m+1$.

Proof of (i). Since G is 4-regular graph and 4-chromatic, then there is a coloring f on G with the color classes, say V_0, V_1, V_2, V_3 corresponding to the color $0, 1, 2, 3$ respectively. Therefore $V(K_m \square G) = \{(u_t, v) : 0 \leq r \leq m-1, v \in V_c, c = 0, 1, 2, 3\}$. Clearly, by 1.1, we have $\chi_{mc}(K_m \square G) \geq \chi(K_m \square G) = max\{m, 4\} = m$. Hence $\chi_{mc}(K_m \square G) \geq m$. Define $g : V(K_m \square G) \to \mathbb{Z}_m$ as follows: for $0 \leq t \leq m-1$, $g((u_t, v)) = (t+c)\ (\bmod m)$ if $v \in V_c, c = 0, 1, 2, 3$.
Then, the color sum corresponding to each vertex $(u_t, v) \in V_0$ are:

$$\phi_g^+((u_t, v)) = \left(\frac{m(m-1)}{2} + 2(t+1) + (t+2) + (t+3) - t\right)(\bmod m).$$

Then the sequence of the color sums are:

$$\begin{aligned}\phi_g^+((u_t, v)) = &\left\{\left(\frac{m}{2} + 3t + 7\right)(\bmod m) \text{ if } m \equiv 0(\bmod 2)\right\}\\ &\cup \{(3t+7)(\bmod m) \text{ if } m \equiv 1(\bmod 2)\}.\end{aligned}$$

The color sum of the vertices $(u_t, v) \in V_1$ are:

$$\phi_g^+((u_t, v)) = \left(\frac{m(m-1)}{2} + 2t + (t+2) + (t+3) - (t+1)\right) \pmod{m}.$$

Then the sequence of the color sums are:

$$\phi_g^+((u_t, v)) = \left\{\left(\frac{m}{2} + 3t + 4\right)(\bmod\, m) \text{ if } m \equiv 0 (\bmod\, 2)\right\}$$
$$\cup \{(3t+4)(\bmod\, m) \text{ if } m \equiv 1(\bmod\, 2)\}.$$

The color sum of the vertices $(u_t, v) \in V_2$ are:

$$\phi_g^+((u_t, v)) = \left(\frac{m(m-1)}{2} + t + (t+1) + 2(t+3) - (t+2)\right) \pmod{m}.$$

Then the sequence of the color sums are:

$$\phi_g^+((u_t, v)) = \left\{\left(\frac{m}{2} + 3t + 5\right)(\bmod\, m) \text{ if } m \equiv 0 (\bmod\, 2)\right\}$$
$$\cup \{(3t+5)(\bmod\, m) \text{ if } m \equiv 1(\bmod\, 2)\}.$$

The color sum of the vertices $(u_t, v) \in V_3$ are:

$$\phi_g^+((u_t, v)) = \left(\frac{m(m-1)}{2} + t + (t+1) + 2(t+2) - (t+3)\right) \pmod{m}.$$

Then the sequence of the color sums are:

$$\phi_g^+((u_t, v)) = \left\{\left(\frac{m}{2} + 3t + 2\right)(\bmod\, m) \text{ if } m \equiv 0 (\bmod\, 2)\right\}$$
$$\cup \{(3t+2)(\bmod\, m) \text{ if } m \equiv 1(\bmod\, 2)\}.$$

It clearly shows that adjacent vertices have different color sum. It follows that $\chi_{mc}(K_m \Box G) \leq m$. Hence $\chi_{mc}(K_m \Box G) = m$.

Proof of (ii). Define $g : V(K_m \Box G) \to \mathbb{Z}_{m+1}$ as follows: $g((u_t, v)) = (t+c)(\bmod (m+1))$if $v \in V_c, c = 1, 2, 3, 4$. Then, the color sum corresponding to each vertex $(u_t, v) \in V_0$ are:

$$\phi_g^+((u_t, v)) = \left(\frac{m(m+1)}{2} + (t+3) + (t+4) + 2(t+2) - (t+1)\right) \pmod{(m+1)}.$$

Then the sequence of the color sums are:

$$\phi_g^+((u_t, v)) = \{(3t+10) \pmod{(m+1)} \text{ if } m \equiv 0(\bmod\, 2)\}$$
$$\cup \left\{\left(\frac{m+1}{2} + 3t + 10\right) \pmod{(m+1)} \text{ if } m \equiv 1(\bmod\, 2)\right\}.$$

The color sum of the vertices $(u_t, v) \in V_1$ are:

$$\phi_g^+((u_t, v)) = \left(\frac{m(m+1)}{2} + m + 2(t+1) + (t+4) + (t+3) - (t+2)\right)(\bmod (m+1)).$$

Then the sequence of the color sums are:

$$\phi_g^+((u_t,v)) = \{(3t+7)\ (\mathrm{mod}(m+1)) \text{ if } m \equiv 0(\mathrm{mod}\,2)\}$$
$$\cup \left\{ \left(\frac{3m+1}{2} + 3t + 7 \right) (\mathrm{mod}(m+1)) \text{ if } m \equiv 1(\mathrm{mod}\,2) \right\}.$$

The color sum corresponding to each vertex $(u_t, v) \in V_2$ are:

$$\phi_g^+((u_t,v)) = \left(\frac{m(m+1)}{2} + 2m + 2(t+4) + (t+1) + (t+2) - (t+3) \right)\ (\mathrm{mod}\ (m+1)).$$

Then the sequence of the color sums are:

$$\phi_g^+((u_t,v)) = \{(3t+8)\ (\mathrm{mod}(m+1)) \text{ if } m \equiv 0(\mathrm{mod}\,2)\}$$
$$\cup \left\{ \left(\frac{5m+1}{2} + 3t + 8 \right) (\mathrm{mod}(m+1)) \text{ if } m \equiv 1(\mathrm{mod}\,2) \right\}.$$

The color sum corresponding to each vertex $(u_t, v) \in V_3$ are:

$$\phi_g^+((u_t,v)) = \left(\frac{m(m+1)}{2} + 3m + 2(t+1) + 2(t+2) - (t+3) \right)\ (\mathrm{mod}\ (m+1)).$$

Then the sequence of the color sums are:

$$\phi_g^+((u_t,v)) = \{(3t+5)\ (\mathrm{mod}\ (m+1)) \text{ if } m \equiv 0(\mathrm{mod}\,2)\}$$
$$\cup \left\{ \left(\frac{7m+1}{2} + 3t + 5 \right)\ (\mathrm{mod}\ (m+1)) \text{ if } m \equiv 1(\mathrm{mod}\,2) \right\}.$$

Therefore it clearly shows that adjacent vertices have different color sum. Hence $\chi_{mc}(K_m \square G) \leq m+1$. □

4. Rooted Product Graphs

In this section, we provide $\chi_{mc}(H \circ G)$, where $H \circ G$ are: cycle with path, cycle with cycle, cycle with complete and complete with complete graphs.

