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Applications of the Arithmetic-Geometric and Holder Inequalities for Unitarily Invariant Norms

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Abstract. Let A_i, B_i, X_i and Y_i be $n \times n$ complex matrices such that X_i and Y_i are positive, $i = 1, 2, ..., n$, $p, q > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \in [0, 1]$ and $r \geq 0$. Then

$$\left|\left|\left|\left|\sum_{i=1}^{n} A_i X_i^{1/2} Y_i^{1/2} B_i^*\right|^r\right|\right|\right| \leq \left|\left|\left|(M(\alpha))^{\frac{rp}{2}}\right|\right|\right|^{1/p} \left|\left|\left|(M(1-\alpha))^{\frac{rq}{2}}\right|\right|\right|^{1/q},$$

where

$$M(\alpha) = \left\|\begin{bmatrix} \sqrt{\alpha} A_1 X_1^{1/2} & \sqrt{\alpha} A_2 X_2^{1/2} & \cdots & \sqrt{\alpha} A_n X_n^{1/2} \\ \sqrt{1-\alpha} B_1 Y_1^{1/2} & \sqrt{1-\alpha} B_2 Y_2^{1/2} & \cdots & \sqrt{1-\alpha} B_n Y_n^{1/2} \end{bmatrix}\right\|^2.$$

Several new results follow as special cases of this general inequality.

1. Introduction

Let $\mathbb{M}_n$ be the space of all $n \times n$ complex matrices. Unitarily invariant norms on $\mathbb{M}_n$ are denoted by $|||.|||$, recall that these norms satisfying $|||UAV||| = |||A|||$ for all $A, U, V \in \mathbb{M}_n$ such that U and V are unitary. For the general theory of unitarily invariant norms, we refer the reader to [8], [10] and [14]. The singular values of $A \in \mathbb{M}_n$ are denoted by $s_1(A) \geq s_2(A) \geq ... \geq s_n(A) \geq 0$. Bhatia and Kittaneh [9] proved the arithmetic-geometric mean inequality for singular values: If $A, B \in \mathbb{M}_n$, then

$$2s_j(AB^*) \leq s_j(A^*A + B^*B) \tag{1.1}$$

for $j = 1, 2, ..., n$. The norm version of this inequality asserts that

$$2\,|||AB^*||| \leq |||A^*A + B^*B|||. \tag{1.2}$$

Horn and Mathisa [12] proved that if $A, B \in \mathbb{M}_n$ and $r \geq 0$, then

$$\left|\left|\left||AB^*|^r\right|\right|\right|^2 \leq \left|\left|\left|(A^*A)^r\right|\right|\right| \left|\left|\left|(B^*B)^r\right|\right|\right|, \tag{1.3}$$

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which is the Cauchy-Schwarz inequality for unitarily invariant norms. Horn and Zhan [13] proved the Holder inequality for unitarily invariant norms, this inequality asserts that if $A, B \in \mathbb{M}_n$, $r \geq 0$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\left|\left|\left| |AB^*|^r \right|\right|\right| \leq \left|\left|\left| (A^*A)^{\frac{rp}{2}} \right|\right|\right|^{1/p} \left|\left|\left| (B^*B)^{\frac{rq}{2}} \right|\right|\right|^{1/q}. \tag{1.4}$$

Inequality (1.4) is a generalization of inequality (1.3). Audenaert [7] proved a unification of inequalities (1.2) and (1.4),

$$\left|\left|\left| |AB^*|^r \right|\right|\right| \leq \left|\left|\left| (Z(\alpha))^{\frac{rp}{2}} \right|\right|\right|^{1/p} \left|\left|\left| (Z(1-\alpha))^{\frac{rq}{2}} \right|\right|\right|^{1/q}, \tag{1.5}$$

where

$$Z(\alpha) = \alpha A^*A + (1-\alpha)B^*B.$$

Audeh [4] proved that if $A_i, B_i, X_i, Y_i \in \mathbb{M}_n$ such that X_i and Y_i are positive, $i = 1, 2, ..., n$. Then

$$2s_j\left(\sum_{i=1}^{n} A_i X_i^{1/2} Y_i^{1/2} B_i^*\right) \leq s_j^2(W) \tag{1.6}$$

for $j = 1, 2, ..., n$, where $W = \begin{bmatrix} A_1X_1^{1/2} & A_2X_2^{1/2} & \dots & A_nX_n^{1/2} \\ B_1Y_1^{1/2} & B_2Y_2^{1/2} & \dots & B_nY_n^{1/2} \end{bmatrix}$. Inequality (1.6) has the following norm version

$$2\left|\left|\left| \sum_{i=1}^{n} A_i X_i^{1/2} Y_i^{1/2} B_i^* \right|\right|\right| \leq |||W|||^2. \tag{1.7}$$

