

## A Novel Subclass of Bi-Univalent Functions Defined by the $q$ -Wright Operator and the $q$ -Analogue of Fibonacci Numbers

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**Abstract.** Inspired by the deep connection between  $q$ -calculus and geometric function theory, this study introduces and examines a novel subclass of bi-univalent functions generated through an operator constructed from the  $q$ -Wright function and subordinated to the  $q$ -analogue of Fibonacci numbers. The core contribution lies in formulating a new  $q$ -differential operator defined via convolution with kernels involving the  $q$ -Wright function. Employing the subordination principle, the bounds are derived for the initial Taylor-Maclaurin coefficients  $|a_2|$  and  $|a_3|$ , along with corresponding Fekete-Szegö type inequalities for the defined class. The presented results not only unify but also generalize various recent developments in the theory of bi-univalent functions, emphasizing the pivotal influence of  $q$ -special functions in constructing new analytic frameworks. Consequently, the findings enhance the theoretical understanding of bi-univalent mappings and open avenues for further exploration in operator theory, convolution techniques, and the broader application of  $q$ -calculus within complex analysis.

### 1. INTRODUCTION

We begin by considering the class  $\mathcal{A}$  consisting of functions that are analytic within the open unit disk  $\mathbb{U}$ , defined by

$$\mathbb{U} = \{z = a + ib \in \mathbb{C} : a, b \in \mathbb{R}, |z| < 1\}.$$

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Geometrically, this region corresponds to the interior of the unit circle centered at the origin in the complex plane, excluding the boundary. Each function  $f \in \mathcal{A}$  satisfies the standard normalization conditions

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1,$$

which remove translational and scaling ambiguities and guarantee that every function is uniquely determined at the origin with a fixed initial slope. This normalization facilitates meaningful comparisons and structural analysis within the class.

Every function  $f \in \mathcal{A}$  admits a Maclaurin series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{U}, \quad (1.1)$$

where the coefficients  $a_n$  describe the nonlinear part of the mapping. The linear term  $z$  arises naturally from the normalization  $f'(0) = 1$ .

A function  $f$  is said to be a *Schwarz function* if it is analytic in  $\mathbb{U}$ , satisfies  $f(0) = 0$ , and remains bounded by unity in modulus, i.e.,  $|f(z)| < 1$  for all  $z \in \mathbb{U}$ . Such functions play a fundamental role in geometric function theory, especially in the study of conformal and univalent mappings.

For two analytic functions  $f_1, f_2 \in \mathcal{A}$ , we say that  $f_1$  is *subordinate* to  $f_2$ , written  $f_1 < f_2$ , if there exists a Schwarz function  $\eta$  such that

$$f_1(z) = f_2(\eta(z)), \quad z \in \mathbb{U}.$$

This concept establishes a functional dependence that preserves analyticity and provides a framework for comparing growth, inclusion, and mapping properties in complex analysis.

Let  $\mathcal{S} \subset \mathcal{A}$  denote the subclass of functions that are univalent (i.e., one-to-one) within  $\mathbb{U}$ . We also define the Carathéodory class  $\mathcal{P}$ , which consists of all functions  $p \in \mathcal{A}$  with strictly positive real parts throughout  $\mathbb{U}$ . Every  $p \in \mathcal{P}$  admits a power series representation of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots, \quad z \in \mathbb{U}, \quad (1.2)$$

where the coefficients satisfy the sharp inequality

$$|p_n| \leq 2, \quad n \geq 1, \quad (1.3)$$

as established by the classical Carathéodory lemma (see [2]). Moreover, a function  $p \in \mathcal{P}$  if and only if it is subordinate to the Möbius transformation  $\frac{1+z}{1-z}$ ; that is,

$$p(z) < \frac{1+z}{1-z}, \quad z \in \mathbb{U}.$$

The class  $\mathbf{P}$  serves as a cornerstone in the development of many important subclasses of analytic functions and is thus of central importance in geometric function theory. For each  $f \in \mathbf{S}$ , there exists an inverse function  $f^{-1}$  defined by

$$z = f^{-1}(f(z)), \quad \xi = f(f^{-1}(\xi)), \quad (r_0(f) \geq 0.25; |\xi| < r_0(f); z \in \mathbb{U}), \tag{1.4}$$

which can be expressed as

$$\chi(\xi) = f^{-1}(\xi) = \xi - a_2\xi^2 + (2a_2^2 - a_3)\xi^3 - (5a_2^3 + a_4 - 5a_2a_3)\xi^4 + \dots \tag{1.5}$$

A function  $f \in \mathbf{S}$  is called *bi-univalent* if both  $f$  and its inverse  $f^{-1}$  are univalent in  $\mathbb{U}$ . The collection of such functions is denoted by  $\Sigma \subset \mathbf{S}$ .

The emergence of  $q$ -calculus—often referred to as the calculus of finite differences—has exerted a profound influence on modern analytic function theory, evolving over the past decades into a coherent and unifying framework through which classical operators, geometric structures, and function classes can be systematically transformed into their  $q$ -analogues. This deformation paradigm not only preserves the fundamental analytic characteristics of the classical theory but also unveils subtler geometric and algebraic features that frequently remain hidden in the non-deformed setting, thereby giving rise to a broad range of new subclasses of analytic and bi-univalent functions and significantly enriching the scope of geometric function theory. At the same time,  $q$ -calculus maintains deep and intrinsic connections with special functions, combinatorial identities, and orthogonal polynomials, extending analytic methodologies to discrete, fractional, and hybrid environments and effectively bridging continuous and discrete analysis within a unified structure. Owing to its flexibility and structural depth, the  $q$ -framework has become an indispensable instrument in both theoretical and applied investigations, fostering sustained developments across diverse areas of mathematics (see, e.g., [10–20, 33, 34]), with further applications to algebraic structures and operator theory discussed in [21–23, 25, 26].

