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Neutrosophic MR-Metric Spaces: A Topos-Theoretic Framework with Applications

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Abstract. This paper introduces and systematically investigates the category of Neutrosophic MR-Metric Spaces (NMR-MS), which generalizes classical metric spaces by incorporating neutrosophic logic to model truth ($\mathcal{T}$), indeterminacy ($\mathcal{I}$), and falsity ($\mathcal{F}$). We define the category NMRMS and construct sheaves of NMR-MS over topological spaces, proving that the category $Sh(X, NMRMS)$ forms an elementary topos. This provides a rich mathematical framework for reasoning about uncertainty, vagueness, and contextual truth in a localized manner. We develop the internal language of this topos as a neutrosophic type theory and establish its soundness and completeness. The framework is applied to diverse fields including manifold theory, dynamic systems, image processing, data fusion, functional analysis, graph theory, differential equations, machine learning, topology optimization, quantum systems, and financial modeling. Our work unifies and extends recent advances in fixed point theory, fractional calculus, and neutrosophic fuzzy metrics within a single, category-theoretic foundation.

1. Introduction

The evolution of metric space theory has been marked by successive generalizations aimed at capturing more complex structural and logical properties. Beginning with the contraction mapping principle of Bakhtin [27] and Czerwik [28] in b-metric spaces, the theory expanded to include G_b-metric spaces [10], M^*-metric spaces [12], and Ω-distance mappings ([7], [8], [9], [13], [14], [15]). A significant advancement was the introduction of MR-metric spaces by Malkawi et al. [2], which extended the ternary relational structure of G-metric spaces and led to numerous fixed point results ([19], [20], [21], [11], [4], [1], [22]). Concurrently, the study of fractional differential equations ([5], [29], [30], [31], [32], [35]) and their solutions via symmetry methods and atomic solutions highlighted the need for spaces accommodating uncertainty and imprecision.

The integration of fuzzy and neutrosophic logic into metric spaces provided a paradigm shift. Building on fuzzy metric spaces ([36], [37], [38], [39], [16], [18]), Malkawi recently introduced

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Neutrosophic MR-Metric Spaces (NMR-MS) ([23], [33]), which combine the ternary metric structure of MR-spaces with the tripartite logic of neutrosophic sets. This allows for the simultaneous quantification of truth, indeterminacy, and falsity in proximity measurements. Recent works have applied these spaces to fixed point theory for fuzzy mappings [33], contraction principles [23], and applications in weighted graphs, deep learning, and measure theory ([24], [25], [26]).

Our paper synthesizes these developments by constructing a category of NMR-MS and demonstrating that the category of sheaves of such spaces over a topological base forms an elementary topos. This result provides a unified semantic framework for a neutrosophic metric logic([40], [41], [42], [43], [44], [45], [46], [47], [48]), internal to the topos, which is both sound and complete. The implications are vast, enabling rigorous, localized reasoning about systems where uncertainty, context-dependence, and approximation are inherent. This work thus bridges metric fixed point theory, neutrosophic logic, category theory, and sheaf theory, offering a powerful new tool for both theoretical and applied mathematics.

Definition 1.1. *[2] Consider a non-empty set $\mathbb{X} \neq \emptyset$ and a real number $\mathbb{R} > 1$. A function*

$$M : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [0, \infty)$$

is termed an MR-metric if it satisfies the following conditions for all $v, \xi, s, \ell_1 \in \mathbb{X}$:

- $M(v, \xi, s) \geq 0$.
- $M(v, \xi, s) = 0$ *if and only if* $v = \xi = s$.
- $M(v, \xi, s)$ *remains invariant under any permutation* $p(v, \xi, s)$, *i.e.,* $M(v, \xi, s) = M(p(v, \xi, s))$.
- *The following inequality holds:*

$$M(v, \xi, s) \leq \mathbb{R}\left[M(v, \xi, \ell_1) + M(v, \ell_1, s) + M(\ell_1, \xi, s)\right].$$

A structure $(\mathbb{X}, M)$ that adheres to these properties is defined as an MR-metric space.