Observation 4.1. Let $\mathscr{G}$ be a simple connected graph of order m and it contains an induced odd cycle. Then, $\chi_{mc}(\mathscr{G}) \geq 3$.

Observation 4.2. Let T be any tree with $\chi_{mc}(T) = 2$. Form a unicylic graph $\mathscr{G}$ by merging one vertex of T with a vertex of the cycle C_m. Then,

$$\chi_{mc}(\mathscr{G}) = \begin{cases} 2 & \text{if } m \text{ is even,} \\ 3 & \text{otherwise.} \end{cases}$$

Observation 4.3. Let T_n be a triangular tree (as defined in the previous section) with $\chi_{mc}(T_n) = 2$. Then $\exists$ a modular 2-coloring g of T_n such that $g(x) = g(y) = 0$, for a pair of non adjacent vertices $x, y \in T_n$, and either $\phi_g^+(x) = 1$ or $\phi_g^+(y) = 1$. Let $\mathscr{G}$ be an unicyclic graph formed by joining an edge between two vertices x and y in the tree T_n. Then, $\chi_{mc}(\mathscr{G}) = 2$.

Let $\underbrace{P_n \cup P_n \cup \cdots \cup P_n}_{m \text{ times}}$ (i.e., $\bigcup\limits_{s=1}^{m} P_n$)be the union of m mutually disjoin copies of P_n with $V(\bigcup\limits_{s=1}^{m} P_n) = \{x_t^s : 1 \leq s \leq m, 1 \leq t \leq n\}$. Then, construct a $C_m \circ P_n$ by joining the vertices as follows $x_1^1 x_1^2 x_1^3 \ldots x_1^m x_1^1$.

Lemma 4.1. *For $m \geq 2$, $n \geq 3$ with $(m,n) \neq (2(\operatorname{mod} 4), 1(\operatorname{mod} 4))$, $\chi_{mc}(C_m \circ P_n)$ is 2 or 3.*

Proof. Case 1. m is even.
Clearly,by 1.1, we have $\chi_{mc}(C_m \circ P_n) \geq \chi(C_m \circ P_n) = 2$. Hence $\chi_{mc}(C_m \circ P_n) \geq 2$. Define $g : V(C_m \circ P_n) \to \mathbb{Z}_2$ as follows:
Let $n \equiv 2, 3 \pmod 4$,

$$g(x_t^s) = \begin{cases} 1 & \text{if } (s,t) \equiv (1 \pmod 2, 1(\operatorname{mod} 4)); \\ & \qquad\qquad \text{and } (s,t) \equiv (0 \pmod 2, 2 \pmod 4)); \\ 0 & \text{otherwise.} \end{cases}$$

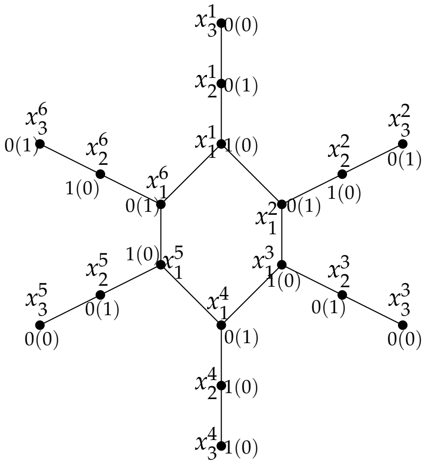

FIGURE 5. $\chi_{mc}(C_6 \circ P_3) = 2$.

Let $n \equiv 0(\operatorname{mod} 4)$,

$$g(x_t^s) = \begin{cases} 1 & \text{if } s \equiv 1 \pmod 2, t \equiv 3(\operatorname{mod} 4); \\ & \qquad\qquad \text{and } s \equiv 0 \pmod 2, t \equiv 2(\operatorname{mod} 4); \\ 0 & \text{otherwise.} \end{cases}$$

Let $m \equiv 0(\operatorname{mod} 4)$ and $n \equiv 1(\operatorname{mod} 4)$,

$$g(x_t^s) = \begin{cases} 1 & \text{if } s \equiv 1 \pmod 2, t \equiv 1(\operatorname{mod} 4); \\ & \qquad\qquad \text{and } s \equiv 0(\operatorname{mod} 2), t \equiv 0(\operatorname{mod} 4); \\ 0 & \text{otherwise.} \end{cases}$$

The color sum of the vertices of the above cases are: $\phi_g^+(x_t^s) = 1 - (t \pmod 2)$ if s is odd; $\phi_g^+(x_t^s) = t \pmod 2$ if s is even. It clearly shows that adjacent vertices have different color sum (For example, see Fig. 5) and $\chi_{mc}(C_m \circ P_n) \leq 2$. Hence $\chi_{mc}(C_m \circ P_n) = 2$.

Case 2. $m \equiv 1 \pmod 2$.

Clearly,by 1.1, we have $\chi_{mc}(C_m \circ P_n) \geq \chi(C_m \circ P_n) = 3$. Hence $\chi_{mc}(C_m \circ P_n) \geq 3$. Define $g : V(C_m \circ P_n) \to \mathbb{Z}_3$ as follows:

Let $n \equiv 0$ (mod4), $g(x_t^m) = 2$ for $t \equiv 2$(mod4). The color sum of the vertices are: $\phi_g^+(x_t^m) = 2(t \pmod 2)$. The color and corresponding color sums of the remaining vertices follows from Case 1 ($n \equiv 0$(mod4)).

Let $n \equiv 1$(mod4),

$$g(x_n^s) = \begin{cases} 2 & \text{if } s \text{ is even,} \\ 0 & \text{if } s = m. \end{cases}$$

The color sum of the vertices are:

$$\phi_g^+(x_{n-1}^s) = \begin{cases} 2 & \text{if } s \text{ is even,} \\ 1 & \text{if } s = m. \end{cases}$$

The color and color sum of the remaining vertices are follows from $n \equiv 0 \pmod 4$ by this case.

