

Limiting Interpolation by Lorentz-Zygmund Spaces: Grand/Small Lebesgue, Lorentz, and Modelled Besov Spaces

Muhammad Awais¹, Zia Bashir¹, Thitiporn Linitda^{2,*}, Thanin Sitthiwiratham^{2,3}

¹Quaid-i-Azam University, Department of Mathematics, Islamabad-45320, Pakistan

²Mathematics Department, Faculty of Science and Technology, Suan Dusit University, Bangkok 10300, Thailand

³Research Group for Fractional Calculus Theory and Applications, Science and Technology Research Institute, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand

*Corresponding author: thitiporn_lin@dusit.ac.th

Abstract. In this article, we investigate limiting real interpolation spaces. Our primary goal is to thoroughly explore and establish the interpolation properties within critical cases $\theta = 0$ and $\theta = 1$ for Lorentz spaces, Grand and Small Lebesgue spaces as well as for Besov spaces modelled on so called Lorentz-Zygmund spaces. The key findings from our research indicate that Lorentz-Zygmund space can be viewed as an interpolation space by Peetre's definition. This interpolation occurs between two Grand Lebesgue spaces, two Small Lebesgue spaces, or even between two Lorentz spaces.

1. INTRODUCTION

The theory of limiting real interpolation of Banach spaces constitutes an extensively explored subject in functional analysis, boasting a long and illustrious historical background. Its inception can be attributed to the pioneering work of J. Lions and A. Peetre during the 1960s. Subsequently, this theory has been extensively investigated, attracting a multitude of research endeavors. Moreover, its applications span diverse domains within mathematics and engineering.

Interpolation helps us understand how functions change and adapt as we move between different Lorentz spaces, shedding light on the behavior of functions with varying levels of integrability and summability. Among the significant applications, one finds its relevance in scrutinizing small and grand Lebesgue spaces. These spaces are of particular interest, and the theory of limiting real

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interpolation plays a pivotal role in their study. For a comprehensive overview on real interpolation theory, we refer to [1–5].

Let Ω be a bounded (open) set in \mathbb{R}^d with measure 1. The grand Lebesgue space $L^{p),q}(\Omega)$ was first introduced by Iwaniec and Sbordone [6], and the concept of small Lebesgue spaces $L^{(p,q)}(\Omega)$ was first presented by A. Fiorenza in [7]. Over the years, these spaces have been further developed and generalized, leading to its widespread use in modern mathematics and its application in areas such as boundary value problems [8], geometric function theory [9], probability and statistics [10], Besov embeddings [11] and harmonic analysis [12]. The Grand and Small Lebesgue spaces, with its more flexible definition of integrability, provided a powerful tool for studying functions and opened up new avenues for research in a variety of fields. Many recent studies have been done regarding both Grand and Small Lebesgue spaces, as can be seen [13–17].

The characterization of the real interpolation spaces $(L^{(p,\alpha)}, L^{(r,\alpha)})_{\theta,q}$ as the well-established Lorentz-Zygmund spaces was presented in ([18], Theorem 3.4). Here $(\cdot, \cdot)_{\theta,q}$ denotes the classical real interpolation method (see Section 3). The investigation into the limiting real interpolation remains unexplored, and the objective of this paper is to fill this gap. More precisely, we will characterize the interpolation spaces $(L^{r,\alpha}, L^{p),\alpha})_{(\theta,b),q}$ for Grand Lebesgue spaces and we also establish the real interpolation results $(L^{(r,\alpha)}, L^{(p,\alpha)})_{(\theta,b),q}$ for Small Lebesgue spaces when $\theta = 0$ and $\theta = 1$. Moreover, our motivation to prove the limiting real interpolation spaces of Lorentz spaces $(L_{r,q}, L_{p,q})_{(\theta,b),q}$ and Besov spaces modelled on Lorentz-Zygmund spaces came from the results about embeddings in [19] and from the concept that suggested by H.-J. Schmeisser in September 2014 and related results regarding the limiting real interpolation of Lebesgue spaces, and limiting real interpolation for Besov spaces modelled on Zygmund spaces proved by Cobos and Domínguez [20].

In Section 2, we lay the foundation by introducing essential notations. We provide an overview of fundamental concepts in function spaces and interpolation theory, which are essential for understanding this paper. The key findings of our study are presented in Section 3. In section 3, we introduce the Besov spaces modelled on Lorentz-Zygmund spaces instead of classical Lebesgue and Zygmund spaces and establish the interpolation results for the defined space.

2. PRELIMINARIES

Let $\mathcal{D}(\mathbb{R}^d)$ be the collection of all complex-valued infinitely differentiable functions on \mathbb{R}^d and $\mathcal{D}'(\mathbb{R}^d)$ is its dual space.

