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Birkhoff Centre of Paradistributive Latticiods

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Abstract. We study the Birkhoff centre $B(V)$ of a Paradistributive Latticoid (PDL) V. Assuming the existence of a greatest element and at least one minimal element, we prove that $B(V)$ forms a relatively complemented sub-PDL and derive a decomposition theorem characterizing its elements via direct products. We establish functoriality of $B(-)$ with respect to products and lattice-quotients enforcing commutativity, and we show a bijection between $B(V)$ and complemented principal ideals of V. For associative PDLs, we obtain a correspondence between $B(V)$ and factor-congruences, hence direct decompositions. The results extend earlier work on almost distributive lattices to the broader framework of PDLs and connect with the theory of normal PDLs.

1. Introduction

The study of lattice-theoretic structures has been a central theme in algebra since the seminal work of Birkhoff on lattice theory [4]. Classical distributive lattices and their generalizations have found applications in universal algebra, logic, and computer science [5]. Among the significant extensions of distributive lattices, the notion of *almost distributive lattices* (ADLs) introduced by Swamy and Rao [12] has provided a flexible framework in which distributivity is relaxed, yet many desirable structural properties are preserved. Further investigations into the Birkhoff centre of ADLs were initiated by Swamy and Ramesh [11], establishing a foundation for decomposition theorems in this more general setting.

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The concept of relative complementation in distributive lattices, studied by Ershov [7], plays a fundamental role in understanding the structural behaviour of such lattices. In parallel, varieties of Birkhoff systems have been examined by Harding and Romanowska [8,9], providing a categorical perspective that enriches the algebraic study of lattices and their generalizations.

Motivated by these developments, the class of *paradistributive latticoids* (PDLs) was recently introduced by Bandaru and Ajjarapu [2], and subsequently enriched through further studies on normality and parapseudo-complementation [1, 3]. PDLs naturally generalize ADLs while incorporating inherent order-theoretic and algebraic features that allow the study of ideals, filters, and congruences in a broader setting. The introduction of normal PDLs [3] extends the earlier work on normal lattices [6,10] and connects with classical investigations on prime ideals and prime filters. Such structures are particularly well-suited for analysing congruences and factorizations, drawing upon techniques from universal algebra [5].

Within this context, the present paper develops a theory of the *Birkhoff centre* $B(V)$ for a PDL V. This continues the line of investigation from ADLs [11] and provides a systematic analysis of the decomposition and factorization properties of PDLs. In particular, we show that $B(V)$ forms a relatively complemented sub-PDL and establish a direct product decomposition theorem. Furthermore, we demonstrate functoriality with respect to products and quotients enforcing commutativity, and obtain bijections between the Birkhoff centre and complemented principal ideals as well as between $B(V)$ and factor-congruences in associative PDLs. These results mirror classical decomposition phenomena studied in lattice theory [4, 6, 7, 10], while situating them within the new framework of PDLs.

The remainder of this paper is organized as follows. In Section 2, we recall the basic axioms and structural properties of PDLs. Section 3 develops the theory of the Birkhoff centre for PDLs, presenting its algebraic characterizations, decomposition theorems, and connections with ideals, filters, and congruences. Section 4 summarizes the results and suggests further directions, including Stone-type dualities and topological representations for special classes of PDLs.

2. Preliminaries on PDLs

Definition 2.1. *[2] An algebra $(V, \vee, \wedge, 1)$ of type $(2, 2, 0)$ is a* Paradistributive Latticoid *(PDL), if for all $x, y, z \in V$:*

$$\begin{aligned}
&(LD\vee) \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z),\\
&(RD\vee) \quad (x \wedge y) \vee z = (x \vee z) \wedge (y \vee z),\\
&(L1) \quad (x \vee y) \wedge y = y, \qquad (L2)\ (x \vee y) \wedge x = x,\\
&(L3) \quad x \vee (x \wedge y) = x, \qquad (I1)\ x \vee 1 = 1.
\end{aligned}$$

We write $x \leq y$ iff $x \wedge y = x$ (equivalently, $x \vee y = y$).

We use ideals and filters in the PDL sense.

Definition 2.2. *[2] A non-empty $U \subseteq V$ is an* ideal *if $x, y \in U \Rightarrow x \vee y \in U$ and $x \in U$, $a \in V \Rightarrow x \wedge a \in U$.*

A non-empty $F \subseteq V$ is a filter *if $x, y \in F \Rightarrow x \wedge y \in F$ and $x \in F, a \in V \Rightarrow a \vee x \in F$.*

For $S \subseteq V$, the ideal generated by S is $(S] = \{(x_1 \vee \cdots \vee x_n) \wedge a : \ x_i \in S, \ a \in V, \ n \geq 1\}$, and the principal ideal/filter are $(a]$ and $[a)$, respectively.

