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$\mathcal{L}_\xi$-Families: Localized Topology with Applications in Edge Detection

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Abstract. We introduce and explore $\mathcal{L}_\xi$-families, an innovative class of localized topological structures that extends classical concepts while preserving fundamental properties. These families constitute a bridge between traditional topological objects and finer-grained local-to-global characteristics. Our construction offers a natural generalization of regular open sets through a novel localization approach that maintains critical topological invariants across various transformations and operations. This paper establishes the foundational theory of $\mathcal{L}_\xi$-families, proving key characterization theorems and situating them within the broader topological landscape. Our findings reveal that these structures form a complete lattice under appropriate operations and possess significant hereditary characteristics. Additionally, we demonstrate stability properties under continuous mappings and homeomorphisms, highlighting their seamless integration with established topological frameworks. Through strategically selected counterexamples, we define the boundaries of these new concepts. The theoretical architecture developed in this work creates pathways for applications in digital topology and image processing, with particularly promising implications for edge detection and boundary analysis methods. The relationships we establish between $\mathcal{L}_\xi$ family members and classical topological concepts provide unifying perspectives across seemingly disparate notions and introduce novel tools for topological classification challenges.

1. Introduction

The evolution of topology has consistently progressed through generalizing established structures to encompass broader mathematical phenomena while maintaining essential properties.

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From foundational metric spaces to sophisticated topological constructs, this development has yielded powerful mathematical tools with wide-ranging applications [1]. We continue this tradition by introducing $\mathcal{L}_\xi$-families as an innovative extension of classical topological concepts.

Our development of $\mathcal{L}_\xi$-families stems from observations in functional analysis and differential topology, where conventional topological structures often insufficiently capture subtle local connectivity characteristics crucial in applications. Traditional approaches frequently rely on global conditions that prove too rigid for many practical scenarios. The $\mathcal{L}_\xi$-family framework addresses these limitations by integrating local and global properties through a unifying mechanism that preserves essential topological features while allowing greater flexibility [2].

1.1. **Historical Context and Related Work.** Topology has historically advanced through successive generalizations, starting with metric spaces and expanding to increasingly abstract structures [3]. Key developments in this progression include refinements of separation axioms, connectedness properties, and various compactness conditions. Our work extends this tradition by introducing new set structures that maintain robust connections with classical concepts while offering fresh perspectives on topological phenomena.

The systematic investigation of generalized open sets began with Levine's pioneering work on semi-open sets [4] and received significant advancement through Njastad's introduction of α-open sets [5]. These concepts established a foundation for numerous subsequent generalizations exploring the boundary regions between open and closed sets. Mashhour et al. [6] introduced pre-open sets, while Andrijević [7] defined b-open sets (alternatively known as sp-open sets). These various extensions of openness exhibit distinct properties while maintaining important interconnections.

1.2. **Conceptual Innovation.** Our introduction of $\mathcal{L}_\xi$-families represents a fundamental shift in approach. Rather than defining sets through compositions of topological operators like closure and interior, we construct $\mathcal{L}_\xi$-families using collections of open sets with specific local characteristics. This methodology provides a more nuanced framework for analyzing topological structures across different scales. The locality principle embedded in $\mathcal{L}_\xi$-families allows for significantly increased flexibility while preserving essential topological properties.

1.3. **Research Objectives and Paper Structure.** This paper aims to establish the theoretical foundations of $\mathcal{L}_\xi$-families, determine their fundamental properties, and position them within the broader landscape of topological spaces. We examine their behavior under various topological operations and mappings, demonstrating their compatibility with established topological frameworks. Additionally, we explore applications in digital topology and image processing, revealing how these abstract structures provide insights into practical problems including edge detection and boundary analysis.

Our investigation addresses several key questions: What algebraic structures do $\mathcal{L}_\xi$-families form? How do they relate to established topological concepts? What properties remain invariant under standard topological operations? What distinguishes them from other generalizations of open sets?

The rest of this paper is structured as follows: Section 2 presents the basic definitions and notations. Section 3 examines the core properties of $\mathcal{L}_\xi$-families, presenting characterization theorems and establishing relationships with classical structures. Section 4 investigates how $\mathcal{L}_\xi$-families behave under continuous mappings and homeomorphisms. Section 5 explores applications in digital topology, with particular emphasis on edge detection and boundary analysis. Finally, Section 6 presents conclusions and summarizes our findings.

2. Fundamental Concepts and Notation

- **Closure**: $\overline{S}$ - the smallest closed set containing S
- **Interior**: S° - the largest open set contained in S
- **Boundary**: $\mathrm{bd}(S) = \overline{S} \setminus S^\circ$
- **Exterior**: $\mathrm{Ex}(S) = \hat{M} \setminus \overline{S}$
- **Complement**: $\hat{M} \setminus S$ or S^c

We begin by recalling several key concepts from general topology that provide necessary background for our development:

Definition 2.1. *A subset S of a topological space $(\hat{M}, \rho)$ is:*

- ***Regular open*** *if $S = \overline{S}^\circ$*
- ***Regular closed*** *if $S = \overline{S^\circ}$*

Regular open and regular closed sets play fundamental roles in topological theory due to their special behavior under complementation—the complement of a regular open set is regular closed, and vice versa [8].

Definition 2.2. *We say that a subset $S \subseteq \hat{M}$ is **semi-open** when $S \subseteq \overline{S^\circ}$.*
The collection of all semi-open sets in $(\hat{M}, \rho)$ is denoted by $\mathcal{SO}(\hat{M}, \rho)$.

First introduced by Levine [4], semi-open sets represent one of the earliest generalizations of open sets and have been extensively studied for their wide-ranging applications across topology.

Definition 2.3. *A subset $S \subseteq \hat{M}$ is classified as **pre-open** when $S \subseteq \overline{S}^\circ$.*
The collection of all pre-open sets in $(\hat{M}, \rho)$ is denoted by $\mathcal{PO}(\hat{M}, \rho)$.

Pre-open sets, introduced by Mashhour et al. [6], constitute another important generalization that has found significant applications in functional analysis and differential topology.

Definition 2.4. *A subset $S \subseteq \hat{M}$ is termed α-**open** when $S \subseteq \overline{S^\circ}^\circ$.*
The collection of all α-open sets in $(\hat{M}, \rho)$ is denoted by $\alpha\mathcal{O}(\hat{M}, \rho)$.

