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A New Approach for Approximate Solutions of Time–Fractional Coupled KdV–Type Equations Using the Constant Proportional Caputo–Fabrizio Operator

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Abstract. Some investigations in the fields such as fluid mechanics, control theory, quantum mechanics, ocean engineering, non-linear optics, biology, economics, plasma physics and electrodynamics highlights that the non-linear partial fractional differential equations (NPFDEs) are important for modeling and analysing the various real-world issues. Obtaining the solution to these equations will help us for the clear interpretation of the non-linear physical phenomena present in our environment. In this study, we have derived the approximate solutions of the non-linear time-fractional Hirota-Satsuma coupled KdV and coupled MKdV equations using the Iterative Laplace Transform Method (ILTM) in conjunction with the Constant Proportional Caputo-Fabrizio (CPCF) differential operator. The conditions for the solution's uniqueness are established by the Banach contraction principle. Further, The Picard's stability conditions are verified to guarantee the stability of obtained approximate solution by the proposed iterative apporoach derived from the fixed-point theorem. These approximate solutions are presented graphically and have compared numerically with the exact solutions. The method ILTM-CPCF serve as effective mathematical tool and seems computationally simple and user-friendly for the analysis of coupled non-linear fractional differential equations.

1. Introduction

The linear systems describes the theoretical approximation of many simpler occurrences. But the fundamental nature of the processes is better captured by nonlinear systems. For this reason, understanding and researching nonlinear phenomena is essential to contemporary science and technology. Many nonlinear phenomena are simplified by solving nonlinear differential equations from the perspective of mathematical physics.

The basic Korteweg–de Vries(KdV) equation, which explains nonlinear wave propagation in dispersive media, is extended by the Hirota–Satsuma coupled KdV(HS-cKdV) and coupled

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MKdV(HS-cMKdV) equation. R. Hirota and J. Satsuma [1] first proposed the Hirota-Satsuma system in 1981 as a model to explain interactions between two long nonlinear waves of varying amplitudes. The generalized HS-cKdV and HS-cMKdV systems represents the fundamental nonlinear equations in physics and mathematics. The applications of HS-cKdV system include the propagation of pulses in nonlinear media in optics, the relationship between two long waves with different amplitudes in shallow water waves, the interaction of ion-acoustic waves in plasma physics, and the evaluation of new nonlinear PDE solution methods in mathematical physics. The HS-cMKdV system explains how two wave modes interact elastically during soliton collisions, forecasts the flow of energy between nonlinear waveguide modes.

The integer order HS-cKdV equation is studied by J. Chena et. al. [2] by adding a Lax pair to show that the HS-cKdV problem which they had given as a matrix generalization, can be reduced to a vector HS-cKdV equation. Using Hirota's bilinear technique, several soliton solutions are found for the vector HS-cKdV problem in a symmetric setting. In [3] C. Zhang et. al. have obtained the one-soliton and two-soliton solutions to integer order HS-cKdV equation by using Hirota's bilinear approach to derive the bilinear structure of HS-cKdV equation. Further, by examining the asymptotic motion of the two-soliton solution the elastic property of the collision among two solitons is demonstrated. In [4], J. feng et. al. uses the modified variational iteration technique to obtain numerical solutions for extended HS-cKdV equation. A trustworthy technique of the New Iterative Method has been employed by M. Jibran et. al. [5] for determining the approximate solutions of the nonlinear interger order HS-cKdV and HS-cMKdV equations.

Over the last few decades, fractional differential equations(FDEs), have become prevalent in a variety of scientific domains. Many important phenomena in fluid mechanics, control theory, quantum mechanics, ocean engineering, non-linear optics, biology, economics, plasma physics and electrodynamics can be clarified more effectively by FDEs. As a result, there has been a lot of interest in the solutions of FDEs. Some time finding the solution to FDEs may be be difficult. Since many FDEs lack precise analytical solution, the intended outcomes must be obtained through the use of approximations and numerical methods. Consequently, efficient computational methods might be required for solving FDEs. The authors Podlubny [6], Baleanu [7], Miller and Ross [8] have been written the various books. These publications offer a detailed examination of fractional calculus, which may assist scholars in understanding the foundational concepts of the discipline. As a results, to solve FDEs, numerous scholars have developed several kinds of effective and trustworthy analytical and approximate techniques viz. fractional complex transform method [9], homotopy perturbation method [10], finite element method [11], Adomian decomposition method [12], Laplace transform method [13], homotopy analysis method [14] and differential transform method [15] etc. The study of FDEs involves the some operators viz, Riemann– Liouville [16],Atangana-Baleanu [17], Katugampola [18] , Caputo operator [19], Caputo-Fabrizio [20]and Hilfer [21] etc.

In [22] H. Ahmed has given a new way to solve the time-fractional cKdV equations with initial boundary conditions. The technique employs a collection of generalized shifted Jacobi polynomials that conform to the designated initial and boundary conditions, effectively transforming the cKdV system into a series of algebraic equations. Two useful methods, the generalized Kudryashov method and the two variables $(G'/G,\ 1/G)-$ expansion method, are introduced by H. Shahadat Ali [23] to construct various analytic wave solution in general and standard forms of the fractional HS- cKdV system. The solutions are converted into solitary waves after being developed in terms of simple functions. Ali Kurt et. al [24] have utilized two distinct approaches; the sub-equation method and the residual power series method to produce novel precise and approximate solutions of the generalized HS-cKdV equation with the conformable sense of the fractional derivative. R. Jena et. al. [25] solved the nonlinear time-fractional HS-cKdV and HS-cMKdV equations using the fractional reduced differential transform method without any discretization, perturbation, transformation or restrictive constraints.

In [26] M. H. Heydari et. al. have employed the Chebyshev–Gauss–Lobatto localization method with the Atangana–Baleanu fractional operator to study the variable-order fractional HS-cKdV system generated in the connection of long waves. A. H. Ganie et. al. [27] have applied two contemporary semi-analytic approaches, the Aboodh residual power series approach and the Aboodh transform iteration approach for resolving the fractional HS-cKdv equation. The generalized 2-dimensional differential transform approach [28] is suggested to solve the time-fractional HS-cKdV and HS-cMKdV systems via the addition of the fractional derivative in the context of Caputo by J. Liu et. al. In order to obtain approximate solutions for the time fractional generalized HS-cKdV system and HS-cMKdV system, A. Shaikh et. al. [29] has used the fractional complex transform approach. In addition to the Analysis of Bifurcation, Chaotic Behaviors and Sensitivity, the logistic technique is suggested by G. Yongyi [30] to produce analytical solutions for the HS-cKdV equation using the Beta fractional derivatives. R. Prajapati [31] has handled the nonlinear parts using He's polynomial and applied the homotopy perturbation method and Laplace transformation to find approximate solutions of the generalized HS-cKdV equation.

In this investigation, by using the Constant Proportional Caputo-Fabrizio differential operator the time fractional-order Hirota- Satsuma Coupled KdV Equation and Coupled MKdV Equation are approximated by the Iterative Laplace transform method. The time fractional generalized HS-cKdV equation is

$$\begin{aligned}
\frac{\partial^\sigma \varphi}{\partial t^\sigma} &= \frac{1}{2}\frac{\partial^3 \varphi}{\partial \xi^3} - 3\varphi\frac{\partial \varphi}{\partial \xi} + 3\psi\frac{\partial \phi}{\partial \xi} + 3\phi\frac{\partial \psi}{\partial \xi},\\
\frac{\partial^\sigma \psi}{\partial t^\sigma} &= -\frac{\partial^3 \psi}{\partial \xi^3} + 3\varphi\frac{\partial \psi}{\partial \xi},\\
\frac{\partial^\sigma \phi}{\partial t^\sigma} &= -\frac{\partial^3 \phi}{\partial \xi^3} + 3\varphi\frac{\partial \phi}{\partial \xi},\ t > 0,\ 0 < \sigma \le 1.
\end{aligned} \tag{1.1}$$

and the time fractional HS- cMKdV equation is

$$\begin{aligned}
\frac{\partial^\sigma \varphi}{\partial t^\sigma} &= \frac{1}{2}\frac{\partial^3 \varphi}{\partial \xi^3} - 3\varphi^2 \frac{\partial \varphi}{\partial \xi} + \frac{3}{2}\frac{\partial^2 \psi}{\partial \xi^2} + 3\varphi \frac{\partial \psi}{\partial \xi} + 3\psi \frac{\partial \varphi}{\partial \xi} - 3\lambda \frac{\partial \varphi}{\partial \xi}, \\
\frac{\partial^\sigma \psi}{\partial t^\sigma} &= -\frac{\partial^3 \psi}{\partial \xi^3} - 3\psi \frac{\partial \psi}{\partial \xi} - 3\frac{\partial \varphi}{\partial \xi}\frac{\partial \psi}{\partial \xi} + 3\varphi^2 \frac{\partial \varphi}{\partial \xi} + 3\lambda \frac{\partial \psi}{\partial \xi}, \qquad 0 < \sigma \leq 1.
\end{aligned} \tag{1.2}$$

The system of equations 1.1 and 1.2 becomes the classical HS-cKdV and HS- cMKdV equation for $\sigma = 1$ [32].

The following is the outline of the current article: Using important definitions, Section 2 introduces some of the basic concepts of fractional calculus. The discussion of the suggested method for the constant proportional Caputo-Fabrizio fractional derivative is the main objective of Section 3. Section 4 presents an explanation of the solution's existence. Section 5 shows the stability criteria for the approximate solution of the HS- cKdV and HS- cMKdV equation which has been derived. The numerical application of the method to the problem, graphs, tables and error estimation to numerically compare the derived approximate solution with previously known solutions is covered in Section 6. whereas the general conclusion and results are summed up in section 7.

2. Basic Definitions of the Fractional Calculus

We provide some essential definitions of Fractional Calculus in this part, which will be useful for further investigation.

Definition 2.1. [33] Let $0 < \sigma < 1$ be the order then Caputo-Fabrizio fractional integral operator is defined as

$$^{CF}I_t^\sigma u(\xi,t) = \frac{2(1-\sigma)}{(2-\sigma)\mathbb{M}(\sigma)}u(t) + \frac{2\sigma}{(2-\sigma)\mathbb{M}(\sigma)}\int_0^t u(\xi,\tau)d\tau \tag{2.1}$$

Definition 2.2. [33] For the order $0 < \sigma < 1$ with $u \in H^1(0, a)$, $a > 0$, the Caputo-Fabrizio fractional differential operator is defined as

$$^{CF}D_t^\sigma u(\xi,t) = \frac{(2-\sigma)\mathbb{M}(\sigma)}{2(1-\sigma)}\int_0^t exp\Big[-\frac{\sigma(t-s)}{1-\sigma}\Big]u'(\tau)d\tau,\ 0 < \sigma < 1,\ t \geq 0, \tag{2.2}$$

here, $\mathbb{M}(\sigma)$ act as a normalization function which depends on σ with $\mathbb{M}(0) = \mathbb{M}(1) = 1$.

Definition 2.3. [20] Let $0 < \sigma < 1$ be the fractional order and $n \in \mathbb{N}$ then the Laplace transform of the Caputo-Fabrizio fractional differential operator is defined as

$$\begin{aligned}
\mathcal{L}\Big(^{CF}D_t^{n+\sigma}u(\xi,t)\Big)(p) &= \frac{1}{1-\sigma}\mathcal{L}\Big(u^{(n+1)}(\xi,t)\Big)\mathcal{L}\Big(exp\Big(-\frac{\sigma}{1-\sigma}t\Big)\Big) \\
&= \frac{p^{n+1}\mathcal{L}(u(\xi,t)) - p^n u(\xi,0) - p^{n-1}u'(\xi,0) - \cdots - u^{(n)}(\xi,0)}{p+\sigma(1-p)}
\end{aligned} \tag{2.3}$$

Specifically, we get

$$\mathcal{L}\Big({}^{CF}D_t^{\sigma}u(\xi,t)\Big)(p) = \frac{p\mathcal{L}(u(\xi,t))}{p+\sigma(1-p)}, \ \ n=0.$$

$$\mathcal{L}\Big({}^{CF}D_t^{\sigma+1}u(\xi,t)\Big)(p) = \frac{p^2\mathcal{L}(u(\xi,t)) - p\,u(\xi,0) - u'(\xi,0)}{p+\sigma(1-p)}, \ \ n=1.$$

Definition 2.4. [34] Let $0 < \sigma < 1$ be the order then Constant proportional Caputo-Fabrizio fractional differential operator is defined as

$$^{CPCF}D_t^{\sigma}u(\xi,t) = \frac{\mathbb{M}(\sigma)}{(1-\sigma)}\int_0^t \Big(D_1(\sigma,\tau)u(t) + D_0(\sigma,\tau)u'(t)\Big)exp\Big[-\frac{\sigma}{1-\sigma}(t-\tau)\Big]d\tau, \ 0<\sigma<1. \quad (2.4)$$

Definition 2.5. [34] Let $0 < \sigma < 1$ be the fractional order then the Laplace transform of the Constant proportional Caputo-Fabrizio fractional differential operator is defined as

$$\mathcal{L}\Big({}^{CPCF}D_t^{\sigma}u(\xi,t)\Big)(p) = \left[\frac{\mathbb{M}(\sigma)D_1(\sigma)}{\sigma+p(1-\sigma)} + \frac{p\,\mathbb{M}(\sigma)D_0(\sigma)}{\sigma+p(1-\sigma)}\right]\mathcal{L}(u(\xi,t)) - \frac{\mathbb{M}(\sigma)D_0(\sigma)}{\sigma+p(1-\sigma)}u(0) \quad (2.5)$$

where $D_0(\sigma) = \sigma\omega^{1-\sigma}, D_1(\sigma) = (1-\sigma)(\omega)^{\sigma}$.

3. Analysis of the Iterative Laplace Transform Method

We have provided a description of the recommended approach in this section for the general FDE in $u(\xi,t)$ so that one can implement it to the other cases. A general non-homogeneous, non-linear Constant Proportional Caputo-Fabrizio time fractional differential equation is

$$^{CPCF}D_t^{\sigma}u(\xi,t) + \mathcal{V}(u(\xi,t)) + \mathcal{W}(u(\xi,t)) = f(\xi,t) \quad (3.1)$$

with the initial condition

$$u(\xi,0) = g(\xi,t) \quad (3.2)$$

here the function $f(\xi,t)$ denotes the source function, $\mathcal{V}$ and $\mathcal{W}$ are the linear and non-linear operators, respectively.