Let $n \equiv 2$(mod4), $g(x_n^s) = 2$ if $s \in \{1, 3, 5, \ldots, m\}$. Then, $\phi_g^+(x_{n-1}^s) = 2$ if s is odd and and the color and color sum of the remaining vertices are follows from $n \equiv 0 \pmod 4$ by this case.

Let $n \equiv 3 \pmod 4$, $g(x_{n-1}^s) = 2$ if s is odd. The color sum of the corresponding vertices are: $\phi_g^+(x_t^s) = 2$ if s is odd and $t \in \{n-2, n\}$, the color and color sum of the remaining vertices are follows from $n \equiv 0 \pmod 4$ by this case.

It clearly, shows that adjacent vertices have different color sum and $\chi_{mc}(C_m \circ P_n) \leq 3$. Hence $\chi_{mc}(C_m \circ P_n) = 3$. □

Lemma 4.2. *For* $(m, n) = (2 \pmod 4, 1 \pmod 4)$, $\chi_{mc}(C_m \circ P_n) \leq 3$.

Proof. Define $g : V(C_m \circ P_n) \to \mathbb{Z}_3$ as follows: Assign the same colors as in Theorem 4.1 for $m \equiv 0 \pmod 4$ and $n \equiv 1 \pmod 4$. The color sum of the vertices are: If s and t both are even(odd), then $\phi_g^+(x_t^s) = 0$; if either s is even(odd) or t is even(odd), then $\phi_g^+(x_t^s) = 2$, and 1 otherwise. It clearly shows that adjacent vertices have different color sum and $\chi_{mc}(C_m \circ P_n) \leq 3$. □

From the above Lemma 4.1 and 4.2 we have the below Theorem:

Theorem 4.1. (i). *For m is even and* $(m, n) \neq (2 \pmod 4, 1 \pmod 4)$, $\chi_{mc}(C_m \circ P_n) = 2$.
(ii). *For* $(m, n) = (2 \pmod 4, 1 \pmod 4)$, $\chi_{mc}(C_m \circ P_n) \leq 3$.
(iii). *For m is odd,* $\chi_{mc}(C_m \circ P_n) = 3$.

The following proposition only shows that: $\chi_{mc}(C_3 \circ C_3) = \chi(C_3 \circ C_3) + 1$.

Proposition 4.1. $\chi_{mc}(C_3 \circ C_3) = 4$.

Proof. Let vertex set of $C_m \circ C_n$ be $V(C_m \circ C_n) = \{x_t^s : 1 \leq s \leq m, 1 \leq t \leq n\}$. To prove that $\chi_{mc}(C_3 \circ C_3) \geq 4$. Let $\bigcup\limits_{s=1}^{3} C_3$ be disjoint union of C_3 with $V(\bigcup\limits_{s=1}^{3} C_3) = \{x_t^s : 1 \leq t \leq 3\}$. Assume that $\chi_{mc}(\bigcup\limits_{s=1}^{3} C_3) \leq 3$. Then $\exists$ a modular coloring $g : V(\bigcup\limits_{s=1}^{3} C_3) \to \mathbb{Z}_3$ as follows $g(x_1^s) = 0, g(x_2^s) = 1$ and $g(x_3^s) = 2$ for $s \in \{1, 2, 3\}$. Using the above colored 3-disjoint cycles C_3, we can construct $C_3 \circ C_3$ by identifying the vertices in 27 possible ways. Among that we consider some cases: suppose we join the colored vertices as follows:

- $x_1^1 x_1^2 x_1^3 x_1^1$, the color sum $\phi_g^+(x_1^1) = \phi_g^+(x_1^2) = \phi_g^+(x_1^3) = 0$.
- $x_1^1 x_1^2 x_2^3 x_1^1$, the color sum $\phi_g^+(x_1^1) = \phi_g^+(x_3^1) = 1$.
- $x_1^1 x_1^2 x_3^3 x_1^1$, the color sum $\phi_g^+(x_1^1) = \phi_g^+(x_2^1) = 2$.
- $x_1^1 x_2^2 x_1^3 x_1^1$, the color sum $\phi_g^+(x_1^1) = \phi_g^+(x_3^1) = 1$ and $\phi_g^+(x_1^3) = \phi_g^+(x_3^3) = 0$.
- $x_1^1 x_2^2 x_2^3 x_1^1$, the color sum $\phi_g^+(x_1^1) = \phi_g^+(x_3^1) = 1$ and $\phi_g^+(x_1^2) = \phi_g^+(x_2^2) = \phi_g^+(x_1^3) = \phi_g^+(x_2^3) = 0$.
- $x_1^1 x_2^2 x_3^3 x_1^1$, the color sum $\phi_g^+(x_2^2) = \phi_g^+(x_3^2) = 1$ and $\phi_g^+(x_1^3) = \phi_g^+(x_2^3) = 0$.
- $x_1^1 x_3^2 x_1^3 x_1^1$, the color sum $\phi_g^+(x_1^1) = \phi_g^+(x_2^1) = \phi_g^+(x_1^3) = \phi_g^+(x_2^3) = 2$.
- $x_1^1 x_3^2 x_2^3 x_1^1$, the color sum $\phi_g^+(x_2^2) = \phi_g^+(x_3^2) = 2$and $\phi_g^+(x_2^3) = \phi_g^+(x_3^3) = 1$.
- $x_1^1 x_3^2 x_3^3 x_1^1$, the color sum $\phi_g^+(x_1^1) = \phi_g^+(x_3^1) = 2$ and $\phi_g^+(x_1^2) = \phi_g^+(x_3^2) = \phi_g^+(x_1^3) = \phi_g^+(x_3^3) = 0$.

Thus, in each case, some adjacent vertices receives the same color sum. Hence we get a contradiction to our assumption. Similarly, we can show the contradiction to the remaining 18 cases. Hence $\chi_{mc}(C_3 \circ C_3) \geq 4$.