In 2008, Shahed and Salem [1] proposed a  $q$ -Wright function, which is defined as follows:

$$\mathcal{W}_{\iota, \beta}(z; q^k) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}} z^n}{[n]_q! \Gamma_{q^k}(n + \rho)} = \frac{1}{\Gamma_{q^k}(\rho)} + \frac{qz}{\Gamma_{q^k}(\iota + \rho)} + \frac{q^3 z^2}{[2]_q! \Gamma_{q^k}(2\iota + \rho)} + \dots \quad (z \in \mathcal{O}), \tag{1.6}$$

where  $\iota = -\log(1 - q)/1 - q^k$ ,  $\rho \in \mathbb{C}$ ,  $0 < q < 1$  and

$$\Gamma_{q^k}(\kappa) = \frac{(q^k, q^k)_{\infty}}{(q^{k\kappa}, q^k)_{\infty}} (1 - q^k)^{1-\kappa}, \quad \kappa \neq 0, -1, -2, \dots$$

**Remark 1.1.** If  $q \rightarrow 1^-$ , then  $\iota = -\frac{\log(1-q)}{1-q^k} \rightarrow -1$  and the  $q$ -Wright function (1.6) tends to the classical case

$$\mathcal{W}_{\iota, \rho}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma(n + \rho)}, \quad (z \in \mathbb{C}). \tag{1.7}$$

It is clear that the  $q$ -Wright function  $\mathcal{W}_{\iota, \rho}(z; q)$  does not belong to the standard class  $\mathcal{A}$  of analytic functions. To ensure analyticity and normalization at the origin, it is therefore natural to introduce

the following normalized form of the  $q$ -Wright function, which serves as a generating kernel for further analytical developments:

$$\mathbb{R}_{\iota, \rho}(z) = z \Gamma_{q^k}(\rho) \mathcal{W}_{\iota, \rho}(z; q) = z + \sum_{n=2}^{\infty} \frac{q^{\frac{n(n-1)}{2}} \Gamma_{q^k}(\rho)}{[n-1]_q! \Gamma_{q^k}(\iota(n-1) + \rho)} z^n, \quad (z \in \mathbb{U}). \quad (1.8)$$

Based on this normalized kernel, we define a linear operator  $\mathcal{V}_{\iota, \rho} : \mathcal{A} \rightarrow \mathcal{A}$  by means of the Hadamard product (or convolution) as follows:

$$\begin{aligned} \mathcal{V}_{\iota, \rho} f(z) &= \mathbb{R}_{\iota, \rho}(z) * f(z) = z + \sum_{n=2}^{\infty} \frac{q^{\frac{n(n-1)}{2}} \Gamma_{q^k}(\rho)}{[n-1]_q! \Gamma_{q^k}(\iota(n-1) + \rho)} a_n z^n \\ &= z + \frac{q \Gamma_{q^k}(\rho)}{\Gamma_{q^k}(\iota + \rho)} a_2 z^2 + \frac{q^3 \Gamma_{q^k}(\rho)}{[2]_q \Gamma_{q^k}(2\iota + \rho)} a_3 z^3 + \dots, \end{aligned} \quad (z \in \mathbb{U}). \quad (1.9)$$

Here,  $\mathbb{R}_{\iota, \rho}(z)$  denotes the generating kernel corresponding to the parameters  $\iota$  and  $\rho$ . The operator  $\mathcal{V}_{\iota, \rho}$  thus acts as a convolution-type transformation preserving analyticity within the domain  $\mathbb{U}$ , and serves as a foundational tool in constructing new subclasses of analytic and bi-univalent functions in the framework of  $q$ -calculus.

In a recent contribution, Alsoboh et al. [5] introduced a distinguished subclass of analytic functions defined via subordination and generated by the  $q$ -analogue of the Fibonacci numbers. The central analytic kernel in their construction is the function

$$Y(z; q) = \frac{1 + q \vartheta_q^2 z^2}{1 - \vartheta_q z - q \vartheta_q^2 z^2}, \quad (1.10)$$

where

$$\vartheta_q = \frac{1 - \sqrt{4q + 1}}{2q}, \quad (1.11)$$

which serves as a  $q$ -extension of the classical Fibonacci generating function and provides a flexible framework for constructing new bi-univalent and starlike classes. Expanding  $Y(z; q) = 1 + \sum_{n=1}^{\infty} \widehat{\rho}_n z^n$ , the coefficients satisfy the recurrence relation

$$\widehat{\rho}_n = \begin{cases} \vartheta_q, & n = 1, \\ (2q + 1) \vartheta_q^2, & n = 2, \\ (3q + 1) \vartheta_q^3, & n = 3, \\ (\varphi_{n+1}(q) + q \varphi_{n-1}(q)) \vartheta_q^n, & n \geq 4, \end{cases}$$

where  $\varphi_n(q)$  denotes the corresponding  $q$ -Fibonacci polynomials, which reduce to the classical Fibonacci numbers as  $q \rightarrow 1^-$ . This formulation highlights a direct structural link between  $\vartheta_q$  and the  $q$ -Fibonacci system, thereby reinforcing the integration of  $q$ -calculus into geometric function theory (see, e.g., [6–8]). The approach unifies several existing analytic subclasses and strengthens

the interplay among  $q$ -calculus, special functions, and operator theory, motivating further investigations into  $q$ -deformed kernels and their applications to coefficient problems, Fekete–Szegő inequalities, and convolution structures (see, for example, [4, 28–, 35–38]).

