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Stability of Cubic Functional Equation in IFN-Space and 2-Banach Space: Direct Method and Fixed-Point Approach

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Abstract. In this paper, the Hyers-Ulam stability of a cubic functional equation in finite-dimensional settings is examined in this study with particular attention to 2-Banach spaces and Intuitionistic Fuzzy Normed (IFN) spaces. We provide strong stability conditions for the cubic functional equation using both fixed-point approaches and direct methods. Our findings show that combinations of norm powers including sums and products may be used to efficiently manage the stability behavior. Moreover, given certain conditions, the related cubic mappings are guaranteed to be unique. We give specific instances that demonstrate the efficacy of our method in order to demonstrate the theoretical conclusions' strength and application.

1. Introduction

Over the past eight decades, there has been a significant rise in interest in the study of stability for functional equations. Ulam [28] initially proposed the idea in 1940 at the University of Wisconsin's Mathematical Colloquium, where he put out a number of difficult and mainly unresolved issues. Among these, the stability of functional equations established a novel and significant line of

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inquiry. Shortly after, in 1941, Hyers [11] established the stability of additive functions via a direct technique, delivering the first affirmative result. One of the most powerful and successful methods for examining the stability of different kinds of functional equations is still this one.

A significant number of research on generalizations of Ulam's problem and Hyers' finding has surfaced after the publication of Hyers' theorem (see, [6]). In 1950, Aoki [2] used powers of norms to reduce the requirements on the Cauchy difference, extending Hyers' theorem to roughly linear transformations in Banach spaces. Maligranda [12] and Rassias [18] separately established the renowned Hyers-Ulam-Aoki-Rassias theorem, which substituted sum-related inequalities for the additive condition. Rassias [19] later developed the theory by substituting products of norm powers for the sum condition. The theory's scope was further expanded by the Ulam-Găvruţă-Rassias stability idea, which was examined in works like [9]. Găvruţă [8] developed what is now known as the generalized Hyers-Ulam-Rassias stability in 1994 by introducing the use of control functions, which generalized all earlier findings.

Many researchers have extensively examined the Hyers-Ulam-Rassias stability of a variety of functional equations during the last three and a half decades (see, for example, [1,7,10,14,25–27]). These advancements remain essential to present functional analysis and its uses. By their embedding in the fuzzy logic framework, IFN-spaces, that integrate the notion of intuitionistic fuzziness with normed linear structures, have greatly expanded the application of classical analysis and algebra. Saadati and Park [24] expanded analytical and topological ideas into intuitionistic fuzzy environments, offering crucial resources for researching compactness, continuity, and convergence in these contexts. The idea of metric spaces has been quite important in analysis in recent years, mostly because of the triangle inequality strength, which offers many methods for examining continuity and convergence (see [3, 16, 17]). This demonstrated the usefulness and relevance of fuzzy approaches in abstract analysis. An improvement of intuitionistic fuzzy normed spaces (IFN-spaces) has greatly facilitated the extension of traditional algebraic theories and analytical into the realm of fuzzy logic by integrating intuitionistic fuzziness with normed linear structures.

The concept of 2-Banach spaces, an extension of traditional Banach spaces, has developed as an effective framework for studying functional equations. In contrast to regular normed spaces, a 2-Banach space has a 2-norm that captures the geometric relationship between each pair of vectors in the space by giving each pair a non-negative real integer. A more flexible and sophisticated examination of stability and functional equation solutions is made possible by this extra structure. 2-Banach spaces, in particular, are ideal for analyzing higher-order functional equations, such as cubic and quartic equations, in which interactions among numerous variables are important. The 2-norm makes it easier to develop direct techniques and generalized fixed-point theorems, which in turn offer useful instruments for proving existence, uniqueness, and other concepts of stability, such as Hyers-Ulam stability. Consequently, 2-Banach spaces remain a stable and developing framework for the analysis of functional equations, both linear and nonlinear.

In this work, we examine the Hyers-Ulam stability of the cubic functional equation

$$f\left(\sum_{1\le i\le m} ia_i\right) = \sum_{1\le i<j<k\le m} f(ia_i + ja_j + ka_k) + (3-m)\sum_{1\le i<j\le m} f(ia_i + ja_j) + \left(\frac{(m^2-5m+6)}{2}\right)\sum_{i=0}^{m-1}(i+1)^3 f(a_{i+1}), \tag{1.1}$$

where $m \ge 3$ is an integer, within the frameworks of 2-Banach spaces and IFN spaces. By employing both the fixed-point method and the direct approach, we establish generalized Hyers-Ulam stability results in these settings, thereby extending the scope of stability theory to more generalized and nonlinear functional-analytic structures.

2. Main Results

In this context, let G be a linear space, $(Z, N'_{\flat_1,\flat_2}, \Lambda)$ represent an intuitionistic fuzzy normed space (IFN-space), and $(T, N_{\flat_1,\flat_2}, \Lambda)$ denote a complete IFN-space.