Definition 1.2. *[33][Neutrosophic MR-Metric Space (NMR-MS)]*
A 9-tuple $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, R, \star)$ is called a Neutrosophic MR-Metric Space if:

(1) $\mathcal{Z}$ *is a non-empty set.*
(2) $M : \mathcal{Z} \times \mathcal{Z} \times \mathcal{Z} \to [0, \infty)$ *is an MR-metric satisfying:*
  (M1) $M(v, \xi, \mathfrak{I}) \geq 0$,
  (M2) $M(v, \xi, \mathfrak{I}) = 0 \iff v = \xi = \mathfrak{I}$,
  (M3) *Symmetry under permutations,*
  (M4) $M(v, \xi, \mathfrak{I}) \leq R\left[M(v, \xi, \ell) \star M(v, \ell, \mathfrak{I}) \star M(\ell, \xi, \mathfrak{I})\right]$, $R > 1$.
(3) $\mathcal{T}, \mathcal{F}, \mathcal{I} : \mathcal{Z} \times \mathcal{Z} \times (0, \infty) \to [0, 1]$ *are neutrosophic functions satisfying:*
  (N1) $\mathcal{T}(v, \xi, \gamma) = 1 \iff v = \xi$ *(Truth-Identity),*
  (N2) $\mathcal{T}(v, \xi, \gamma) = \mathcal{T}(\xi, v, \gamma)$ *(Symmetry),*
  (N3) $\mathcal{T}(v, \xi, \gamma) \bullet \mathcal{T}(\xi, \mathfrak{I}, \rho) \leq \mathcal{T}(v, \mathfrak{I}, \gamma + \rho)$ *(Triangle Inequality),*
  (N4) $\lim_{\gamma \to \infty} \mathcal{T}(v, \xi, \gamma) = 1$ *(Asymptotic Behavior).*
(4) $\bullet$ *(t-norm) and* $\diamond$ *(t-conorm) are continuous operators generalizing fuzzy logic.*

(5) $\star$ *is a binary operation generalizing addition (e.g., weighted sum).*

2. Main Results

This section presents the core mathematical contributions of this work. We begin by defining the category of Neutrosophic MR-Metric Spaces and their morphisms. We then construct sheaves of such spaces over arbitrary topological spaces and prove that the resulting category of sheaves forms an elementary topos. Furthermore, we define the internal language of this topos as a neutrosophic type theory and establish fundamental results about its logic, including soundness and completeness.

Definition 2.1 (Category of Neutrosophic MR-Metric Spaces)**.** *The category NMRMS consists of:*

- *Objects: Neutrosophic MR-Metric Spaces* $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, R, \star)$
- *Morphisms: A morphism* $f : (\mathcal{Z}_1, M_1, \mathcal{T}_1, \mathcal{F}_1, \mathcal{I}_1, \bullet_1, \diamond_1, R_1, \star_1) \to (\mathcal{Z}_2, M_2, \mathcal{T}_2, \mathcal{F}_2, \mathcal{I}_2, \bullet_2, \diamond_2, R_2, \star_2)$ *is a function* $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ *satisfying:*
  (1) *Metric Preservation:* $M_2(f(\upsilon), f(\xi), f(\mathfrak{I})) \leq M_1(\upsilon, \xi, \mathfrak{I})$
  (2) *Neutrosophic Consistency:*

$$\mathcal{T}_2(f(\upsilon), f(\xi), \gamma) \geq \mathcal{T}_1(\upsilon, \xi, \gamma)$$
$$\mathcal{I}_2(f(\upsilon), f(\xi), \gamma) \leq \mathcal{I}_1(\upsilon, \xi, \gamma)$$
$$\mathcal{F}_2(f(\upsilon), f(\xi), \gamma) \leq \mathcal{F}_1(\upsilon, \xi, \gamma)$$

  (3) *Operation Compatibility:* f *preserves the t-norm* $\bullet$ *and t-conorm* $\diamond$ *structures*

Definition 2.2 (Sheaf of NMR-MS)**.** *Let X be a topological space. A sheaf of Neutrosophic MR-Metric Spaces on X is a contravariant functor:*

$$\mathscr{F} : Open(X)^{op} \to NMRMS$$

satisfying the following sheaf conditions:

(1) *Local Identity: If* $U = \bigcup_{i \in I} U_i$ *and* $s, t \in \mathscr{F}(U)$ *with* $s|_{U_i} = t|_{U_i}$ *for all i, then* $s = t$.
(2) *Gluing: If* $U = \bigcup_{i \in I} U_i$ *and* $s_i \in \mathscr{F}(U_i)$ *are sections satisfying* $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ *for all i, j, then there exists* $s \in \mathscr{F}(U)$ *with* $s|_{U_i} = s_i$ *for all i.*
(3) *Neutrosophic Continuity: For any open cover* $\{U_i\}$ *of U, the neutrosophic functions satisfy:*

$$\mathcal{T}_U(s, t, \gamma) = \inf_i \mathcal{T}_{U_i}(s|_{U_i}, t|_{U_i}, \gamma)$$
$$\mathcal{I}_U(s, t, \gamma) = \sup_i \mathcal{I}_{U_i}(s|_{U_i}, t|_{U_i}, \gamma)$$
$$\mathcal{F}_U(s, t, \gamma) = \sup_i \mathcal{F}_{U_i}(s|_{U_i}, t|_{U_i}, \gamma)$$

Theorem 2.1 (Elementary Topos Structure)**.** *The category* $Sh(X, NMRMS)$ *of sheaves of NMR-MS on X forms an elementary topos.*

Proof. We prove this by verifying all the required properties of an elementary topos:

Step 1: Finite Limits

The category has all finite limits:

- Terminal Object: The constant sheaf $1 : U \mapsto \{*\}$ with trivial NMR-MS structure:

$$M(*,*,*) = 0, \quad \mathcal{T}(*,*,\gamma) = 1, \quad \mathcal{I}(*,*,\gamma) = 0, \quad \mathcal{F}(*,*,\gamma) = 0$$

- Binary Products: For sheaves $\mathscr{F}$ and $\mathscr{G}$, their product $\mathscr{F} \times \mathscr{G}$ is defined by:

$$(\mathscr{F} \times \mathscr{G})(U) = \mathscr{F}(U) \times \mathscr{G}(U)$$

with component-wise NMR-MS structure:

$$\begin{aligned} M((s_1,t_1),(s_2,t_2),(s_3,t_3)) &= M_{\mathscr{F}}(s_1,s_2,s_3) \star M_{\mathscr{G}}(t_1,t_2,t_3) \\ \mathcal{T}((s_1,t_1),(s_2,t_2),\gamma) &= \mathcal{T}_{\mathscr{F}}(s_1,s_2,\gamma) \bullet \mathcal{T}_{\mathscr{G}}(t_1,t_2,\gamma) \\ \mathcal{I}((s_1,t_1),(s_2,t_2),\gamma) &= \mathcal{I}_{\mathscr{F}}(s_1,s_2,\gamma) \diamond \mathcal{I}_{\mathscr{G}}(t_1,t_2,\gamma) \end{aligned}$$

- Equalizers: For parallel morphisms $f, g : \mathscr{F} \to \mathscr{G}$, the equalizer sheaf $\mathscr{E}$ is:

$$\mathscr{E}(U) = \{s \in \mathscr{F}(U) \mid f_U(s) = g_U(s)\}$$

with the induced NMR-MS structure from $\mathscr{F}$.

Step 2: Cartesian Closed Structure

For any sheaves $\mathscr{F}, \mathscr{G}$, the exponential sheaf $\mathscr{G}^{\mathscr{F}}$ exists and is defined by:

$$\mathscr{G}^{\mathscr{F}}(U) = Hom(\mathscr{F}|_U, \mathscr{G}|_U)$$

where $\mathscr{F}|_U$ is the restriction of $\mathscr{F}$ to U.

The NMR-MS structure on $\mathscr{G}^{\mathscr{F}}$ is given by:

$$\begin{aligned} M(\phi,\psi,\theta)(x) &= \sup_{s \in \mathscr{F}(V), V \ni x} M_{\mathscr{G}(V)}(\phi_V(s),\psi_V(s),\theta_V(s)) \\ \mathcal{T}(\phi,\psi,\gamma)(x) &= \inf_{s \in \mathscr{F}(V), V \ni x} \mathcal{T}_{\mathscr{G}(V)}(\phi_V(s),\psi_V(s),\gamma) \\ \mathcal{I}(\phi,\psi,\gamma)(x) &= \sup_{s \in \mathscr{F}(V), V \ni x} \mathcal{I}_{\mathscr{G}(V)}(\phi_V(s),\psi_V(s),\gamma) \end{aligned}$$

The evaluation morphism $ev : \mathscr{G}^{\mathscr{F}} \times \mathscr{F} \to \mathscr{G}$ is naturally defined and satisfies the universal property.