## 2. DEFINITION AND ILLUSTRATIVE EXAMPLES

Motivated by the analytical and geometric framework inspired by the theory of  $q$ -Fibonacci numbers, this section introduces a novel subclass of bi-univalent functions defined via a new operator constructed from the  $q$ -Wright function. The formulation is based on a generalized  $q$ -differential operator that unifies and extends several previously studied function families through subordination to an analytic generating kernel  $Y(z; q)$ . This operator framework not only encapsulates existing  $q$ -analytic structures but also provides a broader setting for exploring geometric properties and coefficient problems within the context of  $q$ -calculus.

**Definition 2.1.** Let  $f$  be a bi-univalent function of the form (1.1). Then  $f$  is said to belong to the class  $\mathbb{R}_{\Sigma}^{l, \rho}(\gamma, \delta; q)$  if and only if the following subordinations are satisfied:

$$(1 - \gamma) \frac{\mathcal{V}_{l, \rho} f(z)}{z} + \gamma \partial_q \langle \mathcal{V}_{l, \rho} f(z) \rangle + \delta z \delta_q^2 \langle \mathcal{V}_{l, \rho} f(z) \rangle < Y(z; q), \tag{2.1}$$

and

$$(1 - \gamma) \frac{\mathcal{V}_{l, \rho} \chi(\xi)}{\xi} + \gamma \partial_q \langle \mathcal{V}_{l, \rho} \chi(\xi) \rangle + \delta \xi \delta_q^2 \langle \mathcal{V}_{l, \rho} \chi(\xi) \rangle < Y(\xi; q), \tag{2.2}$$

where  $\delta, \gamma \geq 0, \rho \in \mathbb{C}, \chi = f^{-1}$  denotes the analytic inverse of  $f$  as defined in (1.5), and  $\partial_q$  is the parameter specified by (1.11). The analytic function  $Y(z; q)$  appearing in (2.1) and (2.2) is given by (1.10), and serves as the generating function governing the underlying shell-like geometry of the domain.

This class provides a unified framework connecting  $q$ -calculus with geometric function theory. By adjusting the parameters  $\gamma, \delta$ , and  $q \in (0, 1)$ , one can recover numerous analytic subclasses exhibiting distinct geometric characteristics in the unit disk  $\mathbb{U}$ .

**Example 2.1.** Let  $\delta = 0$  in Definition 2.1. Then the class  $\mathbb{R}_{\Sigma}^{l, \rho}(\gamma, 0; q)$  consists of all bi-univalent functions  $f$  satisfying

$$(1 - \gamma) \frac{\mathcal{V}_{l, \rho} f(z)}{z} + \gamma \delta_q \langle \mathcal{V}_{l, \rho} f(z) \rangle < \frac{1 + q \delta_q^2 z^2}{1 - \partial_q z - q \delta_q^2 z^2},$$

and the analogous subordination for  $\chi = f^{-1}$ .

**Example 2.2.** Let  $\gamma = 1$  and  $\delta = 0$  in Definition 2.1. Then the class  $\mathbb{R}_{\Sigma}^{l, \rho}(1, 0; q)$  satisfies

$$\delta_q \langle \mathcal{V}_{l, \rho} f(z) \rangle < Y(z; q), \quad \delta_q \langle \mathcal{V}_{l, \rho} \chi(\xi) \rangle < Y(\xi; q),$$

which corresponds to a  $q$ -differential subclass where the geometric characterization depends solely on the first  $q$ -derivative of the transformed function.

**Example 2.3.** Let  $\gamma = 0$  and  $\delta = 0$  in Definition 2.1. Then the class  $\mathbf{R}_{\Sigma}^{\ell, \rho}(0, 0; q)$  is characterized by

$$\frac{\mathcal{V}_{\ell, \rho} f(z)}{z} < Y(z; q), \quad \frac{\mathcal{V}_{\ell, \rho} \chi(\xi)}{\xi} < Y(\xi; q),$$

representing the fundamental subclass governed purely by the operator quotient without any  $q$ -differential influence.

**Example 2.4.** In the limit  $q \rightarrow 1^-$ , the Definition 2.1 yields the classical operator-based subclass

$$(1 - \gamma) \frac{\mathcal{V}_{\ell, \rho} f(z)}{z} + \gamma (\mathcal{V}_{\ell, \rho} f(z))' + \delta z (\mathcal{V}_{\ell, \rho} f(z))'' < Y(z; 1) = \frac{1 + \vartheta^2 z^2}{1 - \vartheta z - \vartheta^2 z^2},$$

and the analogous subordination for  $\chi = f^{-1}$ .

### 3. MAIN RESULTS

In this section, we first obtain the estimate of the initial Taylor coefficients  $|a_2|$  and  $|a_3|$  for functions in the class  $\mathbf{R}_{\Sigma}^{\ell, \rho}(\gamma, \delta; q)$  as per Definition 2.1.