It has been shown by the same author that if $A, B, X \in \mathbb{M}_n$ are positive. Then

$$2s_j(M+N) \leq s_j\left((H + |K^*|) \oplus (L + |K|)\right) \tag{1.8}$$

for $j = 1, 2, ..., n$, where $M = A^{1/2}X^{1/2}Y^{1/2}A^{1/2}$, $N = B^{1/2}X^{1/2}Y^{1/2}B^{1/2}$, $H = X^{1/2}AX^{1/2} + Y^{1/2}AY^{1/2}$, $K = X^{1/2}A^{1/2}B^{1/2}X^{1/2} + Y^{1/2}A^{1/2}B^{1/2}Y^{1/2}$ and $L = X^{1/2}BX^{1/2} + Y^{1/2}BY^{1/2}$. In particular, letting $Y = X$, we have

$$s_j\left(A^{1/2}XA^{1/2} + B^{1/2}XB^{1/2}\right) \leq s_j\left((P + |Q^*|) \oplus (R + |Q|)\right)$$

for $j = 1, 2, ..., n$, where $P = X^{1/2}AX^{1/2}$, $Q = X^{1/2}A^{1/2}B^{1/2}X^{1/2}$ and $R = X^{1/2}BX^{1/2}$. Moreover, this author substantiated that if $A, B, X_1, X_2, Y_1, Y_2 \in \mathbb{M}_n$ such that X_1, X_2, Y_1, Y_2 are positive. Then

$$2s_j(E-F) \leq s_j\left((I + |S^*|) \oplus (T + |S|)\right) \tag{1.9}$$

for $j = 1, 2, ..., n$, where $E = AX_1^{1/2}Y_1^{1/2}A^*$, $F = BX_2^{1/2}Y_2^{1/2}B^*$, $I = X_1^{1/2}A^*AX_1^{1/2} + Y_1^{1/2}A^*AY_1^{1/2}$, $S = X_1^{1/2}A^*BX_2^{1/2} - Y_1^{1/2}A^*BY_2^{1/2}$ and $T = X_2^{1/2}B^*BX_2^{1/2} + Y_2^{1/2}B^*BY_2^{1/2}$. Inequalities (1.8) and (1.9) have the following norm versions, respectively,

$$2\,|||M+N||| \leq \left|\left|\left| \left((H + |K^*|) \oplus (L + |K|)\right) \right|\right|\right| \tag{1.10}$$

and

$$2\,|||E-F||| \leq \left|\left|\left| \left((I + |S^*|) \oplus (T + |S|)\right) \right|\right|\right|. \tag{1.11}$$

To follow up recent studies related to generalizations of singular values, unitarily invariant norms and numerical radius for matrices, we refer the reader to [1-**??**] and [**??**-**??**] .

In the second section, we give generalizations of inequalities (1.7), (1.10) and (1.11). Several new results are extracted from these generalizations.

2. Main results

The following lemmas are essential for supporting our main conclusions.

Lemma 2.1. *Let $A \in \mathbb{M}_n$ be self-adjoint. Then*

$$\pm A \leq |A|. \tag{2.1}$$

Lemma 2.2. *Let $A, B, X, Y \in \mathbb{M}_n$, $p, q > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \in [0, 1]$ and $r \geq 0$. Then*

$$|||\, |AXY^*B^*|^r \,||| \leq \left|\left|\left| (F(\alpha))^{\frac{rp}{2}} \right|\right|\right|^{1/p} \left|\left|\left| (F(1-\alpha))^{\frac{rq}{2}} \right|\right|\right|^{1/q} \tag{2.2}$$

where

$$F(\alpha) = \alpha X^*A^*AX + (1-\alpha)Y^*B^*BY$$

Proof. Letting $A = AX$ and $B = BY$ in inequality (1.5), leads to inequality (2.2). □