By applying the Laplace transform on the equation 3.1, we get

$$\mathcal{L}\{u(\xi,t)\} - \left(\frac{D_0(\sigma)}{D_1(\sigma)+p\,D_0(\sigma)}\right)u(\xi,0) + \left(\frac{\sigma+p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma)+p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)\Big(\mathcal{L}\{\mathcal{V}(u(\xi,t))\} + \mathcal{L}\{\mathcal{W}(u(\xi,t))\} - \mathcal{L}\{f(\xi,t)\}\Big) = 0.$$

making rearrangement of the terms, we obtain

$$\mathcal{L}\{u(\xi,t)\} = \frac{1}{p+\dfrac{D_1(\sigma)}{D_0(\sigma)}}g(\xi,t) - \left(\frac{\sigma+p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma)+p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)\Big(\mathcal{L}\{\mathcal{V}(u(x\xi,t)\} + \mathcal{L}\{\mathcal{W}(u(\xi,t)\} - \mathcal{L}\{f(\xi,t)\}\Big)$$

$$= g(\xi,p) - \left(\frac{\sigma+p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma)+p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)\mathcal{L}\Big(\mathcal{V}(u(\xi,t)) + \mathcal{W}(u(\xi,t))\Big) \quad (3.3)$$

where, the term $g(\xi,p)$ is

$$g(\xi,p) = \frac{1}{p+\frac{D_1(\sigma)}{D_0(\sigma)}} g(\xi,t) - \left(\frac{\sigma+p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma)+p\,\mathbb{M}(\sigma)D_0(\sigma)}\right) f(\xi,p).$$

Now applying the inverse Laplace transform on both sides of the equation 3.3, we get

$$u(\xi,t) = exp\left(-\frac{D_1(\sigma)}{D_0(\sigma)}t\right) g(\xi,t) - \mathcal{L}^{-1}\left[\left(\frac{\sigma+p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma)+p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)\mathcal{L}\Big(\mathcal{V}(u(\xi,t)) + \mathcal{W}(u(\xi,t))\Big)\right] \tag{3.4}$$

Further, the New iterative method(NIM) which was formulated by Daftardar-Gejji and Jafari [35] is used to find the infinite series solution.
Let

$$u(\xi,t) = \sum_{n=0}^{\infty} u_n(\xi,t), \tag{3.5}$$

as, $\mathcal{V}$ represents a linear operator,

$$\mathcal{V}\left(\sum_{n=0}^{\infty} u_n(\xi,t)\right) = \sum_{n=0}^{\infty} \mathcal{V}\Big(u_n(\xi,t)\Big) \tag{3.6}$$

The non-linear operator $\mathcal{W}$ is expressed in a decomposed form as

$$\mathcal{W}\left(\sum_{n=0}^{\infty} u_n\right) = \mathcal{W}(u_0(\xi,t)) + \sum_{n=1}^{\infty}\left\{\mathcal{W}\left(\sum_{j=0}^{n} u_j(\xi,t)\right) - \mathcal{W}\left(\sum_{j=0}^{n-1} u_j(\xi,t)\right)\right\} \tag{3.7}$$

By the use of equations 3.5, 3.6 and 3.7, the equation 3.4 becomes

$$\begin{aligned}\sum_{i=0}^{\infty} u_i(\xi,t) = exp\left(-\frac{D_1(\sigma)}{D_0(\sigma)}t\right) g(\xi,t) &- \mathcal{L}^{-1}\left[\left(\frac{\sigma+p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma)+p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)\mathcal{L}\left(\sum_{i=0}^{\infty}\mathcal{V}\Big(u_i(\xi,t)\Big)\right)\right] \\ &- \mathcal{L}^{-1}\left[\left(\frac{\sigma+p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma)+p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)\mathcal{L}\left(\mathcal{W}\Big(u_0(\xi,t)\Big) + \sum_{n=1}^{\infty}\left\{\mathcal{W}\left(\sum_{j=0}^{n} u_j(\xi,t)\right)\right.\right.\right. \\ &\left.\left.\left.- \mathcal{W}\left(\sum_{j=0}^{n-1} u_j(\xi,t)\right)\right\}\right)\right]\end{aligned} \tag{3.8}$$

Further, by defining the recurssive formula as

$$\begin{aligned} u_0(\xi,t) &= g(\xi,t) \\ u_1(\xi,t) &= \mathcal{L}^{-1}\left[\left(\frac{\sigma+p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma)+p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)\mathcal{L}\left(\mathcal{V}\Big(u_0(\xi,t)\Big) + \mathcal{W}\Big(u_0(\xi,t)\Big)\right)\right] \\ &\vdots \\ u_{k+1}(\xi,t) &= \mathcal{L}^{-1}\left[\left(\frac{\sigma+p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma)+p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)\mathcal{L}\left(\mathcal{V}\Big(u_k(\xi,t)\Big) + \left\{\mathcal{W}\left(\sum_{j=0}^{k} u_j(\xi,t)\right)\right.\right.\right.\end{aligned}$$

$$-\mathcal{W}\left(\sum_{j=0}^{k-1} u_j(\xi,t)\right)\Big\}\Big)\Big] \tag{3.9}$$

The $k-$term approximate solution is given as

$$u(x,t) = u_0(\xi,t) + u_1(\xi,t) + u_2(\xi,t) + \cdots + u_{k-1}(\xi,t).$$

The condition for convergence of the above approximate solution obtained by NIM is discussed in [36].

4. Existence of a Solution

Here, by the framework of the Banach contraction principle [37], we have address the criteria that guarantee the existence of unique solution to the equation 1.1

- (M_1) : Consider the function $\Phi : (X \times \mathbb{R}) \to \mathbb{R}$ with $\Phi \in C^{b(1-a)}_{(1-\beta)}[X,\mathbb{R}]$ for any $t \in C^{\beta}_{1-\beta}[X,\mathbb{R}]$.
- (M_2) : for any $y, \bar{y} \in \mathbb{R}$ and $t \in X$ $\exists$ a constant $\Lambda > 0$ as $|\Phi(t,y) - \Phi(t,\bar{y}) \leq \Lambda|y-\bar{y}|$
- (M_3) Suppose that $\Lambda\epsilon < 1$, where,
$$\epsilon = \frac{B(\beta,a)}{q^a\Gamma(a)}\left(|\Omega|\sum_{i=1}^{r} k_i\,(\eta_i - c)^{a+\beta-1} + (\mu - c)^a\right).$$
where,$B(\beta,a)$ denotes the Beta function.

Theorem 4.1. Let $0 < a < 1$, $0 \leq b \leq 1$ and $\beta = a + b - ab$. Suppose that the assumptions (M_1), (M_2) and (M_3) are met. Then problem in equation 1.1 admits the unique solution in the space $C^{\beta}_{1-\beta}[X,\mathbb{R}]$.

Proof: Define the operator $\mathcal{R} : C_{1-\beta}[X,\mathbb{R}] \to C_{1-\beta}[X,\mathbb{R}]$ by

$$\mathcal{R}[\varphi(t)] = \frac{\Delta}{s^a}\Gamma(a)e^{\frac{s-1}{s}(t-c)}(t-c)^{\beta-1}\sum_{i=1}^{r} k_i\int_a^{\eta_i} e^{\frac{s-1}{s}(\eta_i - z)}(\eta_i - z)^{a-1}\Phi(z,\varphi(z))\,dz$$
$$+\frac{1}{s^a\Gamma(a)}\int_c^t e^{\frac{s-1}{s}(t-z)}(t-z)^{a-1}\Phi(z,\varphi(z))\,dz$$

It ensures, $\mathcal{R}$ is well-defined. For any $\varphi_1, \varphi_2 \in C_{1-\beta}[X,\mathbb{R}]$ and $t \in X$,

$$\Big|\{(\mathcal{R}\varphi_1)(t) - (\mathcal{R}\varphi_2)(t)\}(t-c)^{1-\beta}\Big| \leq \frac{|\Delta|}{s^a\Gamma(a)}|e^{\frac{sr-1}{s}(t-c)}|\sum_{i=1}^{r} d_i\int_c^{\eta_i} |e^{\frac{s-1}{s}(\eta_i - z)}|(\eta_i - z)^{a-1}|\Phi(z,\varphi_1(z))$$
$$-\Phi(z,\varphi_2(z))|\,dz + \frac{1}{s^a\Gamma(a)}\int_c^t |e^{\frac{s-1}{s}(t-z)}|(t-z)^{a-1}|\Phi(z,\varphi_1(z))$$
$$-\Phi(z,\varphi_2(z))|\,dz$$

As, $|e^{\frac{(s-1)t}{s}}| < 1$ we obtain,

$$\left|\{(\mathcal{R}\varphi_1)(t) - (\mathcal{R}\varphi_2)(t)\}(t-c)^{1-\beta}\right| \leq \frac{\epsilon|\Delta|}{q^a\Gamma(a)}\left(\sum_{i=1}^{r} d_i \int_{c^+}^{\eta_i} (\eta_i - z)^{a-1}(z-c)^{\beta-1}\, dz\right) \parallel \varphi_1 - \varphi_2 \parallel$$

$$C_{1-\beta}\,[M, \mathbb{R}] + \frac{\epsilon}{s^a\Gamma(a)}(t-c)^{1-\beta}\left(\int_{c^+}^{t} (t-z)^{a-1}(z-c)^{\beta-1}\, dz\right)$$

$$\parallel \varphi_1 - \varphi_2 \parallel C_{1-\beta}\,[X, \mathbb{R}]$$

$$\left|\{(\mathcal{R}\varphi_1)(t) - (\mathcal{R}\varphi_2)(t)\}(t-c)^{1-\beta}\right| \leq \frac{\epsilon|\Delta|}{s^a\Gamma(a)} B(\beta, a) \sum_{i=1}^{r} k_i(\eta_i - c)^{a+\beta-1} \parallel \varphi_1 - \varphi_2 \parallel C_{1-\beta}\,[X, \mathbb{R}]$$

$$+ \frac{\epsilon}{s^a\Gamma(a)}(\mathcal{R} - c)^a B(\beta, a) \parallel \varphi_1 - \varphi_2 \parallel C_{1-\beta}\,[X, \mathbb{R}]$$

Therefore,

$$\parallel \{(\mathcal{R}\varphi_1) - (\mathcal{R}\varphi_2)\} \parallel_{C_{1-\beta}[M,\mathbb{R}]} \leq \frac{\epsilon}{s^a\Gamma(a)} B(\beta, a)\left(|\Delta| \sum_{i=1}^{r} k_i(\eta_i - c)^{a+\beta-1} + (F-c)^a\right) \parallel \varphi_1 - \varphi_2 \parallel C_{1-\beta}\,[X, \mathbb{R}]$$

$$\leq \Lambda\epsilon \parallel \varphi_1 - \varphi_2 \parallel C_{1-\beta}\,[X, \mathbb{R}]$$

This leads to $\mathcal{R}$ being a contraction map. Thus, the Banach contraction principle directly leads to the unique solution $\varphi(\xi, t)$ of equation 1.1. In the same way, we can show the existence of unique $\psi(\xi, t)$, $\phi(\xi, t)$ for equation 1.1 and the existence of unique $\varphi(\xi, t)$, $\psi(\xi, t)$ for equation 1.2.

5. Stability Analysis

This section is mainly associated with the stability analysis of the approximate solution of the time fractional HS- cKdV equation obtained by ILTM-CPCF.

Define T as a self-map of $\mathcal{B}$ in the Banach space $(\mathcal{B}, \|\cdot\|)$. Further, $\zeta_{n+1} = p(T, \zeta_n)$ indicates the exact recurssive process. For a self-map T fixed point set is $\mathbb{U}(T)$. Furthermore, for ζ_n to converge to $s \in \mathbb{U}(T)$, the T must include at least one element. Define $z_n = \|j_{n+1} - p(T, j_n)\|$ with $\{j_n\} \subseteq \mathcal{B}$. The iterative method $\zeta_{n+1} = p(T, \zeta_n)$ is referred to as T-stable if $\lim_{n\to\infty} z^n = 0$ yields $\lim_{n\to\infty} j^n = s$. There is an upper limit for the sequence $\{j_n\}$. This Picard's iteration scheme is known as $T-$ stable if all the conditions for $\zeta_{n+1} = T\zeta_n$ are met.

Theorem 5.1. Let $(\mathcal{B}, \|\cdot\|)$ be a Banach space with the self-map T defined on the $\mathcal{B}$ and satisfying

$$\|T_m - T_n\| \leq \Delta\|m - T_m\| + \epsilon\|m - n\|$$

for all $m,\ n \in \mathcal{B}$, where $0 \leq \epsilon < 1, 0 \leq \Delta$. Assuming Picard T-stability for T. Then the following equations is related to equation 1.1

$$\varphi_{n+1}(\xi,t) = \varphi_n(\xi,t) + \mathcal{L}^{-1}\left[\left(\frac{\sigma + p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma) + p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)\mathcal{L}\left(\frac{1}{2}\frac{\partial^3\varphi_n}{\partial\xi^3} - 3\varphi_n\frac{\partial\varphi_n}{\partial\xi} + 3\psi_n\frac{\partial\phi_n}{\partial\xi} + 3\phi_n\frac{\partial\psi_n}{\partial\xi}\right)\right]$$

$$\psi_{n+1}(\xi,t) = \psi_n(\xi,t) + \mathcal{L}^{-1}\left[\left(\frac{\sigma + p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma) + p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)\mathcal{L}\left(-\frac{\partial^3\psi_n}{\partial\xi^3} + 3\varphi_n\frac{\partial\psi_n}{\partial\xi}\right)\right]$$

$$\phi_{n+1}(\xi,t) = \phi_n(\xi,t) + \mathcal{L}^{-1}\left[\left(\frac{\sigma + p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma) + p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)\mathcal{L}\left(-\frac{\partial^3\phi_n}{\partial\xi^3} + 3\varphi_n\frac{\partial\phi_n}{\partial\xi}\right)\right]$$

where $\dfrac{\sigma + p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma) + p\,\mathbb{M}(\sigma)D_0(\sigma)}$ is denoting the fractional Lagrange multiplier.