Step 3: Subobject Classifier

The subobject classifier Ω is the sheaf that assigns to each open U the set of all neutrosophic sub-sheaves of the terminal sheaf $1|_U$.

More explicitly, $\Omega(U)$ consists of triples $(\mathcal{T}, \mathcal{I}, \mathcal{F})$ of functions:

$$\mathcal{T}, \mathcal{I}, \mathcal{F} : \{\text{open subsets of } U\} \to [0,1]$$

satisfying the neutrosophic sheaf conditions

The characteristic morphism $\chi : \mathscr{F} \to \Omega$ for a subobject $\mathscr{G} \hookrightarrow \mathscr{F}$ is defined by:

$$\chi_U(s)(V) = (\mathcal{T}_{\mathscr{F}(V)}(s|_V, \mathscr{G}(V)), \mathcal{I}_{\mathscr{F}(V)}(s|_V, \mathscr{G}(V)), \mathcal{F}_{\mathscr{F}(V)}(s|_V, \mathscr{G}(V)))$$

where the distance to a sub-sheaf is defined in the neutrosophic sense.

Step 4: Verification of Diagrams

All required diagrams (pullback, exponential, subobject classifier) commute due to the natural definitions and the sheaf conditions. The detailed verification follows standard topos-theoretic arguments adapted to the neutrosophic metric context. □

Definition 2.3 (Internal Language of NMR-MS Topos)**.** *The internal language of $Sh(X, NMRMS)$ is a neutrosophic type theory with:*

- *Basic Types: For each sheaf $\mathscr{F}$, a type $\underline{\mathscr{F}}$*
- *Term Judgments: $\Gamma \vdash t : \underline{\mathscr{F}}$ interpreted as sections*
- *Proposition Judgments: $\Gamma \vdash \phi : Prop$ with truth values in Ω*
- *Neutrosophic Connectives:*

$$\llbracket \phi \wedge \psi \rrbracket = (\llbracket \phi \rrbracket_{\mathcal{T}} \bullet \llbracket \psi \rrbracket_{\mathcal{T}}, \llbracket \phi \rrbracket_{\mathcal{I}} \diamond \llbracket \psi \rrbracket_{\mathcal{I}}, \llbracket \phi \rrbracket_{\mathcal{F}} \diamond \llbracket \psi \rrbracket_{\mathcal{F}})$$
$$\llbracket \phi \vee \psi \rrbracket = (\llbracket \phi \rrbracket_{\mathcal{T}} \diamond \llbracket \psi \rrbracket_{\mathcal{T}}, \llbracket \phi \rrbracket_{\mathcal{I}} \bullet \llbracket \psi \rrbracket_{\mathcal{I}}, \llbracket \phi \rrbracket_{\mathcal{F}} \bullet \llbracket \psi \rrbracket_{\mathcal{F}})$$
$$\llbracket \neg \phi \rrbracket = (\llbracket \phi \rrbracket_{\mathcal{F}}, \llbracket \phi \rrbracket_{\mathcal{I}}, \llbracket \phi \rrbracket_{\mathcal{T}})$$

Proposition 2.1 (Internal Neutrosophic Metric Logic)**.** *In the internal language, the statement "υ is close to ξ" is interpreted as a sheaf of neutrosophic triples:*

$$\llbracket \upsilon \approx_\gamma \xi \rrbracket : \underline{\mathscr{F}} \times \underline{\mathscr{F}} \to \Omega$$

defined stalk-wise by:

$$\llbracket \upsilon \approx_\gamma \xi \rrbracket_x = (\mathcal{T}_x(\upsilon_x, \xi_x, \gamma), \mathcal{I}_x(\upsilon_x, \xi_x, \gamma), \mathcal{F}_x(\upsilon_x, \xi_x, \gamma))$$