Next, to prove that $\chi_{mc}(C_3 \circ C_3) \leq 4$. Define $g : V(C_3 \circ C_3) \to \mathbb{Z}_4$ by $g(x_1^1) = 0, g(x_1^2) = g(x_3^2) = 1$, $g(x_1^3) = g(x_2^1) = g(x_2^2) = g(x_2^3) = 2$, and $g(x_3^1) = g(x_3^3) = 3$. Then the color sum as follows: $\phi_g^+(x_1^1) = \phi_g^+(x_3^3) = 0$, $\phi_g^+(x_1^2) = \phi_g^+(x_2^3) = 1$, $\phi_g^+(x_3^1) = \phi_g^+(x_2^2) = \phi_g^+(x_1^3) = 2$, and $\phi_g^+(x_2^1) = \phi_g^+(x_3^2) = 3$. It clearly shows that adjacent vertices have different color sum and $\chi_{mc}(C_3 \circ C_3) \leq 4$. Hence $\chi_{mc}(C_3 \circ C_3) = 4$. □

Lemma 4.3. *For $m \geq 4$ and $n \geq 4$ are even, $\chi_{mc}(C_m \circ C_n)$ is 2 or 3.*

Proof. Let $\underbrace{C_n \cup C_n \cup \cdots \cup C_n}_{m \text{ times}}$ (i.e., $\bigcup\limits_{s=1}^{m} C_n$) be the union of m mutually disjoin copies of C_n and let $V(\bigcup\limits_{s=1}^{m} C_n) = \{x_t^s : 1 \leq s \leq m, 1 \leq t \leq n\}$. Define $g : V(\bigcup\limits_{s=1}^{m} C_n) \to \mathbb{Z}_2$ as follows:

Let $n \equiv 0 (\operatorname{mod} 4)$,

$$g(x_t^s) = \begin{cases} 1 & \text{if } t \equiv 3 (\operatorname{mod} 4), \\ 0 & \text{elsewhere.} \end{cases}$$

Construct $C_m \circ C_n$ from the above colored m-disjoint union of C_n by line joining the vertices as: $x_1^1 x_2^2 x_1^3 x_2^4 \ldots x_1^{m-1} x_2^m x_1^1$. Therefore the colors and color sum of vertices of $C_m \circ C_n$ are unchanged. Thus $\chi_{mc}(C_m \circ C_n) \leq 2$ and, by Theorem 1.2, $\chi_{mc}(C_m \circ C_n) \geq 2$. Hence $\chi_{mc}(C_m \circ C_n) = 2$.

Let $n \equiv 2 (\operatorname{mod} 4)$.

Define $g : V(\bigcup_{s=1}^{m} C_n) \rightarrow \mathbb{Z}_3$ as follows:

$$g(x_t^s) = \begin{cases} 1 & \text{if } t \equiv 1 (\text{mod } 4) \\ 0 & \text{elsewhere.} \end{cases}$$

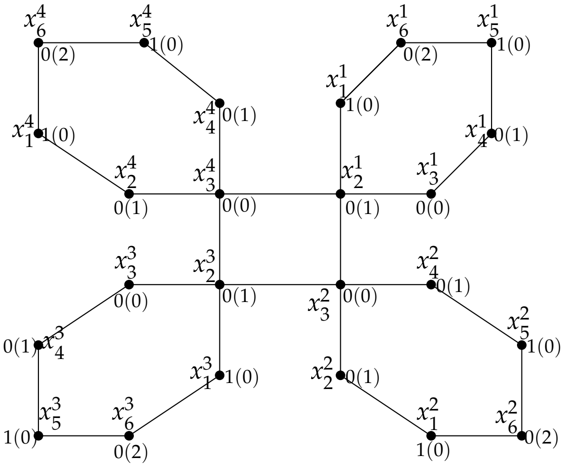

Figure 6. $\chi_{mc}(C_4 \circ C_6) = 3$.

Then,

$$\phi_g^+(x_t^s) = \begin{cases} 2 & \text{if } t = n, \\ 0 & \text{if odd } t, \\ 1 & \text{elsewhere.} \end{cases}$$

Construct $C_m \circ C_n$ from the above colored m-disjoint union of C_n by line joining the vertices as: $x_2^1 x_3^2 x_2^3 x_3^4 \ldots x_2^{m-1} x_3^m x_2^1$. Therefore the colors and color sum of vertices of $C_m \circ C_n$ are unchanged. Therefore $\chi_{mc}(C_m \circ C_n) \leq 3$ and, by Theorem 1.2, $\chi_{mc}(C_m \circ C_n) \geq 3$. It clearly shows that adjacent vertices have different color sum (For example, see Fig. 6). Hence $\chi_{mc}(C_m \circ C_n) = 3$. □

Lemma 4.4. *If $m, n \geq 3$ and both are not even, then $\chi_{mc}(C_m \circ C_n) = 3$.*

Proof. Clearly,by 1.1, we have $\chi_{mc}(C_m \circ C_n) \geq \chi(C_m \circ C_n) = 3$. Hence, $\chi_{mc}(C_m \circ C_n) \geq 3$. Define $g : V(\bigcup_{s=1}^{m} C_n) \rightarrow \mathbb{Z}_3$ as follows:

Let $m \equiv 1 \pmod 2$ and $n \not\equiv 2 (\text{mod } 4)$, $g(x_n^s) = 2$ and

$$g(x_t^s) = \begin{cases} 1 & \text{if } t \equiv 0 \pmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$\phi_g^+(x_n^s) = \begin{cases} 1 & \text{if } n \equiv 1 (\text{mod } 4), \\ 0 & \text{if } n \equiv 0, 3 (\text{mod } 4). \end{cases}$$

and

$$\phi_g^+(x_t^s) = \begin{cases} t \pmod 2 & \text{if } t \neq 1, n-1, \\ 2 & \text{if } t = 1, n-1, \end{cases}$$

Construct $C_m \circ C_n$ from the above colored m-disjoint union of C_n by line joining the vertices as follows:
For m is odd and $n \not\equiv 2(\bmod 4)$, the sequence of vertices joined is $x_1^1 x_2^2 x_1^3 x_2^4 \ldots x_2^{m-1} x_3^m x_1^1$;
For m is even and $n$is odd, the sequence of vertices joined is $x_1^1 x_2^2 x_1^3 x_1^4 \ldots x_1^{m-1} x_2^m x_1^1$. Therefore the colors and color sum of vertices of $C_m \circ C_n$ are unchanged. Clearly, $\chi_{mc}(C_m \circ C_n) \leq 3$ and, by Theorem 1.2, $\chi_{mc}(C_m \circ C_n) \geq 3$.
Let $m \equiv 1 \pmod 2$ and $n \equiv 2(\bmod 4)$.

