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ON A TYPE OF PROJECTIVE SEMI-SYMMETRIC CONNECTION

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Abstract. In the present paper, we have studied some properties of curvature tensors of special projective semi-symmetric connection. We have shown that curvature tensor of such a connection satisfies Bianchi's identities.

1. Introduction

The idea of semi-symmetric connection was introduced by A. Friedmann and J. A. Schouten [2] in 1924. In 1932, H. A. Hayden [4] studied semi-symmetric metric-connection. It was K. Yano [10] who started systematic study of semi-symmetric metric connection and this was further studied by T. Imai [6], R. S. Mishra and S. N. Pandey [9], U. C. De and B. K. De [1] and several other mathematicians ([7], [11]). In 2001, P. Zhao and H. Song [12] studied a semi-symmetric connection which is projectively equivalent to Levi-Civita connection and such a connection is called as projective semi-symmetric connection. They found an invariant under the transformation of projective semi-symmetric connection and showed that this invariant could degenerate into the Weyl projective curvature tensor under certain conditions. After this various papers ([3], [5], [13]) on projective semi-symmetric metric connection have appeared.

The organization of the paper is as follows. After introduction we give some preliminary results in section 2. In sections 3, we present a brief account of special projective semi-symmetric connection. Section 4 is devoted to the study of special projective semi symmetric connection with recurrent curvature tensor.

2. Preliminaries

Let M^n be an n-dimensional $(n > 2)$ Riemannian manifold equipped with a Riemannian metric g and ∇ be the Levi-Civita connection associated with metric g. A linear connection $\bar{\nabla}$ on M^n is called the semi symmetric metric connection [10], if the torsion tensor $\bar{T}$ of the connection $\bar{\nabla}$, given by

$$\bar{T}(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y] \tag{2.1}$$

satisfies the condition

$$\bar{T}(X,Y) = \pi(Y)X - \pi(X)Y \tag{2.2}$$

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and

$$(\bar{\nabla}_X g)(Y, Z) = 0, \tag{2.3}$$

where π is a 1 - form on M^n associated with vector field ρ, i.e.,

$$\pi(X) = g(X, \rho). \tag{2.4}$$

If the geodesic with respect to $\bar{\nabla}$ are always consistent with those of ∇, then $\bar{\nabla}$ is called a connection projectively equivalent to ∇. If $\bar{\nabla}$ is projective equivalent connection to ∇ as well as the semi-symmetric, then $\bar{\nabla}$ is called projective semi-symmetric connection. We also call $\bar{\nabla}$ as projective semi- symmetric transformation.

In this paper, we study a type of projective semi-symmetric connection $\bar{\nabla}$ introduced by P. Zhao and H. Song [12]. The connection is given by

$$\bar{\nabla}_X Y = \nabla_X Y + \psi(Y)X + \psi(X)Y + \phi(Y)X - \phi(X)Y, \tag{2.5}$$

where 1-forms ϕ and ψ are given as

$$\phi(X) = \frac{1}{2}\pi(X) \text{ and } \psi(X) = \frac{n-1}{2(n+1)}\pi(X). \tag{2.6}$$

It is easy to observe that torsion tensor of projective semi- symmetric transformation is same as given by the equation (2.2) and also that

$$(\bar{\nabla}_X g)(Y, Z) = \frac{1}{n+1}[2\pi(X)g(Y, Z) - n\pi(Y)g(Z, X) - n\pi(Z)g(X, Y), \tag{2.7}$$

i.e., the connection $\bar{\nabla}$ is a non metric one.

Let $\bar{R}$ and R be the curvature tensor of the manifold relative to the projective semi-symmetric connection $\bar{\nabla}$ and Levi-Civita connection ∇ respectively. It is known that [12]

$$\bar{R}(X, Y, Z) = R(X, Y, Z) + \beta(X, Y)Z + \alpha(X, Z)Y - \alpha(Y, Z)X, \tag{2.8}$$

where $\beta(X, Y)$ and $\alpha(X, Y)$ are given by the following relations

$$\beta(X, Y) = \Psi'(X, Y) - \Psi'(Y, X) + \Phi'(Y, X) - \Phi'(X, Y), \tag{2.9}$$

$$\alpha(X, Y) = \Psi'(X, Y) + \Phi'(Y, X) - \psi(X)\phi(Y) - \phi(X)\psi(Y), \tag{2.10}$$