Firstly, let

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots,$$

and  $p(z) < Y(z; q)$ . Then there exists  $\varphi \in \mathbf{P}$  such that

$$|p(z)| < 1 \text{ in } \mathbf{U} \text{ and } p(z) = Y(\varphi(z); q).$$

We have

$$h(z) = (1 + \varphi(z))(1 - \varphi(z))^{-1} = 1 + \ell_1 z + \ell_2 z^2 + \dots \in \mathbf{P} \quad (z \in \mathbf{U}). \quad (3.1)$$

Consequently, the function  $\varphi(z)$ , being analytic in  $\mathbf{U}$  and subordinate to  $Y(z; q)$ , admits the following Taylor expansion:

$$\varphi(z) = \frac{\ell_1 z}{2} + \left( \ell_2 - \frac{\ell_1^2}{2} \right) \frac{z^2}{2} + \left( \ell_3 - \ell_1 \ell_2 - \frac{\ell_1^3}{4} \right) \frac{z^3}{2} + \dots, \quad (3.2)$$

and

$$\begin{aligned} Y(\varphi(z); q) &= 1 + \frac{\widehat{p}_1 \ell_1}{2} z + \frac{1}{2} \left[ \left( \ell_2 - \frac{\ell_1^2}{2} \right) \widehat{p}_1 + \frac{\ell_1^2}{2} \widehat{p}_2 \right] z^2 \\ &+ \frac{1}{2} \left[ \left( \ell_3 - \ell_1 \ell_2 + \frac{\ell_1^3}{4} \right) \widehat{p}_1 + \ell_1 \left( \ell_2 - \frac{\ell_1^2}{2} \right) \widehat{p}_2 + \frac{\ell_1^3}{4} \widehat{p}_3 \right] z^3 + \dots. \end{aligned} \quad (3.3)$$

Similarly, there exists an analytic function  $v$  defined on  $\mathbf{U}$ , satisfying  $|v(\xi)| < 1$ , such that  $p(\xi) = Y(v(\xi); q)$ . This allows us to represent the corresponding function

$$\kappa(\xi) = (1 + v(\xi))(1 - v(\xi))^{-1} = 1 + \tau_1 \xi + \tau_2 \xi^2 + \dots \in \mathbf{P}. \quad (3.4)$$

As a result, the Taylor expansion of  $v(\xi)$  takes the form:

$$v(\xi) = \frac{\tau_1 \xi}{2} + \left( \tau_2 - \frac{\tau_1^2}{2} \right) \frac{\xi^2}{2} + \left( \tau_3 - \tau_1 \tau_2 - \frac{\tau_1^3}{4} \right) \frac{\xi^3}{2} + \dots, \quad (3.5)$$

and accordingly, the composition  $Y(v(\xi); q)$  expands as:

$$\begin{aligned}
 Y(v(\xi); q) &= 1 + \frac{\widehat{\rho}_1 \tau_1}{2} \xi + \frac{1}{2} \left[ \left( \tau_2 - \frac{\tau_1^2}{2} \right) \widehat{\rho}_1 + \frac{\tau_1^2}{2} \widehat{\rho}_2 \right] \xi^2 \\
 &+ \frac{1}{2} \left[ \left( \tau_3 - \tau_1 \tau_2 + \frac{\tau_1^3}{4} \right) \widehat{\rho}_1 + \tau_1 \left( \tau_2 - \frac{\tau_1^2}{2} \right) \widehat{\rho}_2 + \frac{\tau_1^3}{4} \widehat{\rho}_3 \right] \xi^3 + \dots
 \end{aligned}
 \tag{3.6}$$

Having established the necessary groundwork and auxiliary results, we are now in a position to derive bounds for the initial coefficients of functions belonging to the newly introduced class  $\mathbb{R}_{\Sigma}^{\iota, \rho}(\gamma, \delta; q)$ . These estimates not only offer insights into the geometric behavior of such bi-univalent functions but also highlight the influence of the deformation parameter  $q$  and the parameter  $\beta$  on the coefficient structure. The following theorem presents the bounds for the second and third coefficients  $|a_2|$  and  $|a_3|$ , respectively.

**Theorem 3.1.** *Let  $f \in \mathbb{R}_{\Sigma}^{\iota, \rho}(\gamma, \delta; q)$ . Then*

$$|a_2| \leq \min \left\{ \frac{\frac{|\vartheta_q| \Gamma_{q^k}(\iota + \rho)}{q |1 + q\gamma + [2]_q \delta| \Gamma_{q^k}(\rho)}, \frac{|\vartheta_q| \Gamma_{q^k}(\iota + \rho) \sqrt{[2]_q \Gamma_{q^k}(2\iota + \rho)}}{q \sqrt{\left| \begin{aligned} &|q \vartheta_q \Gamma_{q^k}(\rho) \Gamma_{q^k}^2(\iota + \rho) (1 + q[2]_q \gamma + [2]_q [3]_q \delta) \\ &- [2]_q \Gamma_{q^k}(2\iota + \rho) ((2q + 1) \vartheta_q - 1) \Gamma_{q^k}^2(\rho) (1 + q\gamma + [2]_q \delta)^2 \end{aligned} \right|}}}{\dots} \right\},$$

and

$$|a_3| \leq \frac{\vartheta_q^2 \Gamma_{q^k}^2(\iota + \rho)}{q^2 (1 + q\gamma + [2]_q \delta)^2 \Gamma_{q^k}^2(\rho)} + \left| \frac{\vartheta_q [2]_q! \Gamma_{q^k}(2\iota + \rho)}{q^3 (1 + q[2]_q \gamma + [2]_q [3]_q \delta) \Gamma_{q^k}(\rho)} \right|.
 \tag{3.7}$$

*Proof.* Let  $f \in \mathbb{R}_{\Sigma}^{\iota, \rho}(\gamma, \delta; q)$  and  $\xi = f^{-1}$ . Considering (2.1) and (2.2) we have

$$(1 - \gamma) \frac{\mathcal{V}_{\iota, \rho} f(z)}{z} + \gamma \partial_q \langle \mathcal{V}_{\iota, \rho} f(z) \rangle + \delta z \delta_q^2 \langle \mathcal{V}_{\iota, \rho} f(z) \rangle = Y(\varphi(z); q), \quad (z \in \mathbb{U}),
 \tag{3.8}$$

and

$$(1 - \gamma) \frac{\mathcal{V}_{\iota, \rho} \chi(\xi)}{\xi} + \gamma \delta_q \langle \mathcal{V}_{\iota, \rho} \chi(\xi) \rangle + \delta \xi \delta_q^2 \langle \mathcal{V}_{\iota, \rho} \chi(\xi) \rangle = Y(v(\xi); q), \quad (\xi \in \mathbb{U}).
 \tag{3.9}$$