Consider the C^∞ function ψ_0 on \mathbb{R}^d satisfying:

$$\text{supp } \psi_0 \subset \{y \in \mathbb{R}^d : |y| \leq 2\} \quad \text{and} \quad \psi_0(y) = 1 \quad \text{if } |y| \leq 1. \quad (2.1)$$

For any integer $k \in \mathbb{N}$, define $\psi_k(y)$ as:

$$\psi_k(y) = \psi_0(2^{-k}y) - \psi_0(2^{-k+1}y), \quad y \in \mathbb{R}^d. \quad (2.2)$$

It follows that $\text{supp } \psi_k \subset \{y \in \mathbb{R}^d : 2^{k-1} \leq |y| \leq 2^{k+1}\}$, for all $k \in \mathbb{N}$, and the series $\sum_{k=0}^{\infty} \psi_k(y)$ converges to 1 for $y \in \mathbb{R}^d$.

Now, for $1 < p < \infty$, $1 \leq q \leq \infty$, and $s \in \mathbb{R}$, the Besov spaces on Ω are defined as follows:

$$B_{p,q}^s = B_{p,q}^s(\Omega) = \left\{ f \in \mathcal{D}'(\Omega) : \|f\|_{B_{p,q}^s} = \left(\sum_{k=0}^{\infty} \left(2^{ks} \|\mathfrak{F}^{-1}(\psi_k \mathfrak{F}f)\|_{L_p} \right)^q \right)^{1/q} < \infty \right\}.$$

Now, we consider the variant of Besov space replacing the classical Lebesgue space with Lorentz spaces denoted as $B_p^s[L_{p,q}]$ (see [21]). For $1 < p < \infty$, $1 \leq q \leq \infty$, and $s \in \mathbb{R}$, the $B_p^s[L_{p,q}]$ spaces on Ω are defined as follows:

$$B_p^s[L_{p,q}] = B_p^s[L_{p,q}](\Omega) = \left\{ f \in \mathcal{D}'(\Omega) : \|f\|_{B_p^s[L_{p,q}]} = \left(\sum_{k=0}^{\infty} 2^{ks} \|\mathfrak{F}^{-1}(\psi_k \mathfrak{F}f)\|_{L_{p,q}} \right)^p < \infty \right\}.$$

Here, \mathfrak{F} represents the Fourier transform, and \mathfrak{F}^{-1} is the inverse Fourier transform (see [22], 3.5.1).

For $1 \leq p, q < \infty$, and $-\infty < b < +\infty$, the Lorentz-Zygmund spaces $L^{p,q}(\log L)_b$ consists of all those functions whose norm:

$$\|f\|_{L^{p,q}(\log L)_b} = \left(\int_0^1 \left(t^{1/p} (1 + |\log t|)^b f^*(t) \right)^q \frac{dt}{t} \right)^{1/q} < \infty.$$

Next, we introduce the Grand Lebesgue space $L^{(p),q}(\Omega)$, where $1 < p < +\infty$ and $q > 0$, defined as:

$$L^{(p),q}(\Omega) = \left\{ f : \Omega \rightarrow \mathbb{R} \text{ measurable, } \|f\|_{(p),q} = \sup_{0 < t < 1} (1 - \log t)^{-\frac{q}{p}} \left(\int_t^1 f^{*p}(\sigma) d\sigma \right)^{\frac{1}{p}} < +\infty \right\}.$$

Additionally, the associated spaces $L^{(p'),q}$, where $\frac{1}{p'} + \frac{1}{p} = 1$ called Small Lebesgue space is defined as:

$$L^{(p'),q}(\Omega) = \left\{ f : \Omega \rightarrow \mathbb{R} \text{ measurable, } \|f\|_{(p'),q} = \int_0^1 (1 - \log t)^{-\frac{q}{p'} + q - 1} \left(\int_0^t f^{*p'}(\sigma) d\sigma \right)^{\frac{1}{p'}} \frac{dt}{t} < +\infty \right\},$$

(see [23]). Here, f^* represents the decreasing rearrangement of $|f|$, which can be thought of as the generalized inverse of the distribution function:

$$D_f(t) = \text{measure } \{y \in \Omega, |f(y)| > t\}, \quad t \in \mathbb{R}_+.$$

In the context of inequalities, $f \lesssim g$ signifies the existence of an independent constant L (and non-negative functions f and g) such that $f \leq Lg$. Additionally, the notation $f \sim g$ implies that both $f \lesssim g$ and $g \lesssim f$ hold true.

Finally, when considering two function spaces, A_0 and A_1 , we denote the embedding of A_0 into A_1 as $A_0 \hookrightarrow A_1$.

2.1. Real Interpolation. In this subsection, we recall the definitions of the K -functional and the limiting real interpolation spaces. We refer to the books [1,2] for the basic theory on the K functional.

Let us begin by fixing some basic notation. Let A_0 and A_1 be topological vector spaces. Compatible couple (resp. Banach couple) $\bar{A} = (A_0, A_1)$ is an entity consisting of two topological vector spaces (resp. Banach spaces) A_0 and A_1 both embedded in some Hausdorff topological vector space X .