We shall use the following congruences [2] frequently. In any PDL, for $a \in V$,

$$\varphi_a := \{(x, y) : \ x \vee a = y \vee a\}$$

is a congruence; if V is *associative* (that is $\wedge$ is associative), then also

$$\theta_a := \{(x, y) : \ x \wedge a = y \wedge a\}$$

is a congruence. Minimal elements are characterized by m is minimal $\Leftrightarrow x \vee m = x$ for all x.

3. Birkhoff Centre for PDLs

In ADLs with 0 and maximal elements, the Birkhoff centre [11] $B(L)$ is defined by the existence of a complement b with $a \wedge b = 0$ and $a \vee b$ maximal. For PDLs a greatest element 1 is built in; however a global least element need not exist. The natural adaptation is:

Definition 3.1. *Let $(V, \vee, \wedge, 1)$ be a PDL that has at least one minimal element m. Define*

$$B(V) := \Big\{a \in V | \text{ there exists } \ b \in V \text{ such that } a \wedge b = m \text{ and } a \vee b = 1\Big\}.$$

If a, b satisfy $a \wedge b = m$ and $a \vee b = 1$, we call a and b complements *(with respect to $(m, 1)$).*

Remark 3.1. *When V has a least element 0 (e.g. on each interval $[m, 1]$ or in finite examples) the condition $a \wedge b = m$ reduces to $a \wedge b = 0$ after translating the base-point; in PDLs with a unique maximal element 1, "$a \vee b$ maximal" is equivalent to $a \vee b = 1$.*

Theorem 3.1. *Let V be a PDL with greatest element 1 and at least one minimal element m. Then $B(V)$ is a subalgebra of V (closed under $\wedge$ and $\vee$) and is relatively complemented: for all $a \leq b$ in $B(V)$ there exists $c \in B(V)$ with $a \wedge c = m$ and $a \vee c = b$.*

Proof. Let $a_1, a_2 \in B(V)$ with complements b_1, b_2 (so $a_i \wedge b_i = m$ and $a_i \vee b_i = 1$). Using the PDL distributivities, one verifies

$$(a_1 \wedge a_2) \wedge (b_1 \vee b_2) = (a_1 \wedge a_2 \wedge b_1) \vee (a_1 \wedge a_2 \wedge b_2) = m,$$

and

$$(a_1 \wedge a_2) \vee (b_1 \vee b_2) = (a_1 \vee b_1 \vee b_2) \wedge (a_2 \vee b_1 \vee b_2) = 1,$$

since $a_i \vee b_i = 1$. Thus $b_1 \vee b_2$ complements $a_1 \wedge a_2$, so $a_1 \wedge a_2 \in B(V)$. A symmetric argument shows $a_1 \vee a_2 \in B(V)$ and yields relative complements inside $B(V)$ as stated. □

Theorem 3.2. *If V_1, V_2 are PDLs each with a greatest element and at least one minimal element, then $B(V_1 \times V_2) = B(V_1) \times B(V_2)$.*

Proof. Minimal (resp. greatest) elements of the product are componentwise minimal (resp. greatest). If $a_i \in B(V_i)$ has complement b_i with $a_i \wedge b_i = m_i$ and $a_i \vee b_i = 1_i$, then (a_1, a_2) and (b_1, b_2) witness that $(a_1, a_2) \in B(V_1 \times V_2)$. The converse is similar. □

Theorem 3.3. *For $a \in V$, the following are equivalent;*

(1) $a \in B(V)$
(2) there exist PDLs V_1, V_2 with greatest and minimal elements and an isomorphism $f : V \to V_1 \times V_2$ such that $f(a) = (1_{V_1}, m_{V_2})$.

Proof. (1) ⇒ (2): If $a \in B(V)$ pick a complement b with $a \wedge b = m$, $a \vee b = 1$. Let $V_1 := (a] = \{a \wedge x : x \in V\}$ and $V_2 := (b] = \{b \wedge x : x \in V\}$. Define $f(x) = (a \wedge x,\ b \wedge x)$. Routine calculations show f is a PDL isomorphism with $f(a) = (a,\ m)$, where a is top of V_1 and m minimal of V_2.

(2) ⇒ (1): If $f(a) = (1, m_2)$ and m_2 is minimal in V_2, pick any element $b \in V$ with $f(b) = (m_1, 1)$ (where m_1 is minimal in V_1). Then $a \wedge b$ maps to (m_1, m_2) (minimal) and $a \vee b$ maps to $(1, 1)$ (top), so $a \in B(V)$. □

Remark 3.2. *The proof mirrors the classical ADL result, with "maximal" replaced by equality to* 1 *and* 0 *replaced by a chosen minimal m.*

Let η be the smallest congruence on V such that V/η is a lattice (i.e. the commutativity conditions forced). Then η identifies those pairs (x, y) with $x \wedge y = y$ and $y \wedge x = x$.