Upon substituting the operator  $\mathcal{V}_{\iota, \rho} f(z)$  defined in (1.9) into equation (3.8), the left-hand side transforms into

$$\begin{aligned}
 (1 - \gamma) \frac{\mathcal{V}_{\iota, \rho} f(z)}{z} + \gamma \partial_q \langle \mathcal{V}_{\iota, \rho} f(z) \rangle + \delta z \delta_q^2 \langle \mathcal{V}_{\iota, \rho} f(z) \rangle &= 1 + \frac{q(1 + q\gamma + [2]_q \delta) \Gamma_{q^k}(\rho)}{\Gamma_{q^k}(\iota + \rho)} a_2 z \\
 &+ \frac{q^3 (1 + q[2]_q \gamma + [2]_q [3]_q \delta) \Gamma_{q^k}(\rho)}{\Gamma_{q^k}(\iota + \rho)} a_3 z^2 + \dots,
 \end{aligned}
 \tag{3.10}$$

and similarly for into equation (3.9), the left-hand side transforms into

$$(1 - \gamma) \frac{\mathcal{V}_{l,\rho} \chi(\xi)}{\xi} + \gamma \delta_q \langle \mathcal{V}_{l,\rho} \chi(\xi) \rangle + \delta \xi \delta_q^2 \langle \mathcal{V}_{l,\rho} \chi(\xi) \rangle = 1 - \frac{q(1 + q\gamma + [2]_q \delta) \Gamma_{q^k}(\rho)}{\Gamma_{q^k}(l + \rho)} a_2 \xi$$

$$+ \frac{q^3 (1 + q[2]_q \gamma + [2]_q [3]_q \delta) \Gamma_{q^k}(\rho)}{\Gamma_{q^k}(l + \rho)} (2a_2^2 - a_3) \xi^2 + \dots \quad (3.11)$$

Substituting (3.3) and (3.10) into (3.9) yields

$$\frac{q(1 + q\gamma + [2]_q \delta) \Gamma_{q^k}(\rho)}{\Gamma_{q^k}(l + \rho)} a_2 z + \frac{q^3 (1 + q[2]_q \gamma + [2]_q [3]_q \delta) \Gamma_{q^k}(\rho)}{\Gamma_{q^k}(l + \rho)} a_3 z^2 + \dots$$

$$= \frac{\widehat{\rho}_1 \ell_1}{2} z + \frac{1}{2} \left[ \left( \ell_2 - \frac{\ell_1^2}{2} \right) \widehat{\rho}_1 + \frac{\ell_1^2}{2} \widehat{\rho}_2 \right] z^2 + \dots \quad (3.12)$$

Besides that, substituting (3.6) and (3.11) into (3.10) yield

$$-\frac{q(1 + q\gamma + [2]_q \delta) \Gamma_{q^k}(\rho)}{\Gamma_{q^k}(l + \rho)} a_2 \xi + \frac{q^3 (1 + q[2]_q \gamma + [2]_q [3]_q \delta) \Gamma_{q^k}(\rho)}{\Gamma_{q^k}(l + \rho)} (2a_2^2 - a_3) \xi^2 + \dots$$

$$= \frac{\widehat{\rho}_1 \tau_1}{2} \xi + \frac{1}{2} \left[ \left( \tau_2 - \frac{\tau_1^2}{2} \right) \widehat{\rho}_1 + \frac{\tau_1^2}{2} \widehat{\rho}_2 \right] \xi^2 + \dots \quad (3.13)$$

Equating the pertinent coefficient in (3.12) and (3.13), we obtain

$$\frac{q(1 + q\gamma + [2]_q \delta) \Gamma_{q^k}(\rho)}{\Gamma_{q^k}(l + \rho)} a_2 = \frac{\widehat{\rho}_1 \ell_1}{2} \quad (3.14)$$

$$-\frac{q(1 + q\gamma + [2]_q \delta) \Gamma_{q^k}(\rho)}{\Gamma_{q^k}(l + \rho)} a_2 = \frac{\widehat{\rho}_1 \tau_1}{2} \quad (3.15)$$

$$\frac{q^3 (1 + q[2]_q \gamma + [2]_q [3]_q \delta) \Gamma_{q^k}(\rho)}{[2]_q! \Gamma_{q^k}(2l + \rho)} a_3 = \frac{1}{2} \left[ \left( \ell_2 - \frac{\ell_1^2}{2} \right) \widehat{\rho}_1 + \frac{\ell_1^2}{2} \widehat{\rho}_2 \right] \quad (3.16)$$

$$\frac{q^3 (1 + q[2]_q \gamma + [2]_q [3]_q \delta) \Gamma_{q^k}(\rho)}{[2]_q! \Gamma_{q^k}(2l + \rho)} (2a_2^2 - a_3) = \frac{1}{2} \left[ \left( \tau_2 - \frac{\tau_1^2}{2} \right) \widehat{\rho}_1 + \frac{\tau_1^2}{2} \widehat{\rho}_2 \right] \quad (3.17)$$