Theorem 5.2. Let T as a self-map given by

$$T(\varphi_n(\xi,t)) = \varphi_{n+1}(\xi,t) = \varphi_n(x,t) + \mathcal{L}^{-1}\left[\left(\frac{\sigma + p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma) + p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)\mathcal{L}\left(\frac{1}{2}\frac{\partial^3\varphi_n}{\partial\xi^3} - 3\varphi_n\frac{\partial\varphi_n}{\partial\xi} + 3\psi_n\frac{\partial\phi_n}{\partial\xi} + 3\phi_n\frac{\partial\psi_n}{\partial\xi}\right)\right]$$

$$T(\psi_n(\xi,t)) = \psi_{n+1}(\xi,t) = \psi_n(x,t) + \mathcal{L}^{-1}\left[\left(\frac{\sigma + p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma) + p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)\mathcal{L}\left(-\frac{\partial^3\psi_n}{\partial\xi^3} + 3\varphi_n\frac{\partial\psi_n}{\partial\xi}\right)\right]$$

$$T(\phi_n(\xi,t)) = \phi_{n+1}(\xi,t) = \phi_n(x,t) + \mathcal{L}^{-1}\left[\left(\frac{\sigma + p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma) + p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)\mathcal{L}\left(-\frac{\partial^3\phi_n}{\partial\xi^3} + 3\varphi_n\frac{\partial\phi_n}{\partial\xi}\right)\right]$$

is $T-$stable in $L^2(m,n)$ if

$$\begin{cases} \Big\{1 + \frac{1}{2}K_1\Pi_1(\sigma) - 3a_1\tau_1\Pi_2(\sigma) - 3a_4\Pi_3(\sigma) + 3a_2\tau_3\Pi_4(\sigma) \\ \quad + 3a_6\Pi_5(\sigma) + 3a_5\Pi_6(\sigma) + 3a_3\tau_2\Pi_7(\sigma)\Big\} < 1, \\ \Big\{1 - K_2\Pi_8(\sigma) + 3a_5\Pi_9(\sigma) + 3a_1\tau_2\Pi_{10}\Big\} < 1, \\ \Big\{1 - K_3\Pi_{11}(\sigma) + 3a_6\Pi_{12} + 3a_1\tau_3\Pi_{13}(\sigma)\Big\} < 1. \end{cases}$$

Proof: Our motive is to show that the map T has a fixed point. For every $(m,\ n) \in \mathbb{N}\times\mathbb{N}$, We consider,

$$\begin{aligned} T(\varphi_m(\xi,t)) - T(\varphi_n(\xi,t)) = \varphi_m(\xi,t) - \varphi_n(\xi,t) &+ \mathcal{L}^{-1}\left[\left(\frac{\sigma + p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma) + p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)\right. \\ &\left.\mathcal{L}\left(\frac{1}{2}\frac{\partial^3\varphi_m}{\partial\xi^3} - 3\varphi_m\frac{\partial\varphi_m}{\partial\xi} + 3\psi_m\frac{\partial\phi_m}{\partial\xi} + 3\phi_m\frac{\partial\psi_m}{\partial\xi}\right)\right] \\ &- \mathcal{L}^{-1}\left[\left(\frac{\sigma + p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma) + p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)\right. \\ &\left.\mathcal{L}\left(\frac{1}{2}\frac{\partial^3\varphi_n}{\partial\xi^3} - 3\varphi_n\frac{\partial\varphi_n}{\partial\xi} + 3\psi_n\frac{\partial\phi_n}{\partial\xi} + 3\phi_n\frac{\partial\psi_n}{\partial\xi}\right)\right] \end{aligned} \tag{5.1}$$

by applying the norm on both sides, we obtain

$$\Big\|T(\varphi_m(\xi,t)) - T(\varphi_n(\xi,t))\Big\| = \Bigg\|\varphi_m(\xi,t) - \varphi_n(\xi,t) + \mathcal{L}^{-1}\Bigg[\left(\frac{\sigma + p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma) + p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)$$
$$\mathcal{L}\left(\frac{1}{2}\frac{\partial^3\varphi_m}{\partial\xi^3} - 3\varphi_m\frac{\partial\varphi_m}{\partial\xi} + 3\psi_m\frac{\partial\phi_m}{\partial\xi} + 3\phi_m\frac{\partial\psi_m}{\partial\xi}\right)\Bigg]$$
$$- \mathcal{L}^{-1}\Bigg[\left(\frac{\sigma + p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma) + p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)$$
$$\mathcal{L}\left(\frac{1}{2}\frac{\partial^3\varphi_n}{\partial\xi^3} - 3\varphi_n\frac{\partial\varphi_n}{\partial\xi} + 3\psi_n\frac{\partial\phi_n}{\partial\xi} + 3\phi_n\frac{\partial\psi_n}{\partial\xi}\right)\Bigg]\Bigg\| \tag{5.2}$$

by using the triangle inequality, we get

$$\Big\|T(\varphi_m(\xi,t)) - T(\varphi_n(\xi,t))\Big\| \le \|\varphi_m(\xi,t) - \varphi_n(\xi,t)\| + \mathcal{L}^{-1}\Bigg\{\left(\frac{\sigma + p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma) + p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)$$
$$\Bigg(\mathcal{L}\Bigg[\left\|\frac{1}{2}\left(\frac{\partial^3\varphi_m}{\partial\xi^3} - \frac{\partial^3\varphi_n}{\partial\xi^3}\right)\right\| + \left\|-3\varphi_n\left(\frac{\partial\varphi_m}{\partial\xi} - \frac{\partial\varphi_n}{\partial\xi}\right)\right\| + \left\|-3\frac{\partial\varphi_m}{\partial\xi}\Big(\varphi_m - \varphi_n\Big)\right\|$$
$$+ \left\|3\psi_m\left(\frac{\partial\phi_m}{\partial\xi} - \frac{\partial\phi_n}{\partial\xi}\right)\right\| + \left\|3\frac{\partial\phi_n}{\partial\xi}\Big(\psi_m - \psi_n\Big)\right\| + \left\|3\frac{\partial\psi_m}{\partial\xi}\Big(\phi_m - \phi_n\Big)\right\|$$
$$+ \left\|3\phi_n\left(\frac{\partial\psi_m}{\partial\xi} - \frac{\partial\psi_n}{\partial\xi}\right)\right\|\Bigg]\Bigg)\Bigg\} \tag{5.3}$$

since all the solutions play the same part, we shall assume that

$$\begin{aligned}
&\|\varphi_m - \varphi_n\| = \|\psi_m - \psi_n\|, && \|\varphi_m - \varphi_n\| = \|\phi_m - \phi_n\|,\\
&\left\|\tfrac{\partial\varphi_m}{\partial\xi} - \tfrac{\partial\varphi_n}{\partial\xi}\right\| = \tau_1\|\varphi_m - \varphi_n\|, && \left\|\tfrac{\partial\psi_m}{\partial\xi} - \tfrac{\partial\psi_n}{\partial\xi}\right\| = \tau_2\|\psi_m - \psi_n\|,\\
&\left\|\tfrac{\partial\phi_m}{\partial\xi} - \tfrac{\partial\phi_n}{\partial\xi}\right\| = \tau_3\|\phi_m - \phi_n\|, && \left\|\tfrac{\partial^3\varphi_m}{\partial\xi^3} - \tfrac{\partial^3\varphi_n}{\partial\xi^3}\right\| = K_1\|\varphi_m - \varphi_n\|.
\end{aligned} \tag{5.4}$$

replacing the terms using Eq. (5.4), we get

$$\Big\|T(\varphi_m(\xi,t)) - T(\varphi_n(\xi,t))\Big\| \le \|\varphi_m(\xi,t) - \varphi_n(\xi,t)\| + \mathcal{L}^{-1}\Bigg\{\left(\frac{\sigma + p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma) + p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)$$
$$\Bigg(\mathcal{L}\Bigg[\left\|\frac{1}{2}K_1\Big(\varphi_m - \varphi_n\Big)\right\| + \left\|-3\varphi_n\tau_1\Big(\varphi_m - \varphi_n\Big)\right\| + \left\|-3\frac{\partial\varphi_m}{\partial\xi}\Big(\varphi_m - \varphi_n\Big)\right\|$$
$$+ \left\|3\psi_m\tau_3\Big(\varphi_m - \varphi_n\Big)\right\| + \left\|3\frac{\partial\phi_m}{\partial\xi}\Big(\varphi_m - \varphi_n\Big)\right\| + \left\|3\frac{\partial\psi_m}{\partial\xi}\Big(\varphi_m - \varphi_n\Big)\right\|$$

$$\left\| 3\phi_m \tau_2 \left(\varphi_n - \varphi_n \right) \right\| \Big] \Big) \Big\} \tag{5.5}$$

here, $\varphi_n, \psi_m, \phi_m, \dfrac{\partial \varphi_m}{\partial \xi}, \dfrac{\partial \psi_m}{\partial \xi}, \dfrac{\partial \phi_m}{\partial \xi}$ are the convergent solution sequence hence; they are bounded. We consider different positive constants a_1, a_2, a_3, a_4, a_5 and a_6 such that

$$\|\varphi_n\| < a_1,\ \|\psi_m\| < a_2,\ \|\phi_m\| < a_3,\ \|\frac{\partial \varphi_m}{\partial \xi}\| < a_4,\ \|\frac{\partial \psi_m}{\partial \xi}\| < a_5,\ \|\frac{\partial \phi_m}{\partial \xi}\| < a_6.$$

we get,

$$\begin{aligned} \left\| T(\varphi_m(\xi,t)) - T(\varphi_n(\xi,t)) \right\| \le \Big\{ & 1 + \frac{1}{2} K_1 \Pi_1(\sigma) - 3a_1 \tau_1 \Pi_2(\sigma) - 3a_4 \Pi_3(\sigma) + 3a_2 \tau_3 \Pi_4(\sigma) \\ & + 3a_6 \Pi_5(\sigma) + 3a_5 \Pi_6(\sigma) + 3a_3 \tau_2 \Pi_7(\sigma) \Big\} \|\varphi_m - \varphi_n\| \end{aligned} \tag{5.6}$$

In the same manner we get,

$$\begin{aligned} \left\| T(\psi_m(\xi,t)) - T(\psi_n(\xi,t)) \right\| &\le \Big\{ 1 - K_2 \Pi_8(\sigma) + 3a_5 \Pi_9(\sigma) + 3a_1 \tau_2 \Pi_{10} \Big\} \|\psi_m - \psi_n\| \\ \left\| T(\phi_m(\xi,t)) - T(\phi_n(\xi,t)) \right\| &\le \Big\{ 1 - K_3 \Pi_{11}(\sigma) + 3a_6 \Pi_{12} + 3a_1 \tau_3 \Pi_{13}(\sigma) \Big\} \|\phi_m - \phi_n\| \end{aligned} \tag{5.7}$$

where the functions Π_1, Π_2, Π_3, Π_4, Π_5, Π_6, Π_7, Π_8, Π_9, Π_{10}, Π_{11}, Π_{12} and Π_{13} are the functions of $\mathcal{L}^{-1}\left\{ \left(\dfrac{\sigma + p(1-\sigma)}{\mathbb{M}(\sigma) D_1(\sigma) + p\, \mathbb{M}(\sigma) D_0(\sigma)} \right) \mathcal{L} \right\}$.

This proves that the self-mapping T has a fixed point. Further, we prove that T satisfies all the criteria given in Theorem 5.1

Let equation 5.6 and 5.7 holds with

$$\epsilon = 0, \quad \Delta = \begin{cases} \Big\{ 1 + \frac{1}{2} K_1 \Pi_1(\sigma) - 3a_1 \tau_1 \Pi_2(\sigma) - 3a_4 \Pi_3(\sigma) + 3a_2 \tau_3 \Pi_4(\sigma) \\ \quad + 3a_6 \Pi_5(\sigma) + 3a_5 \Pi_6(\sigma) + 3a_3 \tau_2 \Pi_7(\sigma) \Big\} < 1, \\ \Big\{ 1 - K_2 \Pi_8(\sigma) + 3a_5 \Pi_9(\sigma) + 3a_1 \tau_2 \Pi_{10} \Big\} < 1, \\ \Big\{ 1 - K_3 \Pi_{11}(\sigma) + 3a_6 \Pi_{12} + 3a_1 \tau_3 \Pi_{13}(\sigma) \Big\} < 1 \end{cases}$$

Thus, T satisfies all the conditions given in Theorem 5.2 Therefore, T is Picard $T-$ stable.

Similarly one can show, the stability results for the approximate solution of the time fractional HS- cMKdV equation obtained by ILTM-CPCF.

6. Numerical Simulation

The principle objective of this section is to derive approximate solutions of the considered HS- cKdV and HS- cMKdV equation with the CPCF fractional derivative. The second one is to do numerical computations and graphical simulations of the derived approximate solution.

The Mathematica software has been used for computations and simulations of the approximate solutions.

6.1 Approximate solution of the time fractional HS- cKdv equation:

Consider the time fractional order HS- cKdV equation given in equation 1.1 with the initial conditions as

$$\begin{aligned}
\varphi(\xi,0) &= \frac{\alpha-2\delta^2}{3}+2\delta^2\tanh^2(\delta\xi),\\
\psi(\xi,0) &= -\frac{4\delta^2c_0(\alpha+\delta^2)}{3c_1^2}+\frac{4\delta^2(\alpha+\delta^2)\tanh(\delta\xi)}{3c_1},\\
\phi(\xi,0) &= c_0+c_1\tanh(\delta\xi)
\end{aligned} \tag{6.1}$$

where, $\delta, c_0, c_1 \neq 0$ and α are real constants.