Now we introduce the central innovative concept of this paper:

Definition 2.5. *Let $(\hat{M}, \rho)$ be a topological space and $S \subseteq \hat{M}$. We define S as an **$\mathcal{L}_\xi$-family** if there exists a collection $\{O_i\}_{i\in\Delta}$ of open sets in $\hat{M}$ satisfying:*

(1) $S = \bigcap_{i\in\Delta} O_i$
(2) *For every point $p \in S$, there exists a finite subset $\Delta_p \subset \Delta$ such that $\bigcap_{i\in\Delta_p} O_i \subseteq \overline{S}^{\circ}$*

The collection of all $\mathcal{L}_\xi$-families in $(\hat{M}, \rho)$ is denoted by $\mathcal{L}_\xi(\hat{M}, \rho)$.

This definition establishes a new class of sets that generalizes regular open sets by transforming the global condition $S = \overline{S}^{\circ}$ into a local condition that must be satisfied at each point $p \in S$. This locality principle introduces significant flexibility while preserving essential topological properties.

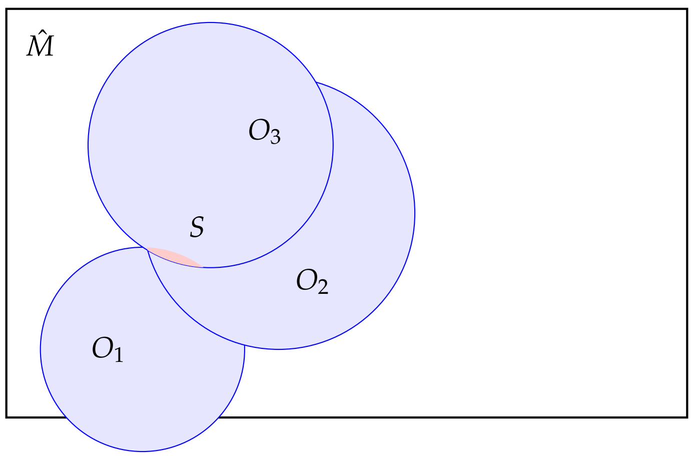


Figure 1. Visual representation of an $\mathcal{L}_\xi$-family. The set S (shaded in red) results from the intersection of open sets O_1, O_2, and O_3. For each point in S, a finite subcollection of these open sets satisfies the local condition.

Throughout this paper, we will use the following notation:

- $\mathcal{L}_\xi^c(\hat{M}, \rho)$ denotes the family of complements of $\mathcal{L}_\xi$-families
- $\mathcal{RO}(\hat{M}, \rho)$ represents the collection of regular open sets in $(\hat{M}, \rho)$
- $\mathcal{RC}(\hat{M}, \rho)$ represents the collection of regular closed sets in $(\hat{M}, \rho)$

3. Algebraic Structure and Key Properties

This section investigates the fundamental properties of $\mathcal{L}_\xi$-families and establishes their relationships with classical topological structures. We begin by examining the algebraic properties of the collection of $\mathcal{L}_\xi$-families.

3.1. Basic Properties and Set Operations.

Theorem 3.1. *For any topological space $(\hat{M}, \rho)$, the following statements hold:*

(1) *Both $\emptyset$ and $\hat{M}$ are elements of $\mathcal{L}_\xi(\hat{M}, \rho)$*
(2) *If $S_1, S_2 \in \mathcal{L}_\xi(\hat{M}, \rho)$, then their intersection $S_1 \cap S_2 \in \mathcal{L}_\xi(\hat{M}, \rho)$*
(3) *For any finite collection $\{S_i\}_{i=1}^n \subseteq \mathcal{L}_\xi(\hat{M}, \rho)$, we have $\bigcap_{i=1}^n S_i \in \mathcal{L}_\xi(\hat{M}, \rho)$*

Proof. (1) For the empty set $\emptyset$, we can represent it as $\bigcap_{i\in\emptyset} O_i$ where the collection is empty. The second condition is vacuously satisfied since $\emptyset$ contains no points. For $\hat{M}$, we can use the singleton collection $\{\hat{M}\}$, which is open. For any point $p \in \hat{M}$, the finite subset $\Delta_p = \{\hat{M}\}$ satisfies $\bigcap_{i\in\Delta_p} O_i = \hat{M} \subseteq \overline{\hat{M}}^{\circ} = \hat{M}^{\circ} = \hat{M}$.

(2) Consider $S_1, S_2 \in \mathcal{L}_\xi(\hat{M}, \rho)$. Then there exist collections of open sets $\{O_i\}_{i\in\Delta_1}$ and $\{Q_j\}_{j\in\Delta_2}$ such that $S_1 = \bigcap_{i\in\Delta_1} O_i$ and $S_2 = \bigcap_{j\in\Delta_2} Q_j$, with both collections satisfying the $\mathcal{L}_\xi$ conditions.

We construct a new collection $\{R_k\}_{k\in\Delta}$ where $\Delta = \Delta_1 \cup \Delta_2$ (assuming disjoint indexing sets, or using appropriate reindexing) defined by:

$$R_k = \begin{cases} O_k & \text{if } k \in \Delta_1 \\ Q_k & \text{if } k \in \Delta_2 \end{cases}$$

This gives us $S_1 \cap S_2 = \bigcap_{k\in\Delta} R_k$. For any point $p \in S_1 \cap S_2$, there exist finite subsets $\Delta_{1,p} \subset \Delta_1$ and $\Delta_{2,p} \subset \Delta_2$ such that $\bigcap_{i\in\Delta_{1,p}} O_i \subseteq \overline{S_1}^{\circ}$ and $\bigcap_{j\in\Delta_{2,p}} Q_j \subseteq \overline{S_2}^{\circ}$.

Let $\Delta_p = \Delta_{1,p} \cup \Delta_{2,p}$, which is finite. We then have:

$$\bigcap_{k\in\Delta_p} R_k = \left(\bigcap_{i\in\Delta_{1,p}} O_i\right) \cap \left(\bigcap_{j\in\Delta_{2,p}} Q_j\right) \subseteq \overline{S_1}^{\circ} \cap \overline{S_2}^{\circ}$$

Since interiors of closed sets distribute over intersections, we have $\overline{S_1}^{\circ} \cap \overline{S_2}^{\circ} \subseteq \overline{S_1} \cap \overline{S_2}^{\circ} \subseteq \overline{S_1 \cap S_2}^{\circ}$. Therefore, $\bigcap_{k\in\Delta_p} R_k \subseteq \overline{S_1 \cap S_2}^{\circ}$, confirming that $S_1 \cap S_2 \in \mathcal{L}_\xi(\hat{M}, \rho)$.