From (3.14) and (3.15), we have

$$\ell_1 = -\tau_1 \iff \ell_1^2 = \tau_1^2, \quad (3.18)$$

and

$$a_2^2 = \frac{\vartheta_q^2 \Gamma_{q^k}^2(l + \rho)}{8q^2 (1 + q\gamma + [2]_q \delta)^2 \Gamma_{q^k}^2(\rho)} (\ell_1^2 + \tau_1^2). \quad (3.19)$$

using (1.3), we have

$$|a_2| \leq \frac{|\vartheta_q| \Gamma_{q^k}(l + \rho)}{q |1 + q\gamma + [2]_q \delta| \Gamma_{q^k}(\rho)}. \quad (3.20)$$

By rearranging the terms in (3.19), we obtain the equivalent relation

$$\ell_1^2 + \tau_1^2 = \frac{8q^2(1 + q\gamma + [2]_q\delta)^2 \Gamma_{q^k}^2(\rho)}{\vartheta_q^2 \Gamma_{q^k}^2(\iota + \rho)} a_2^2. \tag{3.21}$$

By summing (3.16) and (3.17) and then substituting the expression of  $(\ell_1^2 + \tau_1^2)$  from (3.21), a straightforward simplification yields the following explicit representation for  $a_2^2$ :

$$a_2^2 = \frac{(\ell_2 + \tau_2) [2]_q \Gamma_{q^k}(2\iota + \rho) \vartheta_q^2 \Gamma_{q^k}^2(\iota + \rho)}{4q^2 \left\{ \begin{aligned} & q \vartheta_q \Gamma_{q^k}(\rho) \Gamma_{q^k}^2(\iota + \rho) (1 + q[2]_q\gamma + [2]_q[3]_q\delta) \\ & - [2]_q \Gamma_{q^k}(2\iota + \rho) ((2q + 1)\vartheta_q - 1) \Gamma_{q^k}^2(\rho) (1 + q\gamma + [2]_q\delta)^2 \end{aligned} \right\}}. \tag{3.22}$$

Using (1.3) for (3.22), we have

$$|a_2| \leq \frac{|\vartheta_q| \Gamma_{q^k}(\iota + \rho) \sqrt{[2]_q \Gamma_{q^k}(2\iota + \rho)}}{q \sqrt{\left| \begin{aligned} & q \vartheta_q \Gamma_{q^k}(\rho) \Gamma_{q^k}^2(\iota + \rho) (1 + q[2]_q\gamma + [2]_q[3]_q\delta) \\ & - [2]_q \Gamma_{q^k}(2\iota + \rho) ((2q + 1)\vartheta_q - 1) \Gamma_{q^k}^2(\rho) (1 + q\gamma + [2]_q\delta)^2 \end{aligned} \right|}}. \tag{3.23}$$

Now, so as to find the bound on  $|a_3|$ , let's subtract from (3.16) and (3.17) along (3.19), we obtain

$$a_3 = a_2^2 + \frac{\vartheta_q [2]_q! \Gamma_{q^k}(2\iota + \rho)}{4q^3(1 + q[2]_q\gamma + [2]_q[3]_q\delta) \Gamma_{q^k}(\rho)} (\ell_2 - \tau_2). \tag{3.24}$$

Then, in view of (3.20), we obtain

$$|a_3| \leq \frac{\vartheta_q^2 \Gamma_{q^k}^2(\iota + \rho)}{q^2(1 + q\gamma + [2]_q\delta)^2 \Gamma_{q^k}^2(\rho)} + \left| \frac{\vartheta_q [2]_q! \Gamma_{q^k}(2\iota + \rho)}{q^3(1 + q[2]_q\gamma + [2]_q[3]_q\delta) \Gamma_{q^k}(\rho)} \right|. \tag{3.25}$$

This proves (3.7). □

**Theorem 3.2.** For  $\alpha \in \mathbb{C}^*$ , let  $f \in \mathbf{R}_{\Sigma}^{\iota, \rho}(\gamma, \delta; q)$ . Then

$$|a_3 - \alpha a_2^2| \leq \begin{cases} \frac{|\vartheta_q| [2]_q \Gamma_{q^k}(2\iota + \rho)}{q^3 \Gamma_{q^k}(\rho) |1 + q[2]_q\gamma + [2]_q[3]_q\delta|}, & 0 \leq |K(\alpha)| \leq \frac{1}{q(1 + q[2]_q\gamma + [2]_q[3]_q\delta)} \\ \frac{|\vartheta_q| [2]_q \Gamma_{q^k}(2\iota + \rho)}{q^2 \Gamma_{q^k}(\rho)} |K(\alpha)|, & |K(\alpha)| \geq \frac{1}{q(1 + q[2]_q\gamma + [2]_q[3]_q\delta)} \end{cases}$$

where

$$K(\alpha) = \frac{(1 - \alpha) \vartheta_q \Gamma_{q^k}^2(\iota + \rho)}{\left\{ q \vartheta_q \Gamma_{q^k}^2(\iota + \rho) (1 + q[2]_q\gamma + [2]_q[3]_q\delta) - [2]_q \Gamma_{q^k}(\rho) \Gamma_{q^k}(2\iota + \rho) ((2q + 1)\vartheta_q - 1) (1 + q\gamma + [2]_q\delta)^2 \right\}} \tag{3.26}$$