$$\Psi'(X, Y) = (\nabla_X \psi)(Y) - \psi(X)\psi(Y) \tag{2.11}$$

and

$$\Phi'(X, Y) = (\nabla_X \phi)(Y) - \phi(X)\phi(Y). \tag{2.12}$$

Contracting X in the equation (2.8), we get a relation between Ricci tensors $\bar{Ric}(Y, Z)$ and $Ric(Y, Z)$ of manifold with respect to connections $\bar{\nabla}$ and ∇ respectively

$$\bar{Ric}(Y, Z) = Ric(Y, Z) + \beta(Y, Z) - (n-1)\alpha(Y, Z). \tag{2.13}$$

If $\bar{r}$ and r are scalar curvatures of manifold with respect to connection $\bar{\nabla}$ and ∇ respectively, then from the equation (2.13), we get

$$\bar{r} = r + b - (n-1)a, \tag{2.14}$$

where

$$b = \sum_{i=1}^{n} \beta(e_i, e_i) \text{ and } a = \sum_{i=1}^{n} \alpha(e_i, e_i).$$

The Weyl-projective curvature tensor W, conharmonic curvature tensor P and concircular curvature tensor I are given by [9]

$$W(X,Y,Z) = R(X,Y,Z) + \frac{1}{n-1}\{Ric(X,Z)Y - Ric(Y,Z)X\}, \tag{2.15}$$

$$\begin{aligned} P(X,Y,Z) = R(X,Y,Z) - \frac{1}{n-2}[&Ric(Y,Z)X - Ric(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY], \end{aligned} \tag{2.16}$$

where

$$g(QX,Y) = Ric(X,Y) \tag{2.17}$$

and

$$I(X,Y,Z) = R(X,Y,Z) - \frac{r}{n-1}[g(Y,Z)X - g(X,Z)Y]. \tag{2.18}$$

3. Special Projective Semi-Symmetric Connection

In this section, we consider a projective semi-symmetric connection $\bar{\nabla}$ given by the equation (2.5) whose associated 1-form π is closed, i.e.,

$$(\bar{\nabla}_X \pi)Y = (\bar{\nabla}_Y \pi)X. \tag{3.1}$$

Such a connection $\bar{\nabla}$ is called special projective semi-symmetric connection [12]. It is easy to verify that both the 1-forms ϕ and ψ are closed as the 1-form π is closed and also that the tensors Φ' and Ψ' are symmetric. Consequently, we get

$$\beta(X,Y) = 0 \tag{3.2}$$

and

$$\alpha(X,Y) = \alpha(Y,X). \tag{3.3}$$

In view of the equations (3.1) and (3.2), the expressions (2.8), (2.13) and (2.14) reduces to

$$\bar{R}(X,Y,Z) = R(X,Y,Z) + \alpha(X,Z)Y - \alpha(Y,Z)X, \tag{3.4}$$

$$\bar{R}ic(Y,Z) = Ric(Y,Z) - (n-1)\alpha(Y,Z) \tag{3.5}$$

and

$$\bar{r} = r - (n-1)a. \tag{3.6}$$

It is easy to observe that the Ricci tensor $\bar{R}ic(Y,Z)$ is symmetric.

Now, we prove the following theorems:

Theorem 3.1. *Curvature tensor of special projective semi-symmetric connection satisfies Bianchi's first identity.*

Proof : Writing two more equations by cyclic permutations of X, Y and Z from equation (3.4), we get

$$\bar{R}(Y,Z,X) = R(Y,Z,X) + \alpha(Y,X)Z - \alpha(Z,X)Y,$$

and

$$\bar{R}(Z,X,Y) = R(Z,X,Y) + \alpha(Z,Y)X - \alpha(X,Y)Z.$$

Adding these equations to the equation (3.4), we get result.

Theorem 3.2. *Curvature tensor of special projective semi-symmetric connection satisfies Bianchi's second identity if α is parallel tensor with respect to Levi-Civita connection ∇.*

Proof : Suppose α is a parallel tensor with respect to Levi-Civita connection ∇, i.e., $\nabla\alpha = 0$. Now differentiating the equation (3.4) covariantly with respect to the connection ∇, we have

$$(\nabla_X \bar{R})(Y,Z,U) = (\nabla_X R)(Y,Z,U). \tag{3.7}$$

Writing two more equations by cyclic permutations of X, Y and Z in above equation, we get

$$(\nabla_Y \bar{R})(Z,X,U) = (\nabla_Y R)(Z,X,U), \tag{3.8}$$

and

$$(\nabla_Z \bar{R})(X,Y,U) = (\nabla_Z R)(X,Y,U). \tag{3.9}$$

Adding the equations (3.7), (3.8) and (3.9), we get

$$(\nabla_X \bar{R})(Y,Z,U) + (\nabla_Y \bar{R})(Z,X,U) + (\nabla_Z \bar{R})(X,Y,U) = 0.$$

This shows that the curvature tensor of special projective semi-symmetric connection satisfies Bianchi's second identity.