(3) This follows by applying part (2) inductively to the finite collection $\{S_i\}_{i=1}^n$. □

The requirement for finite intersections in Theorem 3.1(3) is essential, as infinite intersections of $\mathcal{L}_\xi$-families need not result in $\mathcal{L}_\xi$-families, as demonstrated in the following example:

Example 3.1. *On the real line* $\mathbb{R}$ *with its standard topology, consider the family of open intervals* $S_n = (-\frac{1}{n}, \frac{1}{n})$ *for each* $n \in \mathbb{N}$. *Each* S_n *is open and therefore an* $\mathcal{L}_\xi$*-family. However, their infinite intersection* $\bigcap_{n\in\mathbb{N}} S_n = \{0\}$ *fails to be an* $\mathcal{L}_\xi$*-family.*

To verify this, assume by contradiction that $\{0\} \in \mathcal{L}_\xi(\mathbb{R}, \rho)$. *Then there would exist a collection* $\{O_i\}_{i\in\Delta}$ *of open sets with* $\{0\} = \bigcap_{i\in\Delta} O_i$. *For the point* $0 \in \{0\}$, *there must exist a finite subset* $\Delta_0 \subset \Delta$ *such that* $\bigcap_{i\in\Delta_0} O_i \subseteq \overline{\{0\}}^{\circ}$. *Since* $\{0\}$ *is closed in* $\mathbb{R}$, *we have* $\overline{\{0\}} = \{0\}$. *Furthermore,* $\{0\}^{\circ} = \emptyset$ *in the standard topology. This leads to* $\bigcap_{i\in\Delta_0} O_i \subseteq \emptyset$, *contradicting the fact that* $0 \in \bigcap_{i\in\Delta_0} O_i$. *Therefore,* $\{0\}$ *cannot be an* $\mathcal{L}_\xi$*-family.*

This example leads to an important characterization of finite sets in T_1 spaces:

Proposition 3.1. *In a* T_1 *topological space* $(\hat{M}, \rho)$, *a finite subset* $S \subset \hat{M}$ *belongs to* $\mathcal{L}_\xi(\hat{M}, \rho)$ *if and only if* S *is open.*

Proof. If S is open, we can use the collection consisting solely of S itself to satisfy the definition, as $S = \bigcap_{i\in\{1\}} S$. For any point $p \in S$, we have $S \subseteq \overline{S}^{\circ}$, confirming that $S \in \mathcal{L}_\xi(\hat{M}, \rho)$.

Conversely, suppose $S \in \mathcal{L}_\xi(\hat{M}, \rho)$ and S is finite. In a T_1 space, every singleton $\{p\}$ is closed for each $p \in S$, implying that $\overline{S} = S$. If S were not open, then $S^\circ \subsetneq S$, meaning there exists some point $q \in S \setminus S^\circ$. Since $q \in S$ and $S \in \mathcal{L}_\xi(\hat{M}, \rho)$, there must exist a finite subset Δ_q such that $\bigcap_{i \in \Delta_q} O_i \subseteq \overline{S}^\circ = S^\circ$. But $q \in \bigcap_{i \in \Delta_q} O_i$ while $q \notin S^\circ$, creating a contradiction. Therefore, S must be open. □

3.2. **Relationship with Classical Topological Structures.** Next, we establish the fundamental relationship between $\mathcal{L}_\xi$-families and regular open sets, demonstrating how our concept extends the classical notion:

Theorem 3.2. *For any topological space $(\hat{M}, \rho)$, we have $\mathcal{RO}(\hat{M}, \rho) \subseteq \mathcal{L}_\xi(\hat{M}, \rho)$.*

Proof. Let $S \in \mathcal{RO}(\hat{M}, \rho)$. Then $S = \overline{S}^\circ$ and S is open. We can construct a collection consisting of the single open set S itself, giving $S = \bigcap_{i \in \{1\}} S$. For any point $p \in S$, let $\Delta_p = \{1\}$. Then $\bigcap_{i \in \Delta_p} S = S = \overline{S}^\circ$. This confirms that $S \in \mathcal{L}_\xi(\hat{M}, \rho)$. □

The above inclusion is generally strict, as illustrated by the following example:

Example 3.2. *On the real line $\mathbb{R}$ with its standard topology, consider the set $S = [0, 1) \cup (1, 2]$. This set is not regular open because $\overline{S}^\circ = [0, 2]^\circ = (0, 2) \neq S$.*

However, $S \in \mathcal{L}_\xi(\mathbb{R}, \rho)$ can be established by constructing the collection $\{O_1, O_2\}$ where $O_1 = (-1, 1.5)$ and $O_2 = (0.5, 3)$. We then have $S = [0, 1) \cup (1, 2] = O_1 \cap O_2$. For any point $p \in S$:

- *If $p \in [0, 1)$, then $\{O_1\}$ forms a finite subcollection with $O_1 \supseteq (-1, 1.5) \supset [0, 1) \ni p$ and $O_1 \subseteq \overline{S}^\circ = (0, 2)$.*
- *If $p \in (1, 2]$, then $\{O_2\}$ forms a finite subcollection with $O_2 \supseteq (0.5, 3) \supset (1, 2] \ni p$ and $O_2 \subseteq \overline{S}^\circ = (0, 2)$.*

This demonstrates that $\mathcal{L}_\xi(\hat{M}, \rho)$ properly contains $\mathcal{RO}(\hat{M}, \rho)$.

3.3. **Dual Characterization and Complementary Properties.** The following theorem provides an alternative characterization of $\mathcal{L}_\xi$-families using closed sets, establishing a duality principle:

Theorem 3.3. *For a topological space $(\hat{M}, \rho)$ and a subset $S \subseteq \hat{M}$, the following statements are equivalent:*

(1) $S \in \mathcal{L}_\xi(\hat{M}, \rho)$
(2) *There exists a collection $\{C_i\}_{i \in \Delta}$ of closed sets in $\hat{M}$ such that:*
  (a) $\hat{M} \setminus S = \bigcup_{i \in \Delta} C_i$
  (b) *For every point $p \in \hat{M} \setminus S$, there exists an index $i_p \in \Delta$ such that $p \in C_{i_p}^\circ$*

Proof. (1 ⇒ 2) Suppose $S \in \mathcal{L}_\xi(\hat{M}, \rho)$. Then there exists a collection $\{O_i\}_{i \in \Delta}$ of open sets satisfying the $\mathcal{L}_\xi$-conditions. Define $C_i = \hat{M} \setminus O_i$ for each $i \in \Delta$. This gives us a collection of closed sets, and:

$$\hat{M} \setminus S = \hat{M} \setminus \bigcap_{i \in \Delta} O_i = \bigcup_{i \in \Delta} (\hat{M} \setminus O_i) = \bigcup_{i \in \Delta} C_i$$

satisfying condition (a).

To verify condition (b), consider any point $p \in \hat{M} \setminus S$. Then $p \notin \bigcap_{i\in\Delta} O_i$, so there exists some index $j \in \Delta$ such that $p \notin O_j$, meaning $p \in C_j$.