*Proof.* Let  $f \in \mathbb{R}_{\Sigma}^{\iota, \rho}(\gamma, \delta; q)$ , from (3.22) and (3.24) we have

$$\begin{aligned}
 a_3 - \alpha a_2^2 &= \frac{(1 - \alpha) \vartheta_q \Gamma_{q^k}^2(\iota + \rho) (\ell_2 + \tau_2)}{\left\{ q \vartheta_q \Gamma_{q^k}^2(\iota + \rho) (1 + q[2]_q \gamma + [2]_q [3]_q \delta) - [2]_q \Gamma_{q^k}(\rho) \Gamma_{q^k}(2\iota + \rho) ((2q + 1) \vartheta_q - 1) (1 + q\gamma + [2]_q \delta)^2 \right\}} \\
 &+ \frac{1}{q(1 + q[2]_q \gamma + [2]_q [3]_q \delta)} (\ell_2 - \tau_2) \tag{3.27} \\
 &= \frac{\vartheta_q [2]_q \Gamma_{q^k}(2\iota + \rho)}{4 q^2 \Gamma_{q^k}(\rho)} \left[ \left( K(\alpha) + \frac{1}{q(1 + q[2]_q \gamma + [2]_q [3]_q \delta)} \right) \ell_2 + \left( K(\alpha) - \frac{1}{q(1 + q[2]_q \gamma + [2]_q [3]_q \delta)} \right) \tau_2 \right],
 \end{aligned}$$

where  $K(\alpha)$  given by (3.26). Then, by taking modulus of (3.27), we get our result. □

#### 4. COROLLARIES

The general coefficient estimates established in Theorems 3.1 and 3.2 give rise to several noteworthy special cases under suitable choices of the parameters  $\gamma, \delta$  and  $q$ , the results simplify to the following corollaries.

**Corollary 4.1.** *Let  $\alpha \in \mathbb{C}^*$  and  $f \in \mathbb{R}_{\Sigma}^{\iota, \rho}(\gamma, 0; q)$ . Then the initial coefficients satisfy*

$$|a_2| \leq \min \left\{ \frac{|\vartheta_q| \Gamma_{q^k}(\iota + \rho)}{q |1 + q\gamma| \Gamma_{q^k}(\rho)}, \frac{|\vartheta_q| \Gamma_{q^k}(\iota + \rho) \sqrt{[2]_q \Gamma_{q^k}(2\iota + \rho)}}{q \sqrt{\left| q \vartheta_q \Gamma_{q^k}(\rho) \Gamma_{q^k}^2(\iota + \rho) (1 + q[2]_q \gamma) - [2]_q \Gamma_{q^k}(2\iota + \rho) ((2q + 1) \vartheta_q - 1) \Gamma_{q^k}^2(\rho) (1 + q\gamma)^2 \right|}} \right\},$$

and

$$|a_3| \leq \frac{\vartheta_q^2 \Gamma_{q^k}^2(\iota + \rho)}{q^2 (1 + q\gamma)^2 \Gamma_{q^k}^2(\rho)} + \left| \frac{\vartheta_q [2]_q! \Gamma_{q^k}(2\iota + \rho)}{q^3 (1 + q[2]_q \gamma) \Gamma_{q^k}(\rho)} \right|.$$

Moreover, for any  $\alpha \in \mathbb{C}^*$ ,

$$|a_3 - \alpha a_2^2| \leq \begin{cases} \frac{|\vartheta_q| [2]_q \Gamma_{q^k}(2\iota + \rho)}{q^3 \Gamma_{q^k}(\rho) |1 + q[2]_q \gamma|}, & 0 \leq |K(\alpha)| \leq \frac{1}{q(1 + q[2]_q \gamma)}, \\ \frac{|\vartheta_q| [2]_q \Gamma_{q^k}(2\iota + \rho)}{q^2 \Gamma_{q^k}(\rho)} |K(\alpha)|, & |K(\alpha)| \geq \frac{1}{q(1 + q[2]_q \gamma)}. \end{cases}$$

**Corollary 4.2.** *Let  $\alpha \in \mathbb{C}^*$  and  $f \in \mathbb{R}_{\Sigma}^{\iota, \rho}(1, 0; q)$  (as in Example 2.2). Then the initial coefficients satisfy*

$$|a_2| \leq \min \left\{ \frac{|\vartheta_q| \Gamma_{q^k}(\iota + \rho)}{q |1 + q| \Gamma_{q^k}(\rho)}, \frac{|\vartheta_q| \Gamma_{q^k}(\iota + \rho) \sqrt{[2]_q \Gamma_{q^k}(2\iota + \rho)}}{q \sqrt{\left| q \vartheta_q \Gamma_{q^k}(\rho) \Gamma_{q^k}^2(\iota + \rho) (1 + q[2]_q) - [2]_q \Gamma_{q^k}(2\iota + \rho) ((2q + 1) \vartheta_q - 1) \Gamma_{q^k}^2(\rho) (1 + q)^2 \right|}} \right\},$$

and

$$|a_3| \leq \frac{\vartheta_q^2 \Gamma_{q^k}^2(\iota + \rho)}{q^2(1 + q)^2 \Gamma_{q^k}^2(\rho)} + \left| \frac{\vartheta_q [2]_q! \Gamma_{q^k}(2\iota + \rho)}{q^3(1 + q[2]_q) \Gamma_{q^k}(\rho)} \right|.$$

Moreover, for any  $\alpha \in \mathbb{C}^*$ ,

$$|a_3 - \alpha a_2^2| \leq \begin{cases} \frac{|\vartheta_q| [2]_q \Gamma_{q^k}(2\iota + \rho)}{q^3 \Gamma_{q^k}(\rho) |1 + q[2]_q|}, & 0 \leq |K(\alpha)| \leq \frac{1}{q(1 + q[2]_q)}, \\ \frac{|\vartheta_q| [2]_q \Gamma_{q^k}(2\iota + \rho)}{q^2 \Gamma_{q^k}(\rho)} |K(\alpha)|, & |K(\alpha)| \geq \frac{1}{q(1 + q[2]_q)}. \end{cases}$$