The exact solution of the equation 1.1 with $c=-\alpha^2$ is

$$\begin{aligned}
\varphi(\xi,t) &= \frac{\alpha-2\delta^2}{3}+2\delta^2\tanh^2(\delta(\xi-ct)),\\
\psi(\xi,t) &= -\frac{4\delta^2c_0(\alpha+\delta^2)}{3c_1^2}+\frac{4\delta^2(\alpha+\delta^2)\tanh(\delta(\xi-ct))}{3c_1},\\
\phi(\xi,t) &= c_0+c_1\tanh(\delta(\xi-ct))
\end{aligned} \tag{6.2}$$

By applying the Laplace Transform to both sides of equation 1.1 , yields

$$\begin{aligned}
\mathcal{L}\{\varphi(\xi,t)\}-\left(\frac{D_0(\sigma)}{D_1(\sigma)+p\,D_0(\sigma)}\right)\varphi(\xi,0) &= \left(\frac{\sigma+p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma)+p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)\mathcal{L}\bigg\{\frac{1}{2}\frac{\partial^3\varphi}{\partial\xi^3}-3\varphi\frac{\partial\varphi}{\partial\xi}\\
&\qquad +3\psi\frac{\partial\phi}{\partial\xi}+3\phi\frac{\partial\psi}{\partial\xi}\bigg\}\\
\mathcal{L}\{\psi(\xi,t)\}-\left(\frac{D_0(\sigma)}{D_1(\sigma)+p\,D_0(\sigma)}\right)\psi(\xi,0) &= \left(\frac{\sigma+p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma)+p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)\mathcal{L}\bigg\{-\frac{\partial^3\psi}{\partial\xi^3}+3\varphi\frac{\partial\psi}{\partial\xi}\bigg\}\\
\mathcal{L}\{\phi(\xi,t)\}-\left(\frac{D_0(\sigma)}{D_1(\sigma)+p\,D_0(\sigma)}\right)\phi(\xi,0) &= \left(\frac{\sigma+p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma)+p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)\mathcal{L}\bigg\{-\frac{\partial^3\phi}{\partial\xi^3}+3\varphi\frac{\partial\phi}{\partial\xi}\bigg\}
\end{aligned} \tag{6.3}$$

Rearranging the terms, we get

$$\begin{aligned}
\mathcal{L}\{\varphi(\xi,t)\} &= \frac{1}{p+\frac{D_1(\sigma)}{D_0(\sigma)}}\varphi(\xi,0)+\left(\frac{\sigma+p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma)+p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)\mathcal{L}\bigg\{\frac{1}{2}\frac{\partial^3\varphi}{\partial\xi^3}-3\varphi\frac{\partial\varphi}{\partial\xi}\\
&\qquad +3\psi\frac{\partial\phi}{\partial\xi}+3\phi\frac{\partial\psi}{\partial\xi}\bigg\}\\
\mathcal{L}\{\psi(\xi,t)\} &= \frac{1}{p+\frac{D_1(\sigma)}{D_0(\sigma)}}\psi(\xi,0)+\left(\frac{\sigma+p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma)+p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)\mathcal{L}\bigg\{-\frac{\partial^3\psi}{\partial\xi^3}+3\varphi\frac{\partial\psi}{\partial\xi}\bigg\}\\
\mathcal{L}\{\phi(\xi,t)\} &= \frac{1}{p+\frac{D_1(\sigma)}{D_0(\sigma)}}\phi(\xi,0)+\left(\frac{\sigma+p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma)+p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)\mathcal{L}\bigg\{-\frac{\partial^3\phi}{\partial\xi^3}+3\varphi\frac{\partial\phi}{\partial\xi}\bigg\}
\end{aligned} \tag{6.4}$$

Now, using the Inverse Laplace transform to both sides of equation 6.4, we can obtain

$$\begin{aligned}\varphi(\xi,t) &= exp\Big(-\frac{D_1(\sigma)}{D_0(\sigma)}t\Big)\varphi(\xi,0) + \mathcal{L}^{-1}\Big\{\Big(\frac{\sigma+p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma)+p\,\mathbb{M}(\sigma)D_0(\sigma)}\Big)\mathcal{L}\Big\{\frac{1}{2}\frac{\partial^3\varphi}{\partial\xi^3} - 3\varphi\frac{\partial\varphi}{\partial\xi} \\ &\qquad + 3\psi\frac{\partial\phi}{\partial\xi} + 3\phi\frac{\partial\psi}{\partial\xi}\Big\}\Big\} \\ \psi(\xi,t) &= exp\Big(-\frac{D_1(\sigma)}{D_0(\sigma)}t\Big)\psi(\xi,0) + \mathcal{L}^{-1}\Big\{\Big(\frac{\sigma+p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma)+p\,\mathbb{M}(\sigma)D_0(\sigma)}\Big)\mathcal{L}\Big\{-\frac{\partial^3\psi}{\partial\xi^3} + 3\varphi\frac{\partial\psi}{\partial\xi}\Big\}\Big\} \\ \phi(\xi,t) &= exp\Big(-\frac{D_1(\sigma)}{D_0(\sigma)}t\Big)\phi(\xi,0) + \mathcal{L}^{-1}\Big\{\Big(\frac{\sigma+p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma)+p\,\mathbb{M}(\sigma)D_0(\sigma)}\Big)\mathcal{L}\Big\{-\frac{\partial^3\phi}{\partial\xi^3} + 3\varphi\frac{\partial\phi}{\partial\xi}\Big\}\Big\}\end{aligned} \tag{6.5}$$

Let the series form of the approximate solution is

$$\begin{aligned}\varphi(\xi,t) &= \sum_{n=0}^{\infty}\varphi_n(\xi,t), \\ \psi(\xi,t) &= \sum_{n=0}^{\infty}\psi_n(\xi,t), \\ \phi(\xi,t) &= \sum_{n=0}^{\infty}\phi_n(\xi,t)\end{aligned} \tag{6.6}$$

By the use of initial conditions as $\varphi_0(\xi,\ t) = \varphi(\xi,0)$, $\psi_0(\xi,\ t) = \psi(\xi,0)$ and $\phi_0(\xi,\ t) = \phi(\xi,0)$ resulting recursive relation is as follows

$$\begin{aligned}\varphi_{n+1}(\xi,t) &= exp\Big(-\frac{D_1(\sigma)}{D_0(\sigma)}t\Big)\varphi_0(\xi,\ t) + \mathcal{L}^{-1}\Big\{\Big(\frac{\sigma+p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma)+p\,\mathbb{M}(\sigma)D_0(\sigma)}\Big)\mathcal{L}\Big\{\frac{1}{2}\frac{\partial^3\varphi_n}{\partial\xi^3} \\ &\quad -3\Big(\sum_{i=0}^{n}\varphi_i\frac{\partial\varphi_i}{\partial\xi} - \sum_{i=0}^{(n-1)}\varphi_i\frac{\partial\varphi_i}{\partial\xi}\Big) + 3\Big(\sum_{i=0}^{n}\psi_i\frac{\partial\phi_i}{\partial\xi} - \sum_{i=0}^{(n-1)}\psi_i\frac{\partial\phi_i}{\partial\xi}\Big) + 3\Big(\sum_{i=0}^{n}\phi_i\frac{\partial\psi_i}{\partial\xi} - \sum_{i=0}^{(n-1)}\phi_i\frac{\partial\psi_i}{\partial\xi}\Big)\Big\}\Big\} \\ \psi_{n+1}(\xi,t) &= exp\Big(-\frac{D_1(\sigma)}{D_0(\sigma)}t\Big)\psi_0(\xi,t) + \mathcal{L}^{-1}\Big\{\Big(\frac{\sigma+p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma)+p\,\mathbb{M}(\sigma)D_0(\sigma)}\Big)\mathcal{L}\Big\{-\frac{\partial^3\psi_n}{\partial\xi^3} \\ &\quad + 3\Big(\sum_{i=0}^{n}\varphi_i\frac{\partial\psi_i}{\partial\xi} - \sum_{i=0}^{(n-1)}\varphi_i\frac{\partial\psi_i}{\partial\xi}\Big)\Big\}\Big\} \\ \phi_{n+1}(\xi,t) &= exp\Big(-\frac{D_1(\sigma)}{D_0(\sigma)}t\Big)\phi_0(\xi,t) = \mathcal{L}^{-1}\Big\{\Big(\frac{\sigma+p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma)+p\,\mathbb{M}(\sigma)D_0(\sigma)}\Big)\mathcal{L}\Big\{-\frac{\partial^3\phi_n}{\partial\xi^3} \\ &\quad + 3\Big(\sum_{i=0}^{n}\varphi_i\frac{\partial\phi_i}{\partial\xi} - \sum_{i=0}^{(n-1)}\varphi_i\frac{\partial\phi_i}{\partial\xi}\Big)\Big\}\Big\}\end{aligned} \tag{6.7}$$

The approximate solution with the $k-$ terms is

$$\begin{aligned}\varphi(x,t) &= \varphi_0(\xi,t) + \varphi_1(\xi,t) + \varphi_2(\xi,t) + \cdots + \varphi_{k-1}(\xi,t), \\ \psi(x,t) &= \psi_0(\xi,t) + \psi_1(\xi,t) + \psi_2(\xi,t) + \cdots + \psi_{k-1}(\xi,t), \\ \phi(x,t) &= \phi_0(\xi,t) + \phi_1(\xi,t) + \phi_2(\xi,t) + \cdots + \phi_{k-1}(\xi,t).\end{aligned} \tag{6.8}$$

Hence, by using equation 6.7 first few terms of the series solution of equation 1.1 are as:

$$\varphi_0(\xi,t) = \frac{\alpha - 2\delta^2}{3} + \frac{2\delta^2\left(e^{2\delta\xi}-1\right)^2}{\left(e^{2\delta\xi}+1\right)^2},$$

$$\psi_0(\xi,t) = -\frac{4\delta^2 c_0(\alpha+\delta^2)}{3c_1^2} + \frac{4\delta^2(\alpha+\delta^2)\left(e^{2\delta\xi}-1\right)}{3c_1\left(e^{2\delta\xi}+1\right)}, \tag{6.9}$$

$$\phi_0(\xi,t) = c_0 + \frac{\left(e^{2\delta\xi}-1\right)}{\left(e^{2\delta\xi}+1\right)}c_1$$

next,

$$\varphi_1(\xi,t) = \frac{1}{\sigma\left(e^{2\delta\xi}+1\right)^3}\Big(16\alpha\delta^3\omega^{-\sigma-1}e^{2\delta\xi}\Big(e^{2\delta\xi}-1\Big)\Big((\sigma-1)^2\omega^{2\sigma}e^{\frac{(\sigma-1)t\omega^{2\sigma-1}}{\sigma}} - \sigma^2\omega\Big(e^{\frac{(\sigma-1)t\omega^{2\sigma-1}}{\sigma}}-1\Big)\Big)\Big),$$

$$\psi_1(\xi,t) = \frac{1}{3c_1\sigma\left(e^{2\delta\xi}+1\right)^2}\Big(16\alpha\delta^3\left(\alpha+\delta^2\right)\omega^{-\sigma-1}e^{2\delta\xi}\Big((\sigma-1)^2\omega^{2\sigma}e^{\frac{(\sigma-1)t\omega^{2\sigma-1}}{\sigma}} - \sigma^2\omega\Big(e^{\frac{(\sigma-1)t\omega^{2\sigma-1}}{\sigma}}-1\Big)\Big)\Big),$$

$$\phi_1(\xi,t) = \frac{1}{\sigma\left(e^{2\delta\xi}+1\right)^2}\Big(4\alpha c_1\delta\omega^{-\sigma-1}e^{2\delta\xi}\Big((\sigma-1)^2\omega^{2\sigma}e^{\frac{(\sigma-1)t\omega^{2\sigma-1}}{\sigma}} - \sigma^2\omega\Big(e^{\frac{(\sigma-1)t\omega^{2\sigma-1}}{\sigma}}-1\Big)\Big)\Big) \tag{6.10}$$

and

$$\varphi_2(\xi,t) = \frac{-1}{\left(1+e^{2\xi\delta}\right)^7\sigma^3}\Big(32e^{2\xi\delta}\alpha^2\delta^4\omega^{-3(\sigma+1)}\Big)\Big(\sigma^5\,\omega^{3+\sigma} + \mathrm{e}^{10\xi\delta}\,\sigma^5\,\omega^{3+\sigma} + 8\,\mathrm{e}^{6\xi\delta}\,\sigma^5\,\omega^3\Big(\alpha\,\delta\,\sigma$$

$$+ 31\,\delta^3\sigma - \omega^\sigma\Big) - 8\,\mathrm{e}^{4\xi\delta}\,\sigma^5\,\omega^3\Big(\alpha\,\delta\,\sigma + 31\,\delta^3\sigma + \omega^\sigma\Big) - \mathrm{e}^{8\xi\delta}\,\sigma^5\,\omega^3 - 8\,\alpha\,\delta\,\sigma + \cdots\Big),$$

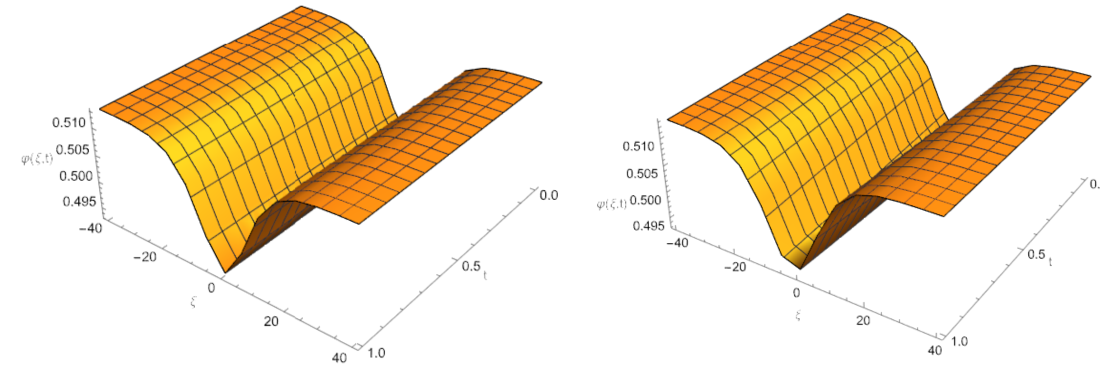
$$\psi_2(\xi,t) = \frac{-1}{3c_1\sigma^3\left(e^{2\delta\xi}+1\right)^6}\Big(32\alpha^2\delta^4\left(\alpha+\delta^2\right)\omega^{-3(\sigma+1)}e^{2\delta\xi}\Big(e^{2\delta\xi}-1\Big)\Big(-48\delta^3\sigma^6\omega^3e^{2\delta\xi}\left(e^{2\delta\xi}-1\right)$$

$$\Big(e^{\frac{2(\sigma-1)t\omega^{2\sigma-1}}{\sigma}}-1\Big) + 96\delta^3(\sigma-1)\sigma^4\omega^{2\sigma+2}\Big(e^{2\delta\xi}-1\Big)\Big(2(\sigma-1)e^{\frac{(\sigma-1)t\omega^{2\sigma-1}}{\sigma}} + \cdots\Big)$$

$$\phi_2(\xi,t) = \frac{-1}{\sigma^3\left(e^{2\delta\xi}+1\right)^6}\Big(8\alpha^2c_1\delta^2\omega^{-3(\sigma+1)}e^{2\delta\xi}\Big(e^{2\delta\xi}-1\Big)\Big(-48\delta^3\sigma^6\omega^3e^{2\delta\xi}\Big(e^{2\delta\xi}-1\Big)\Big(e^{\frac{2(\sigma-1)t\omega^{2\sigma-1}}{\sigma}}-1\Big)$$