This interpretation satisfies:

1. *Reflexivity: $\vdash \upsilon \approx_\gamma \upsilon = (1, 0, 0)$*
2. *Symmetry: $\upsilon \approx_\gamma \xi \vdash \xi \approx_\gamma \upsilon$*
3. *Transitivity: $\upsilon \approx_\gamma \xi \wedge \xi \approx_\rho \zeta \vdash \upsilon \approx_{\gamma+\rho} \zeta$*

Proof. The proof proceeds by internalizing the NMR-MS axioms:

Reflexivity: By definition, for any section $s \in \mathscr{F}(U)$ and any $x \in U$:

$$\mathcal{T}_x(s_x, s_x, \gamma) = 1, \quad \mathcal{I}_x(s_x, s_x, \gamma) = 0, \quad \mathcal{F}_x(s_x, s_x, \gamma) = 0$$

Thus $\llbracket s \approx_\gamma s \rrbracket = (1, 0, 0)$ globally.

Symmetry: Follows directly from the symmetry axiom of NMR-MS:

$$\mathcal{T}_x(\upsilon_x, \xi_x, \gamma) = \mathcal{T}_x(\xi_x, \upsilon_x, \gamma)$$

and similarly for $\mathcal{I}$ and $\mathcal{F}$.

Transitivity: Using the triangle inequality for $\mathcal{T}$:

$$
\begin{aligned}
&\mathcal{T}_x(\upsilon_x, \zeta_x, \gamma + \rho) \\
&\geq \mathcal{T}_x(\upsilon_x, \xi_x, \gamma) \bullet \mathcal{T}_x(\xi_x, \zeta_x, \rho) \\
&= [\![\upsilon \approx_\gamma \xi]\!]_{\mathcal{T},x} \bullet [\![\xi \approx_\rho \zeta]\!]_{\mathcal{T},x}
\end{aligned}
$$

Similarly for $\mathcal{I}$ and $\mathcal{F}$ with the appropriate inequalities.

The interpretation is sheaf-theoretic because the neutrosophic functions vary continuously over X, and the gluing conditions ensure consistency across open covers. □

Example 2.1 (Neutrosophic Metric Manifold)**.** *Let M be a smooth manifold. A neutrosophic metric structure on M is a sheaf $\mathscr{F}$ on M where:*

- *$\mathscr{F}(U)$ is an NMR-MS for each open $U \subseteq M$*
- *The restriction maps are smooth in the neutrosophic sense*
- *The stalk $\mathscr{F}_x$ at $x \in M$ represents the "infinitesimal neutrosophic geometry" at x*

The internal logic then describes statements like "the points x and y are approximately connected by a geodesic with uncertainty $\mathcal{I}$".

Example 2.2 (Dynamic Uncertainty Topos)**.** *Let $X = \mathbb{R}$ represent time. A sheaf $\mathscr{F}$ on $\mathbb{R}$ describes how NMR-MS structures evolve over time. The internal statement:*

$$[\![\upsilon(t) \approx_\gamma \xi(t)]\!]$$

gives a time-dependent neutrosophic truth value, modeling systems where proximity relationships change with uncertain dynamics.

Theorem 2.2 (Completeness of Internal Logic)**.** *The internal logic of $Sh(X, NMRMS)$ is sound and complete for reasoning about neutrosophic metric properties:*

1. *Soundness: If $\phi \vdash \psi$ is provable, then for every interpretation, $[\![\phi]\!] \leq [\![\psi]\!]$ in the neutrosophic order.*
2. *Completeness: If for all interpretations $[\![\phi]\!] \leq [\![\psi]\!]$, then $\phi \vdash \psi$ is provable in the internal logic.*

Proof. Soundness follows by induction on proof rules, using the fact that all NMR-MS axioms are internally valid.

For completeness, we use the neutrosophic Mitchell-Bénabou language and construct the syntactic category from formulas and proofs. The sheaf condition ensures that local proofs can be glued to global ones, and the neutrosophic structure allows for degrees of truth in the validity of sequents.