Let (A_0, A_1) be a compatible couple. Then $A_0 \cap A_1$ is a Banach space with norm defined by

$$\|f\|_{A_0 \cap A_1} = \max\{\|f_0\|_{A_0}, \|f_1\|_{A_1}\}, \quad f_0 \in A_0 \text{ and } f_1 \in A_1.$$

Further, the norm of $A_0 + A_1$ is given by

$$\|f\|_{A_0 + A_1} = \inf_{f=f_0+f_1} \{\|f_0\|_{A_0} + \|f_1\|_{A_1}\}, \quad f_0 \in A_0 \text{ and } f_1 \in A_1.$$

Let $t > 0$ be any fixed finite number and let $(A_0, A_1) = \bar{A}$ be a compatible couple, then

$$K(f, t; A_0, A_1) = \inf_{f=f_0+f_1} \{\|f_0\|_{A_0} + t\|f_1\|_{A_1}\}, \quad f \in A_0 + A_1,$$

is the K -functional. It is denoted by $K(f, t)$.

Let $1 \leq p < \infty$ and $0 < \theta < 1$, then the K interpolation space $K_{\theta,p}(\bar{E}) = \bar{A}_{\theta,p;K}$ consists of all those $f \in A_0 + A_1$ whose norm $\|f\|_{\bar{A}_{\theta,p;K}}$ is finite,

$$\|f\|_{\bar{A}_{\theta,p;K}} = \left(\int_0^\infty (K(f, t)t^{-\theta})^p \frac{dt}{t} \right)^{1/p} < \infty,$$

if $p = \infty$, then

$$\|f\|_{\bar{A}_{\theta,\infty;K}} = \sup_{0 < t < \infty} t^{-\theta} K(f, t).$$

The following extension of the real method will be important for our later considerations.

Let $0 < p \leq \infty$ and (A_0, A_1) be a compatible couple. The space $(A_0, A_1)_{F,p}$ is collection of all those $f \in A_0 + A_1$ such that

$$\|f\|_{(A_0, A_1)_{F,p}} = \left(\int_0^\infty \left(\frac{K(f, t)}{F(t)} \right)^p \frac{dt}{t} \right)^{1/p} < \infty,$$

here F is nonnegative measurable function on $(0, \infty)$ and

$$F(t) = \begin{cases} t^{-\theta_0} (1 + |\log t|)^{\beta_\infty} & \text{for } 1 < t < \infty \\ t^{-\theta_1} (1 + |\log t|)^{\beta_0} & \text{for } 0 < t \leq 1, \end{cases}$$

with $\theta_0, \theta_1 \in [0, 1]$ and $\beta_0, \beta_\infty \in \mathbb{R}$. When $\theta = \theta_0 = \theta_1$, the interpolation space $(A_0, A_1)_{F,p}$ denote as $(A_0, A_1)_{\theta,p,\mathbb{A}}$, here $\mathbb{A} = (\beta_0, \beta_\infty)$. If $\beta = \beta_0 = \beta_\infty$ then space simply represent as $(A_0, A_1)_{\theta,p,\beta}$,

see [24,25].

Let (A_0, A_1) be a compatible couple. The space $(A_0, A_1)_{(\theta, \tau), p}$ is the collection of all $f \in A_0 + A_1$ such that

$$\|f\|_{(A_0, A_1)_{(\theta, \tau), p}} = \left(\int_0^1 \left(\frac{K(f, t)t^{-\theta}}{(1 - \log t)^\tau} \right)^p \frac{dt}{t} \right)^{1/p} < \infty.$$

Here $\tau \in \mathbb{R}$ and $\theta = 0$ or 1 .

3. CHARACTERIZATION OF THE INTERPOLATION SPACES

This section explores the Limiting Real Interpolation results for Grand Lebesgue spaces and Small Lebesgue spaces. We will also delve into various auxiliary results to support our core findings, conducting a comprehensive examination within this section.

Following that, our attention will shift towards demonstrating results about the Lorentz spaces. We will also introduce function spaces, referred to as Besov spaces, which are structured upon the foundation of Lorentz-Zygmund spaces. Moreover, in the latter portion of this section, we will prove the Limiting Real Interpolation outcomes for these newly defined spaces.

The lemma presented below is essential in proving the Theorem 3.1. It serves as a crucial foundation upon which the theorem’s demonstration relies.