Assign the same color identically to the case described in Lemma 4.3, where in m is even and $n \equiv 2(\bmod 4)$. Construct $C_m \circ C_n$ from the above colored m-disjoint union of C_n by line joining the vertices as $x_2^1, x_3^2, x_2^3, x_3^4, \ldots, x_n^m, x_2^1$. Therefore the colors and color sum of vertices of $C_m \circ C_n$ are unchanged. Clearly, $\chi_{mc}(C_m \circ C_n) \leq 3$ and, by Theorem 1.2, $\chi_{mc}(C_m \circ C_n) \geq 3$. In all the cases, it clearly shows that adjacent vertices have different color sum. Hence $\chi_{mc}(C_m \circ C_n) = 3$. □

From the above discussion, we form the following theorem.

Theorem 4.2. *For $m, n \geq 3$,*

$$\chi_{mc}(C_m \circ C_n) = \begin{cases} 2 & \text{if } m \equiv 0 \pmod 2, n \equiv 0(\bmod 4), \\ 4 & \text{if } m = n = 3, \\ 3 & \text{otherwise.} \end{cases}$$

Proof. The proof directly obtained from Lemmas 4.3 and 4.4 together with Proposition 4.1. □

Theorem 4.3. *For $m \geq 3, n \geq 4$ with $(m,n) \neq (3,3)$, $\chi_{mc}(C_m \circ K_n) = n+1$.*

Proof. Let$\underbrace{K_n \cup K_n \cup \cdots \cup K_n}_{m \text{ times}}$ (i.e., $\bigcup\limits_{s=1}^{m} K_n$)be the union of m mutually disjoin copies of K_n with $V(\bigcup\limits_{s=1}^{m} K_n) = \{x_r^s : 1 \leq r \leq n\}$. Assume that $\chi_{mc}(\bigcup\limits_{s=1}^{m} K_n) \leq n$. Then $\exists$ a modular coloring $g : V(\bigcup\limits_{s=1}^{m} K_n) \rightarrow \mathbb{Z}_n$ with $g(v_r^s) = r-1$ if $r \in \{1, 2, \ldots, n\}$. Using the colored disjoint K_n, construct $C_m \circ K_n$ by identifying the vertices in n^m possible ways. Among that we consider some cases:

- $x_1^1, x_1^2, x_1^3, \ldots, x_1^m, x_1^1$, the color sum $\phi_g^+(x_1^1) = \phi_g^+(x_1^2) = \cdots =$ $\phi_g^+(x_1^m) = \frac{n(n-1)}{2}(\bmod n)$;
- $x_1^1, x_2^2, x_1^3, \ldots, x_1^m, x_1^1$, the color sum $\phi_g^+(x_1^4) = \phi_g^+(x_1^5) = \cdots = \phi_g^+(x_1^{m-1}) =$ $\frac{n(n-1)}{2}(\bmod n)$;
- $x_1^1, x_3^2, x_1^3, \ldots, x_1^m, x_1^1$, the color sum $\phi_g^+(x_1^1) = \phi_g^+(x_1^5) = \cdots = \phi_g^+(x_1^{m-1}) = \phi_g^+(x_1^m) = \frac{n(n-1)}{2} (\bmod n)$;

- $x_1^1, x_4^2, x_1^3, \dots, x_1^m, x_1^1$, the color sum
  $\phi_g^+(x_1^1) = \phi_g^+(x_1^2) = \cdots = \phi_g^+(x_1^m) = \frac{n(n+1)}{2} (\bmod n)$.

Thus, in each above cases, adjacent vertices have same color sum, we get a contradiction to our assumption. Similarly, the remaining possibilities also lead to a contradiction. Hence $\chi_{mc}(C_m \circ K_n) \geq n+1$.

Next, we prove that $\chi_{mc}(C_m \circ K_n) \leq n+1$. Define a coloring $g : V(C_m \circ K_n) \to \mathbb{Z}_{n+1}$ as follows: Let$m$ is even, $g(x_1^s) = 0$ and

$$g(x_r^s) = \begin{cases} r-1 & \text{if } s \equiv 1(\bmod 2), r \neq 1, \\ r & \text{if } s \equiv 0(\bmod 2), r \neq 1. \end{cases}$$

Then,

$$\phi_g^+(x_t^s) = \begin{cases} \left(\frac{n(n-1)}{2} - (r-1)\right)(\bmod(n+1)) & \text{if } s \equiv 1(\bmod 2), \\ \left(\frac{n(n+1)}{2} - (r+1)\right)(\bmod(n+1)) & \text{if } s \equiv 0(\bmod 2), \\ \left(\frac{n(n+1)}{2} - 1\right)(\bmod(n+1)) & \text{if } s \equiv 0(\bmod 2), r = 1. \end{cases}$$

Let m is odd,

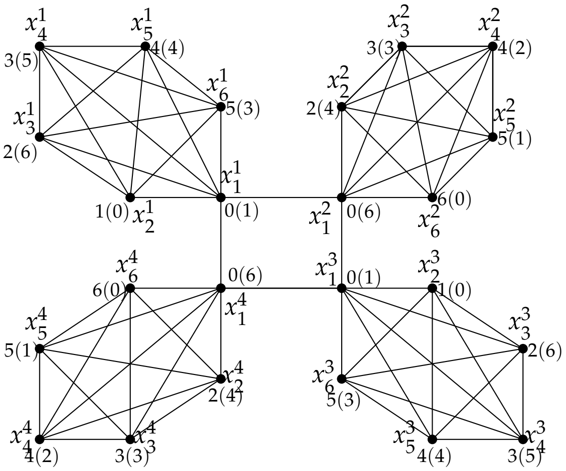

FIGURE 7. $\chi_{mc}(C_4 \circ K_6) = 7$.

$$g(x_r^m) = \begin{cases} t & \text{if } r \neq 1, 2, \\ 0 & \text{if } r = 1, \\ 1 & \text{if } r = 2. \end{cases}$$

Then,

$$\phi_g^+(x_r^m) = \begin{cases} \left(\frac{n(n+1)}{2} - (r+2)\right)(\bmod(n+1)) & \text{if } r \neq 1, 2 \\ \left(\frac{n(n+1)}{2} - 2\right)(\bmod(n+1)) & \text{if } r = 1, \\ \left(\frac{n(n+1)}{2} - 3\right)(\bmod(n+1)) & \text{if } r = 2. \end{cases}$$