Lemma 2.3. *Let $A, B, X, Y \in \mathbb{M}_n$ such that X and Y are positive, $p, q > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \in [0, 1]$ and $r \geq 0$. Then*

$$\left|\left|\left| \left|AX^{1/2}Y^{1/2}B^*\right|^r \right|\right|\right| \leq \left|\left|\left| (H(\alpha))^{\frac{rp}{2}} \right|\right|\right|^{1/p} \left|\left|\left| (H(1-\alpha))^{\frac{rq}{2}} \right|\right|\right|^{1/q}, \tag{2.3}$$

where

$$H(\alpha) = \alpha X^{1/2}A^*AX^{1/2} + (1-\alpha)Y^{1/2}B^*BY^{1/2}.$$

Proof. Substituting X by $X^{1/2}$ and Y by $Y^{1/2}$ in inequality (2.2), leads to inequality (2.3). □

Lemma 2.4. *Let $A, B, X \in \mathbb{M}_n$ such that X and Y are positive, $p, q > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \in [0, 1]$ and $r \geq 0$. Then*

$$|||\, |AXB^*|^r \,||| \leq \left|\left|\left| (R(\alpha))^{\frac{rp}{2}} \right|\right|\right|^{1/p} \left|\left|\left| (R(1-\alpha))^{\frac{rq}{2}} \right|\right|\right|^{1/q}. \tag{2.4}$$

where

$$R(\alpha) = \alpha X^{1/2}A^*AX^{1/2} + (1-\alpha)X^{1/2}B^*BX^{1/2}$$

Proof. Letting $Y = X$ in inequality (2.3), leads to inequality (2.4). □

Lemma 2.5. *Let $A, B, X, Y \in \mathbb{M}_n$ be positive, $p, q > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \in [0, 1]$ and $r \geq 0$. Then*

$$\left|\left|\left| \left|A^{1/2}X^{1/2}Y^{1/2}B^{1/2}\right|^r \right|\right|\right| \leq \left|\left|\left| (G\alpha))^{\frac{rp}{2}} \right|\right|\right|^{1/p} \left|\left|\left| (G(1-\alpha))^{\frac{rq}{2}} \right|\right|\right|^{1/q} \tag{2.5}$$

where

$$G(\alpha) = \alpha X^{1/2}AX^{1/2} + (1-\alpha)Y^{1/2}BY^{1/2}$$

Proof. The result is deduced from inequality (2.3) by letting $A = A^{1/2}$ and $B = B^{1/2}$. □

At this stage of our discussion, we present the main result of this paper, which is a generalization of inequality (1.7).

Theorem 2.1. *Let $A_i, B_i, X_i, Y_i \in \mathbb{M}_n$ such that X_i and Y_i are positive, $i = 1, 2, ..., n$, $p, q > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \in [0, 1]$ and $r \geq 0$. Then*

$$\left|\left|\left|\left|\sum_{i=1}^{n} A_i X_i^{1/2} Y_i^{1/2} B_i^*\right|^r\right|\right|\right| \leq \left|\left|\left|(M(\alpha))^{\frac{rp}{2}}\right|\right|\right|^{1/p} \left|\left|\left|(M(1-\alpha))^{\frac{rq}{2}}\right|\right|\right|^{1/q}, \tag{2.6}$$

where

$$M(\alpha) = \left|\begin{bmatrix} \sqrt{\alpha} A_1 X_1^{1/2} & \sqrt{\alpha} A_2 X_2^{1/2} & \cdots & \sqrt{\alpha} A_n X_n^{1/2} \\ \sqrt{1-\alpha} B_1 Y_1^{1/2} & \sqrt{1-\alpha} B_2 Y_2^{1/2} & \cdots & \sqrt{1-\alpha} B_n Y_n^{1/2} \end{bmatrix}\right|^2.$$

Proof. On $\oplus_{j=1}^{n} H$, Define

$$A = \begin{bmatrix} A_1 & A_2 & \dots & A_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, B = \begin{bmatrix} B_1 & B_2 & \dots & B_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$X = \begin{bmatrix} X_1 & 0 & \dots & 0 \\ 0 & X_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_n \end{bmatrix} \text{ and } Y = \begin{bmatrix} Y_1 & 0 & \dots & 0 \\ 0 & Y_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Y_n \end{bmatrix}.$$

