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DHAGE ITERATION METHOD FOR GENERALIZED QUADRATIC FUNCTIONAL INTEGRAL EQUATIONS

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ABSTRACT. In this paper we prove the existence as well as approximations of the solutions for a certain nonlinear generalized quadratic functional integral equation. An algorithm for the solutions is developed and it is shown that the sequence of successive approximations starting at a lower or upper solution converges monotonically to the solutions of related quadratic functional integral equation under some suitable mixed hybrid conditions. We rely our main result on Dhage iteration method embodied in a recent hybrid fixed point theorem of Dhage (2014) in partially ordered normed linear spaces. An example is also provided to illustrate the abstract theory developed in the paper.

1. INTRODUCTION

The quadratic integral equations have been a topic of interest since long time because of their occurrence in the problems of some natural and physical processes of the universe. See Argyros [1], Deimling [3], Chandrasekher [2] and the references therein. The study gained momentum after the formulation of the hybrid fixed point principles in Banach algebras due to Dhage [4, 5, 6, 7, 8]. The existence results for such quadratic operators equations are generally proved under the mixed Lipschitz and compactness type conditions together with a certain growth condition on the nonlinearities involved in the quadratic operator or functional equations. The hybrid fixed point theorems in Banach algebras find numerous applications in the theory of nonlinear quadratic differential and integral equations. See Dhage [5, 6, 7] and the references therein. The Lipschitz and compactness hypotheses are considered to be very strong conditions in the theory of nonlinear differential and integral equations but which still do not yield any algorithm to determine the numerical solutions. Therefore, it is of interest to relax or weaken these condition in the existence and approximation theory of quadratic integral equations. This is the main motivation of the present paper. In this paper we prove the existence as well as approximations of the solutions of a certain generalized quadratic integral equation via an algorithm based on successive approximations under partially Lipschitz and compactness type conditions.

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Given a closed and bounded interval $J = [0, T]$ of the real line $\mathbb{R}$ for some $T > 0$, we consider the quadratic functional integral equation (in short QFIE)

$$x(t) = k(t, x(t)) + \big[f(t, x(t))\big] \left(q(t) + \int_0^t v(t,s) g(s, x(s))\, ds \right), \ t \in J, \tag{1.1}$$

where $q : J \to \mathbb{R}$, $v : J \times J \to \mathbb{R}$ and $f, g, k : J \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

By a *solution* of the QFIE (1.1) we mean a function $x \in C(J, \mathbb{R})$ that satisfies the equation (1.1) on J, where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on J.

The QFIE (1.1) is well-known in the literature and studied earlier in the work of Dhage [4]. If $f(t, x) = 0$ for all $t \in J$ and $x \in \mathbb{R}$ the QFIE (1.1) reduces to the nonlinear functional equation

$$x(t) = k(t, x(t)), \ \ t \in J, \tag{1.2}$$

and if $k(t, x) = 0$ and $f(t, x) = 1$ for all $t \in J$ and $x \in \mathbb{R}$, it is reduced to nonlinear usual Volterra integral equation

$$x(t) = q(t) + \int_0^t v(t,s) g(s, x(s))\, ds, \ \ t \in J. \tag{1.3}$$

Therefore, the QFIE (1.1) is general and the results of this paper include the existence and approximations results for above nonlinear functional and Volterra integral equations as special cases.

The paper is organized as follows: In the following section we give the preliminaries and auxiliary results needed in the subsequent part of the paper. The main result is included in Section 3. In Section 4 some concluding remarks are presented.

2. Auxiliary Results

Unless otherwise mentioned, throughout this paper that follows, let E denote a partially ordered real normed linear space with an order relation $\preceq$ and the norm $\|\cdot\|$. It is known that E is **regular** if $\{x_n\}_{n\in\mathbb{N}}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. Clearly, the partially ordered Banach space $C(J, \mathbb{R})$ is regular and the conditions guaranteeing the regularity of any partially ordered normed linear space E may be found in Heikkilä and Lakshmikantham [13] and the references therein.

We need the following definitions in the sequel.