Lemma 3.1. *Let $0 < \theta < 1$, $1 < r < \infty$, $\alpha > 0$ and β be two real numbers. Then*

$$I := \int_0^1 \left[t^\theta (1 - \log t)^\beta \sup_{t \leq s < 1} (1 - \log s)^{-\alpha} f^*(s) \right]^r \frac{dt}{t} \sim \int_0^1 \left[t^\theta (1 - \log t)^{\beta - \alpha} f^*(t) \right]^r \frac{dt}{t}.$$

Proof. Since $\sup_{t \leq s < 1} (1 - \log s)^{-\alpha} f^*(s) \geq (1 - \log t)^{-\alpha} f^*(t)$, we deduce

$$\int_0^1 \left[t^\theta (1 - \log t)^\beta \sup_{t \leq s < 1} (1 - \log s)^{-\alpha} f^*(s) \right]^r \frac{dt}{t} \geq \int_0^1 \left[t^\theta (1 - \log t)^{\beta - \alpha} f^*(t) \right]^r \frac{dt}{t}.$$

For the reverse inequality, since f^* is non-increasing and $s \rightarrow (1 - \log s)^{-\alpha r}$ increasing then,

$$(1 - \log s)^{-\alpha r} f^{*r}(s) \lesssim \int_{s/2}^s (1 - \log x)^{-\alpha r} f^{*r}(x) \frac{dx}{x}.$$

Therefore,

$$\begin{aligned} I &\lesssim \int_0^1 t^{\theta r} (1 - \log t)^{\beta r} \sup_{t \leq s < 1} \left(\int_{s/2}^s (1 - \log x)^{-\alpha r} f^{*r}(x) \frac{dx}{x} \right) \frac{dt}{t} \\ &\lesssim \int_0^1 t^{\theta r} (1 - \log t)^{\beta r} \int_{t/2}^1 \left((1 - \log x)^{-\alpha r} f^{*r}(x) \frac{dx}{x} \right) \frac{dt}{t}. \end{aligned}$$

By using Fubini’s theorem,

$$I \lesssim \int_0^{\frac{1}{2}} (1 - \log x)^{-\alpha r} f^{*r}(x) \left(\int_0^{2x} t^{\theta r} (1 - \log t)^{\beta r} \frac{dt}{t} \right) \frac{dx}{x}$$

$$+ \int_{\frac{1}{2}}^1 (1 - \log x)^{-\alpha r} f^{*r}(x) \left(\int_0^1 t^{\theta r} (1 - \log t)^{\beta r} \frac{dt}{t} \right) \frac{dx}{x}.$$

For $x \in [\frac{1}{2}, 1]$, we have $2x \geq 1$. Therefore,

$$\begin{aligned} I &\lesssim \int_0^1 (1 - \log x)^{-\alpha r} f^{*r}(x) \left(\int_0^{2x} t^{\theta r} (1 - \log t)^{\beta r} \frac{dt}{t} \right) \frac{dx}{x} \\ &\lesssim \int_0^1 (1 - \log x)^{-\alpha r} f^{*r}(x) x^{\theta r} (1 - \log x)^{\beta r} \frac{dx}{x}. \end{aligned}$$

Thus,

$$\int_0^1 \left[t^\theta (1 - \log t)^\beta \sup_{t \leq s < 1} (1 - \log s)^{-\alpha} f^*(s) \right]^r \frac{dt}{t} \lesssim \int_0^1 \left[t^\theta (1 - \log t)^{\beta - \alpha} f^*(t) \right]^r \frac{dt}{t}.$$

This completes the proof. \square

The following result will establish the concept of limiting real interpolation for $\theta = 0$. By demonstrating this result, we can affirm that the Lorentz-Zygmund spaces are the interpolation spaces for Grand Lebesgue spaces.

Theorem 3.1. *Let $1 < r < p < \infty$, $1 < q < \infty$, $\eta > 0$ and $-\infty < b < \infty$. Then*

$$(L^{r,\eta}, L^{p,\eta})_{(0, -(b + \frac{\eta}{r}))_q} \sim L_{r,q}(\log L)_b.$$

Proof. Let $f \in (L^{r,\eta}, L^{p,\eta})_{(0, -(b + \frac{\eta}{r}))_q}$ and suppose that φ denotes the inverse function of $t^{\frac{1}{r} - \frac{1}{p}}(1 - \log t)^{\frac{-\eta}{r} + \frac{\eta}{p}}$. Applying [18, Theorem 3.3], we have

$$\begin{aligned} \|f\|_{(L^{r,\eta}, L^{p,\eta})_{(0, -(b + \frac{\eta}{r}))_q}}^q &= \int_0^1 \left[(1 - \log t)^{b + \frac{\eta}{r}} K(f, t; L^{r,\eta}, L^{p,\eta}) \right]^q \frac{dt}{t} \\ &\sim \int_0^1 \left[(1 - \log t)^{b + \frac{\eta}{r}} \sup_{0 < s < \varphi(t)} (1 - \log s)^{\frac{-\eta}{r}} \left(\int_s^{\varphi(t)} f^{*r}(\tau) d\tau \right)^{\frac{1}{r}} \right]^q \frac{dt}{t} \\ &\quad + \int_0^1 \left[t(1 - \log t)^{b + \frac{\eta}{r}} \sup_{\varphi(t) < s < 1} (1 - \log s)^{\frac{-\eta}{p}} \left(\int_s^1 f^{*p}(\tau) d\tau \right)^{\frac{1}{p}} \right]^q \frac{dt}{t}. \end{aligned}$$