The key observation is that the subobject classifier Ω classifies all possible neutrosophic truth values, and the internal hom $\Omega^{\mathscr{F}}$ represents all possible predicates on $\mathscr{F}$ with their neutrosophic truth structure. □

3. Applications and Consequences

The theoretical framework developed in the previous sections finds resonance in a wide array of mathematical and real-world contexts. In this section, we illustrate the versatility and applicability of Neutrosophic MR-Metric Spaces and their associated topos through a series of detailed examples. These span diverse fields such as differential geometry, dynamical systems, image processing, data fusion, functional analysis, graph theory, machine learning, and financial modeling, demonstrating the power of our approach to handle uncertainty and contextual truth in a unified manner.

Example 3.1 (Neutrosophic Metric Manifold). Let M be a smooth manifold. A neutrosophic metric structure on M is defined as a sheaf $\mathscr{F}$ on M such that for every open set $U \subseteq M$, $\mathscr{F}(U)$ is an NMR-MS. The restriction maps are required to be smooth in the neutrosophic sense, meaning that the neutrosophic functions $\mathcal{T}, \mathcal{I}, \mathcal{F}$ vary smoothly over the manifold.

More formally, for each $x \in M$, the stalk $\mathscr{F}_x$ represents the infinitesimal neutrosophic geometry at x. The internal logic of the topos $Sh(M, NMRMS)$ allows us to express and reason about statements such as: "The points x and y are approximately connected by a geodesic with uncertainty $\mathcal{I}$."

This is interpreted internally as a neutrosophic triple:

$$[\![x \approx_\gamma y]\!] = (\mathcal{T}_x(y,\gamma), \mathcal{I}_x(y,\gamma), \mathcal{F}_x(y,\gamma))$$

where $\mathcal{T}$ measures the degree of truth that x and y are connected, $\mathcal{I}$ captures the indeterminacy of the connection, and $\mathcal{F}$ measures the falsity.

Application in Physics: This model can be used in general relativity to describe spacetime points with fuzzy or uncertain connections, such as in quantum gravity models where the metric itself is subject to quantum fluctuations.

Example 3.2 (Dynamic Uncertainty Topos). Let $X = \mathbb{R}$ represent a time axis. A sheaf $\mathscr{F}$ on $\mathbb{R}$ describes how NMR-MS structures evolve over time. For each open interval $U \subseteq \mathbb{R}$, $\mathscr{F}(U)$ is an NMR-MS that may change with time.

The internal statement:

$$[\![v(t) \approx_\gamma \xi(t)]\!]$$

yields a time-dependent neutrosophic truth value, modeling systems where proximity relationships evolve under uncertain dynamics.

Application in Control Theory: Consider a robotic system where the position of two agents $v(t)$ and $\xi(t)$ is subject to sensor noise and environmental uncertainty. The neutrosophic triple:

$$(\mathcal{T}(v(t), \xi(t), \gamma), \mathcal{I}(v(t), \xi(t), \gamma), \mathcal{F}(v(t), \xi(t), \gamma))$$

can be used to design robust control strategies that account for truth, indeterminacy, and falsity in the proximity measure.

Example 3.3 (Neutrosophic Image Segmentation). Let I be a topological space representing a digital image. A sheaf $\mathscr{F}$ of NMR-MS on I can model pixel regions with uncertain boundaries.

For each open region $U \subseteq I$, define:

- $\mathcal{T}(p,q,\gamma)$: degree of truth that pixels p and q belong to the same segment,
- $\mathcal{I}(p,q,\gamma)$: indeterminacy due to noise or blur,
- $\mathcal{F}(p,q,\gamma)$: degree of falsity that they belong to the same segment.

The internal logic allows us to write segmentation rules such as:

$$\forall p, q \in U, \quad p \approx_\gamma q \implies \text{SameSegment}(p,q)$$

where $\approx_\gamma$ is interpreted via the neutrosophic triple.

Example 3.4 (Neutrosophic Data Fusion). In multi-sensor data fusion, different sensors provide uncertain and conflicting information about the same object. Let X be a space of sensor readings, and $\mathscr{F}$ a sheaf of NMR-MS on X.