Theorem 3.3. *The Weyl-projective curvature tensor of Riemannian manifold with respect to the special projective semi-symmetric connection $\bar{\nabla}$ satisfies*

$$\overline{W}(X,Y,Z) + \overline{W}(Y,Z,X) + \overline{W}(Z,X,Y) = 0.$$

Proof : The Weyl-projective curvature tensor of Riemannian Manifold with respect to special projective semi-symmetric connection $\bar{\nabla}$ is given by

$$\overline{W}(X,Y,Z) = \bar{R}(X,Y,Z) - \frac{1}{n-1}[\bar{R}ic(Y,Z)X - \bar{R}ic(X,Z)Y]. \tag{3.10}$$

Writing two more equations by cyclic permutations of X, Y and Z in above equation, we get

$$\overline{W}(Y,Z,X) = \bar{R}(Y,Z,X) - \frac{1}{n-1}[\bar{R}ic(Z,X)Y - \bar{R}ic(Y,X)Z], \tag{3.11}$$

$$\overline{W}(Z,X,Y) = \bar{R}(Z,X,Y) - \frac{1}{n-1}[\bar{R}ic(X,Y)Z - \bar{R}ic(Z,Y)X]. \tag{3.12}$$

Adding the equations (3.10), (3.11) and (3.12), we get

$$\overline{W}(X,Y,Z) + \overline{W}(Y,Z,X) + \overline{W}(Z,X,Y) = 0.$$

4. Special Projective Semi-Symmetric Connection with Recurrent Curvature Tensor

In this section, we consider a special projective semi-symmetric connection $\bar{\nabla}$ whose curvature tensor $\bar{R}$ is recurrent with respect to the Levi-Civita connection ∇, i.e.,

$$(\nabla_U \bar{R})(X, Y, Z) = B(U)\bar{R}(X, Y, Z), \tag{4.1}$$

where B is a non-zero 1-form.
Differentiating the equation (3.4) covariantly with respect to the Levi-Civita connection ∇, we get

$$(\nabla_U \bar{R})(X, Y, Z) = (\nabla_U R)(X, Y, Z) + (\nabla_U \alpha)(X, Z)Y - (\nabla_U \alpha)(Y, Z)X. \tag{4.2}$$

Contracting X in the above equation, we have

$$(\nabla_U \bar{R}ic)(Y, Z) = (\nabla_U Ric)(Y, Z) - (n-1)(\nabla_U \alpha)(Y, Z). \tag{4.3}$$

Putting $Y = Z = e_i$ in the above equation and taking summation over i, $1 \leq i \leq n$, we get

$$(\nabla_U \bar{r}) = (\nabla_U r) - (n-1)(\nabla_U a). \tag{4.4}$$

Now the equations (3.4) and (4.2) together give

$$\begin{aligned}(\nabla_U \bar{R})(X, Y, Z) - B(U)\bar{R}(X, Y, Z) =& (\nabla_U R)(X, Y, Z) - B(U)R(X, Y, Z)\\ &+ [(\nabla_U \alpha)(X, Z) - B(U)\alpha(X, Z)]Y\\ &- [(\nabla_U \alpha)(Y, Z) - B(U)\alpha(Y, Z)]X,\end{aligned} \tag{4.5}$$

which, in view of the equation (4.1), reduces to

$$\begin{aligned}(\nabla_U R)(X, Y, Z) - B(U)R(X, Y, Z) =& [(\nabla_U \alpha)(Y, Z) - B(U)\alpha(Y, Z)]X\\ &- [(\nabla_U \alpha)(X, Z) - B(U)\alpha(X, Z)]Y.\end{aligned} \tag{4.6}$$

Contracting X in above, we get

$$(\nabla_U Ric)(Y, Z) - B(U)Ric(Y, Z) = (n-1)\{(\nabla_U \alpha)(Y, Z) - B(U)\alpha(Y, Z)\}. \tag{4.7}$$