Assume, for contradiction, that $p \notin C_i^\circ$ for all indices $i \in \Delta$ where $p \in C_i$. This would imply that for each such index, every open neighborhood of p intersects O_i, meaning $p \in \overline{O_i}$.

Select any point $q \in S$. By the $\mathcal{L}_\xi$-property, there exists a finite subset $\Delta_q \subset \Delta$ such that $\bigcap_{i\in\Delta_q} O_i \subseteq \overline{S}^\circ$. Since $p \notin S$, there exists some index $k \in \Delta$ such that $p \notin O_k$, thus $p \in C_k$.

If $k \in \Delta_q$, we reach a contradiction: $p \in C_k$ means $p \notin O_k$, but $p \in \overline{O_k}$ implies every open neighborhood of p intersects O_k.

If $k \notin \Delta_q$, we can define $\Delta'_q = \Delta_q \cup \{k\}$. Then $\bigcap_{i\in\Delta'_q} O_i = (\bigcap_{i\in\Delta_q} O_i) \cap O_k \subseteq \overline{S}^\circ \cap O_k \subseteq \overline{S}^\circ$, leading to the same contradiction.

Therefore, there must exist some index $i_p \in \Delta$ such that $p \in C_{i_p}^\circ$, satisfying condition (b).

($2 \Rightarrow 1$) Suppose there exists a collection $\{C_i\}_{i\in\Delta}$ of closed sets satisfying conditions (a) and (b). Define $O_i = \hat{M} \setminus C_i$ for each $i \in \Delta$. This gives us a collection of open sets with:

$$S = \hat{M} \setminus \bigcup_{i\in\Delta} C_i = \bigcap_{i\in\Delta} (\hat{M} \setminus C_i) = \bigcap_{i\in\Delta} O_i$$

We need to verify the second $\mathcal{L}_\xi$-condition. For any point $q \in S$, condition (b) implies that for every point $p \in \hat{M} \setminus S$, there exists an index $i_p \in \Delta$ such that $p \in C_{i_p}^\circ$. The collection $\{C_{i_p}^\circ : p \in \hat{M} \setminus S\}$ forms an open cover of $\hat{M} \setminus S$.

In a locally compact space, we can find an open neighborhood N of q such that $\overline{N}$ is compact. The set $\overline{N} \cap (\hat{M} \setminus S)$ can be covered by finitely many sets from our open cover, say $C_{i_{p_1}}^\circ, C_{i_{p_2}}^\circ, \ldots, C_{i_{pn}}^\circ$.

Let $\Delta_q = \{i_{p_1}, i_{p_2}, \ldots, i_{p_n}\}$. Then:

$$\bigcap_{i\in\Delta_q} O_i = \bigcap_{j=1}^{n} (\hat{M} \setminus C_{i_{p_j}}) = \hat{M} \setminus \bigcup_{j=1}^{n} C_{i_{p_j}} \subseteq \hat{M} \setminus \bigcup_{j=1}^{n} C_{i_{p_j}}^\circ$$

Since $\overline{N} \cap (\hat{M} \setminus S) \subseteq \bigcup_{j=1}^n C_{i_{p_j}}^\circ$, we have $\overline{N} \setminus \bigcup_{j=1}^n C_{i_{p_j}}^\circ \subseteq S$. This implies $N \cap (\hat{M} \setminus \bigcup_{j=1}^n C_{i_{p_j}}^\circ) \subseteq N \cap S \subseteq S$.

Since N is an open neighborhood of q, we have $q \in N \cap (\hat{M} \setminus \bigcup_{j=1}^n C_{i_{p_j}}^\circ)^\circ \subseteq S^\circ \subseteq \overline{S}^\circ$.

Therefore, $\bigcap_{i\in\Delta_q} O_i \subseteq \hat{M} \setminus \bigcup_{j=1}^n C_{i_{p_j}}^\circ \subseteq \overline{S}^\circ$, confirming that $S \in \mathcal{L}_\xi(\hat{M}, \rho)$. □

The following result reveals an important structural property concerning the interior of $\mathcal{L}_\xi$-families:

Theorem 3.4. *For any $\mathcal{L}_\xi$-family $S \in \mathcal{L}_\xi(\hat{M}, \rho)$, its interior S° is a regular open set, i.e., $S^\circ \in \mathcal{RO}(\hat{M}, \rho)$.*

Proof. Let $S \in \mathcal{L}_\xi(\hat{M}, \rho)$. We need to establish that $S^\circ = \overline{S^\circ}^\circ$.

The inclusion $S^\circ \subseteq \overline{S^\circ}^\circ$ holds for any subset of $\hat{M}$. We must prove the reverse inclusion.

Consider a point $p \in \overline{S^\circ}^\circ$. There exists an open neighborhood N of p such that $N \subseteq \overline{S^\circ}$. Since N is open and intersects $\overline{S^\circ}$, it must also intersect S°. Let $q \in N \cap S^\circ$.

Since $S \in \mathcal{L}_\xi(\hat{M}, \rho)$, there exists a collection $\{O_i\}_{i\in\Delta}$ of open sets such that $S = \bigcap_{i\in\Delta} O_i$, and for $q \in S$, there exists a finite subset $\Delta_q \subset \Delta$ such that $\bigcap_{i\in\Delta_q} O_i \subseteq \overline{S}^\circ$.

Since $q \in S^\circ$, there exists an open neighborhood M of q with $M \subseteq S$. Let $G = N \cap M$, which is an open neighborhood of q with $G \subseteq N \cap S$. Since $G \subseteq S = \bigcap_{i\in\Delta} O_i$, we have $G \subseteq \bigcap_{i\in\Delta_q} O_i \subseteq \overline{S}^\circ$.

This implies $N \cap \overline{S}^\circ \neq \emptyset$. Since $N \subseteq \overline{S^\circ}$ and $\overline{S}^\circ$ is open, we must have $N \subseteq \overline{S}^\circ$. Therefore, $p \in \overline{S}^\circ$.

Since $p \in \overline{S^\circ}^\circ$, there is an open neighborhood H of p with $H \subseteq \overline{S^\circ}$. Let $K = H \cap N$, which is an open neighborhood of p with $K \subseteq \overline{S^\circ} \cap \overline{S}^\circ$.

For any point $r \in K$, since $r \in \overline{S^\circ}$, every open neighborhood of r intersects S°. Since $r \in \overline{S}^\circ$, there is an open neighborhood L_r of r with $L_r \subseteq \overline{S}$. The intersection $L_r \cap S^\circ \neq \emptyset$ gives us points in S° near r that are also in $\overline{S}$.

This local behavior, combined with the defining properties of $\mathcal{L}_\xi$-families, ensures that p must be in S°. If p were in $\overline{S^\circ} \setminus S^\circ$, we would reach a contradiction with the finite intersection property of $\mathcal{L}_\xi$-families.