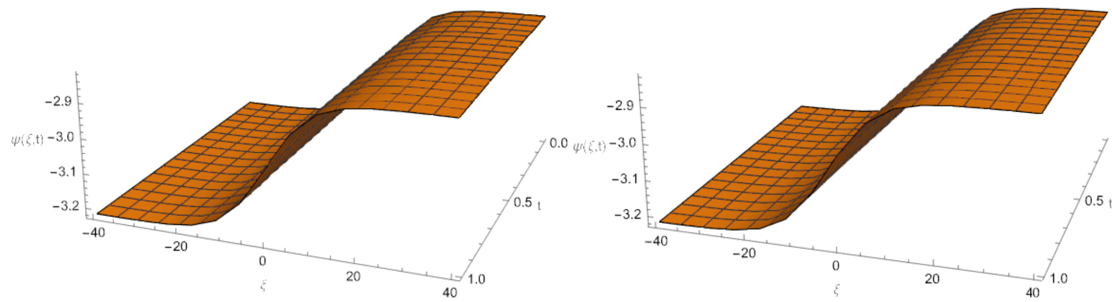
$$+ 96\delta^3(\sigma-1)\sigma^4\omega^{2\sigma+2}\left(e^{2\delta\xi}-1\right)\Big(2(\sigma-1)e^{\frac{(\sigma-1)t\omega^{2\sigma-1}}{\sigma}} + \sigma(t-2) + 2\Big) + \cdots\Big)\Big) \tag{6.11}$$

Continuing in this way, further terms of the approximate solution are obtained from the iteration formula given in equation 6.7.



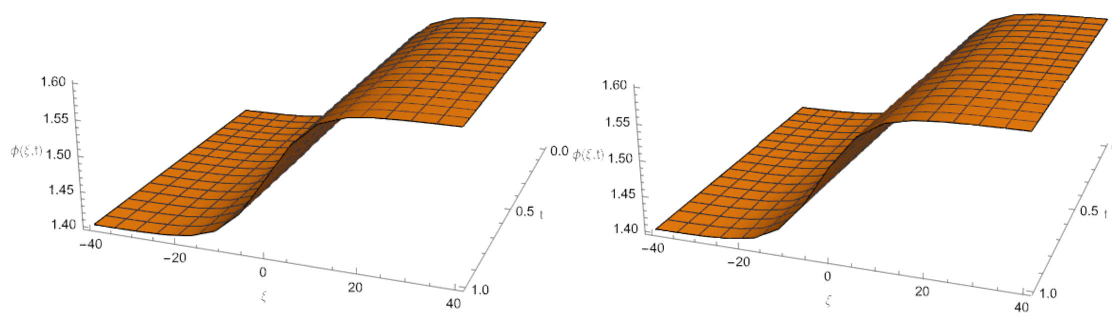
(a) (b)

Fig. 1: 3-D Surface of solution $\varphi(\xi,t)$ of equation 1.1 when $\alpha = 1.5,\ \delta = 0.1,\ c_0 = 1.5,\ c_1 = 0.1$ (a) approximate solution $\varphi(\xi,t)$ by ILTM-CPCF with $\sigma = 1,\ \omega = 0.5$ (b) the exact solution.



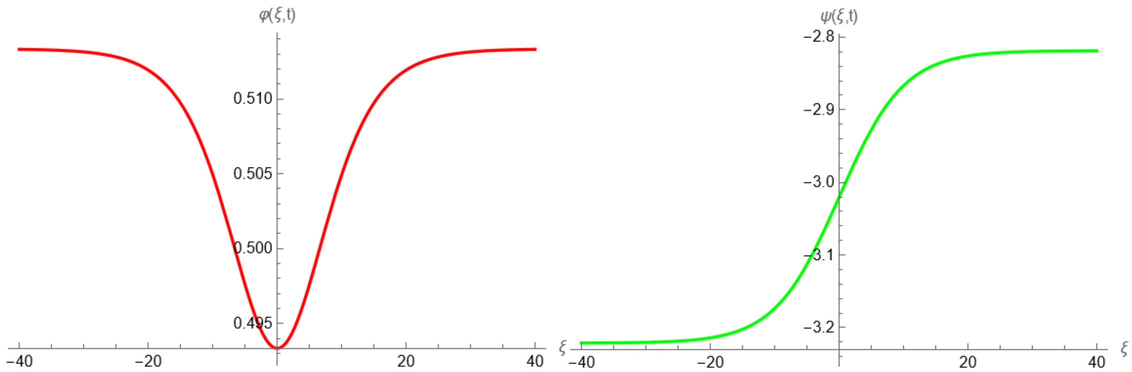
(c) (d)

Fig. 2: 3-D Surface of solution $\psi(\xi,t)$ of equation 1.1 when $\alpha = 1.5,\ \delta = 0.1,\ c_0 = 1.5,\ c_1 = 0.1$ (c) approximate solution $\psi(\xi,t)$ by ILTM-CPCF with $\sigma = 1,\ \omega = 0.5$ (d) the exact solution.



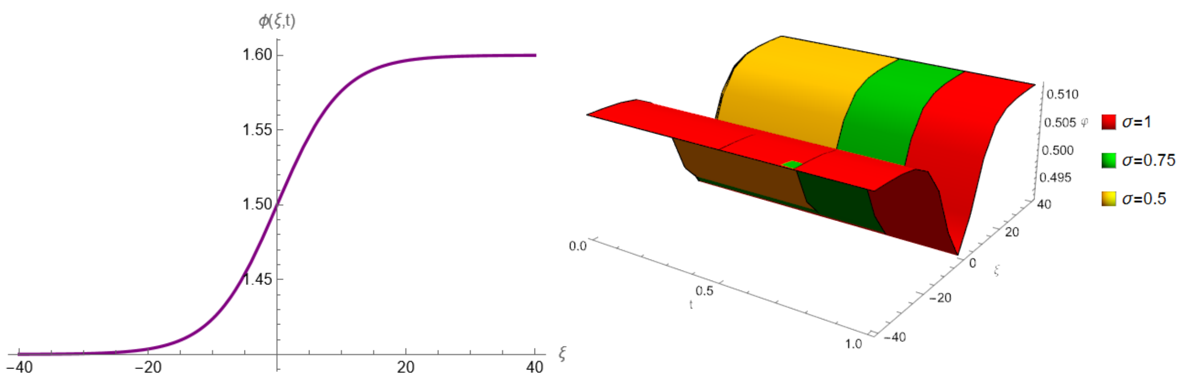
(e) (f)

Fig. 3: 3-D Surface of solution $\phi(\xi,t)$ of equation 1.1 when $\alpha = 1.5,\ \delta = 0.1,\ c_0 = 1.5,\ c_1 = 0.1$ (e) approximate solution $\phi(\xi,t)$ by ILTM-CPCF with $\sigma = 1,\ \omega = 0.5$ (f) the exact solution.



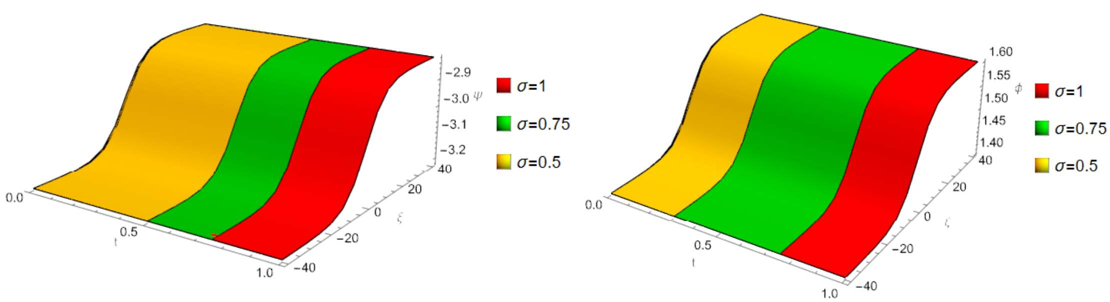
(g) (h)

Fig. 4: Plot of solutions of equation 1.1 when $\sigma = 1,\ \alpha = 1.5,\ \delta = 0.1,\ c_0 = 1.5,\ c_1 = 0.1,\ \omega = 0.5$ at $t = 0.5$ (g) plot of $\varphi(\xi, t)$ (h) plot of $\psi(\xi, t)$.



(i) (j)

Fig. 5: (i) Plot of solution $\phi(\xi, t)$ of equation 1.1 when $\sigma = 1,\ \alpha = 1.5,\ \delta = 0.1,\ c_0 = 1.5,\ c_1 = 0.1,\ \omega = 0.5$ at $t = 0.5$ (j) 3-D plot of solution $\varphi(\xi, t)$ of equation when $\sigma = 1,\ \sigma = 0.75,\ \sigma = 0.50$.



(k) (l)

Fig. 6: 3-D plot of solutions 1.1 of equation when $\sigma = 1,\ \sigma = 0.75,\ \sigma = 0.50$ at $\alpha = 1.5,\ \delta = 0.1,\ c_0 = 1.5,\ c_1 = 0.1,\ \omega = 0.5$ (k) Plot of solution $\psi(\xi, t)$ (l) plot of solution $\phi(\xi, t)$.

The Figure−1, 2 and 3 shows, $3-D$ surfaces of approximate and exact solution of equation 1.1 obtained by ILTM-CPCF when $\sigma = 1,\ \alpha = 1.5,\ \delta = 0.1,\ c_0 = 1.5,\ c_1 = 0.1$. The surface exhibits a bell form for $\varphi(\xi, t)$ and a kink-type shape for $\psi(\xi, t)$ and $\phi(\xi, t)$. Each surface exhibits wave-like behavior in both space(ξ) and time(t). The wavefronts and curvature are maintained, confirming the employed approach. The approximate solution closely resembles the exact solution, hence validating the efficacy of the ILTM-CPCF in managing coupled dynamics.

The Figure−4 and 5(i) shows, graphs of approximate solution of equation 1.1 obtained by ILTM-CPCF when $\sigma = 1,\ \alpha = 1.5,\ \delta = 0.1,\ c_0 = 1.5,\ c_1 = 0.1$ and $t = 0.5$. $\varphi(\xi, t)$ exhibits a symmetrical $V-$shape, maybe indicating a localized dip. $\psi(\xi, t)$ and $\phi(\xi, t)$ exhibit a sigmoid-like curve, signifying a gradual transition. This graph facilitates the visualization of cross-sectional behavior at a specific moment. The difference among the linear graphs illustrates the distinct roles and physical meanings of $\varphi,\ \psi$ and ϕ within the system.

The Figure−5(j) and 6 shows, $3-D$ surfaces of approximate solution of equation 1.1 obtained by ILTM-CPCF when $\sigma = 1,\ \sigma = 0.75, \sigma = 0.50$ at $\alpha = 1.5,\ \delta = 0.1,\ c_0 = 1.5,\ c_1 = 0.1$. Under various σ values, each plot illustrates how solutions change across time and place. The comparative perspective suggests different dynamics or coupling effects by highlighting the interaction between dispersion and nonlinearity in forming the solution. This demonstrates how σ affects wave steepness and dispersion, highlighting parameter sensitivity.

t	ξ	Exact Solution $\varphi(\xi, t)$	App. Solu for $\sigma = 1$	for $\sigma = 0.75$	for $\sigma = 0.5$	Absolute error $\lvert\varphi_{exact} - \varphi_{apprx}\rvert$ for $\sigma = 1$
0.2	0	0.49337377	0.49333333	0.49335302	0.49378333	0.0000404454
	0.5	0.49351275	0.49338325	0.49344829	0.49403045	0.000129503
	5	0.49826881	0.49760437	0.49794224	0.49921813	0.000664433
	10	0.50549680	0.50493384	0.50522038	0.50612555	0.000562959
	20	0.51203792	0.51192031	0.51198018	0.51215154	0.000117609
0.4	0	0.49349446	0.49333333	0.49335302	0.49378333	0.000161129
	0.5	0.49372026	0.49338325	0.49344829	0.49403045	0.000337017
	5	0.49894912	0.49760437	0.49794224	0.49921813	0.00013474
	10	0.50603362	0.50493384	0.50522038	0.50612555	0.00209978
	20	0.51214605	0.51192031	0.51198018	0.51215154	0.00125741
3	0	0.50025431	0.50877341	0.49363969	0.49383467	0.0085191
	0.5	0.50102125	0.50233327	0.49388715	0.49416833	0.00131201
	5	0.50697450	0.50037388	0.49916228	0.49924652	0.00660062
	10	0.51071363	0.50135839	0.50614592	0.50658410	0.00935523
	20	0.51295706	0.51582503	0.51216240	0.51216249	0.00286797

Table 1. The numerical results for the comparison between exact solution $\varphi(\xi, t)$ of equation 1.1 and solution at various σ values; the absolute error between the exact and the approximate solution obtained by ILTM-CPCF for $\sigma = 1$ with $\alpha = 1.5,\ \delta = 0.1,\ c_0 = 1.5,\ c_1 = 0.1,\ \omega = 0.5$