Theorem 3.4. *The canonical map induces a natural isomorphism*

$$B(V)/\eta|_{B(V)} \ \cong\ B(V/\eta).$$

Proof. If $a/\eta \in B(V/\eta)$, pick b/η with $a/\eta \wedge b/\eta = m/\eta$ and $a/\eta \vee b/\eta = 1/\eta$. Then $a \wedge b$ is η-equivalent to a minimal element and $a \vee b$ to 1, implying $a \in B(V)$. The map $a \mapsto a/\eta$ is surjective from $B(V)$ onto $B(V/\eta)$ with kernel $\eta \cap (B(V) \times B(V))$, yielding the stated isomorphism by the homomorphism theorem. □

Let us recall $(a]$ is the principal ideal [2] generated by a, and $[a)$ is the principal filter [2]. In PDLs we have a dual isomorphism between principal ideals and principal filters given by $(a] \leftrightarrow [a)$, and $(a] \subseteq (b] \iff b \wedge a = a \iff b \vee a = b$. We use this to connect $B(V)$ with complemented principal ideals.

Definition 3.2. *A principal ideal $(a]$ is* complemented *if there exists some b such that*

$$(a] \cap (b] = (\,m] \qquad \text{and} \qquad [a) \vee [b) = [1),$$

with m a minimal element and 1 *the greatest element.*

Theorem 3.5. *For a PDL V with* 1 *and a minimal m, the map*

$$\Phi :\ B(V) \longrightarrow \{\textit{complemented principal ideals}\}, \qquad \Phi(a) = (a]$$

is a bijection. Its inverse sends a complemented principal ideal $(a]$ to a.

Proof. If $a \in B(V)$, choose b with $a \wedge b = m$ and $a \vee b = 1$. Then $(a] \cap (b] = (a \wedge b] = (m]$ and $[a) \vee [b) = [a \wedge b)$, so $(a]$ is complemented. Conversely, if $(a]$ is complemented by some $(b]$ in the stated sense, then $a \wedge b = m$ and $a \vee b = 1$, so $a \in B(V)$. Uniqueness is immediate. □

Assume V is an *associative* PDL [2] so that $\theta_c := \{(x, y) : x \wedge c = y \wedge c\}$ is a congruence for each c. A congruence Θ is a *factor congruence* if there exists Φ with $\Theta \cap \Phi = \Delta$ and $\Theta \circ \Phi = V \times V$, equivalently $V \cong V/\Theta \times V/\Phi$.

Theorem 3.6. *Let V be an associative PDL with 1 and a minimal m. Then a congruence Θ is a factor-congruence iff $\Theta = \theta_a$ for some $a \in B(V)$. Consequently, elements of $B(V)$ are in bijection with direct decompositions of V up to isomorphism.*

Proof. ($\Rightarrow$) Suppose Θ is a factor-congruence. Pick Φ with $\Theta \cap \Phi = \Delta$ and $\Theta \circ \Phi = V \times V$. Choose b with $(1, b) \in \Phi$ and a with $(a, 1) \in \Theta$. Then $(a \wedge b, m) \in \Theta \cap \Phi$, forcing $a \wedge b = m$. From $(1, b) \in \Phi$ and $(a, 1) \in \Theta$ one deduces $(1, a \vee b) \in \Theta \cap \Phi$, hence $a \vee b = 1$. Thus $a \in B(V)$. For any x, y, if $(x, y) \in \theta_a$ then $a \wedge x = a \wedge y$; using $(a, 1) \in \Theta$ gives $(x, y) \in \Theta$, so $\theta_a \subseteq \Theta$. Conversely, if $(x, y) \in \Theta$ then $(a \wedge x, a \wedge y) \in \Theta$, but also in Φ via $(m, a \wedge x)$ and $(m, a \wedge y)$; hence $a \wedge x = a \wedge y$, so $(x, y) \in \theta_a$. Thus $\Theta = \theta_a$.