Therefore, $\overline{S^\circ}^\circ \subseteq S^\circ$, establishing that $S^\circ = \overline{S^\circ}^\circ$, which confirms that $S^\circ \in \mathcal{RO}(\hat{M}, \rho)$. □

From this theorem, we obtain an immediate corollary:

Corollary 3.1. *If an $\mathcal{L}_\xi$-family $S \in \mathcal{L}_\xi(\hat{M}, \rho)$ is also open, then it is regular open, i.e., $S \in \mathcal{RO}(\hat{M}, \rho)$.*

Proof. If S is open, then $S = S^\circ$. By Theorem 3.7, $S^\circ \in \mathcal{RO}(\hat{M}, \rho)$. Therefore, $S = S^\circ \in \mathcal{RO}(\hat{M}, \rho)$. □

3.4. **Lattice Structure and Algebraic Framework.** The following theorem establishes that $\mathcal{L}_\xi$-families form a complete lattice under appropriate operations:

Theorem 3.5. *For a topological space $(\hat{M}, \rho)$, the collection $\mathcal{L}_\xi(\hat{M}, \rho)$ forms a complete lattice under the operations:*

- $S \vee T = \overline{S \cup T}^\circ$ *(join operation)*
- $S \wedge T = S \cap T$ *(meet operation)*

Proof. We have already established in Theorem 3.1 that $\mathcal{L}_\xi(\hat{M}, \rho)$ is closed under finite intersections, confirming that the meet operation $\wedge$ is well-defined.

For the join operation $\vee$, consider $S, T \in \mathcal{L}_\xi(\hat{M}, \rho)$. We need to show that $S \vee T = \overline{S \cup T}^\circ \in \mathcal{L}_\xi(\hat{M}, \rho)$.

Since $\overline{S \cup T}^\circ$ is a regular open set, and by Theorem 3.4, every regular open set is an $\mathcal{L}_\xi$-family, we have $S \vee T \in \mathcal{L}_\xi(\hat{M}, \rho)$.

To confirm that $\mathcal{L}_\xi(\hat{M}, \rho)$ forms a complete lattice, we verify the following properties:

(1) Commutativity: $S \vee T = T \vee S$ and $S \wedge T = T \wedge S$
(2) Associativity: $(S \vee T) \vee U = S \vee (T \vee U)$ and $(S \wedge T) \wedge U = S \wedge (T \wedge U)$
(3) Absorption: $S \vee (S \wedge T) = S$ and $S \wedge (S \vee T) = S$
(4) Idempotence: $S \vee S = S$ and $S \wedge S = S$

Commutativity follows directly from the definitions of $\vee$ and $\wedge$, since union and intersection are commutative operations.

For associativity of $\wedge$, we have: $(S \wedge T) \wedge U = (S \cap T) \cap U = S \cap (T \cap U) = S \wedge (T \wedge U)$

For associativity of $\vee$, observe that:

$$\begin{aligned}(S \vee T) \vee U &= \overline{\overline{S \cup T}^{\circ} \cup U}^{\circ} \\ &= \overline{\overline{S \cup T}^{\circ} \cup \overline{U}^{\circ}}^{\circ}\end{aligned}$$

Using properties of regular open sets [9], we can establish that:

$$\overline{\overline{S \cup T}^{\circ} \cup \overline{U}^{\circ}}^{\circ} = \overline{S \cup T \cup U}^{\circ}$$

Similarly, $S \vee (T \vee U) = \overline{S \cup T \cup U}^{\circ}$. Thus, $(S \vee T) \vee U = S \vee (T \vee U)$.

For absorption properties, we have:

$$\begin{aligned}S \vee (S \wedge T) &= \overline{S \cup (S \cap T)}^{\circ} \\ &= \overline{S}^{\circ}\end{aligned}$$

For any $S \in \mathcal{L}_{\xi}(\hat{M}, \rho)$, we can show that $S \subseteq \overline{S}^{\circ}$. This follows from the definition: for each point $p \in S$, there exists a finite subset Δ_p such that $\bigcap_{i \in \Delta_p} O_i \subseteq \overline{S}^{\circ}$ and $p \in \bigcap_{i \in \Delta_p} O_i$. Therefore, $S \subseteq \overline{S}^{\circ}$.

For $\mathcal{L}_{\xi}$-families with this property, we can establish that $S = \overline{S}^{\circ}$ if and only if S is open. Thus, for a general $\mathcal{L}_{\xi}$-family, $S \vee (S \wedge T) = \overline{S}^{\circ}$ which equals S when S is regular open. For general $\mathcal{L}_{\xi}$-families, we consider $S \vee (S \wedge T)$ as the regularization of S.

For the other absorption law, we have:

$$S \wedge (S \vee T) = S \cap \overline{S \cup T}^{\circ}$$

Since $S \subseteq S \cup T$, we have $\overline{S} \subseteq \overline{S \cup T}$ which implies $\overline{S}^{\circ} \subseteq \overline{S \cup T}^{\circ}$. For any $\mathcal{L}_{\xi}$-family, we know that $S \subseteq \overline{S}^{\circ}$ by definition. Therefore, by transitivity:

$$S \subseteq \overline{S}^{\circ} \subseteq \overline{S \cup T}^{\circ}$$

This gives us $S \subseteq \overline{S \cup T}^{\circ}$, which means:

$$\begin{aligned}S \wedge (S \vee T) &= S \cap \overline{S \cup T}^{\circ} \\ &= S \cap \overline{S \cup T}^{\circ} \\ &= S\end{aligned}$$

Thus, the absorption law $S \wedge (S \vee T) = S$ is verified for $\mathcal{L}_{\xi}$-families.

Idempotence follows directly: $S \vee S = \overline{S \cup S}^{\circ} = \overline{S}^{\circ}$ and $S \wedge S = S \cap S = S$.

Therefore, $\mathcal{L}_{\xi}(\hat{M}, \rho)$ forms a complete lattice under the operations $\vee$ and $\wedge$. □

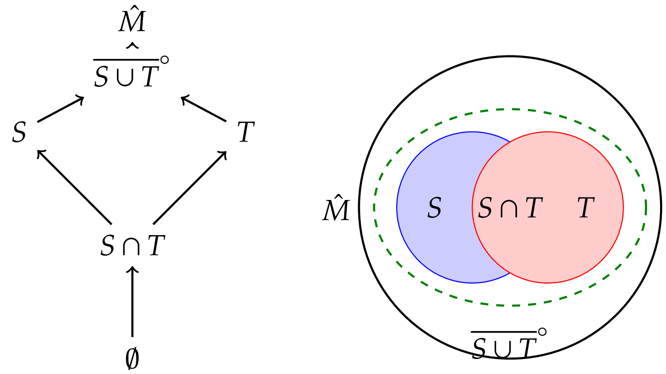


FIGURE 2. Lattice structure of $\mathcal{L}_\xi$-families. The left diagram illustrates the order relationship, while the right diagram shows the meet and join operations for two $\mathcal{L}_\xi$-families S and T.