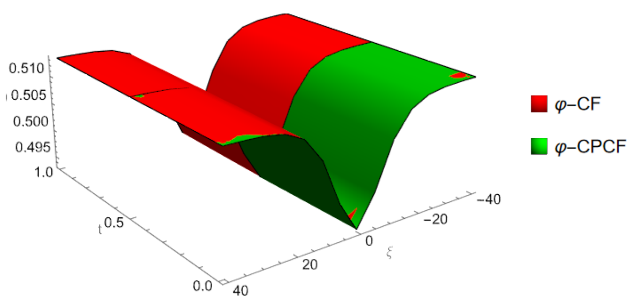
In Table 1, 2 and 3, by comparing the absolute errors between the approximate solution of equation 1.1 and the exact solution, the series solutions are reviewed when $\sigma = 1$ with $\alpha = 1.5$, $\delta = 0.1$, $c_0 = 1.5$, $c_1 = 0.1$, $\omega = 0.5$. Also the solution $\varphi(\xi, t)$, $\psi(\xi, t)$ and $\phi(\xi, t)$ are evaluated at $\sigma = 0.5$ and $\sigma = 0.75$ this shows the method ILTM-CPCF handles changes in σ. The wave structure is preserved, which is important for soliton-type equations like HS-cKdV, as suggested by the consistency of the $\varphi(\xi, t)$, $\psi(\xi, t)$ and $\phi(\xi, t)$ values. These findings show that there is a good agreement between the approximate series solutions and the exact solutions.

t	ξ	Exact Solution $\psi(\xi, t)$	App. Solu for $\sigma = 1$	for $\sigma = 0.75$	for $\sigma = 0.5$	Absolute error $\lvert\psi_{\text{exact}} - \psi_{\text{apprx}}\rvert$ for $\sigma = 1$
	0	-3.01094611	-3.02	-3.01539093	-2.99864537	0.00905389
	0.5	-3.00093066	-3.00994171	-3.00535403	-2.98886622	0.00901105
0.2	5	-2.91998496	-2.92696041	-2.92340772	-2.91181753	0.00697545
	10	-2.86298922	-2.86666570	-2.86479346	-2.85915016	0.00367649
	20	-2.82529602	-2.82590911	-2.82559698	-2.82470910	0.000613086
	0	-3.00192876	-3.02	-3.01539093	-2.99864537	0.0080712
	0.5	-2.99199605	-3.00994171	-3.00535403	-2.98886622	0.0079457
0.4	5	-2.91331435	-2.92696041	-2.92340772	-2.91181753	0.0036461
	10	-2.85956186	-2.86666570	-2.86479346	-2.85915016	0.00710384
	20	-2.82473404	-2.82590911	-2.82559698	-2.82470910	0.00117507
	0	-2.90156380	-2.90835103	-2.99921398	-2.99947294	0.00678723
	0.5	-2.89517396	-2.90283653	-2.98936132	-2.99902481	0.00766257
3	5	-2.85372515	-2.85840217	-2.91173684	-2.92290451	0.004677
	10	-2.83231511	-2.83816304	-2.85892464	-2.88940184	0.00584793
	20	-2.82056955	-2.82590319	-2.82465071	-2.83018327	0.00533956

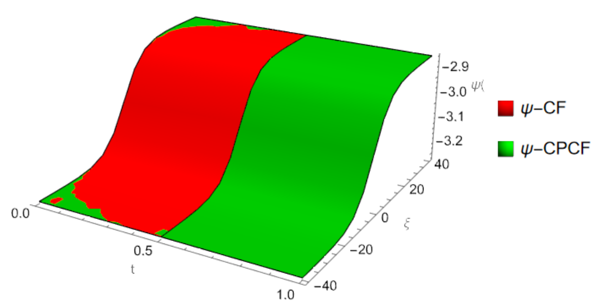
TABLE 2. The numerical results for the comparison between exact solution $\psi(\xi, t)$ of equation 1.1 and solution at various σ values; the absolute error between the exact and the approximate solution obtained by ILTM-CPCF for $\sigma = 1$ with $\alpha = 1.5$, $\delta = 0.1$, $c_0 = 1.5$, $c_1 = 0.1$, $\omega = 0.5$

t	ξ	Exact Solution $\phi(\xi,t)$	App. Solu for $\sigma=1$	for $\sigma=0.75$	for $\sigma=0.5$	Absolute error $\lvert\phi_{\text{exact}}-\phi_{\text{apprx}}\rvert$ for $\sigma=1$
0.2	0	1.50449696	1.5	1.50228926	1.51060660	0.00449696
	0.5	1.50947152	1.50499583	1.50727448	1.51546379	0.00447569
	5	1.54967634	1.54621171	1.54797629	1.55373301	0.00346463
	10	1.57798548	1.57615941	1.57708933	1.57989230	0.00182607
	20	1.59670727	1.59640275	1.59091240	1.59206297	0.000304513
0.4	0	1.50897577	1.5	1.50228926	1.51060660	0.00897578
	0.5	1.51390924	1.50499583	1.50727448	1.51546379	0.00891341
	5	1.55298956	1.54621171	1.54797629	1.55373301	0.00677785
	10	1.57968781	1.57615941	1.57708933	1.57989230	0.0035284
	20	1.59698640	1.59640275	1.59091240	1.59206297	0.000583644
3	0	1.55882592	1.557247	1.51032417	1.51875044	0.00157893
	0.5	1.56199968	1.56480128	1.51521788	1.52682401	0.0028016
	5	1.58258684	1.58629043	1.55377308	1.55859032	0.00370359
	10	1.59322096	1.58986730	1.58000431	1.59894857	0.00335366
	20	1.59905485	1.59473602	1.59702779	1.59899462	0.00431883

TABLE 3. The numerical results for the comparison between exact solution $\phi(\xi,t)$ of equation 1.1 and solution at various σ values; the absolute error between the exact and the approximate solution obtained by ILTM-CPCF for $\sigma=1$ with $\alpha=1.5,\ \delta=0.1,\ c_0=1.5,\ c_1=0.1,\ \omega=0.5$

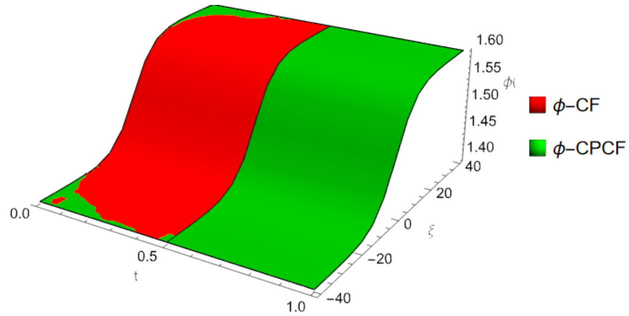


(m)



(n)

Fig. 7: 3-D plot of solutions of equation 1.1 when $\sigma=1,\ \alpha=1.5,\ \delta=0.1, c_0=1.5,\ c_1=0.1,\ \omega=0.5$ (m) Plot of solution $\varphi(\xi,t)$ (n) plot of solution $\psi(\xi,t)$.



(o)

Fig. 8: $3-D$ plot of solutions of equation 1.1 when $\sigma = 1,\ \alpha = 1.5,\ \delta = 0.1, c_0 = 1.5,\ c_1 = 0.1,\ \omega = 0.5$ (o) Plot of solution $\phi(\xi, t)$.

t	ξ	$\varphi(\xi,t)-$ CF	$\varphi(\xi,t)-$ CPCF	$\psi(\xi,t)-$ CF	$\psi(\xi,t)-$ CPCF	$\phi(\xi,t)-$ CF	$\phi(\xi,t)-$ CPCF
0.2	0	0.49335133	0.49333333	-3.01396	-3.02	1.503	1.5
	0.5	0.49346079	0.49338325	-3.00392581	-3.00994171	1.50798386	1.50499583
	5	0.49804516	0.49760437	-2.92227616	-2.92696041	1.54853833	1.54621171
	10	0.50531188	0.50493384	-2.86418705	-2.86666570	1.57739053	1.57615941
	20	0.51199976	0.51192031	-2.82549472	-2.82590911	1.59660857	1.59640275
0.4	0	0.49340533	0.49333333	-3.00792	-3.02	1.506	1.5
	0.5	0.49357354	0.49338325	-2.99792798	-3.00994171	1.51096291	1.50499583
	5	0.49849362	0.49760437	-2.91772384	-2.92696041	1.55079941	1.54621171
	10	0.50567757	0.50493384	-2.86182448	-2.86666570	1.57856399	1.57615941
	20	0.51207462	0.51192031	-2.82510502	-2.82590911	1.59680213	1.59640275
3	0	0.49738333	0.50877341	-2.92940000	-2.90835103	1.545	1.557247
	0.5	0.49804190	0.50233327	-2.92160198	-2.90283653	1.54887318	1.56480128
	5	0.50388235	0.50037388	-2.87065456	-2.85840217	1.57417819	1.58629043
	10	0.50878482	0.50135839	-2.84175642	-2.83816304	1.58853157	1.58986730
	20	0.51261195	0.51582503	-2.82228950	-2.82590319	1.59820057	1.59473602

Table 4. The numerical results for the comparison between solutions of equation 1.1 obtained by ILTM-CF and ILTM-CPCF for $\sigma = 1$ with $\alpha = 1.5,\ \delta = 0.1,\ c_0 = 1.5,\ c_1 = 0.1,\ \omega = 0.5$

The Figure−7 and 8 shows, $3-D$ surfaces of approximate solution of equation 1.1 obtained by ILTM-CF and ILTM-CPCF when $\sigma = 1$ at $\alpha = 1.5,\ \delta = 0.1,\ c_0 = 1.5,\ c_1 = 0.1$. The CF and CPCF both solutions exhibit smooth space and time evolution in the plot of $\varphi(\xi, t)$. Surfaces exhibit small amplitude variations but overlaps closely. Compared to the CF operator, the CPCF operator slightly modifies the solution, suggesting that both methods capture identical qualitative behavior. The CF and CPCF solutions for the plot of $\psi(\xi, t)$ are similar, although there are some variations in height. A more precise adjustment is offered by CPCF, indicating sensitivity in this solution component of HS- cKdv equation. The CF and CPCF solutions for $\phi(\xi, t)$ are extremely similar, with just minor vertical changes on the surface. The solutions remains stable for both operators, with CPCF modifying the amplitude somewhat.

In Table 4 the series approximate solution of equation 1.1 obtained by ILTM-CF and ILTM-CPCF when $\sigma = 1$ with $\alpha = 1.5,\ \delta = 0.1,\ c_0 = 1.5,\ c_1 = 0.1,\ \omega = 0.5$ has been compared. All three components of the solution are very near between CF and CPCF. This demonstrates the ILTM's stability and reliability when used with both operators. The CPCF reduces amplitudes slightly, implying that it smoothes the solution more than CF.

6.2 Approximate solution of the time fractional HS- cMKdv equation:

Consider the time fractional order HS- cMKdv equation given in equation 1.2 with the initial conditions as

$$\begin{aligned} \varphi(\xi,0) &= \frac{\delta(-1+e^{2\delta\xi})}{(1+e^{2\delta\xi})}, \\ \psi(\xi,0) &= \frac{4\delta^2+\lambda}{2} - \frac{2\delta^2(-1+e^{2\delta\xi})^2}{(1+e^{2\delta\xi})^2} \end{aligned} \tag{6.12}$$

where, δ and λ are real constants.

The exact solution of the equation 1.2 is

$$\begin{aligned} \varphi(\xi,t) &= \frac{\delta(-1+e^{2\delta(\xi-t\delta^2-\frac{3t\lambda}{2})})}{(1+e^{2\delta(\xi-t\delta^2-\frac{3t\lambda}{2})})}, \\ \psi(\xi,t) &= \frac{4\delta^2+\lambda}{2} - \frac{2\delta^2(-1+e^{2\delta(\xi-t\delta^2-\frac{3t\lambda}{2})})^2}{(1+e^{2\delta(\xi-t\delta^2-\frac{3t\lambda}{2})})^2} \end{aligned} \tag{6.13}$$

By applying the Laplace Transform to both sides of equation , yields

$$\begin{aligned} \mathcal{L}\{\varphi(\xi,t)\} - \left(\frac{D_0(\sigma)}{D_1(\sigma)+p\,D_0(\sigma)}\right)\varphi(\xi,0) &= \left(\frac{\sigma+p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma)+p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)\mathcal{L}\left\{\frac{1}{2}\frac{\partial^3\varphi}{\partial\xi^3} - 3\varphi^2\frac{\partial\varphi}{\partial\xi}\right. \\ &\quad \left. + \frac{3}{2}\frac{\partial^2\psi}{\partial\xi^2} + 3\varphi\frac{\partial\psi}{\partial\xi} + 3\psi\frac{\partial\varphi}{\partial\xi} - 3\lambda\frac{\partial\varphi}{\partial\xi}\right\} \\ \mathcal{L}\{\psi(\xi,t)\} - \left(\frac{D_0(\sigma)}{D_1(\sigma)+p\,D_0(\sigma)}\right)\psi(\xi,0) &= \left(\frac{\sigma+p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma)+p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)\mathcal{L}\left\{-\frac{\partial^3\psi}{\partial\xi^3} - 3\psi\frac{\partial\psi}{\partial\xi}\right. \\ &\quad \left. - 3\frac{\partial\varphi}{\partial\xi}\frac{\partial\psi}{\partial\xi} + 3\varphi^2\frac{\partial\varphi}{\partial\xi} + 3\lambda\frac{\partial\psi}{\partial\xi}\right\} \end{aligned} \tag{6.14}$$