The remaining vertices of color and color sum are similar to the above Case m is even. Construct $C_m \circ K_n$ from the above colored m-disjoint union of K_n by line joining the vertices as $x_1^1 x_1^2 x_1^3 x_1^4 \dots x_1^{m-1} x_1^m x_1^1$. Therefore the colors and color sum of vertices of $C_m \circ K_n$ are unchanged. It clearly shows that adjacent vertices have different color sum (For example, see Fig. 7) and $\chi_{mc}(C_m \circ K_n) \leq n+1$. Hence $\chi_{mc}(C_m \circ K_n) = n+1$. □

Theorem 4.4. *For $m, n \geq 3$, with $(m,n) \neq (3,3)$, $\chi_{mc}(K_m \circ C_n) = m$.*

Proof. Let $\underbrace{C_n \cup C_n \cup \cdots \cup C_n}_{m \text{ times}}$ (i.e., $\bigcup\limits_{s=1}^{m} C_n$)be the union of m mutually disjoin copies of C_n and let $V(\bigcup\limits_{s=1}^{m} C_n) = \{x_t^s : 1 \leq s \leq m, 1 \leq t \leq n\}$. Clearly,by 1.1, we have $\chi_{mc}(K_m \circ C_n) \geq \chi(K_m \circ C_n) = m$. Hence $\chi_{mc}(K_m \circ C_n) \geq m$. Define a coloring $g : V(K_m \circ C_n) \rightarrow \mathbb{Z}_m$ as follows:

Case 1. $n \equiv 0, 1 (\mathrm{mod}\, 4)$.

$$g(x_n^s) = \begin{cases} 2 & \text{if } s = 1,2,3, \\ s-1 & \text{if } s \neq 1,2,3. \end{cases}$$

Assign the color for the remaining vertices are: $g(x_t^s) = 1$ if $t \equiv 1 (\mathrm{mod}\, 4)$ and 0 otherwise.

The color sum of the corresponding vertices of $K_m \circ C_n$ are:

Let $n \equiv 0 (\mathrm{mod}\, 4)$,

$$\phi_g^+(x_t^s) = \begin{cases} 1 - (t \ (\mathrm{mod}\ 2)) & \text{if } t \neq 1, n-1, \\ s-1 & \text{if } s \neq 1,2,3, t \neq 1, n-1, \\ 2 & \text{if } s = 1,2,3, t = 1, n-1. \end{cases}$$

Let $n \equiv 1 (\mathrm{mod}\, 4)$, $\phi_g^+(x_n^s) = 1$ and the remaining color sums are similar to the above case $n \equiv 0 (\mathrm{mod}\, 4)$. Construct $K_m \circ C_n$ from the above colored m-disjoint union of C_n by all possible line joining between these vertices $x_2^1, x_3^2, x_{n-1}^3, x_{n-1}^4, \dots, x_{n-1}^m$. Which means that these vertices induces a clique of size m. Therefore the colors and color sum of vertices of $K_m \circ C_n$ are unchanged.

Let $n \equiv 2 (\mathrm{mod}\, 4)$,

$$g(x_n^s) = \begin{cases} 0 & \text{if } s = 1,2,3, \\ s-2 & \text{if } s \neq 1,2,3. \end{cases}$$

$g(x_t^s) = 1$ if $s \neq 1,2,3$ and $t \equiv 2 (\mathrm{mod}\, 4)$ and assign the color for the remaining vertices are similar to the above case $n \equiv 0, 1 (\mathrm{mod}\, 4)$.

Then, the color sum of the corresponding vertices of $K_m \circ C_n$ are:

$$\phi_g^+(x_t^s) = \begin{cases} 2 & \text{if } s = 1,2,3 \text{ and } t = n, \\ 1 - (t \ (\mathrm{mod}\ 2)) & \text{if } s = 1,2,3 \text{ and } t \neq n, \\ s-1 & \text{if } s \neq 1,2,3 \text{ and } t = 1, \\ s-2 & \text{if } s \neq 1,2,3 \text{ and } t = n-1, \\ t \ (\mathrm{mod}\ 2) & \text{if } s \neq 1,2,3 \text{ and } t \neq 1, n-1. \end{cases}$$

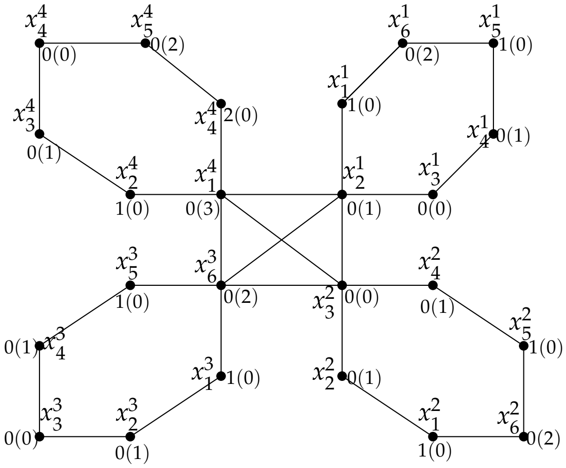

FIGURE 8. $\chi_{mc}(K_4 \circ C_6) = 4$.

and the color sum of the remaining vertices are similar to the above case as mentioned in $n \equiv 0, 1 (\bmod 4)$.

Construct $K_m \circ C_n$ from the above colored m-disjoint union of C_n by all possible line joining between these vertices $x_2^1, x_3^2, x_n^3, x_1^4, x_1^5, x_1^6, \ldots, x_1^{m-1}, x_1^m$. Which means that these vertices induces a clique of size m. Therefore the colors and color sum of vertices of $K_m \circ C_n$ are unchanged.

Case 3. $n \equiv 3 (\bmod 4)$.

$$g(x_{n-1}^s) = \begin{cases} 2 & \text{if } s = 1, 2, 3, \\ s-1 & \text{if } s \neq 1, 2, 3. \end{cases}$$

$g(x_t^s) = 1$ if $t \equiv 2 (\bmod 4)$ and 0 otherwise.

The color sum of the corresponding vertices of $K_m \circ C_n$ are:

$$\phi_g^+(x_t^s) = \begin{cases} t \ (\bmod 2) & \text{if } t \neq n, n-2, \\ s-1 & \text{if } s \neq 1, 2, 3 \text{ and } t \neq n, n-2, \\ 2 & \text{if } s = 1, 2, 3 \text{ and } t = n, n-2. \end{cases}$$

Construct $K_m \circ C_n$, from the above colored m-disjoint union of C_n, by all possible line joining between these vertices $x_3^1, x_4^2, x_n^3, x_n^4, x_n^5, x_n^6, \ldots, x_n^{m-1}, x_n^m$. Which means that these vertices induces a clique of size m. Therefore the colors and color sum of vertices of $K_m \circ C_n$ are unchanged.