Further, we obtain

$$(\nabla_U r) - B(U)r = (n-1)\{(\nabla_U a) - B(U)a\}. \tag{4.8}$$

Also, from the equation (2.17), we have

$$g((\nabla_U Q)X, Y) = (\nabla_U Ric)(X, Y), \tag{4.9}$$

which can be written as

$$g((\nabla_U Q)X - B(U)QX, Y) = (\nabla_U Ric)(X, Y) - B(U)Ric(X, Y). \tag{4.10}$$

Now we prove following theorems:

Theorem 4.1. *If the curvature tensor of special projective semi-symmetric connection on a Riemannian manifold M^n is recurrent with respect to the Levi-Civita connection then manifold M^n is projectively recurrent.*

Proof : Differentiating the projective curvature tensor W given by (2.15) covariantly with respect to Levi-Civita connection ∇, we have
(4.11)

$$(\nabla_U W)(X, Y, Z) = (\nabla_U R)(X, Y, Z) + \frac{1}{n-1}\{(\nabla_U Ric)(X, Z)Y - (\nabla_U Ric)(Y, Z)X\}.$$

The above equation gives

(4.12)
$$(\nabla_U W)(X,Y,Z) - B(U)W(X,Y,Z) = (\nabla_U R)(X,Y,Z) - B(U)R(X,Y,Z) + \frac{1}{n-1}[\{(\nabla_U Ric)(X,Z) - B(U)Ric(X,Z)\}Y - \{(\nabla_U Ric)(Y,Z) - B(U)Ric(Y,Z)\}X].$$

Using equation (4.6) and (4.7) in above, we get

$$(\nabla_U W)(X,Y,Z) = B(U)W(X,Y,Z),$$

which proves the statement.

Theorem 4.2. *: A Riemannian manifold M^n admitting a special projective semi-symmetric connection whose curvature tensor and tensor α are recurrent with respect to the Levi-Civita connection, is conharmonically recurrent.*

Proof: Differentiating covariantly the equation (2.16) with respect to the Levi-Civita connection, we get

(4.13)
$$(\nabla_U P)(X,Y,Z) = (\nabla_U R)(X,Y,Z) - \frac{1}{n-2}[(\nabla_U Ric)(Y,Z)X - (\nabla_U Ric)(X,Z)Y. + g(Y,Z)(\nabla_U Q)X - g(X,Z)(\nabla_U Q)Y],$$

From above, we have

(4.14)
$$(\nabla_U P)(X,Y,Z) - B(U)P(X,Y,Z) = (\nabla_U R)(X,Y,Z) - B(U)R(X,Y,Z) - \frac{1}{n-2}[\{(\nabla_U Ric)(Y,Z) - B(U)Ric(Y,Z)\}X - \{(\nabla_U Ric)(X,Z) - B(U)Ric(X,Z)\}Y + g(Y,Z)\{(\nabla_U Q)X - B(U)QX\} - g(X,Z)\{(\nabla_U Q)Y - B(U)QY\}].$$

If the tensor α and the curvature tensor of the special projective semi-symmetric connection $\bar{\nabla}$ are recurrent with respect to the Levi-Civita connection ∇, then from the equations (4.6), (4.7) and (4.10), we get

$$(\nabla_U P)(X,Y,Z) = B(U)P(X,Y,Z),$$

which shows that manifold is conharmonically recurrent.

Theorem 4.3. *A Riemannian manifold M^n admitting a special projective semi-symmetric connection whose curvature tensor and tensor α are recurrent with respect to Levi-Civita connection, is concircular recurrent.*

Proof: Differentiating the concircular curvature tensor I of M^n given by the equation (2.18) covariantly with respect to the Levi- Civita connection ∇, we have

$$(\nabla_U I)(X,Y,Z) = (\nabla_U R)(X,Y,Z) - \frac{\nabla_U r}{(n-1)}\{g(Y,Z)X - g(X,Z)Y\}. \quad (4.15)$$

From this, we have

(4.16)
$$(\nabla_U I)(X,Y,Z) - B(U)I(X,Y,Z) = (\nabla_U R)(X,Y,Z) - B(U)R(X,Y,Z) - \frac{\nabla_U r - B(U)r}{(n-1)}\{g(Y,Z)X - g(X,Z)Y\}.$$