Rearranging the terms, we get

$$\begin{aligned} \mathcal{L}\{\varphi(\xi,t)\} &= \frac{1}{p+\frac{D_1(\sigma)}{D_0(\sigma)}}\varphi(\xi,0) + \left(\frac{\sigma+p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma)+p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)\mathcal{L}\left\{\frac{1}{2}\frac{\partial^3\varphi}{\partial\xi^3} - 3\varphi^2\frac{\partial\varphi}{\partial\xi} + \frac{3}{2}\frac{\partial^2\psi}{\partial\xi^2}\right. \\ &\quad \left. + 3\varphi\frac{\partial\psi}{\partial\xi} + 3\psi\frac{\partial\varphi}{\partial\xi} - 3\lambda\frac{\partial\varphi}{\partial\xi}\right\} \end{aligned}$$

$$\mathcal{L}\{\psi(\xi,t)\} = \frac{1}{p+\frac{D_1(\sigma)}{D_0(\sigma)}}\psi(\xi,0) + \left(\frac{\sigma+p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma)+p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)\mathcal{L}\Bigg\{-\frac{\partial^3\psi}{\partial\xi^3} - 3\psi\frac{\partial\psi}{\partial\xi} - 3\frac{\partial\varphi}{\partial\xi}\frac{\partial\psi}{\partial\xi} + 3\varphi^2\frac{\partial\varphi}{\partial\xi} + 3\lambda\frac{\partial\psi}{\partial\xi}\Bigg\} \tag{6.15}$$

Now, using the Inverse Laplace transform to both sides of Eq. (6.15), we can obtain

$$\begin{aligned}\varphi(\xi,t) &= exp\left(-\frac{D_1(\sigma)}{D_0(\sigma)}t\right)\varphi(\xi,0) + \mathcal{L}^{-1}\Bigg\{\left(\frac{\sigma+p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma)+p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)\mathcal{L}\Bigg\{\frac{1}{2}\frac{\partial^3\varphi}{\partial\xi^3} - 3\varphi^2\frac{\partial\varphi}{\partial\xi} \\ &\quad + \frac{3}{2}\frac{\partial^2\psi}{\partial\xi^2} + 3\varphi\frac{\partial\psi}{\partial\xi} + 3\psi\frac{\partial\varphi}{\partial\xi} - 3\lambda\frac{\partial\varphi}{\partial\xi}\Bigg\}\Bigg\} \\ \psi(\xi,t) &= exp\left(-\frac{D_1(\sigma)}{D_0(\sigma)}t\right)\psi(\xi,0) + \mathcal{L}^{-1}\Bigg\{\left(\frac{\sigma+p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma)+p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)\mathcal{L}\Bigg\{-\frac{\partial^3\psi}{\partial\xi^3} - 3\psi\frac{\partial\psi}{\partial\xi} \\ &\quad - 3\frac{\partial\varphi}{\partial\xi}\frac{\partial\psi}{\partial\xi} + 3\varphi^2\frac{\partial\varphi}{\partial\xi} + 3\lambda\frac{\partial\psi}{\partial\xi}\Bigg\}\Bigg\}\end{aligned} \tag{6.16}$$

Let the series form of the approximate solution is

$$\begin{aligned}\varphi(\xi,t) &= \sum_{n=0}^{\infty}\varphi_n(\xi,t), \\ \psi(\xi,t) &= \sum_{n=0}^{\infty}\psi_n(\xi,t)\end{aligned} \tag{6.17}$$

By the use of initial conditions as $\varphi_0(\xi,\ t) = \varphi(\xi,0)$ and $\psi_0(\xi,\ t) = \psi(\xi,0)$ resulting recursive relation is as follows

$$\begin{aligned}\varphi_{n+1}(\xi,t) &= exp\left(-\frac{D_1(\sigma)}{D_0(\sigma)}t\right)\varphi_0(\xi,\ t) + \mathcal{L}^{-1}\Bigg\{\left(\frac{\sigma+p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma)+p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)\mathcal{L}\Bigg\{\frac{1}{2}\frac{\partial^3\varphi_n}{\partial\xi^3} \\ &\quad - 3\left(\sum_{i=0}^{n}\varphi_i^2\frac{\partial\varphi_i}{\partial\xi} - \sum_{i=0}^{(n-1)}\varphi_i^2\frac{\partial\varphi_i}{\partial\xi}\right) + \frac{3}{2}\frac{\partial^2\psi_n}{\partial\xi^2} + 3\left(\sum_{i=0}^{n}\varphi_i\frac{\partial\psi_i}{\partial\xi} - \sum_{i=0}^{(n-1)}\varphi_i\frac{\partial\psi_i}{\partial\xi}\right) + \\ &\quad 3\left(\sum_{i=0}^{n}\psi_i\frac{\partial\varphi_i}{\partial\xi} - \sum_{i=0}^{(n-1)}\psi_i\frac{\partial\varphi_i}{\partial\xi}\right) - 3\lambda\frac{\partial\varphi_n}{\partial\xi}\Bigg\}\Bigg\}\end{aligned} \tag{6.18}$$

$$\begin{aligned}\psi_{n+1}(\xi,t) &= exp\left(-\frac{D_1(\sigma)}{D_0(\sigma)}t\right)\psi_0(\xi,t) + \mathcal{L}^{-1}\Bigg\{\left(\frac{\sigma+p(1-\sigma)}{\mathbb{M}(\sigma)D_1(\sigma)+p\,\mathbb{M}(\sigma)D_0(\sigma)}\right)\mathcal{L}\Bigg\{-\frac{\partial^3\psi_n}{\partial\xi^3} \\ &\quad - 3\left(\sum_{i=0}^{n}\psi_i\frac{\partial\psi_i}{\partial\xi} - \sum_{i=0}^{(n-1)}\psi_i\frac{\partial\psi_i}{\partial\xi}\right) - 3\left(\sum_{i=0}^{n}\frac{\partial\varphi_i}{\partial\xi}\frac{\partial\psi_i}{\partial\xi} - \sum_{i=0}^{(n-1)}\frac{\partial\varphi_i}{\partial\xi}\frac{\partial\psi_i}{\partial\xi}\right)\end{aligned}$$

$$+3\Bigg(\sum_{i=0}^{n}\varphi_i^2\frac{\partial\varphi_i}{\partial\xi}-\sum_{i=0}^{(n-1)}\varphi_i^2\frac{\partial\varphi_i}{\partial\xi}\Bigg)+3\lambda\frac{\partial\psi_n}{\partial\xi}\Bigg\}\Bigg\}$$

The approximate solution with the $k-$ terms is

$$\begin{aligned}\varphi(x,t)&=\varphi_0(\xi,t)+\varphi_1(\xi,t)+\varphi_2(\xi,t)+\cdots+\varphi_{k-1}(\xi,t),\\ \psi(x,t)&=\psi_0(\xi,t)+\psi_1(\xi,t)+\psi_2(\xi,t)+\cdots+\psi_{k-1}(\xi,t)\end{aligned}\tag{6.19}$$

Hence, by using equation 6.18 first few terms of the series solution of equation 1.2 are as:

$$\begin{aligned}\varphi_0(\xi,t)&=\frac{\delta(-1+e^{2\delta\xi})}{(1+e^{2\delta\xi})},\\ \psi_0(\xi,t)&=\frac{4\delta^2+\lambda}{2}-\frac{2\delta^2(-1+e^{2\delta\xi})^2}{(1+e^{2\delta\xi})^2}\end{aligned}\tag{6.20}$$

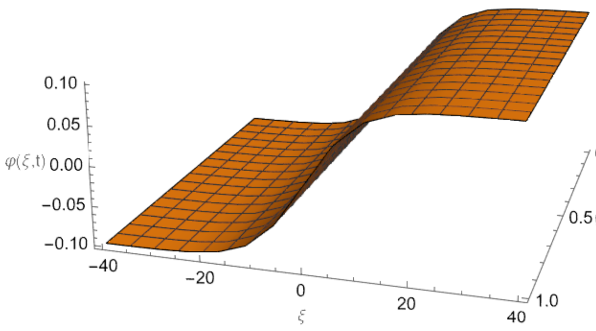
next,

$$\begin{aligned}\varphi_1(\xi,t)&=\frac{1}{\sigma\left(e^{2\delta\xi}+1\right)^2}\Bigg(2\delta^2\left(2\delta^2+3\lambda\right)\omega^{-\sigma-1}e^{2\delta\xi}\Big(\sigma^2\omega\Big(e^{\frac{(\sigma-1)t\omega^{2\sigma-1}}{\sigma}}-1\Big)-(\sigma-1)^2\omega^{2\sigma}e^{\frac{(\sigma-1)t\omega^{2\sigma-1}}{\sigma}}\Big)\Bigg),\\ \psi_1(\xi,t)&=\frac{1}{\sigma\left(e^{2\delta\xi}+1\right)^5}\Bigg(8\delta^3\omega^{-\sigma-1}e^{4\delta\xi}\Big(e^{2\delta x}-1\Big)\Big((\sigma-1)^2\omega^{2\sigma}e^{\frac{(\sigma-1)t\omega^{2\sigma-1}}{\sigma}}-\sigma^2\omega\Big(e^{\frac{(\sigma-1)t\omega^{2\sigma-1}}{\sigma}}-1\Big)\Big)\\ &\quad\Big(-8\delta^2-6\lambda+2\left(8\delta^2-3\lambda\right)+3\Big)\Bigg)\end{aligned}\tag{6.21}$$

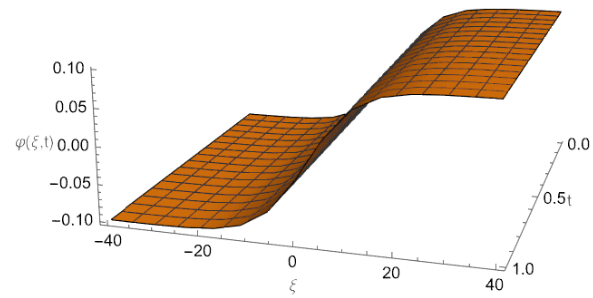
and

$$\begin{aligned}\varphi_2(\xi,t)&=\frac{1}{\left(1+e^{2\xi\delta}\right)^8\sigma^4}\Bigg(2e^{2\xi\delta}\delta^3\omega^{-4(\sigma+1)}\Big(24e^{\frac{2t(\sigma-1)\omega^{2\sigma-1}}{\sigma}+10\xi\delta}\delta^3\sigma^5\Big(68\Big(-8\sigma\omega^{2\sigma}+4\omega^{2\sigma}\\ &\quad+\sigma^2\left(4\omega^{2\sigma}-\omega\right)\Big)\delta^4+12\left(-8\sigma\omega^{2\sigma}+4\omega^{2\sigma}+\sigma^2\left(4\omega^{2\sigma}-\omega\right)\right)\delta^3+\cdots\Bigg)\\ \psi_2(\xi,t)&=\frac{-1}{\left(1+e^{2\xi\delta}\right)^{11}\sigma^4}\Bigg(24e^{6\xi\delta}\delta^7\omega^{-4(\sigma+1)}\Big(8\Big(-1+e^{2\xi\delta}\Big)\Big(1+e^{2\xi\delta}\Big)^4\Big(56e^{\frac{t(\sigma-1)\omega^{2\sigma-1}}{\sigma}}\Big(-1\\ &\quad+3e^{\frac{2t(\sigma-1)\omega^{2\sigma-1}}{\sigma}}\Big)\sigma^3\omega^{8\sigma}-28e^{\frac{t(\sigma-1)\omega^{2\sigma-1}}{\sigma}}\Big(-1+3e^{\frac{2t(\sigma-1)\omega^{2\sigma-1}}{\sigma}}\Big)\sigma^2\omega^{8\sigma}+\cdots\Bigg)\end{aligned}\tag{6.22}$$

Continuing in this way, further terms of the approximate solution are obtained from the iteration formula given in equation 6.18.

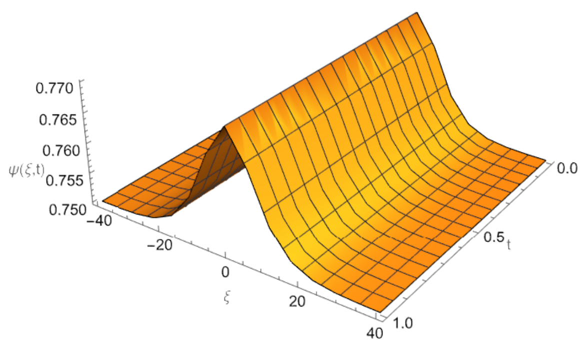


(p)

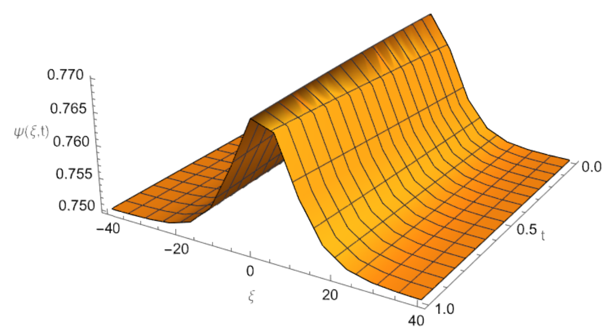


(q)

Fig. 9: 3-D Surface of solution $\varphi(\xi, t)$ of equation 1.2 when $\lambda = 1.5,\ \delta = 0.1$
(p) approximate solution $\varphi(\xi, t)$ by ILTM-CPCF with $\sigma = 1,\ \omega = 0.5$ (q) the exact solution.

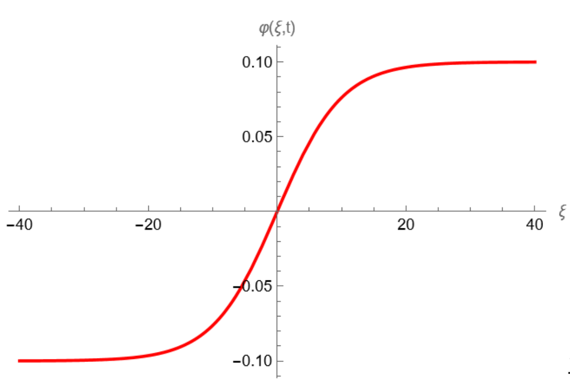


(r)

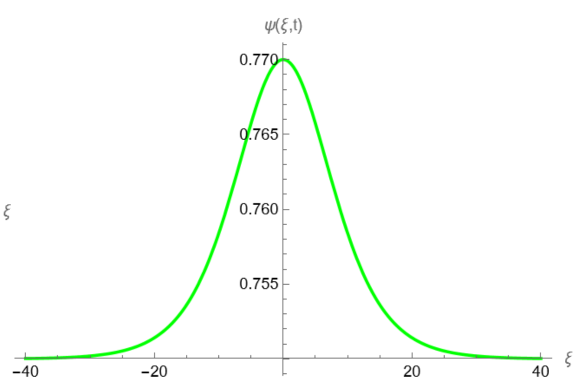


(s)

Fig. 10: 3-D Surface of solution $\psi(\xi, t)$ of equation 1.2 when $\lambda = 1.5,\ \delta = 0.1$
(r) approximate solution $\psi(\xi, t)$ by ILTM-CPCF with $\sigma = 1,\ \omega = 0.5$ (s) the exact solution.