**Corollary 4.3.** Let  $\alpha \in \mathbb{C}^*$  and suppose  $f \in \mathbf{R}_{\Sigma}^{\iota, \rho}(0, 0; q)$ . (Equivalently, by Example 2.3. Then the initial coefficients satisfy

$$|a_2| \leq \min \left\{ \frac{|\vartheta_q| \Gamma_{q^k}(\iota + \rho)}{q \Gamma_{q^k}(\rho)}, \frac{|\vartheta_q| \Gamma_{q^k}(\iota + \rho) \sqrt{[2]_q \Gamma_{q^k}(2\iota + \rho)}}{q \sqrt{|q \vartheta_q \Gamma_{q^k}(\rho) \Gamma_{q^k}^2(\iota + \rho) - [2]_q \Gamma_{q^k}(2\iota + \rho) ((2q + 1) \vartheta_q - 1) \Gamma_{q^k}^2(\rho)|}} \right\},$$

and

$$|a_3| \leq \frac{\vartheta_q^2 \Gamma_{q^k}^2(\iota + \rho)}{q^2 \Gamma_{q^k}^2(\rho)} + \left| \frac{\vartheta_q [2]_q! \Gamma_{q^k}(2\iota + \rho)}{q^3 \Gamma_{q^k}(\rho)} \right|.$$

Moreover, for the Fekete–Szegő functional we have

$$|a_3 - \alpha a_2^2| \leq \begin{cases} \frac{|\vartheta_q| [2]_q \Gamma_{q^k}(2\iota + \rho)}{q^3 \Gamma_{q^k}(\rho)}, & 0 \leq |K_0(\alpha)| \leq \frac{1}{q}, \\ \frac{|\vartheta_q| [2]_q \Gamma_{q^k}(2\iota + \rho)}{q^2 \Gamma_{q^k}(\rho)} |K_0(\alpha)|, & |K_0(\alpha)| \geq \frac{1}{q}, \end{cases}$$

where

$$K_0(\alpha) = \frac{(1 - \alpha) \vartheta_q \Gamma_{q^k}^2(\iota + \rho)}{q \vartheta_q \Gamma_{q^k}^2(\iota + \rho) - [2]_q \Gamma_{q^k}(\rho) \Gamma_{q^k}(2\iota + \rho) ((2q + 1) \vartheta_q - 1) \Gamma_{q^k}(\rho)}.$$

**Corollary 4.4.** Let  $\alpha \in \mathbb{C}^*$  and suppose  $f \in \mathbf{R}_{\Sigma}^{\iota, \rho}(\gamma, \delta)$  as in Example 2.4. Then, in the limit  $q \rightarrow 1^-$  (so that  $[k]_q \rightarrow k$ ,  $\Gamma_{q^k}(\cdot) \rightarrow \Gamma(\cdot)$ , and  $\vartheta_q \rightarrow \vartheta$ ), the initial coefficients satisfy

$$|a_2| \leq \min \left\{ \frac{|\vartheta| \Gamma(\iota + \rho)}{|1 + \gamma + 2\delta| \Gamma(\rho)}, \frac{|\vartheta| \Gamma(\iota + \rho) \sqrt{2\Gamma(2\iota + \rho)}}{\sqrt{|\vartheta \Gamma(\rho) \Gamma^2(\iota + \rho) (1 + 2\gamma + 6\delta) - 2\Gamma(2\iota + \rho) (3\vartheta - 1) \Gamma^2(\rho) (1 + \gamma + 2\delta)^2|}} \right\},$$

and

$$|a_3| \leq \frac{\vartheta^2 \Gamma^2(\iota + \rho)}{(1 + \gamma + 2\delta)^2 \Gamma^2(\rho)} + \left| \frac{2 \vartheta \Gamma(2\iota + \rho)}{(1 + 2\gamma + 6\delta) \Gamma(\rho)} \right|.$$

Moreover, the Fekete–Szegő functional obeys

$$|a_3 - \alpha a_2^2| \leq \begin{cases} \frac{2|\vartheta|\Gamma(2\iota + \rho)}{\Gamma(\rho)|1 + 2\gamma + 6\delta|}, & 0 \leq |K_{cl}(\alpha)| \leq \frac{1}{1 + 2\gamma + 6\delta}, \\ \frac{2|\vartheta|\Gamma(2\iota + \rho)}{\Gamma(\rho)} |K_{cl}(\alpha)|, & |K_{cl}(\alpha)| \geq \frac{1}{1 + 2\gamma + 6\delta}. \end{cases}$$

#### CONCLUSION AND FUTURE WORK

In this paper, we introduced and investigated a new subclass of bi-univalent functions associated with shell-like domains generated by the  $q$ -Wright function and the  $q$ -analogue of Fibonacci numbers. By constructing a convolution-based  $q$ -derivative operator and employing the principle of subordination, we derived the bounds for the initial Taylor–Maclaurin coefficients  $|a_2|$  and  $|a_3|$ , along with corresponding Fekete–Szegő type inequalities. The obtained results unify and extend several previously studied subclasses, demonstrating the effectiveness of  $q$ -special functions in producing enriched geometric structures within geometric function theory. Future research may focus on extending the proposed operator to multivalent and fractional  $q$ -frameworks, investigating higher-order coefficient problems, and exploring generalizations involving other  $q$ -special functions such as the  $q$ -Mittag–Leffler,  $q$ -Bessel, and  $q$ -Rabotnov functions. Further analytical and computational studies of the associated shell-like domains may also provide deeper insight into the geometric behavior governed by the  $q$ -parameters, thereby strengthening the connection between  $q$ -calculus and modern operator-based approaches in analytic function theory.

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