Case 4. $n = 3$.

$$g(x_1^s) = s - 1.$$

Let m is even,

$$g(x_t^s)=\begin{cases}1 & \text{if } n\equiv 0(\text{mod}4), t=2,\\ 2 & \text{if } n\equiv 2(\text{mod}4), t=2,\\ n-1 & \text{if } n\equiv 0(\text{mod}4), t=3,\\ n-2 & \text{if } n\equiv 2(\text{mod}4), t=3.\end{cases}$$

Let m is odd,

$$g(x_t^s)=\begin{cases}1 & \text{if } s\neq \frac{m+1}{2},\frac{n+3}{2}, t=2,\\ 2 & \text{if } s=\frac{m+1}{2},\frac{n+3}{2}, t=2,\\ n-1 & \text{if } s\neq \frac{m+1}{2},\frac{n+3}{2}, t=3,\\ n-2 & \text{if } s=\frac{m+1}{2},\frac{n+3}{2}, t=3.\end{cases}$$

Then, $\phi_g^+(x_t^s)=\Big(\big(\sum_{t=1}^{3} g(x_t^s)\big)-g(x_t^s)\Big) \pmod m$.
It clearly shows that adjacent vertices have different color sum (For example, see Fig. 8) and $\chi_{mc}(K_m\circ C_n)\leq m$. Hence $\chi_{mc}(K_m\circ C_n)=m$ □

Theorem 4.5. *For $n\geq 4, m\geq 3$ with $m\leq n$, $\chi_{mc}(K_m\circ K_n)=n+1$.*

Proof. Let$\underbrace{K_n\cup K_n\cup\cdots\cup K_n}_{m \text{ times}}$ (i.e., $\bigcup\limits_{s=1}^{m} K_n$) be disjoint union of m copies of K_n with $V(\bigcup\limits_{s=1}^{m} K_n)=\{x_t^s : 1\leq t\leq n\}$. To prove that $\chi_{mc}(K_m\circ K_n)\geq n+1$. Assume that $\chi_{mc}(\bigcup\limits_{s=1}^{m} K_n)\leq n$. Then $\exists$ a modular coloring $g: V(\bigcup\limits_{s=1}^{m} K_n)\to \mathbb{Z}_n$ with $g(v_t^s)=t-1$ if $t\in\{1,2,\ldots,n\}$. From the colored disjoint copies of K_n, we can construct $K_m\circ K_n$ by joining the vertices in n^m possible ways. Among that, we consider some cases:

- $x_1^1, x_1^2, x_1^3, \ldots, x_1^m, x_1^1$, the color sum $\phi_g^+(x_1^1)=\phi_g^+(x_1^2)=\phi_g^+(x_n^1)=\frac{n(n+1)}{2}(\text{mod}\, n)$;
- $x_2^1, x_2^2, x_2^3, \ldots, x_2^m$, the color sum
  $\phi_g^+(x_2^1)=\phi_g^+(x_2^2)=\phi_g^+(x_2^3)=\cdots=\phi_g^+(x_2^m)=\Big(\frac{n(n+1)}{2}+(m-1)\Big)(\text{mod}\, n)$;
- $x_3^1, x_3^2, x_3^3, \ldots, x_3^m$, the color sum
  $\phi_g^+(x_3^1)=\phi_g^+(x_3^2)=\cdots=\phi_g^+(x_3^{m-1})=\phi_g^+(x_3^m)=\Big(\frac{n(n+1)}{2}+2(m-1)\Big)(\text{mod}\, n)$;
- $x_4^1, x_4^2, x_4^3, \ldots, x_4^m$, the color sum $\phi_g^+(x_4^1)=\phi_g^+(x_4^2)=\cdots=\phi_g^+(x_4^m)=\Big(\frac{n(n+1)}{2}+3(m-1)\Big)(\text{mod}\, n)$.

Thus, in each case, adjacent vertices have different color sum and we get a contradiction to our assumption. Similarly, we can show the contradiction to the remaining cases also. Hence $\chi_{mc}(K_m\circ K_n)\geq n+1$.
Next, to prove that $\chi_{mc}(K_m\circ K_n)\leq n+1$. Define a coloring $g: V(K_m\circ K_n)\to\mathbb{Z}_{n+1}$ as follows:

$$g(x_t^s)=\begin{cases}((t+s-3)(\text{mod}\, n))+1 & \text{if } t\neq 1,\\ 0 & \text{if } t=1.\end{cases}$$

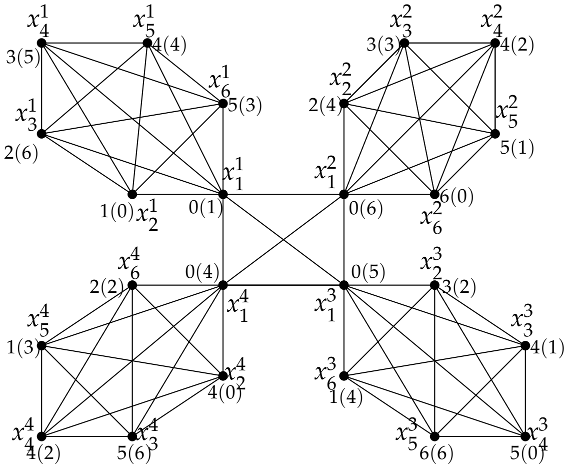

FIGURE 9. $\chi_{mc}(K_4 \circ K_6) = 7$.