($\Leftarrow$) If $a \in B(V)$ pick b with $a \wedge b = m$ and $a \vee b = 1$. Then $\theta_a \cap \theta_b = \Delta$ and $\theta_a \circ \theta_b = V \times V$ by a routine verification using the PDL identities and the complement equations. Hence θ_a is a factor-congruence and we have a product representation $V \cong V/\theta_a \times V/\theta_b$. □

Example 3.1. *Let $V = \{0, 1, 2, 3, 4\}$ with operations $\vee, \wedge$ given by the tables*

$\vee$	0	1	2	3	4
0	0	1	0	3	3
1	1	1	1	1	1
2	2	1	2	4	4
3	3	1	3	3	3
4	4	1	4	4	4

$\wedge$	0	1	2	3	4
0	0	0	2	0	2
1	0	1	2	3	4
2	0	2	2	0	2
3	0	3	2	3	4
4	0	4	2	3	4

This is an associative PDL with greatest element 1. The element 0 is minimal. One checks; $0 \wedge 1 = 0, 0 \vee 1 = 1, 1 \wedge 0 = 0, 1 \vee 0 = 1$, so $0, 1 \in B(V)$. No other a has a complement b with $a \wedge b = 0$ and $a \vee b = 1$ (by inspecting the tables), hence $B(V) = \{0, 1\}$. By Theorem 3.6, the only nontrivial factor-congruences are θ_0 and $\theta_1 = \Delta$, yielding the trivial product factorization.

Example 3.2. *If V_1 and V_2 are finite associative PDLs each with minimal 0_i and top 1_i, then $B(V_1 \times V_2) = B(V_1) \times B(V_2)$. By Theorem 3.2. In particular, if $B(V_i) = \{0_i, 1_i\}$ then*

$$B(V_1 \times V_2) = \{(0_1, 0_2), (0_1, 1_2), (1_1, 0_2), (1_1, 1_2)\}.$$

Definition 3.3. *[3] A PDL V is called normal if every prime filter of V contains a unique minimal prime filter of V.*

Theorem 3.7. *Let $f : V \to W$ be a surjective PDL homomorphism with V normal. Then W is normal and $f(B(V)) = B(W)$.*

Proof. Let $K = \ker(f)$ and $\pi : V \to V/K$ be the canonical projection. Since f is surjective, there exists an isomorphism $\varphi : V/K \to W$ such that $f = \varphi \circ \pi$. To prove normality of W, let Q be a prime filter of W. Then $f^{-1}(Q)$ is a prime filter of V. As V is normal, $f^{-1}(Q)$ contains a unique minimal prime M. Thus $\varphi(\pi(M))$ is the unique minimal prime of W contained in Q. Hence W is normal.

Now, we will prove $f(B(V)) = B(W)$, if $a \in B(V)$, then there exists $b \in V$ such that $a \wedge b = 0$ and $a \vee b = 1$. Applying f gives $f(a) \wedge f(b) = 0$ and $f(a) \vee f(b) = 1$, hence $f(a) \in B(W)$. Thus $f(B(V)) \subseteq B(W)$.

Conversely, if $c \in B(W)$, then c has a complement $d \in W$. Choose $a, b \in V$ with $f(a) = c$, $f(b) = d$. Consider the ideals $I = f^{-1}((c])$ and $J = f^{-1}((d])$ in V. Since $(c]$ and $(d]$ are complementary in W, normality of V ensures I and J contain complementary principal ideals $(a_0]$ and $(b_0]$. Thus $a_0 \in B(V)$ and $f(a_0) = c$. Hence $B(W) \subseteq f(B(V))$. Therefore $f(B(V)) = B(W)$. □

We have $S^e = \{[a) \mid a \in S\}$ for $S \subseteq V$ and $T^c = \{a \in V \mid [a) \in T\}$ for $T \subseteq \mathrm{PF}(V)$.

Theorem 3.8. *If $J \subseteq V$ is a filter regarded as a sub-PDL and V is normal, then J is normal if and only if the image $J^e = \{[a) \mid a \in J\}$ is normal inside $PF(V)$. In this case $B(J) = J \cap B(V)$.*

Proof. It is easy to observe that $(-)^e$ and $(-)^c$ are inverse bijections between filters of J and filters of J^e, preserving primality and minimal primes. Hence J is normal $\Leftrightarrow$ J^e is normal.

To prove $B(J) = J \cap B(V)$. If $a \in B(J)$, then $a \in J$ and a has a complement $b \in J$, so also $a \in B(V)$. Thus $B(J) \subseteq J \cap B(V)$.

If $a \in J \cap B(V)$, then a has a complement $b \in V$ with $[a) \vee [b) = [1)$ in $\mathrm{PF}(V)$. Since $[a) \in J^e$ and J^e is normal, the complement $[b)$ must also lie in J^e, i.e. $b \in J$. Hence $a \in B(J)$. Therefore $B(J) = J \cap B(V)$. □

4. Conclusions and Further Directions

We introduced the Birkhoff centre $B(V)$ of a Paradistributive Latticoid and established that it is a relatively complemented sub-PDL, preserved under products and quotients, and in bijection with complemented principal ideals. For associative PDLs, $B(V)$ was shown to correspond to factor-congruences and direct decompositions, extending classical results on almost distributive lattices. Future research may explore Stone-type dualities, categorical properties of the functor $B(-)$, the role of normal PDLs, extensions to parapseudo-complemented structures, and computational aspects of PDLs.

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