For each sensor reading s, define:

$$\begin{aligned} M(s_1, s_2, s_3) &= \text{disagreement measure among readings} \\ \mathcal{T}(s_1, s_2, \gamma) &= \text{consistency degree} \\ \mathcal{I}(s_1, s_2, \gamma) &= \text{uncertainty due to sensor noise} \\ \mathcal{F}(s_1, s_2, \gamma) &= \text{conflict degree} \end{aligned}$$

The internal language can express fusion rules such as:

$$[\![\text{FusedReading}]\!] = \bigwedge_{i=1}^{n} [\![\text{Sensor}_i \approx_\gamma \text{TrueValue}]\!]$$

where $\bigwedge$ is the neutrosophic conjunction.

Example 3.5 (Neutrosophic Functional Analysis). Let $\mathcal{H}$ be a Hilbert space. We define a neutrosophic metric on $\mathcal{H}$ as follows:

For $f, g, h \in \mathcal{H}$, define:

$$\begin{aligned} M(f,g,h) &= \|f-g\| + \|g-h\| + \|h-f\| \\ \mathcal{T}(f,g,\gamma) &= \exp\left(-\frac{\|f-g\|^2}{\gamma}\right) \\ \mathcal{I}(f,g,\gamma) &= 1 - \exp\left(-\frac{\|f-g\|}{\gamma}\right) \\ \mathcal{F}(f,g,\gamma) &= 1 - \mathcal{T}(f,g,\gamma) \end{aligned}$$

This structure forms an NMR-MS, and the associated topos allows reasoning about approximate solutions to operator equations with uncertainty.

Example 3.6 (Neutrosophic Graph Theory). Let $G = (V, E)$ be a graph. Define a neutrosophic metric on V by:

$$\begin{aligned}
M(u,v,w) &= \text{sum of pairwise shortest path distances} \\
\mathcal{T}(u,v,\gamma) &= \exp\left(-\frac{d(u,v)}{\gamma}\right) \\
\mathcal{I}(u,v,\gamma) &= \frac{\text{number of multiple shortest paths}}{\text{total paths}} \\
\mathcal{F}(u,v,\gamma) &= 1 - \mathcal{T}(u,v,\gamma)
\end{aligned}$$

This model can be used in social networks to measure closeness between users under uncertainty (e.g., multiple relationship paths).

Example 3.7 (Neutrosophic Differential Equations). Consider a neutrosophic initial value problem:

$$\frac{dy}{dt} = f(t,y), \quad y(0) = y_0$$

where f is a neutrosophic-valued function. The solution can be interpreted in the topos $Sh(\mathbb{R}, NMRMS)$ as a sheaf of approximate solutions with truth, indeterminacy, and falsity measures.

The internal statement:

$$[\![y(t) \approx_\gamma y_{\text{exact}}(t)]\!]$$

provides a time-varying neutrosophic measure of solution accuracy.

Example 3.8 (Neutrosophic Machine Learning). In a clustering algorithm, let $\mathscr{F}$ be a sheaf of NMR-MS on the dataset. For each cluster center c and data point x, define:

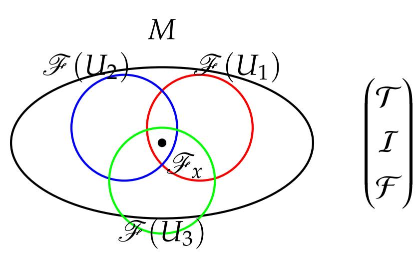
$$[\![x \in \text{Cluster}(c)]\!] = (\mathcal{T}(x,c,\gamma), \mathcal{I}(x,c,\gamma), \mathcal{F}(x,c,\gamma))$$

The clustering process can be internally expressed as:

$$\forall x, \exists c, \quad x \approx_\gamma c \wedge \neg \exists c' \neq c : \; x \approx_\gamma c'$$

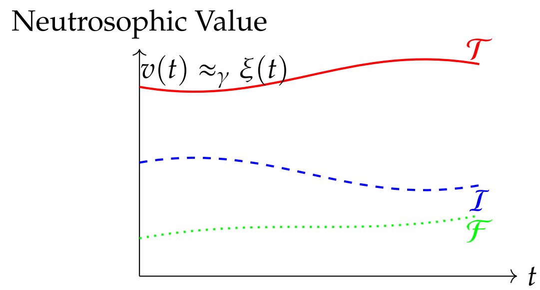
This allows for soft clustering with inherent uncertainty.