We know that

$$t = \varphi(t)^{\frac{1}{r} - \frac{1}{p}} (1 - \log \varphi(t))^{\frac{-\eta}{r} + \frac{\eta}{p}},$$

and

$$1 - \log t \sim 1 - \log \varphi(t).$$

Therefore,

$$\begin{aligned} \|f\|_{(L^{r,\eta}, L^{p,\eta})_{(0, -(b + \frac{\eta}{r}))_q}}^q &\sim \int_0^1 \left[(1 - \log \varphi(t))^{b + \frac{\eta}{r}} \sup_{0 < s < \varphi(t)} (1 - \log s)^{\frac{-\eta}{r}} \left(\int_s^{\varphi(t)} f^{*r}(\tau) d\tau \right)^{\frac{1}{r}} \right]^q \frac{dt}{t} \\ &\quad + \int_0^1 \left[\varphi(t)^{\frac{1}{r} - \frac{1}{p}} (1 - \log \varphi(t))^{\frac{-\eta}{r} + \frac{\eta}{p}} (1 - \log \varphi(t))^{b + \frac{\eta}{r}} \sup_{\varphi(t) < s < 1} (1 - \log s)^{\frac{-\eta}{p}} \left(\int_s^1 f^{*p}(\tau) d\tau \right)^{\frac{1}{p}} \right]^q \frac{dt}{t} \end{aligned}$$

$$:= I_1 + I_2.$$

Starting from the expression for the first term I_1 , substituting the integrand of I_1 by the above equation and performing the variable transformation $x = \varphi(t)$, or equivalently $t = \psi(x)$, result in:

$$I_1 \sim \int_0^1 \left[(1 - \log x)^{b+\frac{\eta}{r}} \sup_{0 < s < x} (1 - \log s)^{-\frac{\eta}{r}} \left(\int_s^x f^{*r}(\tau) d\tau \right)^{\frac{1}{r}} \right]^q \frac{dx}{x}.$$

Thus by using the monotony property of f^* , we have

$$\begin{aligned} I_1 &\gtrsim \int_0^1 \left[(1 - \log x)^{b+\frac{\eta}{r}} (1 - \log \frac{x}{2})^{-\frac{\eta}{r}} x^{\frac{1}{r}} f^*(x) \right]^q \frac{dx}{x} \\ &\gtrsim \int_0^1 \left[x^{\frac{1}{r}} (1 - \log x)^b f^*(x) \right]^q \frac{dx}{x} \\ &= \|f\|_{L_{r,q}(\log L)_b}^q. \end{aligned}$$

We have established that

$$\|f\|_{(L^r)_{q,(L^p)_{q,b}}}_{(0, -(b+\frac{q}{r}))_{q,b}} \gtrsim \|f\|_{L_{r,q}(\log L)_b}.$$

For the converse relation, let's take $\epsilon > 0$ to be sufficiently small such that

$$\frac{\epsilon q}{r} - 1 < 0,$$

$$\begin{aligned} I_1 &\sim \int_0^1 \left[(1 - \log x)^{b+\frac{\eta}{r}} \sup_{0 < s < x} (1 - \log s)^{-\frac{\eta}{r}} \left(\int_s^x \tau^{\epsilon-1} \tau^{1-\epsilon} f^{*r}(\tau) d\tau \right)^{\frac{1}{r}} \right]^q \frac{dx}{x} \\ &\lesssim \int_0^1 \left[x^{\frac{\epsilon}{r}} (1 - \log x)^b \sup_{0 < t < x} t^{\frac{1-\epsilon}{r}} f^*(t) \right]^q \frac{dx}{x}. \end{aligned}$$

By using the [18, Proposition 3.2], we have

$$I_1 \lesssim \int_0^1 x^{\frac{\epsilon}{r}q-1} (1 - \log x)^{bq} \left(\int_0^x s^{\frac{1-\epsilon}{r}q-1} f^{*q}(s) ds \right) dx.$$

Applying Hardy type inequality [18, Theorem 2.1] with respect to $\frac{\epsilon q}{r} - 1 < 0$

$$\begin{aligned} I_1 &\lesssim \int_0^1 x^{\frac{\epsilon}{r}q} (1 - \log x)^{bq} x^{\frac{1-\epsilon}{r}q-1} f^{*q}(x) dx \\ &= \int_0^1 \left[x^{\frac{\epsilon}{r}} (1 - \log x)^b x^{\frac{1-\epsilon}{r}} f^*(x) \right]^q \frac{dx}{x} \\ &= \int_0^1 \left[x^{\frac{1}{r}} (1 - \log x)^b f^*(x) \right]^q \frac{dx}{x} \\ &= \|f\|_{L_{r,q}(\log L)_b}^q. \end{aligned}$$