Let $V_{ij} = \alpha X_i^{1/2} A_i^* A_j X_j^{1/2} + (1-\alpha) Y_i^{1/2} B_i^* B_j Y_j^{1/2}$. Then

$$AX^{1/2}Y^{1/2}B^* = \begin{bmatrix} \sum\limits_{i=1}^{n} A_i X_i^{1/2} Y_i^{1/2} B_i^* & 0 & \dots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{2.7}$$

and

$$\begin{aligned} H(\alpha) &= \alpha X^{1/2} A^* A X^{1/2} + (1-\alpha) Y^{1/2} B^* B Y^{1/2} && (2.8) \\ &= \begin{bmatrix} V_{11} & \cdots & V_{1n} \\ V_{21} & \cdots & V_{2n} \\ \vdots & \ddots & \vdots \\ V_{n1} & \cdots & V_{nn} \end{bmatrix} && (2.9) \\ &= \begin{bmatrix} \sqrt{\alpha} X_1^{1/2} A_1^* & \sqrt{1-\alpha} Y_1^{1/2} B_1^* \\ \sqrt{\alpha} X_2^{1/2} A_2^* & \sqrt{1-\alpha} Y_2^{1/2} B_2^* \\ \cdots & \cdots \\ \sqrt{\alpha} X_n^{1/2} A_n^* & \sqrt{1-\alpha} Y_n^{1/2} B_n^* \end{bmatrix} \times \end{aligned}$$

$$\begin{aligned} & \begin{bmatrix} \sqrt{\alpha}A_1X_1^{1/2} & \sqrt{\alpha}A_2X_2^{1/2} & \cdots & \sqrt{\alpha}A_nX_n^{1/2} \\ \sqrt{1-\alpha}B_1Y_1^{1/2} & \sqrt{1-\alpha}B_2Y_2^{1/2} & \cdots & \sqrt{1-\alpha}B_nY_n^{1/2} \end{bmatrix} \\ = & \left| \begin{bmatrix} \sqrt{\alpha}A_1X_1^{1/2} & \sqrt{\alpha}A_2X_2^{1/2} & \cdots & \sqrt{\alpha}A_nX_n^{1/2} \\ \sqrt{1-\alpha}B_1Y_1^{1/2} & \sqrt{1-\alpha}B_2Y_2^{1/2} & \cdots & \sqrt{1-\alpha}B_nY_n^{1/2} \end{bmatrix} \right|^2 \\ = & M(\alpha), \end{aligned}$$

similarly,

$$H(1-\alpha) = \left| \begin{bmatrix} \sqrt{1-\alpha}A_1X_1^{1/2} & \sqrt{1-\alpha}A_2X_2^{1/2} & \cdots & \sqrt{1-\alpha}A_nX_n^{1/2} \\ \sqrt{\alpha}B_1Y_1^{1/2} & \sqrt{\alpha}B_2Y_2^{1/2} & \cdots & \sqrt{\alpha}B_nY_n^{1/2} \end{bmatrix} \right|^2 \tag{2.10}$$

Substituting (2.7), (2.8) and (2.10) in inequality (2.3), we get inequality (2.6). □

Remark 2.1. *Letting $r = 1$, $p = q = 2$, $\alpha = \frac{1}{2}$ in inequality (2.6), we get inequality (1.7).*

We will give a considerable special case of inequality (2.6).

Corollary 2.1. *Let $A_i, B_i, X_i, Y_i \in \mathbb{M}_n$ such that X_i and Y_i are positive, $i = 1, 2$, $p, q > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \in [0, 1]$ and $r \geq 0$. Then*

$$\left|\left|\left| |D|^r \right|\right|\right| \leq \left|\left|\left| (E(\alpha))^{\frac{rp}{2}} \right|\right|\right|^{1/p} \left|\left|\left| (E(1-\alpha))^{\frac{rq}{2}} \right|\right|\right|^{1/q}, \tag{2.11}$$

where

$$D = A_1X_1^{1/2}Y_1^{1/2}B_1^* + A_2X_2^{1/2}Y_2^{1/2}B_2^*$$

and

$$E(\alpha) = \left| \begin{bmatrix} \sqrt{\alpha}A_1X_1^{1/2} & \sqrt{\alpha}A_2X_2^{1/2} \\ \sqrt{1-\alpha}B_1Y_1^{1/2} & \sqrt{1-\alpha}B_2Y_2^{1/2} \end{bmatrix} \right|^2.$$

Proof. Inequality (2.11) follows from inequality (2.6) by letting $n = 2$. □

Corollary 2.2. *Let $A_i, B_i \in \mathbb{M}_n$, $i = 1, 2$, $p, q > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \in [0, 1]$ and $r \geq 0$. Then*

$$\left|\left|\left| |J|^r \right|\right|\right| \leq \left|\left|\left| (K(\alpha))^{\frac{rp}{2}} \right|\right|\right|^{1/p} \left|\left|\left| (K(1-\alpha))^{\frac{rq}{2}} \right|\right|\right|^{1/q}, \tag{2.12}$$

where

$$J = A_1B_1^* + A_2B_2^*$$

and

$$K(\alpha) = \left| \begin{bmatrix} \sqrt{\alpha}A_1 & \sqrt{\alpha}A_2 \\ \sqrt{1-\alpha}B_1 & \sqrt{1-\alpha}B_2 \end{bmatrix} \right|^2.$$