Definition 2.1. *[5] Let $\flat_1$ and $\flat_2$ denote respectively the membership and non-membership functions of an intuitionistic fuzzy set, defined from $W \times (0,+\infty)$ into $[0,1]$, such that for every $a \in W$ and $t > 0$, the condition*

$$(\flat_1)_a(t) + (\flat_2)_a(t) \le 1$$

holds. The structure $(W, N_{\flat_1,\flat_2}, \Upsilon)$ is called an IFN-space if the upcoming requirements are fulfilled:

i. *W is a vector space,*
ii. *Υ is a continuous t-norm,*
iii. *$N_{\flat_1,\flat_2} : W \times (0,+\infty) \to L^*$,*

and for all $a_1, a_2 \in W$ and $t, s > 0$, the axioms of an intuitionistic fuzzy norm are fulfilled.

(IFN1) $N_{\flat_1,\flat_2}(a_1, 0) = 0_{L^*}$;
(IFN2) $N_{\flat_1,\flat_2}(a_1, t) = 1_{L^*}$ *if and only if* $a_1 = 0$;
(IFN3) $N_{\flat_1,\flat_2}(\flat a_1, t) = N_{\flat_1,\flat_2}\left(a_1, \frac{t}{|\flat|}\right)$, *for every* $\flat \ne 0$;
(IFN4) $N_{\flat_1,\flat_2}(a_1 + a_2, t+s) \ge_{L^*} \Upsilon(N_{\flat_1,\flat_2}(a_1, t), N_{\flat_1,\flat_2}(a_2, s))$.

In this case, $N_{\flat_1,\flat_2}$ is called an IFN, where, $N_{\flat_1,\flat_2}(a_1, t) = ((\flat_1)_{a_1}(t), (\flat_2)_{a_1}(t))$.

Theorem 2.1. *[23]*

Let (G, d) be a generalized complete metric space, and let $\Omega : G \to G$ be a contraction function with $L < 1$ (Lipschitz constant). Then, for any initial point $a_1 \in G$, one of the following statements is true:

$$d\left(\Omega^r a_1, \Omega^{r+1} a_1\right) = \infty \quad \text{for every } r \ge r_0,$$

or there is $r_0 > 0$ such that:

(1) $d\left(\Omega^r a_1, \Omega^{r+1} a_1\right) < \infty$ *for any* $r \ge r_0$;
(2) *the sequence* $\{\Omega^r a_1\}_{r\in\mathbb{N}} \to a_1^*$ *to a fixed point of* Ω;

(3) a_1^* *is the precisely one fixed point of* Ω *in*

$$G^* = \left\{a_2 \in G \,\middle|\, d(\Omega^{r_0} a_1, a_2) < \infty\right\};$$

(4) *for every* $a_2 \in G^*$*, the inequality holds*

$$d(a_2, a_1^*) \leq \frac{1}{1-L} d(\Omega a_2, a_2).$$

2.1. **Analysis of Hyers-Ulam Stability in IFN-spaces.**

2.1.1. *Direct Technique.*

Theorem 2.2. *Let* G *be a linear space, and let* $(Z, N'_{\flat_1,\flat_2}, \Lambda)$ *denote an IFN-space. Assume that a mapping* $g : G^m \to Z$ *is given such that* $0 < \frac{\psi}{2^3} < 1$.

$$N'_{\flat_1,\flat_2}(g(2a, 2a, 0, \ldots, 0), \epsilon) \geq_{L^*} N'_{\flat_1,\flat_2}(\psi\, g(a, a, 0, \ldots, 0), \epsilon), \tag{2.1}$$

with

$$\lim_{l\to\infty} N'_{\flat_1,\flat_2}\Big(g(2^l a_1, 2^l a_2, \ldots, 2^l a_m), 2^{3l}\epsilon\Big) = 1_{L^*}, \tag{2.2}$$

for every $a, a_1, a_2, \ldots, a_m \in G$, $\epsilon > 0$*. If a mapping* $f : G \to T$ *satisfies*

$$N_{\flat_1,\flat_2}(Df(a_1, a_2, \ldots, a_m), \epsilon) \geq_{L^*} N'_{\flat_1,\flat_2}(g(a_1, a_2, \ldots, a_m), \epsilon), \tag{2.3}$$

then the limit

$$N_{\flat_1,\flat_2}\left(Q_3(a) - \frac{f(2^l a)}{2^{3l}}, \epsilon\right) \longrightarrow 1_{L^*} \quad \text{as } l \to \infty \tag{2.4}$$

exists. Moreover, there exists a unique cubic solution $Q_3 : G \to T$ *satisfying equation* (1.1)*, and*

$$N_{\flat_1,\flat_2}(g(a) - Q_3(a), \epsilon) \geq_{L^*} N'_{\flat_1,\flat_2}\Big(g(a, a, 0, \cdots, 0), (m^2 - 5m + 6)\epsilon(2^3 - \psi)\Big), \tag{2.5}$$

for any $a \in G$ *and each* $\epsilon > 0$.