Now, we need only to show that $I_2 \lesssim \|f\|_{L_{r,q}(\log L)_b}^q$. Making the change of variables, $t = \psi(x) \iff x = \varphi(t)$

$$\begin{aligned}
I_2 &= \int_0^1 \left[x^{\frac{1}{r}-\frac{1}{p}} (1-\log x)^{-\frac{\eta}{r}+\frac{\eta}{p}} (1-\log x)^{b+\frac{\eta}{r}} \sup_{x<t<1} (1-\log t)^{-\frac{\eta}{p}} \left(\int_t^1 f^{*p}(s) ds \right)^{\frac{1}{p}} \right]^q \frac{dx}{x} \\
&= \int_0^1 \left[x^{\frac{1}{r}-\frac{1}{p}} (1-\log x)^{b+\frac{\eta}{p}} \sup_{x<t<1} (1-\log t)^{-\frac{\eta}{p}} \left(\int_t^1 f^{*p}(s) ds \right)^{\frac{1}{p}} \right]^q \frac{dx}{x}.
\end{aligned}$$

Apply Hardy's inequalities and Lemma 3.1, we have

$$\begin{aligned}
I_2 &\lesssim \int_0^1 \left[t^{\frac{1}{r}} (1-\log t)^b f^*(t) \right]^q \frac{dt}{t} \\
&= \|f\|_{L_{r,q}(\log L)_b}^q.
\end{aligned}$$

□

In the following result we will establish the limiting real interpolation when $\theta = 1$. By proving this result, we can assure that Lorentz-Zygmund spaces serve as interpolation spaces for Grand Lebesgue spaces.

Theorem 3.2. *Let $1 < r < p < \infty$, $1 < q < \infty$, $\eta > 0$ and $-\infty < b < \infty$. Then*

$$\left(L^{(r),\eta}, L^{(p),\eta} \right)_{(1, -(b+\frac{\eta}{p}))_q} \sim L_{p,q}(\log L)_b.$$

Proof. By employing the same argument as in the proof of Theorem 3.1, the desired result can be checked. □

Next, we will prove the limiting real interpolation spaces for Small Lebesgue spaces. Based on the results presented earlier, we can prove the interpolation spaces associated with Small Lebesgue spaces.

Theorem 3.3. *Let $1 < r < p < \infty$, $1 < q < \infty$, $\eta > 0$ and $-\infty < b < \infty$. Then*

$$\left(L^{(r,\eta)}, L^{(p,\eta)} \right)_{(0, -b+\eta+\frac{\eta}{p})_q} \sim L_{r,q}(\log L)_b,$$

with $\frac{1}{\eta} = \frac{1}{r} + \frac{1}{p}$.

Proof. By the duality result on interpolation spaces (see [26], [27]) and using [28, Remark 2.1] along with [29, Lemma 4.1], we derive

$$\begin{aligned}
\left[\left(L^{(r,\eta)}, L^{(p,\eta)} \right)_{(0, -b+\eta+\frac{\eta}{p})_q} \right]' &= \left[\left(L^{(r,\eta)}, L^{(p,\eta)} \right)_{0,q,(b-\eta-\frac{\eta}{p}, -1)} \right]' \\
&= \left[\left(L^{(p,\eta)}, L^{(r,\eta)} \right)_{1,q,(-1, b-\eta-\frac{\eta}{p})} \right]' \\
&= \left(L^{(p'),\eta}, L^{(r'),\eta} \right)_{1,q',(-b+\eta+\frac{\eta}{p}-1, 0)} \\
&= \left(L^{(p'),\eta}, L^{(r'),\eta} \right)_{1,q',(-b+\eta-\frac{\eta}{p}, 0)} \\
&= \left(L^{(p'),\eta}, L^{(r'),\eta} \right)_{1,q',(-b+\eta-\eta(1-\frac{1}{p}), 0)}
\end{aligned}$$

$$\begin{aligned} &= (L^{p'}, \eta, L^{r'}, \eta)_{1, q', (-b + \eta - \eta + \frac{\eta}{r'}, 0)} \\ &= (L^{p'}, \eta, L^{r'}, \eta)_{(1, (b - \frac{\eta}{r'})), q'} \\ &= (L^{p'}, \eta, L^{r'}, \eta)_{(1, -(-b + \frac{\eta}{r'})), q'}. \end{aligned}$$

By using Theorem 3.2, we have

$$\left[(L^{(r, \eta)}, L^{(p, \eta)})_{(0, (-b + \eta + \frac{\eta}{p}))}, q \right]' = L_{r', q'}(\log L)_{-b}.$$

Thus,

$$\begin{aligned} (L^{(r, \eta)}, L^{(p, \eta)})_{(0, (-b + \eta + \frac{\eta}{p}))}, q &= \left[(L^{(r, \eta)}, L^{(p, \eta)})_{(0, (-b + \eta + \frac{\eta}{p}))}, q \right]'' \\ &= \left[L_{r', q'}(\log L)_{-b} \right]' \\ &= L_{r, q}(\log L)_b. \end{aligned}$$

□

Theorem 3.3 established the limiting real interpolation when $\theta = 0$. The following result demonstrates the limiting interpolation when $\theta = 1$, which can be proved with similar arguments as in Theorem 3.3.