Proof. Letting $X_i = Y_i = I$, $i = 1, 2$ in inequality (2.11), we get inequality (2.12). □

Corollary 2.3. *Let $A, B \in \mathbb{M}_n$, $\alpha \in [0, 1]$. Then*

$$|||AB^*||| \leq \left|\left|\left| Z(\alpha) \right|\right|\right|^{1/2} \left|\left|\left| Z(1-\alpha) \right|\right|\right|^{1/2} \tag{2.13}$$

where

$$Z(\alpha) = \alpha A^*A + (1-\alpha)B^*B.$$

Proof. Letting $A_2 = B_2 = 0$, $r = 1$, $p = q = 2$ in inequality (2.12), we give

$$\begin{aligned} |||AB^*||| &\leq \left|\left|\left| \begin{bmatrix} \sqrt{\alpha}A & 0 \\ \sqrt{1-\alpha}B & 0 \end{bmatrix} \right|^2 \right|\right|^{1/2} \left|\left|\left| \begin{bmatrix} \sqrt{1-\alpha}A & 0 \\ \sqrt{\alpha}B & 0 \end{bmatrix} \right|^2 \right|\right|^{1/2} \\ &= \left|\left|\left| \alpha A^*A + (1-\alpha)B^*B \right|\right|\right|^{1/2} \left|\left|\left| (1-\alpha)A^*A + \alpha B^*B \right|\right|\right|^{1/2}. \end{aligned}$$

□

Remark 2.2. *Letting $\alpha = \frac{1}{2}$ in inequality (2.13), we get the arithmetic-geometric mean inequality (1.2).*

Another special case of inequality (2.6) is the following result.

Corollary 2.4. *Let $A, B, X, Y \in \mathbb{M}_n$ such that X and Y are positive, $p, q > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \in [0, 1]$ and $r \geq 0$. Then*

$$\left|\left|\left| |G|^r \right|\right|\right| \leq \left|\left|\left| (H(\alpha))^{\frac{rp}{2}} \right|\right|\right|^{1/p} \left|\left|\left| (H(1-\alpha))^{\frac{rq}{2}} \right|\right|\right|^{1/q}, \tag{2.14}$$

where

$$G = AX^{1/2}Y^{1/2}B^* + BX^{1/2}Y^{1/2}A^*$$

and

$$H(\alpha) = \left| \begin{bmatrix} \sqrt{\alpha}AX^{1/2} & \sqrt{\alpha}BX^{1/2} \\ \sqrt{1-\alpha}BY^{1/2} & \sqrt{1-\alpha}AY^{1/2} \end{bmatrix} \right|^2.$$

Proof. Substituting $n = 2$, $A_1 = B_2 = A$, $A_2 = B_1 = B$, $X_1 = X_2 = X$ and $Y_1 = Y_2 = Y$ in inequality (2.6), we get inequality (2.14). □

Corollary 2.5. *Let $A, B, X \in \mathbb{M}_n$ such that X is positive, $p, q > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \in [0, 1]$ and $r \geq 0$. Then*

$$\begin{aligned} &\left|\left|\left| |AXB^* + BXA^*|^r \right|\right|\right| \\ \leq\; &\left|\left|\left| (J(\alpha))^{\frac{rp}{2}} \right|\right|\right|^{1/p} \left|\left|\left| (J(1-\alpha))^{\frac{rq}{2}} \right|\right|\right|^{1/q}, \end{aligned} \tag{2.15}$$

where

$$J(\alpha) = \left| \begin{bmatrix} \sqrt{\alpha}AX^{1/2} & \sqrt{\alpha}BX^{1/2} \\ \sqrt{1-\alpha}BX^{1/2} & \sqrt{1-\alpha}AX^{1/2} \end{bmatrix} \right|^2.$$