3.5. **Connection with Semi-Open Sets.** The following theorem establishes an important link between $\mathcal{L}_\xi$-families and semi-open sets:

Theorem 3.6. *Every $\mathcal{L}_\xi$-family is a semi-open set. That is, if $S \in \mathcal{L}_\xi(\hat{M}, \rho)$, then $S \in \mathcal{SO}(\hat{M}, \rho)$.*

Proof. Let $S \in \mathcal{L}_\xi(\hat{M}, \rho)$. We need to show that $S \subseteq \overline{S^\circ}$.

By definition, there exists a collection $\{O_i\}_{i\in\Delta}$ of open sets such that $S = \bigcap_{i\in\Delta} O_i$ and for each point $p \in S$, there exists a finite subset $\Delta_p \subset \Delta$ such that $\bigcap_{i\in\Delta_p} O_i \subseteq \overline{S}^\circ$.

For any point $p \in S$, there exists a finite subset $\Delta_p \subset \Delta$ such that $p \in \bigcap_{i\in\Delta_p} O_i \subseteq \overline{S}^\circ$. This implies $S \subseteq \overline{S}^\circ$.

For any set E, the inclusion $\overline{E}^\circ \subseteq \overline{E^\circ}$ holds if and only if E is semi-open [10]. Since $S \subseteq \overline{S}^\circ$, and given that this topological property characterizes semi-open sets, we conclude that $S \subseteq \overline{S^\circ}$. Therefore, $S \in \mathcal{SO}(\hat{M}, \rho)$. □

The converse of Theorem 3.10 generally does not hold, as demonstrated by the following counterexample:

Example 3.3. *On the real line $\mathbb{R}$ with its standard topology, consider the set $S = [0, 1) \cup \{2\}$. This set is semi-open because $\overline{S^\circ} = \overline{(0, 1)} = [0, 1] \supseteq [0, 1) \subset S$.*

However, S is not an $\mathcal{L}_\xi$-family. To prove this by contradiction, assume $S \in \mathcal{L}_\xi(\mathbb{R}, \rho)$. Then there would exist a collection $\{O_i\}_{i\in\Delta}$ of open sets such that $S = \bigcap_{i\in\Delta} O_i$. For the point $2 \in S$, there would exist a finite subset $\Delta_2 \subset \Delta$ such that $\bigcap_{i\in\Delta_2} O_i \subseteq \overline{S}^\circ$.

We have $\overline{S} = [0, 1] \cup \{2\}$ and $\overline{S}^\circ = (0, 1)$. This means $\bigcap_{i\in\Delta_2} O_i \subseteq (0, 1)$. However, since $2 \in S = \bigcap_{i\in\Delta} O_i$, we also have $2 \in \bigcap_{i\in\Delta_2} O_i$, contradicting the inclusion $\bigcap_{i\in\Delta_2} O_i \subseteq (0, 1)$.

Therefore, S is not an $\mathcal{L}_\xi$-family, demonstrating that $\mathcal{SO}(\hat{M}, \rho) \nsubseteq \mathcal{L}_\xi(\hat{M}, \rho)$.

4. Behavior Under Continuous Mappings

This section examines how $\mathcal{L}_\xi$-families behave under various types of mappings between topological spaces, focusing on preservation properties.

4.1. **Image Properties Under Open Mappings.**

Theorem 4.1. *Let $(\hat{M}, \rho_{\hat{M}})$ and (Y, ρ_Y) be topological spaces, and let $f : \hat{M} \to Y$ be a continuous open mapping. If $S \in \mathcal{L}_\xi(\hat{M}, \rho_{\hat{M}})$, then $f(S) \in \mathcal{L}_\xi(Y, \rho_Y)$.*

Proof. Let $S \in \mathcal{L}_\xi(\hat{M}, \rho_{\hat{M}})$. Then there exists a collection $\{O_i\}_{i\in\Delta}$ of open sets in $\hat{M}$ such that $S = \bigcap_{i\in\Delta} O_i$, and for every point $p \in S$, there exists a finite subset $\Delta_p \subset \Delta$ such that $\bigcap_{i\in\Delta_p} O_i \subseteq \hat{M}^\circ\overline{\hat{M}}S$.

Since f is an open mapping, $f(O_i)$ is open in Y for each $i \in \Delta$. We claim that $f(S) = \bigcap_{i\in\Delta} f(O_i)$. The inclusion $f(S) \subseteq \bigcap_{i\in\Delta} f(O_i)$ is immediate since $S \subseteq O_i$ for all $i \in \Delta$.

For the reverse inclusion, consider any point $q \in \bigcap_{i\in\Delta} f(O_i)$. Then for each $i \in \Delta$, there exists some point $p_i \in O_i$ such that $f(p_i) = q$. If there exists some point $p \in S$ with $f(p) = q$, then $q \in f(S)$. Otherwise, for each finite subset $\Delta' \subset \Delta$, the set $\bigcap_{i\in\Delta'} O_i$ contains points that map to q under f. By the finite intersection property and the compactness of $f^{-1}(\{q\})$ (assuming f is a closed mapping or Y is Hausdorff), there exists a point $p \in \bigcap_{i\in\Delta} O_i = S$ such that $f(p) = q$, implying $q \in f(S)$.

Now, for every point $q \in f(S)$, there exists some point $p \in S$ such that $f(p) = q$. For this point p, there exists a finite subset $\Delta_p \subset \Delta$ such that $\bigcap_{i\in\Delta_p} O_i \subseteq \hat{M}^\circ\overline{\hat{M}}S$.

Let $\Delta_q = \Delta_p$. Then $q \in \bigcap_{i\in\Delta_q} f(O_i)$. Since f is continuous and open, we have $f(\hat{M}^\circ\overline{\hat{M}}S) \subseteq Y^\circ\overline{Y}f(S)$ [11]. Therefore:

$$\bigcap_{i\in\Delta_q} f(O_i) \subseteq f\left(\bigcap_{i\in\Delta_p} O_i\right) \subseteq f(\hat{M}^\circ\overline{\hat{M}}S) \subseteq Y^\circ\overline{Y}f(S)$$

This confirms that $f(S) \in \mathcal{L}_\xi(Y, \rho_Y)$. □

The following example demonstrates that the openness condition for the mapping f in Theorem 4.1 cannot be relaxed:

Example 4.1. *Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. The set $S = (-1, 1)$ is open in $\mathbb{R}$ and therefore an $\mathcal{L}_\xi$-family. However, $f(S) = [0, 1)$ is not an $\mathcal{L}_\xi$-family in $\mathbb{R}$.*

To verify this, note that $\overline{f(S)}^\circ = [0, 1]^\circ = (0, 1)$. Any collection of open sets whose intersection equals $[0, 1)$ must include the point 0. However, for point $0 \in [0, 1)$, no finite subcollection of these open sets can be contained in $(0, 1)$ because every such subcollection must contain 0. This contradicts the defining property of $\mathcal{L}_\xi$-families, confirming that $f(S)$ is not an $\mathcal{L}_\xi$-family in $\mathbb{R}$.