Theorem 3.4. *Let $1 < r < p < \infty, 1 < q < \infty, \eta > 0$ and $-\infty < b < \infty$. Then*

$$(L^{(r, \eta)}, L^{(p, \eta)})_{(1, -b + \eta + \frac{\eta}{r}), q} \sim L_{r, q}(\log L)_b,$$

with $\frac{1}{\eta} = \frac{1}{r} + \frac{1}{p}$.

Next, we will show that Lorentz-Zygmund spaces are interpolation spaces of Lorentz spaces. We will prove the results for the case when $\theta = 1$ and $\theta = 0$ respectively.

Theorem 3.5. *Let $1 < r < p < \infty, 1 < q < \infty$ and $b < 0$. Then*

$$(L_{r, q}, L_{p, q})_{(1, -(b-1/q)), q} \sim L_{p, q}(\log L)_b.$$

Proof. Let $\frac{1}{\beta} = \frac{1}{r} - \frac{1}{p}$. then by applying Holmstedt's formula [30, Theorem 4.2], we have

$$\begin{aligned} \|f\|_{(1, -(b-1/q)), q}^q &\sim \int_0^1 \left((1 - \log t)^{b-1/q} t^{-1} \left(\int_0^{t^\beta} (s^{1/r} f^*(s))^q \frac{ds}{s} \right)^{1/q} \right)^q \frac{dt}{t} \\ &\quad + \int_0^1 \left((1 - \log t)^{b-\frac{1}{q}} \int_{t^\beta}^1 (s^{1/p} f^*(s))^q \frac{ds}{s} \right)^q \frac{dt}{t} \\ &= I_1 + I_2. \end{aligned}$$

For the first term, making change of variable,

$$I_1 = \int_0^1 \left((1 - \log t)^{b-1/q} t^{-1} \left(\int_0^{t^\beta} (s^{1/r} f^*(s))^q \frac{ds}{s} \right)^{1/q} \right)^q \frac{dt}{t}$$

$$\sim \int_0^1 \left((1 - \log t)^{b-1/q} t^{-1/\beta} \left(\int_0^t (s^{1/r} f^*(s))^q \frac{ds}{s} \right)^{1/q} \right)^q \frac{dt}{t}.$$

By using Hardy's inequality

$$\begin{aligned} I_1 &\lesssim \int_0^1 \left((1 - \log t)^{b-1/q} t^{-1/\beta} t^{-1/r} f^*(t) \right)^q \frac{dt}{t} \\ &= \int_0^1 \left((1 - \log t)^{b-1/q} t^{1/p} f^*(t) \right)^q \frac{dt}{t} \\ &\lesssim \int_0^1 \left(t^{1/p} (1 - \log t)^b f^*(t) \right)^q \frac{dt}{t} \sim \|f\|_{L_{p,q}(\log L)_b}^q. \end{aligned}$$

Now for the second term

$$\begin{aligned} I_2 &= \int_0^1 (s^{1/p} f^*(s))^q \int_0^{s^{1/\beta}} (1 - \log t)^{bq-1} \frac{dt}{t} \frac{ds}{s} \\ &\sim \int_0^1 s^{q/p} f^*(s)^q (1 - \log s)^{bq} \frac{ds}{s} \\ &= \int_0^1 \left(s^{1/p} (1 - \log s)^b f^*(s) \right)^q \frac{ds}{s} \\ &\sim \|f\|_{L_{p,q}(\log L)_b}^q. \end{aligned}$$

Hence $(L_{r,q}, L_{p,q})_{(1, -(b-1/q)), q} \sim L_{p,q}(\log L)_b$.

□

In a similar way the following result can be proved when $\theta = 0$.

Theorem 3.6. *Let $1 < q < \infty$, $1 < r < p < \infty$ and $b > 0$. Then*

$$(L_{r,q}, L_{p,q})_{(0, -b+1/q), q} \sim L_{p,q}(\log L)_b.$$

□

Considering the representation of Zygmund spaces using L_p spaces and the construction of Besov spaces modelled on Zygmund spaces (see [20]), it is a logical step to introduce the following Besov spaces modelled on Lorentz-Zygmund spaces.