Proof. Letting $Y = X$ in inequality (2.14), we get inequality (2.15). □

Corollary 2.6. *Let $A, B, X \in \mathbb{M}_n$, $\alpha \in [0, 1]$. Then*

$$|||AB^* + BA^*||| \leq \left|\left|\left| (M(\alpha)) \right|\right|\right|^{1/2} \left|\left|\left| (M(1-\alpha)) \right|\right|\right|^{1/2}, \tag{2.16}$$

where

$$M(\alpha) = \begin{bmatrix} \alpha A^*A + (1-\alpha)B^*B & \alpha A^*B + (1-\alpha)B^*A \\ \alpha B^*A + (1-\alpha)A^*B & (1-\alpha)A^*A + \alpha B^*B \end{bmatrix}.$$

Proof. Letting $X = I$, $r = 1$, $p = q = 2$ in inequality (2.15), we get inequality (2.16). □

Remark 2.3. *Letting $\alpha = \frac{1}{2}$ in inequality (2.16), we give*

$$2\,|||\,|AB^* + BA^*|\,||| \leq \left|\left|\left| \begin{bmatrix} A^*A + B^*B & A^*B + B^*A \\ A^*B + B^*A & A^*A + B^*B \end{bmatrix} \right|\right|\right|.$$

Using Lemma 2.1 and inequality (2.6), lead to the following general inequality, which is a generalization of inequality (1.10).

Corollary 2.7. *Let $A, B, X, Y \in \mathbb{M}_n$ be positive, $p, q > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \in [0,1]$ and $r \geq 0$. Then*

$$\left|\left|\left| |M + N|^r \right|\right|\right| \leq |||Z_1|||^{1/p}\,|||Z_2|||^{1/q}, \tag{2.17}$$

where $M = A^{1/2}X^{1/2}Y^{1/2}A^{1/2}$, $N = B^{1/2}X^{1/2}Y^{1/2}B^{1/2}$,

$$Z_1 = (\Big(H(\alpha) + \big|K^*(\alpha)\big|\Big) \oplus \Big(L(\alpha) + \big|K(\alpha)\big|)\Big)^{\frac{rp}{2}},$$

$$Z_2 = (\Big(H(1-\alpha) + \big|K^*(1-\alpha)\big|\Big) \oplus \Big(L(1-\alpha) + \big|K(1-\alpha)\big|)\Big)^{\frac{rq}{2}}.$$

$H(\alpha) = \alpha X^{1/2}AX^{1/2} + (1-\alpha)Y^{1/2}AY^{1/2}$, $K(\alpha) = \alpha X^{1/2}A^{1/2}B^{1/2}X^{1/2} + (1-\alpha)Y^{1/2}A^{1/2}B^{1/2}Y^{1/2}$ and $L(\alpha) = \alpha X^{1/2}BX^{1/2} + (1-\alpha)Y^{1/2}BY^{1/2}$. In particular, letting $Y = X$, we have

$$\left|\left|\left| \left(A^{1/2}XA^{1/2} + B^{1/2}XB^{1/2}\right)^r \right|\right|\right| \leq \left|\left|\left| W^{\frac{rp}{2}} \right|\right|\right|^{1/p} \left|\left|\left| W^{\frac{rq}{2}} \right|\right|\right|^{1/q} \tag{2.18}$$

for $j = 1, 2, \dots$, where $W = ((P + |Q^*| \oplus (R + |Q|))$, $P = X^{1/2}AX^{1/2}$, $Q = X^{1/2}A^{1/2}B^{1/2}X^{1/2}$ and $R = X^{1/2}BX^{1/2}$. In addition, letting $X = I$, we have

$$\left|\left|\left| (A+B)^r \right|\right|\right| \leq \left|\left|\left| V^{\frac{rp}{2}} \right|\right|\right|^{1/p} \left|\left|\left| V^{\frac{rq}{2}} \right|\right|\right|^{1/q} \tag{2.19}$$

for $j = 1, 2, \dots$, where $V = \Big(\Big(A + \big|B^{1/2}A^{1/2}\big|\Big) \oplus \Big(B + \big|A^{1/2}B^{1/2}\big|\Big)\Big)$. Moreover, letting $r = 1$, $p = q = 2$, leads to the well-known result

$$|||A + B||| \leq \left|\left|\left| \Big(A + \big|B^{1/2}A^{1/2}\big|\Big) \oplus \Big(B + \big|A^{1/2}B^{1/2}\big|\Big) \right|\right|\right|.$$