Definition 2.1. *A mapping $\mathcal{T} : E \to E$ is called* **isotone** *or* **nondecreasing** *if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$.*

Definition 2.2 (Dhage [9])**.** *A mapping $\mathcal{T} : E \to E$ is called* **partially continuous** *at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{T}$ called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is partially continuous on E, then it is continuous on every chain C contained in E.*

Definition 2.3. *A mapping $\mathcal{T} : E \to E$ is called* **partially bounded** *if $\mathcal{T}(C)$ is bounded for every chain C in E. $\mathcal{T}$ is called* **uniformly partially bounded** *if all* *chains $\mathcal{T}(C)$ in E are bounded by a unique constant. $\mathcal{T}$ is called* **bounded** *if $\mathcal{T}(E)$ is a bounded subset of E.*

Definition 2.4. *A mapping $\mathcal{T} : E \to E$ is called* **partially compact** *if $\mathcal{T}(C)$ is a relatively compact subset of E for all totally ordered sets or chains C in E. $\mathcal{T}$ is called* **uniformly partially compact** *if $\mathcal{T}(C)$ is a uniformly partially bounded and partially compact on E. $\mathcal{T}$ is called* **partially totally bounded** *if for any totally ordered and bounded subset C of E, $\mathcal{T}(C)$ is a relatively compact subset of E. If $\mathcal{T}$ is partially continuous and partially totally bounded, then it is called* **partially completely continuous** *on E.*

Definition 2.5 (Dhage [9])**.** *The order relation $\preceq$ and the metric d on a non-empty set E are said to be* ***compatible*** *if $\{x_n\}_{n\in\mathbb{N}}$ is a monotone, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ converges to x^* implies that the whole sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible.*

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function $|\cdot|$ has this property. Similarly, the finite dimensional Euclidean space $\mathbb{R}^n$ with usual componentwise order relation and the standard norm possesses the compatibility property.

Definition 2.6 (Dhage [6])**.** *A upper semi-continuous and nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$-function provided $\psi(0) = 0$. Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear space. A mapping $\mathcal{T} : E \to E$ is called* ***partially nonlinear $\mathcal{D}$-Lipschitz*** *if there exists a $\mathcal{D}$-function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi(\|x - y\|) \tag{2.1}$$

for all comparable elements $x, y \in E$. If $\psi(r) = k\,r$, $k > 0$, then $\mathcal{T}$ is called a partially Lipschitz with a Lipschitz constant k.

Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear algebra. Denote

$$E^+ = \{x \in E \mid x \succeq \theta, \text{ where } \theta \text{ is the zero element of } E\}$$

and

$$\mathcal{K} = \{E^+ \subset E \mid uv \in E^+ \text{ for all } u, v \in E^+\}. \tag{2.2}$$

The elements of $\mathcal{K}$ are called the positive vectors of the normed linear algebra E. The following lemma follows immediately from the definition of the set $\mathcal{K}$ and which is often times used in the applications of hybrid fixed point theory in Banach algebras.

Lemma 2.7 (Dhage [7])**.** *If $u_1, u_2, v_1, v_2 \in \mathcal{K}$ are such that $u_1 \preceq v_1$ and $u_2 \preceq v_2$, then $u_1 u_2 \preceq v_1 v_2$.*

Definition 2.8. *An operator $\mathcal{T} : E \to E$ is said to be positive if the range $R(\mathcal{T})$ of $\mathcal{T}$ is such that $R(\mathcal{T}) \subseteq \mathcal{K}$.*

The Dhage iteration principle or method (in short DIP or DIM) developed in Dhage [9, 10, 11] may be formulated as "monotonic convergence of the sequence of successive approximations to the solutions of a nonlinear equation beginning with a lower or an upper solution of the equation as its initial or first approximation" and which is a powerful tool in the existence theory of nonlinear analysis. It is clear that Dhage iteration method is different from the usual Picard's successive iteration method and embodied in the following applicable hybrid fixed point theorems proved in Dhage [10] which forms a useful key tool for our work contained in this paper. A few other hybrid fixed point theorems involving the Dhage iteration method may be found in Dhage [9, 10, 11, 12].