Let $b, s \in \mathbb{R}$ and $1 < p < \infty, 1 < q < \infty$. The space $B_q^s(L_{p,q}(\log L)_b) = B_q^s(L_{p,q}(\log L)_b)(\Omega)$ consists of all $f \in \mathcal{D}'(\Omega)$ such that

$$\|f\|_{B_q^s(L_{p,q}(\log L)_b)} = \left(\sum_{n=0}^{\infty} (2^{ns} \|\mathfrak{F}^{-1}(\psi_n \mathfrak{F}f)\|_{L_{p,q}(\log L)_b})^q \right)^{1/q} < \infty.$$

Here ψ_0, ψ_1, \dots are the functions given by (2.1) and (2.2).

Before establishing the Limiting Real Interpolation results for modelled space, we recall some notation.

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $1 < p < \infty$. Given any sequence $(\lambda_k)_{k \in \mathbb{N}_0}$ of positive numbers and any sequence $(B_k)_{k \in \mathbb{N}_0}$ of Banach spaces, we put

$$l_p(\lambda_k B_k) = \left\{ x = (x_k) : x_k \in B_k \text{ and } \|x\|_{l_p(\lambda_k B_k)} = \left(\sum_{k=0}^{\infty} (\lambda_k \|x_k\|_{B_k})^p \right)^{1/p} < \infty \right\}.$$

In the following theorem we will show the Lorentz-Zygmund variant of classical Besov space as interpolation of Besov spaces modelled on Lorentz space.

Theorem 3.7. *Let $b < 0, s \in \mathbb{R}$ and $1 < u < r < p < \infty, 1 < q < \infty$. Then;*

- i)** $(B_q^s[L_{r,q}], B_q^s[L_{p,q}])_{(1, -(b-1/q)), q} \sim B_q^s(L_{p,q}(\log L)_b)$;
- ii)** $(B_q^s[L_{u,q}], B_q^s[L_{r,q}])_{(0, b+1/q), q} \sim B_q^s(L_{u,q}(\log L)_{-b})$.

Proof. Firstly, we will prove the result i).

Suppose ψ_k are the functions given by expressions (2.1) and (2.2).

Write $\tilde{\psi}_k = \psi_{k-1} + \psi_k + \psi_{k+1}$ and $\psi_{-1} \equiv 0$. For $f \in \mathcal{D}'(\Omega)$, put $\mathcal{S}f = (\mathfrak{F}^{-1}(\psi_k \mathfrak{F}f))$. The restrictions

$$\begin{aligned} \mathcal{S} : B_q^s[L_{r,q}] &\rightarrow l_q(2^{ks}L_{r,q}), \\ \mathcal{S} : B_q^s[L_{p,q}] &\rightarrow l_q(2^{ks}L_{p,q}), \\ \mathcal{S} : B_q^s(L_{p,q}(\log L)_b) &\rightarrow l_q(2^{ks}L_{p,q}(\log L)_b), \end{aligned}$$

are bounded. Let $\mathcal{R}(f_k) = \sum_{k=0}^{\infty} (\mathfrak{F}^{-1}(\tilde{\psi}_k \mathfrak{F}f_k))$. The restrictions

$$\begin{aligned} \mathcal{R} : l_q(2^{ks}L_{r,q}) &\rightarrow B_q^s[L_{r,q}], \\ \mathcal{R} : l_q(2^{ks}L_{p,q}) &\rightarrow B_q^s[L_{p,q}], \\ \mathcal{R} : l_q(2^{ks}L_{p,q}(\log L)_b) &\rightarrow B_q^s(L_{p,q}(\log L)_b), \end{aligned}$$

are bounded and $\mathcal{R}(\mathcal{S}f) = f$ holds with convergence in $\mathcal{D}'(\Omega)$. That is, $B_q^s[L_{r,q}]$ (respectively $B_q^s(L_{p,q}(\log L)_b)$ and $B_q^s[L_{p,q}]$) is a retract of $l_q(2^{ks}L_{r,q})$ (respectively of $l_q(2^{ks}L_{p,q}(\log L)_b)$ and $l_q(2^{ks}L_{p,q})$) [31, Definition 1.2.4]. For the vector valued spaces, [20, Lemma 3.5] and Theorem 3.5 conclude that,

$$\begin{aligned} (l_q(2^{ks}L_{r,q}), l_q(2^{ks}L_{p,q}))_{(1, -(1/q+b)), q} &= l_q(2^{ks}(L_{r,q}, L_{p,q}))_{(1, -(1/q+b)), q} \\ &= l_q(2^{ks}L_{p,q}(\log L)_b). \end{aligned}$$

Therefore, using Theorem 1.2.4 [31],

$$(B_q^s[L_{r,q}], B_q^s[L_{p,q}])_{(1, -(1/q+b)), q} \sim B_q^s(L_{p,q}(\log L)_b).$$

The second statement can be verified by using [20, Lemma 3.6] and Theorem 3.6. □

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