Then,

$$\phi_g^+(x_t^s) = \begin{cases} \left(\frac{n(n+1)}{2} - (s-1) - g(x_t^s)\right)(\mathrm{mod}(n+1)) & \text{if } s \neq 1, \\ \left(\frac{n(n-1)}{2} - (t-1)\right)(\mathrm{mod}(n+1)) & \text{if } s = 1. \end{cases}$$

Construct $K_m \circ K_n$ from the above colored m-disjoint union of K_n by all possible line joining between these vertices $x_1^1, x_1^2, x_1^3, x_1^4, x_1^5, x_1^6, \ldots, x_1^{m-1}, x_1^m$. Which means that these vertices induces a clique of size m. Therefore, the colors and the color sum of the vertices in $K_m \circ K_n$ remain unchanged. It clearly shows that adjacent vertices have different color sum (For example, see Fig. 9) and $\chi_{mc}(K_m \circ K_n) \leq n+1$. Hence $\chi_{mc}(K_m \circ K_n) = n+1$. □

5. $K_1 \vee mG$

This section demonstrates the modular chromatic number of $K_1 \vee mG$ where G is some classes of graphs. Construct the graph, for $n \geq 3$ and $m \geq 2$, $G = K_1 \vee mC_n$ as follows: consider m-disjoint copies of C_n with $V(\bigcup_{s=1}^{m} C_n) = \{x_{rs} \mid 1 \leq s \leq m, 1 \leq r \leq n\}$. Then join the each vertices of m-disjoint copies of C_n to K_1. Consider $V(K_1 \vee mC_n) = \{v, x_{rs} \mid 1 \leq s \leq m, 1 \leq r \leq n\}$. Clearly,by 1.1, we have $\chi_{mc}(K_1 \vee mC_n) \geq \chi(K_1 \vee mC_n) = 1 + \chi(C_n)$.

Theorem 5.1. *For $n \geq 3$ and $m \geq 2$, $\chi(G) = \chi_{mc}(K_1 \vee mC_n)) = \chi(C_n) + 1$.*

Proof. Let n is even, the proof follows from Proposition 1.1.

Let n is odd, $\chi_{mc}(G) \geq 1 + \chi(C_n) = 1 + 3 = 4$. Hence $\chi_{mc}(G) \geq 4$. Define $g : V(G) \to \mathbb{Z}_4$ as follows: assign the color for the vertices x_{rs} is same as mentioned in the Proposition 1.1 and we assign the color vertex v that are:

$$g(v) = \left(m \sum_{r=1}^{k} g(x_{rs}) + 1\right)(\mathrm{mod}\, 4).$$

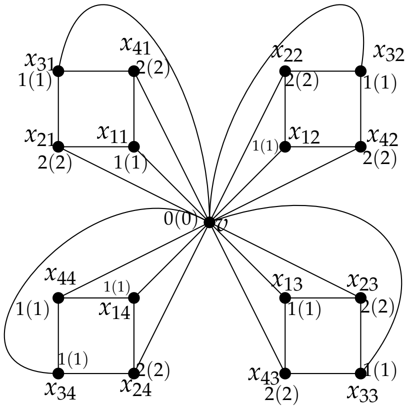

FIGURE 10. $\chi_{mc}(K_1 + 4C_4) = 3$.

Then, $\phi_g^+(v) = g(v) - 1(\bmod 4)$ and for $s \in \{1, 2, \ldots, m\}$,

$$\phi_g^+(x_{rs}) = \begin{cases} g(v) + 2 & \text{if } r \in \{1, n-1\}, \\ g(v) & \text{if } r \equiv 0(\bmod 2), \\ g(v) + 1 & \text{if } r \equiv 1(\bmod 2). \end{cases}$$

Implies that $\chi_{mc}(G) = 1 + \chi(C_n) = 1 + 3 = 4$. Therefore $\chi_{mc}(G) = 4$. It clearly shows that adjacent vertices have different color sum (For example, see Fig.10) and $\chi_{mc}(G) \le \chi(C_n) + 1$. Hence $\chi_{mc}(G) = \chi(C_n) + 1$. □

For $n \ge 3$ and $m \ge 2$, $G = K_1 \vee mK_n$. Clearly, by 1.1, we have $\chi_{mc}(K_1 \vee mK_n) \ge \chi(K_1 \vee mK_n) = 1 + n$. Hence $\chi_{mc}(K_1 \vee mK_n) \ge n + 1$.

Theorem 5.2. *For $n \ge 3$ and $m \ge 2$, $\chi_{mc}(G) = \chi_{mc}(K_1 \vee mK_n) = n + 1$.*

Proof. Let $V(G) = \{v, x_{rs} \mid 1 \le r \le n, 1 \le s \le m\}$. Define $g : V(G) \to \mathbb{Z}_{n+1}$ as follows:

$$g(v) = \begin{cases} \frac{n+1}{2} & \text{if } m \equiv 1(\bmod 2), \\ 0 & \text{elsewhere.} \end{cases}$$

For n is even, $g(x_{rs}) = r$.
For n is odd,

$$g(x_{rs}) = \begin{cases} r & \text{if } r \ne \frac{n+1}{2}, \\ 0 & \text{if } r = \frac{n+1}{2}. \end{cases}$$

Then, $\phi_g^+(v) = 0$; for n is even, $\phi_g^+(x_{rs}) = \left(\frac{n(n+1))}{2} - r\right) \pmod{(n+1)}$ and

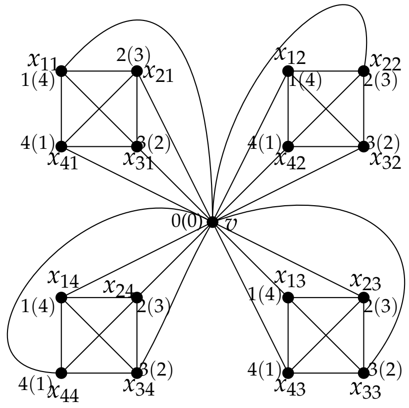

Figure 11. $\chi_{mc}(K_1 + 4K_4) = 5$.

for n is odd,

$$\phi_g^+(x_{rs}) = \begin{cases} \left(\frac{n(n+1)}{2} - r\right) \pmod{(n+1)} & \text{if } r \neq \frac{n+1}{2}, \\ \frac{n+1}{2} & \text{if } r = \frac{n+1}{2}. \end{cases}$$

Then the sequence of the color sums are: $\{(n+1-r) \pmod{(n+1)}\}$ for n is even; $\{\left(\frac{n+1}{2} - r\right) \pmod{(n+1)}\}$ for n is odd and $r \neq \frac{n+1}{2}$; and $\{\left(\frac{n+1}{2}\right) \pmod{(n+1)}\}$ n is odd and $r = \frac{n+1}{2}$. It clearly shows that adjacent vertices have different color sum (For example, see Fig. 11) and $\chi_{mc}(G) \leq n+1$. Hence $\chi_{mc}(G) = n+1$. □

6. Conclusion

Through the detailed analysis, this study determines the modular chromatic number for several distinct graph classes, including trees, cartesian product, rooted product and join graphs.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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