Theorem 2.9 (Dhage [10]). *Let $\big(E, \preceq, \|\cdot\|\big)$ be a regular partially ordered complete normed linear algebra such that the order relation $\preceq$ and the norm $\|\cdot\|$ in E are compatible in every compact chain of E. Let $\mathcal{A}, \mathcal{B} : E \to \mathcal{K}$ and $\mathcal{B} : E \to E$ be three nondecreasing operators such that*

(a) *$\mathcal{A}$ and $\mathcal{C}$ are partially bounded and partially nonlinear $\mathcal{D}$-Lipschitz with $\mathcal{D}$-functions $\psi_{\mathcal{A}}$ and $\psi_{\mathcal{C}}$ respectively,*
(b) *$\mathcal{B}$ is partially continuous and uniformly partially compact, and*
(c) *$M\psi_{\mathcal{A}}(r)+\psi_{\mathcal{C}}(r) < r$, $r > 0$, where $M = \sup\{\|\mathcal{B}(C)\| : C$ is a chain in $E\}$, and*
(d) *there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{A}x_0\,\mathcal{B}x_0 + \mathcal{C}x_0$ or $x_0 \succeq \mathcal{A}x_0\,\mathcal{B}x_0 + \mathcal{C}x_0$.*

Then the operator equation

$$\mathcal{A}x\,\mathcal{B}x + \mathcal{C}x = x \tag{2.3}$$

has a solution x^* in E and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n\,\mathcal{B}x_n + \mathcal{C}x_n$, $n = 0, 1, \ldots,$ converges monotonically to x^*.

Remark 2.10. The compatibility of the order relation $\preceq$ and the norm $\|\cdot\|$ in every compact chain of E holds if every partially compact subset of E possesses the compatibility property with respect to $\preceq$ and $\|\cdot\|$.

3. Main Result

The QFIE (1.1) is considered in the function space $C(J,\mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\le$ in $C(J,\mathbb{R})$ by

$$\|x\| = \sup_{t\in J} |x(t)| \tag{3.1}$$

and

$$x \le y \iff x(t) \le y(t) \tag{3.2}$$

for all $t \in J$ respectively. Clearly, $C(J,\mathbb{R})$ is a Banach algebra with respect to above supremum norm and is also partially ordered w.r.t. the above partially order relation $\le$. It is known that the partially ordered Banach algebra $C(J,\mathbb{R})$ has some nice properties w.r.t. the above order relation in it. The following lemma follows by an application of Arzellá-Ascolli theorem.

Lemma 3.1. *Let $\big(C(J,\mathbb{R}), \le, \|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\le$ defined by (3.1) and (3.2) respectively. Then $\|\cdot\|$ and $\le$ are compatible in every partially compact subset of $C(J,\mathbb{R})$.*

Proof. The lemma mentioned in Dhage [10], but the proof appears in Dhage [11]. Since the proof is not well-known, we give the details of the proof. Let S be a partially compact subset of $C(J,\mathbb{R})$ and let $\{x_n\}_{n\in\mathbb{N}}$ be a monotone nondecreasing sequence of points in S. Then we have

$$x_1(t) \le x_2(t) \le \cdots \le x_n(t) \le \cdots, \tag{3.3}$$

for each $t \in \mathbb{R}_+$.

Suppose that a subsequence $\{x_{n_k}\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ is convergent and converges to a point x in S. Then the subsequence $\{x_{n_k}(t)\}_{n\in\mathbb{N}}$ of the monotone real sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is convergent. By monotone characterization, the whole sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is convergent and converges to a point $x(t)$ in $\mathbb{R}$ for each $t \in \mathbb{R}_+$. This shows that the sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ converges point-wise in S. To show the convergence is uniform, it is enough to show that the sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is equicontinuous. Since S is partially compact, every chain or totally ordered set and consequently $\{x_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence by Arzelá-Ascoli theorem. Hence $\{x_n\}_{n\in\mathbb{N}}$ is convergent and converges uniformly to x. As a result $\|\cdot\|$ and $\le$ are compatible in S. This completes the proof. □

We need the following definition in what follows.

Definition 3.2. *A function $u \in C(J,\mathbb{R})$ is said to be a lower solution of the QFIE (1.1) if it satisfies*

$$u(t) \le k(t,u(t)) + \big[f(t,u(t))\big]\left(q(t) + \int_0^t v(t,s)g(s,u(s))\,ds\right) \tag{$*$}$$

for all $t \in J$. Similarly, a function $v \in C(J,\mathbb{R})$ is said to be an upper solution of the QFIE (1.1) if it satisfies the above inequalities with reverse sign.

We consider the following set of assumptions in what follows:

(A_1) f defines a function $f : J\times\mathbb{R} \to \mathbb{R}_+$.

(A_2) There exists a constant $M_f > 0$ such that $f(t,x) \le M_f$ for all $t \in J$ and $x \in \mathbb{R}$.