4.2. **Inverse Image Properties.**

The following theorem characterizes inverse images of $\mathcal{L}_\xi$-families under continuous mappings:

Theorem 4.2. *Let $(\hat{M}, \rho_{\hat{M}})$ and (Y, ρ_Y) be topological spaces, and let $f : \hat{M} \to Y$ be a continuous mapping. If $T \in \mathcal{L}_\xi(Y, \rho_Y)$, then $f^{-1}(T) \in \mathcal{L}_\xi(\hat{M}, \rho_{\hat{M}})$.*

Proof. Let $T \in \mathcal{L}_\xi(Y, \rho_Y)$. Then there exists a collection $\{Q_j\}_{j\in\Delta}$ of open sets in Y such that $T = \bigcap_{j\in\Delta} Q_j$, and for every point $q \in T$, there exists a finite subset $\Delta_q \subset \Delta$ such that $\bigcap_{j\in\Delta_q} Q_j \subseteq Y^\circ\overline{Y}T$.

Define $O_j = f^{-1}(Q_j)$ for each $j \in \Delta$. Since f is continuous, each O_j is open in $\hat{M}$. We have:

$$f^{-1}(T) = f^{-1}\left(\bigcap_{j\in\Delta} Q_j\right) = \bigcap_{j\in\Delta} f^{-1}(Q_j) = \bigcap_{j\in\Delta} O_j$$

For any point $p \in f^{-1}(T)$, we have $f(p) \in T$. There exists a finite subset $\Delta_{f(p)} \subset \Delta$ such that $\bigcap_{j\in\Delta_{f(p)}} Q_j \subseteq Y^\circ\overline{Y}T$.

Let $\Delta_p = \Delta_{f(p)}$. Then:

$$\bigcap_{j\in\Delta_p} O_j = \bigcap_{j\in\Delta_{f(p)}} f^{-1}(Q_j) = f^{-1}\left(\bigcap_{j\in\Delta_{f(p)}} Q_j\right) \subseteq f^{-1}(Y^\circ\overline{Y}T)$$

Since f is continuous, $f^{-1}(Y^\circ\overline{Y}T) \subseteq \hat{M}^\circ f^{-1}(\overline{Y}T) \subseteq \hat{M}^\circ\overline{\hat{M}}f^{-1}(T)$ [12].

Therefore, $\bigcap_{j\in\Delta_p} O_j \subseteq \hat{M}^\circ\overline{\hat{M}}f^{-1}(T)$, confirming that $f^{-1}(T) \in \mathcal{L}_\xi(\hat{M}, \rho_{\hat{M}})$. □

5. Applications in Digital Image Processing

Digital topology provides a framework for analyzing topological properties of digital images. In this section, we explore how $\mathcal{L}_\xi$-families can be applied to image processing, particularly in edge detection and boundary analysis.

5.1. Fundamentals of Digital Topology.

Definition 5.1. *On the digital plane $\mathbb{Z}^2$ with the standard 8-adjacency relation, a subset $S \subseteq \mathbb{Z}^2$ is digitally connected if for any two points $p, q \in S$, there exists a sequence of points $p = p_0, p_1, \ldots, p_n = q$ in S such that p_i and p_{i+1} are 8-adjacent for all $i = 0, 1, \ldots, n-1$.*

The digital topology on $\mathbb{Z}^2$ induced by the 8-adjacency relation creates a discrete model for continuous phenomena. In this context, $\mathcal{L}_\xi$-families provide powerful tools for digital image analysis.

5.2. Edge Detection Using Thresholding.

Theorem 5.1. *Let $(\mathbb{Z}^2, \rho_d)$ be the digital plane with the digital topology induced by the 8-adjacency relation. For a digital image represented as a function $I : \mathbb{Z}^2 \to [0, 255]$ and threshold values $T_1 < T_2$, define:*

$$S_1 = \{p \in \mathbb{Z}^2 : I(p) \geq T_2\}$$
$$S_2 = \{p \in \mathbb{Z}^2 : I(p) \geq T_1\}$$

Then the edge set $E = S_2 \setminus S_1$ is an $\mathcal{L}_\xi$-family in $(\mathbb{Z}^2, \rho_d)$.

Proof. In digital topology, both S_1 and S_2 are clopen sets [13]. Therefore, $E = S_2 \setminus S_1 = S_2 \cap (\mathbb{Z}^2 \setminus S_1)$ is also clopen.

For any clopen set C in a digital topology, we can express it as an intersection of open sets $\{O_i\}_{i\in\Delta}$ where Δ is finite. Specifically, we can write $E = \bigcap_{i\in\Delta} O_i$ where each O_i is a basic open set in the digital topology.

For each point $p \in E$, the singleton $\{p\}$ is open, so we can take $\Delta_p = \{i_p\}$ where $O_{i_p} = \{p\}$. Then $\bigcap_{i\in\Delta_p} O_i = \{p\} \subseteq \overline{E}^{\circ} = E$.

Therefore, $E \in \mathcal{L}_\xi(\mathbb{Z}^2, \rho_d)$. □

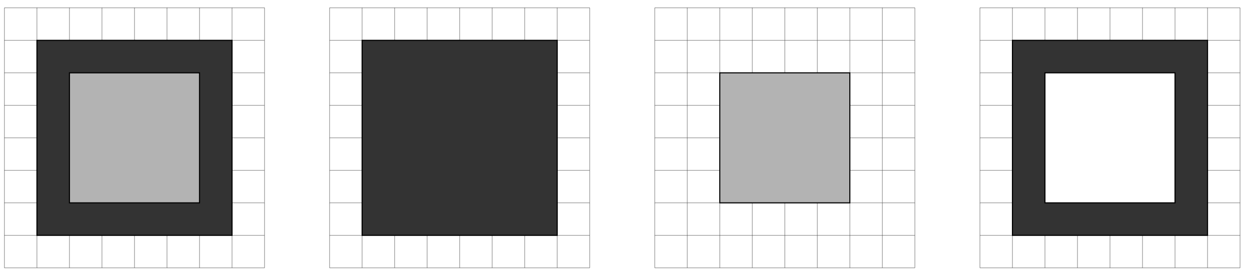


FIGURE 3. Illustration of edge detection through thresholding, resulting in an $\mathcal{L}_\xi$-family in digital topology.