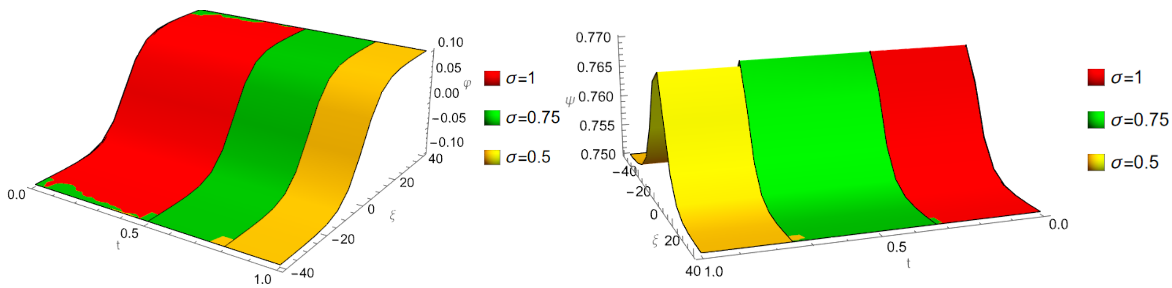


(t)



(u)

Fig. 11: Plot of solutions of equation 1.2 when $\sigma = 1,\ \lambda = 1.5,\ \delta = 0.1,\ \omega = 0.5$ at $t = 0.5$
(t) plot of $\varphi(\xi, t)$ (u) plot of $\psi(\xi, t)$.



(v) (w)

Fig. 12: 3-D plot of solutions of equation 1.2 when $\sigma = 1$, $\sigma = 0.75$, $\sigma = 0.50$ at $\lambda = 1.5$, $\delta = 0.1$, $\omega = 0.5$ (v) Plot of solution $\varphi(\xi, t)$ (w) plot of solution $\psi(\xi, t)$.

The Figure−9 and 10 shows, $3 - D$ surfaces of approximate solution of equation 1.2 obtained by ILTM-CPCF when $\sigma = 1$, $\lambda = 1.5$, $\delta = 0.1$, $\omega = 0.5$. The surface exhibits a kink-type shape for $\varphi(\xi, t)$ and a bell shape for $\psi(\xi, t)$. The transitions in curvature and gradient imply a traveling wave or soliton-like configuration. In Figure 10, the approximation of $\psi(\xi, t)$ effectively captures both the peak and decay, thereby validating the numerical scheme. The approximation closely aligns with the exact solution, demonstrating the high accuracy of the ILTM-CPCF method.

The Figure−11 shows, graphs of approximate solution of equation 1.2 obtained by ILTM-CPCF when $\sigma = 1$, $\lambda = 1.5$, $\delta = 0.1$, $\omega = 0.5$ and $t = 0.5$. A sigmoid-like curve of $\varphi(\xi, t)$, potentially exhibiting a shock wave, along with a bell-shaped curve of $\psi(\xi, t)$, illustrates the characteristic form of a soliton. These plots provide a more distinct representation of the wave profile at a certain moment in time. $\varphi(\xi, t)$ exhibits monotonic growth, whereas function $\psi(\xi, t)$ is symmetric and localized.

The Figure 12 shows, $3 - D$ surfaces of approximate solution of equation 1.2 obtained by ILTM-CPCF when $\sigma = 1$, $\sigma = 0.75, \sigma = 0.50$ at $\lambda = 1.5$, $\delta = 0.1$, $\omega = 0.5$. As σ decreases from 1 to 0.5, the amplitude of both $\varphi(\xi, t)$ and $\psi(\xi, t)$ reduces. This indicates that σ directly affects the effectiveness or intensity of the solution. The wave-like configuration suggests that σ acts as a traveling wave. Over time, the wave either disperses or stabilizes based on σ. The form of $\psi(\xi, t)$ may suggest an alternative physical or mathematical function. $\psi(\xi, t)$ may denote a secondary field or a response function, responding to the identical governing equation and exhibiting distinct properties. σ acts as a damping or scaling factor. Higher σ produces more energetic solutions, whereas lower σ smoothes or flattens the wave.

In Tables 5 and 6, the series solutions of the HS-cMKdv equation are tested by analyzing the absolute errors between the approximate and exact solutions when $\sigma = 1$ with $\lambda = 1.5$, $\delta = 0.1$, $\omega = 0.5$. Also the solution $\varphi(\xi, t)$ and $\psi(\xi, t)$ are evaluated at $\sigma = 0.5$ and $\sigma = 0.75$ this demonstrates that ILTM-CPCF manages change in σ for this system too. The error remains minimal, even throughout an domain, indicating that the approach is robust and operates effectively under different circumstances. The results indicate that the approximate series solutions coincide closely with the exact answers.

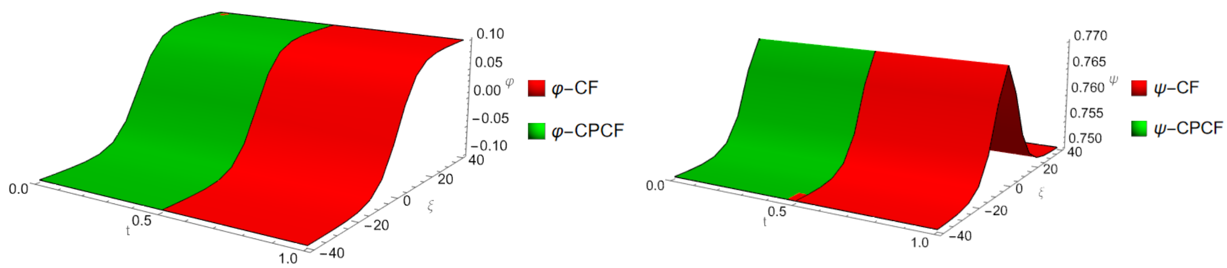
t	ξ	Exact Solution $\varphi(\xi,t)$	App. Solu for $\sigma=1$	for $\sigma=0.75$	for $\sigma=0.5$	Absolute error $\lvert\varphi_{\text{exact}}-\varphi_{\text{apprx}}\rvert$ for $\sigma=1$
0.2	0	-0.00451692	-0.00496785	-0.00344882	-0.01596005	0.000450931
	0.5	0.00047999	0.000019223	0.00154566	-0.01115226	0.000460773
	5	0.04258366	0.04214586	0.04342464	0.03194097	0.000437797
	10	0.07419484	0.07392721	0.07464239	0.06787948	0.000437797
	20	0.09606910	0.09601959	0.09614395	0.09492803	0.0000495093
0.4	0	-0.00901545	-0.00496785	-0.00344882	-0.01596005	0.0040476
	0.5	-0.00403780	0.000019223	0.00154566	-0.01115226	0.00405703
	5	0.03881329	0.04214586	0.04342464	0.03194097	0.00333256
	10	0.07209402	0.07392721	0.07464239	0.06787948	0.00183319
	20	0.09570518	0.09601959	0.09614395	0.09492803	0.000314414
3	0	-0.05902176	-0.05733239	-0.01554346	-0.00246185	0.00168937
	0.5	-0.05566735	-0.05250390	-0.01066390	-0.00729478	0.00316345
	5	-0.01761436	-0.01099289	0.03281938	0.02796205	0.00662147
	10	0.03113142	0.03742343	0.06855963	0.06260925	0.00629202
	20	0.08672804	0.09485746	0.09506847	0.09378351	0.00812942

Table 5. The numerical results for the comparison between exact solution $\varphi(\xi,t)$ of equation 1.2 and solution at various σ values; the absolute error between the exact and the approximate solution obtained by ILTM-CPCF for $\sigma=1$ with $\lambda=1.5,\ \delta=0.1,\ \omega=0.5$

t	ξ	Exact Solution $\psi(\xi,t)$	App. Solu for $\sigma=1$	for $\sigma=0.75$	for $\sigma=0.5$	Absolute error $\lvert\psi_{\text{exact}}-\psi_{\text{apprx}}\rvert$ for $\sigma=1$
0.2	0	0.76995919	0.77	0.76995611	0.76900153	0.0000408052
	0.5	0.76999953	0.76995008	0.76983862	0.76864671	0.000049456
	5	0.76637326	0.76572895	0.76522938	0.76320112	0.000644309
	10	0.75899025	0.75839948	0.75799055	0.75675699	0.000590764
	20	0.75154145	0.75141301	0.75132947	0.75112491	0.000128438
0.4	0	0.76983744	0.77	0.76995611	0.76900153	0.000162557
	0.5	0.76996739	0.76995008	0.76983862	0.76864671	0.000173091
	5	0.76698705	0.76572895	0.76522938	0.76320112	0.0012581
	10	0.75960490	0.75839948	0.75799055	0.75675699	0.00120542
	20	0.75168103	0.75141301	0.75132947	0.75112491	0.0026802
3	0	0.76303286	0.76382237	0.76931944	0.76958250	0.000789508
	0.5	0.76380229	0.76338693	0.76896975	0.76872593	0.000415359
	5	0.76937946	0.76492640	0.76334668	0.76289354	0.00445306
	10	0.76806166	0.75823853	0.75670829	0.75639053	0.00982313
	20	0.75495649	0.76324595	0.75109628	0.75093682	0.00828946

Table 6. The numerical results for the comparison between exact solution $\psi(\xi,t)$ of equation 1.2 and solution at various σ values; the absolute error between the exact and the approximate solution obtained by ILTM-CPCF for $\sigma=1$ with $\lambda=1.5,\ \delta=0.1,\ \omega=0.5$

The Figure 13 shows, $3-D$ surfaces of approximate solution of equation 1.2 obtained by ILTM-CF and ILTM-CPCF when $\sigma = 1$ at $\lambda = 1.5,\ \delta = 0.1,\ \omega = 0.5$. The plot of $\varphi(\xi, t)$ exhibits wave-like features in both CF and CPCF solution surfaces, which may be solitonic or dispersive in character. The surface plot of $\psi(\xi, t)$ exhibits varying amplitude or frequency characteristics. The peaks and troughs indicate localized energy or concentration moving through space and time. The green surface (CPCF) may exhibit smoother or marginally altered behavior compared to the red surface (CF), suggesting that CPCF may provide enhanced stability or precision in capturing the dynamics.



(x) (y)

Fig. 13: 3-D plot of solutions of equation 1.2 when $\sigma = 1,\ \lambda = 1.5,\ \delta = 0.1,\ \omega = 0.5$
(x) Plot of solution $\varphi(\xi, t)$ (y) plot of solution $\psi(\xi, t)$.

t	ξ	$\varphi(\xi,t)-$ CF	$\varphi(\xi,t)-$ CPCF	$\psi(\xi,t)-$ CF	$\psi(\xi,t)-$ CPCF
	0	-0.00451969	-0.00496785	0.76995988	0.77
	0.5	0.00047831	0.000019223	0.76982123	0.76995008
0.2	5	0.04258890	0.04214586	0.76507893	0.76572895
	10	0.07419856	0.07392721	0.75785808	0.75839948
	20	0.09606959	0.09601959	0.75130138	0.75141301
	0	-0.00473831	-0.00496785	0.76983970	0.77
	0.5	0.000405633	0.000019223	0.76961315	0.76995008
0.4	5	0.03882991	0.04214586	0.76440751	0.76572895
	10	0.07211242	0.07392721	0.75734205	0.75839948
	20	0.09570879	0.09601959	0.75119969	0.75141301
	0	-0.06735066	-0.05733239	0.76110298	0.76382237
	0.5	-0.06424903	-0.05250390	0.75980957	0.76338693
3	5	-0.02247728	-0.01099289	0.75374974	0.76492640
	10	0.03347609	0.03742343	0.75292045	0.75823853
	20	0.08849692	0.09485746	0.75078008	0.76324595

Table 7. The numerical results for the comparison between solutions of equation 1.2 obtained by ILTM-CF and ILTM-CPCF for $\sigma = 1$ with $\lambda = 1.5,\ \delta = 0.1,\ \omega = 0.5$

In Table 7 the series approximate solution of equation 1.2 obtained by ILTM-CF and ILTM-CPCF when $\sigma = 1$ with $\lambda = 1.5,\ \delta = 0.1,\ \omega = 0.5$ has been compared. The values of $\varphi(\xi, t)$ and $\psi(\xi, t)$ by CF and CPCF methods are extremely close. This suggests that both ILTM-CF and ILTM-CPCF approaches exhibit high accuracy and consistency; however, CPCF may provide significantly smoother or more stable outcomes.

7. Conclusion

In this investigation, the Constant Proportional Caputo-Fabrizio derivative along with the Iterative Laplace Transform Method has been utilized to estimate the solution of time fractional Hirota-Satsuma Coupled KdV Equation and Coupled MKdV Equation. The criteria for the existence of solution and the conditions of the stable approximate solution of HS- cKdV and HS- cMKdv equation are obtained, which demonstrates the usefullness of the considered fractional differential operator. ILTM-CPCF provides the good numerical behavior for less fractional order. The figures and tables indicate that the fractional derivative CPCF incorporates a memory effect, resulting in a deceleration of wave propagation and altering the solution's characteristics with time. The CPCF approach may be advantageous for scenarios when memory effects are physically significant. The comparison shows that CPCF generalizes CF while maintaining consistency, making it a promising use for fractional modeling. The numerical analysis indicates that the method is computationally efficient, reliable, and very effective in deriving solutions for a broad spectrum of fractional models in applied sciences. This research contributes to address the various fractional-order coupled equations that involve the CPCF derivative.

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