(A_3) There exists a $\mathcal{D}$-function ψ_f such that

$$0 \le f(t,x) - f(t,y) \le \psi_f(x-y),$$

for all $t \in J$ and $x, y \in \mathbb{R}$, $x \ge y$.

(B_0) q defines a continuous function $q : J \to \mathbb{R}_+$.

(B_1) v defines a continuous and nonnegative function on $J \times J$.

(B_2) g defines a function $g : J\times\mathbb{R} \to \mathbb{R}_+$.

(B_3) There exists a constant $M_g > 0$ such that $g(t,x) \le M_g$ for all $t \in J$ and $x \in \mathbb{R}$.

(B_4) $g(t,x)$ is nondecreasing in x for all $t \in J$.

(C_1) There exists a constant $M_k > 0$ such that $|k(t,x)| \le M_k$ for all $t \in J$ and $x \in \mathbb{R}$.

(C_2) There exists a $\mathcal{D}$-function ψ_k, such that

$$0 \le k(t,x) - k(t,y) \le \psi_k(x-y),$$

for all $t \in J$ and $x, y \in \mathbb{R}$, $x \ge y$.

(C_3) The QFIE (1.1) has a lower solution $u \in C(J,\mathbb{R})$.

Theorem 3.3. *Assume that hypotheses (A_1)-(A_3), (B_0)-(B_4) and (C_1)-(C_3) hold. Furthermore, assume that*

$$\big(\|q\| + M_g\,T\big)\psi_f(r) + \psi_k(r) < r,\ \ r > 0, \tag{3.4}$$

then the QFIE (1.1) has a solution x^* *defined on* J *and the sequence* $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ *of successive approximations defined by*

$$x_{n+1}(t) = k(t, x_n(t)) + \big[f(t, x_n(t))\big]\left(q(t) + \int_{t_0}^{t} v(t,s)g(s, x_n(s))\,ds\right), \tag{3.5}$$

for all $t \in J$*, where* $x_0 = u$*, converges monotonically to* x^**.*

Proof. Set $E = C(J, \mathbb{R})$. Then, from Lemma 3.1 it follows that every compact chain in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ in E.

Define two operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ on E by

$$\mathcal{A}x(t) = f(t, x(t)),\ \ t \in J, \tag{3.6}$$

$$\mathcal{B}x(t) = q(t) + \int_{t_0}^{t} v(t,s)g(s, x(s))\,ds,\ \ t \in J, \tag{3.7}$$

and

$$\mathcal{C}x(t) = k(t, x(t)),\ \ t \in J. \tag{3.8}$$

From the continuity of the integral and the hypotheses (A_1) and (B_0)-(B_2), it follows that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ define the maps $\mathcal{A}, \mathcal{B} : E \to \mathcal{K}$ and $\mathcal{C} : E \to E$. Now by definitions of the operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, the QFIE (1.1) is equivalent to the quadratic operator equation

$$\mathcal{A}x(t)\mathcal{B}x(t) + \mathcal{C}x(t) = x(t),\ \ t \in J. \tag{3.9}$$

We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.9. This is achieved in the series of following steps.

Step I: *$\mathcal{A}$, $\mathcal{B}$ and $\mathcal{B}$ are nondecreasing on E.*

Let $x, y \in E$ be such that $x \geq y$. Then by hypothesis (A_3) and (C_2), we obtain

$$\mathcal{A}x(t) = f(t, x(t)) \geq f(t, y(t)) = \mathcal{A}y(t),$$

and

$$\mathcal{C}x(t) = k(t, x(t)) \geq k(t, y(t)) = \mathcal{C}y(t),$$

for all $t \in J$. This shows that $\mathcal{A}$ and $\mathcal{C}$ are nondecreasing operators on E into E. Similarly, using hypothesis (B_4),

$$\begin{aligned} \mathcal{B}x(t) &= q(t) + \int_0^t v(t,s)g(s, x(s))\,ds \\ &\leq q(t) + \int_0^t v(t,s)g(s, y(s))\,ds \\ &= \mathcal{B}y(t) \end{aligned}$$

for all $t \in J$. Hence, it is follows that the operator $\mathcal{B}$ is also nondecreasing on E into itself. Thus, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are nondecreasing operators on E into itself.