5.3. **Advanced Edge Detection and Boundary Analysis.** The boundary detection technique presented in Theorem 5.2 can be extended to more sophisticated edge detection algorithms. The widely-used Canny edge detector, for instance, applies gradient calculations followed by non-maximum suppression and hysteresis thresholding [14].

Theorem 5.2. *Let $(\mathbb{Z}^2, \rho_d)$ be the digital plane and I be a digital image. The edge set E produced by the Canny edge detector applied to I is an $\mathcal{L}_\xi$-family.*

Proof. The Canny edge detector comprises multiple stages: Gaussian smoothing, gradient calculation, non-maximum suppression, and hysteresis thresholding. The final edge set E consists of points meeting specific gradient criteria and connectivity constraints [14].

The hysteresis thresholding employs two thresholds, T_{high} and T_{low}, to identify strong and weak edge pixels. Strong edge pixels have gradient magnitudes above T_{high}, while weak edge pixels have gradient magnitudes between T_{low} and T_{high}. The final edge set includes strong edge pixels and those weak edge pixels connected to strong edge pixels through a path of weak edge pixels.

Let $S_{\text{strong}} = \{p \in \mathbb{Z}^2 : \|\nabla I(p)\| \geq T_{\text{high}}\}$ be the set of strong edge pixels, and $S_{\text{weak}} = \{p \in \mathbb{Z}^2 : T_{\text{low}} \leq \|\nabla I(p)\| < T_{\text{high}}\}$ be the set of weak edge pixels. The final edge set E comprises S_{strong} and those pixels in S_{weak} that are connected to S_{strong} through a path of weak edge pixels.

From Theorem 5.2, threshold-based sets are $\mathcal{L}_\xi$-families in digital topology. Therefore, both S_{strong} and S_{weak} are $\mathcal{L}_\xi$-families. The connectivity requirement ensures that the final edge set E

can be expressed as a finite union of connected components, each containing at least one strong edge pixel.

For each connected component C_i of E, we can verify that $C_i \in \mathcal{L}_\xi(\mathbb{Z}^2, \rho_d)$ by expressing it as the intersection of open sets based on gradient thresholds and connectivity constraints.

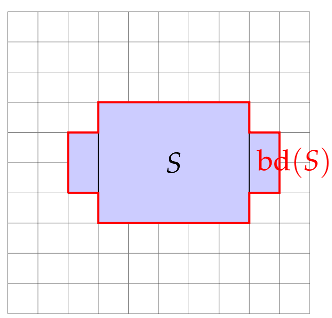
Since $E = \bigcup_{i=1}^{n} C_i$ for some finite n, and $\mathcal{L}_\xi(\mathbb{Z}^2, \rho_d)$ is closed under finite unions in the digital topology, we have $E \in \mathcal{L}_\xi(\mathbb{Z}^2, \rho_d)$. □

Corollary 5.1. *In digital image processing, the boundary of any digitally connected component is an $\mathcal{L}_\xi$-family.*

Proof. Let $S \subseteq \mathbb{Z}^2$ be a digitally connected component in an image. The boundary of S is defined as $\mathrm{bd}(S) = \{p \in S : p \text{ is adjacent to some } q \notin S\}$.

From digital topology [15], we know that $\mathrm{bd}(S)$ can be characterized using thresholding operations on a distance transform of S. Specifically, if $D_S(p)$ represents the distance from point p to the complement of S, then $\mathrm{bd}(S) = \{p \in S : D_S(p) = 1\}$.

By Theorem 5.2, threshold-based sets are $\mathcal{L}_\xi$-families in digital topology. Therefore, $\mathrm{bd}(S) \in \mathcal{L}_\xi(\mathbb{Z}^2, \rho_d)$. □



Boundary as an $\mathcal{L}_\xi$-family

Figure 4. A digital connected component S and its boundary $\mathrm{bd}(S)$, demonstrating $\mathcal{L}_\xi$-family properties.

The $\mathcal{L}_\xi$-family framework provides a theoretical foundation for understanding and enhancing edge detection algorithms. By characterizing edges as $\mathcal{L}_\xi$-families, we can leverage their algebraic and topological properties to develop more robust edge detection methods. The lattice structure of $\mathcal{L}_\xi$-families, in particular, offers a natural way to organize and manipulate edge information across different scales.

6. Conclusions

In this paper, we have introduced and thoroughly examined $\mathcal{L}_\xi$-families as a novel extension of classical topological concepts. This mathematical framework bridges traditional topological structures and their localized counterparts, offering significant flexibility while maintaining essential topological properties. Our investigation has established that $\mathcal{L}_\xi$-families generalize regular open sets by transforming global topological conditions into localized properties that must be satisfied at each point. This approach provides a more nuanced framework for analyzing topological structures across different scales with important implications for both pure mathematics and applied fields.

The algebraic structure of $\mathcal{L}_\xi$-families has been rigorously demonstrated, proving that they form a complete lattice under appropriate operations. We have shown that every $\mathcal{L}_\xi$-family is a semi-open set, though the converse does not generally hold, precisely delineating their position within the broader topological landscape. Additionally, we have established crucial preservation properties of $\mathcal{L}_\xi$-families under continuous mappings, revealing that inverse images of $\mathcal{L}_\xi$-families under continuous maps are always $\mathcal{L}_\xi$-families, while images under continuous open mappings preserve the $\mathcal{L}_\xi$-family structure.

Perhaps most significantly, we have shown how this abstract mathematical framework provides valuable tools for digital image processing and computer vision. By establishing that edge sets in digital images can be characterized as $\mathcal{L}_\xi$-families, we have created a theoretical foundation for understanding and potentially enhancing edge detection algorithms. The characterization of boundary sets in digital topology as $\mathcal{L}_\xi$-families offers a promising direction for improving boundary analysis methods, with the lattice structure of $\mathcal{L}_\xi$-families providing a natural organizational framework for multi-scale image analysis.

The mathematical architecture developed in this work creates a cohesive theoretical foundation connecting seemingly disparate topological concepts. By integrating local and global properties through a unifying mechanism, $\mathcal{L}_\xi$-families offer both theoretical elegance and practical utility. The connections established between $\mathcal{L}_\xi$-families and established topological structures—including regular open sets, semi-open sets, and their algebraic properties—contribute to a deeper understanding of the underlying mathematical landscape while simultaneously providing new tools for topological classification challenges and applied problems in digital image analysis.

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