Proof. Letting $n = 2$, $A_1 = B_1 = A^{1/2}$, $A_2 = B_2 = B^{1/2}$, $X_1 = X_2 = X$, $Y_1 = Y_2 = Y$ in inequality (2.6), leads to

$$\left|\left|\left| \left|A^{1/2}X^{1/2}Y^{1/2}A^{1/2} + B^{1/2}X^{1/2}Y^{1/2}B^{1/2}\right|^r \right|\right|\right|$$

$$\leq \left|\left|\left|\left(\left|\begin{bmatrix} \sqrt{\alpha}A^{1/2}X^{1/2} & \sqrt{\alpha}B^{1/2}X^{1/2} \\ \sqrt{1-\alpha}A^{1/2}Y^{1/2} & \sqrt{1-\alpha}B^{1/2}Y^{1/2} \end{bmatrix}\right|^2\right)^{\frac{rp}{2}}\right|\right|\right|^{1/p} \times$$

$$\left|\left|\left|\left(\left|\begin{bmatrix} \sqrt{1-\alpha}A^{1/2}X^{1/2} & \sqrt{1-\alpha}B^{1/2}X^{1/2} \\ \sqrt{\alpha}A^{1/2}Y^{1/2} & \sqrt{\alpha}B^{1/2}Y^{1/2} \end{bmatrix}\right|^2\right)^{\frac{rq}{2}}\right|\right|\right|^{1/q}$$

$$= \left|\left|\left|\left(\begin{bmatrix} H(\alpha) & K(\alpha) \\ K^*(\alpha) & L(\alpha) \end{bmatrix}\right)^{\frac{rp}{2}}\right|\right|\right|^{1/p} \times$$

$$\left|\left|\left|\left(\begin{bmatrix} H(1-\alpha) & K(1-\alpha) \\ K^*(1-\alpha) & L(1-\alpha) \end{bmatrix}\right)^{\frac{rq}{2}}\right|\right|\right|^{1/q}$$

Using Lemma 2.1, we give

$$\left|\left|\left|\left(\begin{bmatrix} H(\alpha) & K(\alpha) \\ K^*(\alpha) & L(\alpha) \end{bmatrix}\right)^{\frac{rp}{2}}\right|\right|\right| =$$

$$\begin{aligned} & \left|\left|\left|\left(\begin{bmatrix} H(\alpha) & 0 \\ 0 & L(\alpha) \end{bmatrix} + \begin{bmatrix} 0 & K(\alpha) \\ K^*(\alpha) & 0 \end{bmatrix}\right)^{\frac{rp}{2}}\right|\right|\right| \\ \leq & \left|\left|\left|\left(\begin{bmatrix} H(\alpha) & 0 \\ 0 & L(\alpha) \end{bmatrix} + \left|\begin{bmatrix} 0 & K(\alpha) \\ K^*(\alpha) & 0 \end{bmatrix}\right|\right)^{\frac{rp}{2}}\right|\right|\right| \\ = & \left|\left|\left|\left(\begin{bmatrix} H(\alpha) & 0 \\ 0 & L(\alpha) \end{bmatrix} + \begin{bmatrix} |K^*(\alpha)| & 0 \\ 0 & |K(\alpha)| \end{bmatrix}\right)^{\frac{rp}{2}}\right|\right|\right| \\ = & \left|\left|\left|\left(\begin{bmatrix} H(\alpha) + |K^*(\alpha)| & 0 \\ 0 & L(\alpha) + |K(\alpha)| \end{bmatrix}\right)^{\frac{rp}{2}}\right|\right|\right| \end{aligned} \tag{2.20}$$

Similarly,

$$\left|\left|\left|\left(\begin{bmatrix} H(1-\alpha) & K(1-\alpha) \\ K^*(1-\alpha) & L(1-\alpha) \end{bmatrix}\right)^{\frac{rq}{2}}\right|\right|\right| \leq \tag{2.21}$$

$$\left|\left|\left|\left(\begin{bmatrix} H(1-\alpha) + |K^*(1-\alpha)| & 0 \\ 0 & L(1-\alpha) + |K(1-\alpha)| \end{bmatrix}\right)^{\frac{rp}{2}}\right|\right|\right|$$

Thus, inequality (2.17) follows from inequalities (2.20) and (2.21). □

Remark 2.4. *Letting $\alpha = \frac{1}{2}$, $r = 1$, $p = q = 2$ in inequality (2.17), we give inequality (1.10). In that sense, inequality (2.17) is a generalization of inequality (1.10).*

Using Lemma 2.1 and inequality (2.6), lead to the following general inequality, which is a generalization of inequality (1.11).