Step II: *$\mathcal{A}$ and $\mathcal{C}$ are partially bounded and partially $\mathcal{D}$-Lipschitz on E.*

Let $x \in E$ be arbitrary. Then by (A_2),

$$|\mathcal{A}x(t)| \leq \big|f(t, x(t))\big| \leq M_f,$$

for all $t \in J$. Taking supremum over t, we obtain $\|\mathcal{A}x\| \leq M_f$ and so, $\mathcal{A}$ is bounded. This further implies that $\mathcal{A}$ is partially bounded on E. Similarly, using hypothesis (C_1), it is shown that $\|\mathcal{C}x\| \leq M_k$ and consequently $\mathcal{C}$ is partially bounded on E.

Next, let $x, y \in E$ be such that $x \geq y$. Then, by hypothesis (A_3),

$$|\mathcal{A}x(t) - \mathcal{A}y(t)| = \big|f(t, x(t)) - f(t, y(t))\big| \leq \psi_f(|x(t) - y(t)|) \leq \psi_f(\|x - y\|),$$

for all $t \in J$. Taking supremum over t, we obtain

$$\|\mathcal{A}x - \mathcal{A}y\| \leq \psi_f(\|x - y\|)$$

for all $x, y \in E$ with $x \geq y$. Similarly, by hypothesis (C_2),

$$\|\mathcal{C}x - \mathcal{C}y\| \leq \psi_k(\|x - y\|)$$

for all $x, y \in E$ with $x \geq y$. Hence $\mathcal{A}$ and $\mathcal{C}$ are partially nonlinear $\mathcal{D}$-Lipschitz operators on E which further implies that they are also a partially continuous on E into itself.

Step III: *$\mathcal{B}$ is a partially continuous on E.*

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a chain C of E such that $x_n \to x$ for all $n \in \mathbb{N}$. Then, by dominated convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{B}x_n(t) &= \lim_{n\to\infty} q(t) + \lim_{n\to\infty} \int_0^t v(t, s)g(s, x_n(s))\, ds \\
&= q(t) + \int_0^t v(t, s) \Big[\lim_{n\to\infty} g(s, x_n(s))\Big]\, ds \\
&= q(t) + \int_0^t v(t, s)g(s, x(s))\, ds \\
&= \mathcal{B}x(t),
\end{aligned}$$

for all $t \in J$. This shows that $\mathcal{B}x_n$ converges monotonically to $\mathcal{B}x$ pointwise on J.

Next, we will show that $\{\mathcal{B}x_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence of functions in E. Let $t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$. Then

$$\begin{aligned}
&|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| \\
&\quad \leq \big|q(t_1) - q(t_2)\big| + \left|\int_0^{t_2} v(t_1, s)g(s, x_n(s))\, ds - \int_0^{t_1} v(t_1, s)g(s, x_n(s))\, ds\right| \\
&\quad \leq \big|q(t_1) - q(t_2)\big| + \left|\int_0^{t_2} v(t_2, s)g(s, x_n(s))\, ds - \int_0^{t_2} v(t_1, s)g(s, x_n(s))\, ds\right| \\
&\qquad + \left|\int_0^{t_2} v(t_1, s)g(s, x_n(s))\, ds - \int_0^{t_1} v(t_1, s)g(s, x_n(s))\, ds\right| \\
&\quad \leq \big|q(t_1) - q(t_2)\big| + \left|\int_0^{t_2} |v(t_2, s) - v(t_1, s)|\, |g(s, x_n(s))|\, ds\right| \\
&\qquad + \left|\int_{t_1}^{t_2} |v(t_1, s)|\, |g(s, x_n(s))|\, ds\right|
\end{aligned}$$

$$\leq \left|q(t_1) - q(t_2)\right| + \left|\int_0^T |v(t_2,s) - v(t_1,s)|\, M_g\, ds\right|$$

$$+ V M_g |t_2 - t_1|. \tag{3.10}$$

Since the function q is continuous on compact interval J and v is continuous on compact set $J \times J$, they are uniformly continuous there. Therefore, from above inequality (3.10) it follows that

$$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| \to 0 \quad \text{as} \quad n \to \infty$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{B}x_n \to \mathcal{B}x$ is uniform and hence $\mathcal{B}$ is partially continuous on E.