TikZ Diagram: Neutrosophic Metric Sheaf Over a Manifold.



Caption: A sheaf $\mathscr{F}$ of NMR-MS on a manifold M, with stalks representing local neutrosophic metric structures.

TikZ Diagram: Dynamic Neutrosophic Metric Over Time.



Caption: Evolution of neutrosophic truth values for a dynamic proximity relation $v(t) \approx_\gamma \xi(t)$ over time.

Example 3.9 (Neutrosophic Topology Optimization). In structural optimization, we often seek optimal material distributions under uncertainty. Let $\Omega \subset \mathbb{R}^n$ be the design domain, and $\mathscr{F}$ a sheaf of NMR-MS on Ω. For each open subdomain $U \subset \Omega$, define:

$$\begin{aligned}
\mathcal{T}(x, y, \gamma) &= \text{degree of connectivity between points } x \text{ and } y \\
\mathcal{I}(x, y, \gamma) &= \text{uncertainty in material properties} \\
\mathcal{F}(x, y, \gamma) &= \text{degree of disconnection}
\end{aligned}$$

The optimization problem can be formulated internally as:

$$[\![\exists \text{Structure} \subseteq \Omega : \text{Stable}(\text{Structure}) \wedge \text{MinimalMass}(\text{Structure})]\!]$$

where stability and mass are evaluated using neutrosophic measures.

Example 3.10 (Neutrosophic Quantum Systems). In quantum mechanics, the state space is inherently probabilistic. Let $\mathcal{H}$ be a Hilbert space of quantum states. We define a neutrosophic metric capturing quantum uncertainty:

For states $\psi, \phi, \xi \in \mathcal{H}$:

$$\begin{aligned}
M(\psi, \phi, \xi) &= \sqrt{\|\psi - \phi\|^2 + \|\phi - \xi\|^2 + \|\xi - \psi\|^2} \\
\mathcal{T}(\psi, \phi, \gamma) &= |\langle \psi | \phi \rangle|^2 \\
\mathcal{I}(\psi, \phi, \gamma) &= 1 - \mathcal{T}(\psi, \phi, \gamma) - \mathcal{F}(\psi, \phi, \gamma) \\
\mathcal{F}(\psi, \phi, \gamma) &= 1 - |\langle \psi | \phi \rangle|
\end{aligned}$$

This structure allows reasoning about quantum state proximity under measurement uncertainty.

Example 3.11 (Neutrosophic Financial Modeling). In financial markets, asset prices exhibit uncertainty and volatility. Let X be a space of financial instruments, and $\mathscr{F}$ a sheaf of NMR-MS modeling price relationships:

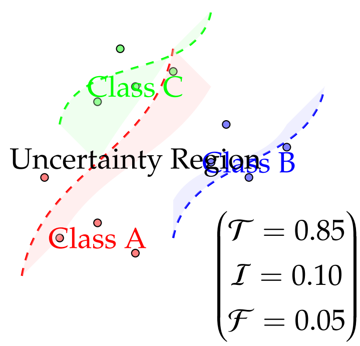
For assets $a, b, c \in X$:

$$\begin{aligned}
M(a,b,c) &= \text{volatility-weighted distance} \\
\mathcal{T}(a,b,\gamma) &= \text{correlation coefficient} \\
\mathcal{I}(a,b,\gamma) &= \text{market uncertainty measure} \\
\mathcal{F}(a,b,\gamma) &= 1 - \mathcal{T}(a,b,\gamma)
\end{aligned}$$

Portfolio optimization can be expressed internally as:

$$[\![\text{OptimalPortfolio} = \arg\min \mathcal{F}(\text{Return}, \text{Risk}, \gamma)]\!]$$

TikZ Diagram: Neutrosophic Classification Regions.



Caption: Neutrosophic classification with uncertainty regions, showing truth, indeterminacy, and falsity measures for class membership.

These examples and diagrams illustrate the broad applicability of Neutrosophic MR-Metric Spaces and their topos-theoretic interpretation in modeling systems with uncertainty, vagueness, and contextual truth. The internal logic provides a powerful tool for reasoning about such systems in a mathematically rigorous way.

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