Corollary 2.8. *Let $A, B, X_1, X_2, Y_1, Y_2 \in \mathbb{M}_n$ such that X_1, X_2, Y_1, Y_2 are positive, $p, q > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \in [0,1]$ and $r \geq 0$. Then*

$$2\left|\left|\left||E-F|^r\right|\right|\right| \leq \tag{2.22}$$

$$\left|\left|\left|\left(\left(H(\alpha)+\left|K^*(\alpha)\right|\right)\oplus\left(L(\alpha)+\left|K(\alpha)\right|\right)\right)^{\frac{rp}{2}}\right|\right|\right|^{1/p}\times$$

$$\left|\left|\left|\left(\left(H(1-\alpha)+\left|K^*(1-\alpha)\right|\right)\oplus\left(L(1-\alpha)+\left|K(1-\alpha)\right|\right)\right)^{\frac{rq}{2}}\right|\right|\right|^{1/q},$$

where $E = AX_1^{1/2}Y_1^{1/2}A^*$, $F = BX_2^{1/2}Y_2^{1/2}B^*$, $H(\alpha) = \alpha X_1^{1/2}A^*AX_1^{1/2} + (1-\alpha)Y_1^{1/2}A^*AY_1^{1/2}$, $K(\alpha) = \alpha X_1^{1/2}A^*BX_2^{1/2} - (1-\alpha)Y_1^{1/2}A^*BY_2^{1/2}$ *and* $L(\alpha) = \alpha X_2^{1/2}B^*BX_2^{1/2} + (1-\alpha)Y_2^{1/2}B^*BY_2^{1/2}$.

Proof. Letting $n = 2$, $A_1 = B_1 = A$, and $A_2 = -B_2 = B$ in inequality (2.6), leads to

$$\left|\left|\left|\left|AX_1^{1/2}Y_1^{1/2}A^* - BX_2^{1/2}Y_2^{1/2}B^*\right|^r\right|\right|\right|$$

$$\leq \left|\left|\left|\left(\left|\begin{bmatrix}\sqrt{\alpha}AX_1^{1/2} & \sqrt{\alpha}BX_2^{1/2}\\ \sqrt{1-\alpha}AY_1^{1/2} & -\sqrt{1-\alpha}BY_2^{1/2}\end{bmatrix}\right|^2\right)^{\frac{rp}{2}}\right|\right|\right|^{1/p}\times$$

$$\left|\left|\left|\left(\left|\begin{bmatrix}\sqrt{1-\alpha}AX_1^{1/2} & \sqrt{1-\alpha}BX_2^{1/2}\\ \sqrt{\alpha}AY_1^{1/2} & -\sqrt{\alpha}BY_2^{1/2}\end{bmatrix}\right|^2\right)^{\frac{rq}{2}}\right|\right|\right|^{1/q}$$

$$= \left|\left|\left|\left(\begin{bmatrix}H(\alpha) & K(\alpha)\\ K^*(\alpha) & L(\alpha)\end{bmatrix}\right)^{\frac{rp}{2}}\right|\right|\right|^{1/p}\times$$

$$\left|\left|\left|\left(\begin{bmatrix}H(1-\alpha) & K(1-\alpha)\\ K^*(1-\alpha) & L(1-\alpha)\end{bmatrix}\right)^{\frac{rq}{2}}\right|\right|\right|^{1/q}.$$

Applying the same steps used in the proof of Corollary 2.7, we give the desired result. □

Remark 2.5. *Letting $\alpha = \frac{1}{2}$, $r = 1$, $p = q = 2$ in inequality (2.22), we give inequality (1.11). In that sense, inequality (2.22) is a generalization of inequality (1.11).*

Corollary 2.9. *Let $A, B \in \mathbb{M}_n$ be positive, $\alpha \in [0,1]$. Then*

$$|||A-B||| \leq \left|\left|\left|\left(H_1(\alpha)+\left|K_1^*(\alpha)\right|\right)\oplus\left(L_1(\alpha)+\left|K_1(\alpha)\right|\right)\right|\right|\right|, \tag{2.23}$$

where $H_1(\alpha) = \alpha A + (1-\alpha)A$, $K_1(\alpha) = \alpha A^{1/2}B^{1/2} - (1-\alpha)A^{1/2}B^{1/2}$ *and* $L_1(\alpha) = \alpha B + (1-\alpha)B$. *In particular, letting* $\alpha = \frac{1}{2}$, *we have*

$$|||A-B||| \leq |||A \oplus B|||.$$

Proof. Letting $r = 1$, $p = q = 2$, $X = Y = I$ in inequality (2.22), we get inequality (2.23). □

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