Step IV: *$\mathcal{B}$ is a uniformly partially compact operator on E.*

Let C be an arbitrary chain in E. We show that $\mathcal{B}(C)$ is a uniformly bounded and equicontinuous set in E. First we show that $\mathcal{B}(C)$ is uniformly bounded. Let $y \in \mathcal{B}(C)$ be any element. Then there is an element $x \in C$ be such that $y = \mathcal{B}x$. Now, by hypothesis (B_2),

$$|y(t)| \leq |q(t)| + \int_0^t v(t,s)|g(s,x(s))|\, ds \leq \|q\| + V\, M_g\, T \leq r,$$

for all $t \in J$. Taking supremum over t, we obtain $\|y\| = \|\mathcal{B}x\| \leq r$ for all $y \in \mathcal{B}(C)$. Hence, $\mathcal{B}(C)$ is a uniformly bounded subset of E. Moreover, $\|\mathcal{B}(C)\| \leq r$ for all chains C in E. Hence, $\mathcal{B}$ is a uniformly partially bounded operator on E.

Next, we will show that $\mathcal{B}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$. Then, for any $y \in \mathcal{B}(C)$, one has

$$\begin{aligned}
|y(t_2) - y(t_1)| &= |\mathcal{B}x(t_2) - \mathcal{B}x(t_1)| \\
&\leq \left|q(t_1) - q(t_2)\right| + \left|\int_0^{t_2} v(t_1,s)g(s,x(s))\, ds - \int_0^{t_1} v(t_1,s)g(s,x(s))\, ds\right| \\
&\leq \left|q(t_1) - q(t_2)\right| + \left|\int_0^{t_2} v(t_2,s)g(s,x(s))\, ds - \int_0^{t_2} v(t_1,s)g(s,x(s))\, ds\right| \\
&\quad + \left|\int_0^{t_2} v(t_1,s)g(s,x(s))\, ds - \int_0^{t_1} v(t_1,s)g(s,x(s))\, ds\right| \\
&\leq \left|q(t_1) - q(t_2)\right| + \left|\int_0^{t_2} |v(t_2,s) - v(t_1,s)|\, |g(s,x(s))|\, ds\right| \\
&\quad + \left|\int_{t_1}^{t_2} |v(t_1,s)|\, |g(s,x(s))|\, ds\right| \\
&\leq \left|q(t_1) - q(t_2)\right| + \left|\int_0^T |v(t_2,s) - v(t_1,s)|\, M_g\, ds\right| \\
&\quad + V M_g |t_2 - t_1| \\
&\to 0 \quad \text{as} \quad n \to \infty,
\end{aligned}$$

uniformly for all $y \in \mathcal{B}(C)$. Hence $\mathcal{B}(C)$ is an equicontinuous subset of E. Now, $\mathcal{B}(C)$ is a uniformly bounded and equicontinuous set of functions in E, so it is compact. Consequently, $\mathcal{B}$ is a uniformly partially compact operator on E into itself.

Step V: *u satisfies the operator inequality $u \leq \mathcal{A}u\,\mathcal{B}u + \mathcal{C}u$.*

By hypothesis (C_3), the QFIE (1.1) has a lower solution u defined on J. Then, we have

$$u(t) \leq k(t, u(t)) + \big[f(t, u(t))\big] \left(q(t) + \int_0^t v(t,s) g(s, u(s))\, ds \right) \tag{3.11}$$

for all $t \in J$. From definitions of the operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ it follows that $u(t) \leq \mathcal{A}u(t)\,\mathcal{B}u(t) + \mathcal{C}u(t)$ for all $t \in J$. Hence $u \leq \mathcal{A}u\,\mathcal{B}u + \mathcal{C}u$.

Step VI: *The $\mathcal{D}$-functions $\psi_{\mathcal{A}}$ and $\psi_{\mathcal{C}}$ satisfy the growth condition*

$$M\psi_{\mathcal{A}}(r) + \psi_{\mathcal{C}}(r) < r, \ r > 0.$$

Finally, the $\mathcal{D}$-function ϕ of the operator $\mathcal{A}$ satisfies the inequality given in hypothesis (d) of Theorem 2.9, viz.,

$$M\psi_{\mathcal{A}}(r) + \psi_{\mathcal{C}}(r) \leq (\|q\| + V\, M_g\, T)\,\psi_f(r) + \psi_k(r) < r$$

for all $r > 0$.

Thus $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ satisfy all the conditions of Theorem 2.9 and we conclude that the operator equation $\mathcal{A}x\,\mathcal{B}x + \mathcal{C}x = x$ has a solution. Consequently the integral equation and the QFIE (1.1) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}_{n\in\mathbb{N}}$ of successive approximations defined by (3.5) converges monotonically to x^*. This completes the proof. □

The conclusion of Theorems 3.3 also remains true if we replace the hypothesis (C_3) with the following one:

(C_3') The QFIE (1.1) has an upper solution $v \in C(J, \mathbb{R})$.