If the tensor α and the curvature tensor of the special projective semi-symmetric connection $\bar{\nabla}$ are recurrent with respect to the Levi-Civita connection ∇, then from the equations (4.6), (4.7) and (4.8), we get

$$(\nabla_U I)(X,Y,Z) = B(U)I(X,Y,Z).$$

Theorem 4.4. *Let M^n be a Riemannian manifold admitting a special projective semi-symmetric connection whose Ricci-tensor is recurrent with respect to the Levi-Civita connection. If the manifold is projectively recurrent with respect to Levi-Civita connection, then the curvature tensor of the special projective semi-symmetric connection is recurrent.*

Proof: Let the manifold M^n be projectively recurrent with respect to Levi Civita connection ∇. Then from the equation (4.12), we have

(4.17)
$$(\nabla_U R)(X,Y,Z) - B(U)R(X,Y,Z) = \frac{1}{n-1}[\{(\nabla_U Ric)(Y,Z) - B(U)Ric(Y,Z)X\} - \{(\nabla_U Ric)(X,Z) - B(U)Ric(X,Z)Y\}].$$

Now, from equations (3.5) and (4.3), we get

$$(\nabla_U \bar{R}ic)(Y,Z) - B(U)\bar{R}ic(Y,Z) = (\nabla_U Ric)(Y,Z) - B(U)Ric(Y,Z) - (n-1)\{(\nabla_U \alpha)(Y,Z) - B(U)\alpha(Y,Z)\}. \tag{4.18}$$

Since the Ricci tensor of the special projective semi-symmetric connection $\bar{\nabla}$ is recurrent with respect to the Levi-Civita connection ∇, hence the above equation gives

$$(\nabla_U Ric)(Y,Z) - B(U)Ric(Y,Z) = (n-1)\{(\nabla_U \alpha)(Y,Z) - B(U)\alpha(Y,Z)\}. \tag{4.19}$$

Thus, from the equations (4.17) and (4.19), we get

$$(\nabla_U R)(X,Y,Z) - B(U)R(X,Y,Z) = \{(\nabla_U \alpha)(Y,Z) - B(U)\alpha(Y,Z)\}X - \{(\nabla_U \alpha)(X,Z) - B(U)\alpha(X,Z)\}Y, \tag{4.20}$$

which, on using in the equation (4.5), gives

$$(\nabla_U \bar{R})(X,Y,Z) = B(U)\bar{R}(X,Y,Z). \tag{4.21}$$

This proves the statement.

Theorem 4.5. *Let M^n be a Riemannian manifold admitting a special projective semi-symmetric connection whose Ricci-tensor is recurrent with respect to the Levi-Civita connection. If the manifold is of constant curvature, then the curvature tensor of the special projective semi-symmetric connection is recurrent with respect to the Levi-Civita connection.*

Proof: If the Riemannian manifold M^n is of constant curvature, then we have [9]

$$R(X,Y,Z) = \frac{1}{n-1}\{Ric(Y,Z)X - Ric(X,Z)Y\}. \tag{4.22}$$

Using the above equation in the equation (3.4), we have

$$\bar{R}(X,Y,Z) = \frac{1}{n-1}[\{Ric(Y,Z)-(n-1)\alpha(Y,Z)\}X - \{Ric(X,Z)-(n-1)\alpha(X,Z)\}Y], \tag{4.23}$$

which, on using the equation (3.5), gives

$$\bar{R}(X,Y,Z) = \frac{1}{n-1}\{\bar{R}ic(Y,Z)X - \bar{R}ic(X,Z)Y\}. \tag{4.24}$$

Differentiating the above equation covariantly with respect to the Levi-Civita connection, we have

$$(\nabla_U \bar{R})(X,Y,Z) = \frac{1}{n-1}\{(\nabla_U \bar{R}ic)(Y,Z)X - (\nabla_U \bar{R}ic)(X,Z)Y\},$$

which can be written as

$$\begin{aligned}(\nabla_U \bar{R})(X,Y,Z) - B(U)\bar{R}(X,Y,Z) = & \frac{1}{n-1}[\{(\nabla_U \bar{R}ic)(Y,Z) - B(U)\bar{R}ic(Y,Z)\}X \\ & -\{(\nabla_U \bar{R}ic)(X,Z) - B(U)\bar{R}ic(X,Z)\}Y].\end{aligned} \tag{4.25}$$

Since the Ricci tensor of special projective semi-symmetric connection is recurrent with respect to the Levi-Civita connection ∇, hence from the above equation, we have

$$(\nabla_U \bar{R})(X,Y,Z) = B(U)\bar{R}(X,Y,Z),$$

which proves the statement.

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