The proof of Theorem 3.3 under this new hypothesis is similar and can be obtained by closely observing the same arguments with appropriate modifications.

Example 3.4. *Given a closed and bounded interval $J = [0, 1]$, consider the QFIE,*

$$\begin{aligned} x(t) \;&=\; \frac{1}{2}\big[2 + \tan^{-1} x(t)\big] \left(\frac{t}{t+1} + \int_0^t \frac{1}{t^2+1} \cdot \frac{[1 + \tanh x(s)]}{4}\, ds \right) \\ &\quad + \frac{1}{2}\tan^{-1} x(t) \end{aligned} \tag{3.12}$$

for $t \in J$.

Here, $q(t) = \dfrac{t}{t+1}$ and $v(t,s) = \dfrac{1}{t^2+1}$ which are continuous and $\|q\| = \dfrac{1}{2}$ and $V = 1$. Similarly, the functions k, f and g are defined by $k(t,x) = \dfrac{1}{2}\tan^{-1} x$, $f(t,x) = \dfrac{1}{2}\big[2 + \tan^{-1} x(t)\big]$ and $g(t,x) = \dfrac{1 + \tanh x}{4}$.

The function f satisfies the hypothesis (A_3) with $\psi_f(r) = \dfrac{1}{2} \cdot \dfrac{r}{1+\xi^2}$ for each $0 < \xi < r$. To see this, we have

$$0 \leq f(t,x) - f(t,y) \leq \frac{1}{2} \cdot \frac{1}{1+\xi^2} \cdot (x - y)$$

for all $x, y \in \mathbb{R}$, $x \geq y$ and $x > \xi > y$. Moreover, the function f is nonnegative and bounded on $J \times \mathbb{R}$ with bound $M_f = 2$ and so the hypothesis (A_2) is satisfied.

Again, since g is nonnegative and bounded on $J \times \mathbb{R}$ by $M_g = \frac{1}{2}$, the hypothesis (B_3) holds. Furthermore, $g(t, x)$ is nondecreasing in x for all $t \in J$, and thus hypothesis (B_4) is satisfied.

Similarly, the function k satisfies the hypothesis (C_2) with $\psi_k(r) = \frac{1}{2} \cdot \frac{r}{1+\xi^2}$ for every $0 < \xi < r$. To see this, we have

$$0 \leq k(t,x) - k(t,y) \leq \frac{1}{2} \cdot \frac{1}{1+\xi^2} \cdot (x-y)$$

for all $x, y \in \mathbb{R}$, $x \geq y$ and $x > \xi > y$. Moreover, the function k is bounded on $J \times \mathbb{R}$ with bound $M_k = \frac{\pi}{4}$ and so the hypothesis (C_1) is satisfied.

Also we have

$$\big(\|q\| + M_g V T\big)\psi_f(r) + \psi_k(r) \leq \frac{r}{1+\xi^2} < r$$

for every $r > 0$. Thus, condition (3.4) of Theorem 3.3 is held. Finally, the QFIE (3.12) has a lower solution $u(t) = 0$ on J. Thus all the hypotheses of Theorem 3.3 are satisfied. Hence we apply Theorem 3.3 and conclude that the QFIE (3.12) has a solution x^* defined on J and the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by

$$\begin{aligned} x_{n+1}(t) &= \frac{1}{2}\big[2 + \tan^{-1} x_n(t)\big] \left(\frac{t}{t+1} + \int_0^t \frac{1}{t^2+1} \cdot \frac{[1 + \tanh x_n(s)]}{4}\, ds \right) \\ &\quad + \frac{1}{2} \tan^{-1} x_n(t), \end{aligned} \tag{3.13}$$

for all $t \in J$, where $x_0 = 0$, converges monotonically to x^*.

4. Conclusion

Finally, while concluding this paper we mention that the generalized quadratic integral equation considered here is of very simple nature for which we have illustrated the Dhage iteration method to obtain the algorithms for the solutions under weaker partially Lipschitz and compactness conditions. However, an analogous study could also be made for other complex quadratic integral equations as well as other different types of quadratic integral equations using similar method with appropriate modifications. Some of the results along this line will be reported elsewhere.

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