Proof. Fix $a \in G$ and let $\epsilon > 0$. Substituting $(a_1, a_2, a_3, \ldots, a_m) = (a, a, 0, \ldots, 0)$ into (2.3), we obtain

$$N_{\flat_1,\flat_2}\Big((m^2 - 5m + 6)f(2a) - 2^3(m^2 - 5m + 6)f(a), \epsilon\Big) \geq_{L^*} N'_{\flat_1,\flat_2}(g(a, a, 0, \ldots, 0), \epsilon). \tag{2.6}$$

Next, replacing a with $2^l a$ in (2.6) and applying (IFN3), we deduce

$$N_{\flat_1,\flat_2}\left(\frac{f(2^{l+1}a)}{2^3} - f(2^l a), \frac{\epsilon}{2^3(m^2 - 5m + 6)}\right) \geq_{L^*} N'_{\flat_1,\flat_2}\Big(g(2^l a, 2^l a, 0, 0, \cdots, 0), \epsilon\Big). \tag{2.7}$$

Now, by combining condition (2.1) with (IFN3) in (2.7), it follows that

$$N_{\flat_1,\flat_2}\left(\frac{f(2^{l+1}a)}{2^3} - f(2^l a), \frac{\epsilon}{2^3(m^2 - 5m + 6)}\right) \geq_{L^*} N'_{\flat_1,\flat_2}\left(g(a, a, 0, \ldots, 0), \frac{\epsilon}{\psi^l}\right). \tag{2.8}$$

Finally, from (2.8), we clearly deduce that

$$N_{\flat_1,\flat_2}\left(\frac{f(2^{l+1}a)}{2^{3(l+1)}} - \frac{f(2^l a)}{2^{3l}}, \frac{\epsilon}{2^{3(l+1)}(m^2 - 5m + 6)}\right) \geq_{L^*} N'_{\flat_1,\flat_2}\left(g(a, a, 0, \ldots, 0), \frac{\epsilon}{\psi^l}\right). \tag{2.9}$$

Replacing ϵ with $\psi^l\epsilon$ in (2.9), we obtain

$$N_{\flat_1,\flat_2}\left(\frac{\mathsf{f}(2^{l+1}\mathsf{a})}{2^{3(l+1)}}-\frac{\mathsf{f}(2^l\mathsf{a})}{2^{3l}},\left(\frac{\psi^l\epsilon}{2^{3(l+1)}(\mathsf{m}^2-5\mathsf{m}+6)}\right)\right)\geq_{L^*} N'_{\flat 1,\flat_2}\left(\mathsf{g}(\mathsf{a},\mathsf{a},0,\cdots,0),\epsilon\right). \tag{2.10}$$

Clearly,

$$\frac{\mathsf{f}(2^l\mathsf{a})}{2^{3l}}-\mathsf{f}(\mathsf{a})=\sum_{i=0}^{l-1}\left(\frac{\mathsf{f}(2^{i+1}\mathsf{a})}{2^{3(i+1)}}-\frac{\mathsf{f}(2^i\mathsf{a})}{2^{3i}}\right). \tag{2.11}$$

Using (2.10) together with (2.11), we deduce that

$$\begin{aligned}
&N_{\flat_1,\flat_2}\left(\frac{\mathsf{f}(2^l\mathsf{a})}{2^{3l}}-\mathsf{f}(\mathsf{a}),\sum_{i=0}^{l-1}\frac{\psi^i\epsilon}{2^{3(i+1)}(\mathsf{m}^2-5\mathsf{m}+6)}\right)\\
\geq_L\quad & \Lambda_{i=0}^{l-1}\left\{N'_{\flat 1,\flat_2}\left(\frac{\mathsf{f}(2^{i+1}\mathsf{a})}{2^{3(i+1)}}-\frac{\mathsf{f}(2^i\mathsf{a})}{2^{3i}},\frac{\psi^i\epsilon}{2^{3(i+1)}(\mathsf{m}^2-5\mathsf{m}+6)}\right)\right\}\\
\geq_{L^*}\quad & N'_{\flat 1,\flat_2}\left(\mathsf{g}(\mathsf{a},\mathsf{a},0,\cdots,0),\epsilon\right).
\end{aligned} \tag{2.12}$$

Next, substituting a by 2^jv in (2.12) and applying (2.1), we arrive at

$$N_{\flat_1,\flat_2}\left(\frac{\mathsf{f}(2^{l+j}\mathsf{a})}{2^{3(l+j)}}-\frac{\mathsf{f}(2^j\mathsf{a})}{2^{3j}},\sum_{i=0}^{l-1}\frac{\psi^i\epsilon}{2^{3(i+j)}(\mathsf{m}^2-5\mathsf{m}+6)}\right)\geq_{L^*} N'_{\flat 1,\flat_2}\left(\mathsf{g}(\mathsf{a},\mathsf{a},0,\cdots,0),\epsilon\psi^{-j}\right), \tag{2.13}$$

valid for all j and l are greater than or equal to 0. Finally, setting $\epsilon=\psi^j\epsilon$ in (2.13), we obtain

$$N_{\flat_1,\flat_2}\left(\frac{\mathsf{f}(2^{l+j}\mathsf{a})}{2^{3(l+j)}}-\frac{\mathsf{f}(2^j\mathsf{a})}{2^{3j}},\sum_{i=j}^{l+j-1}\frac{\psi^i\epsilon}{2^{3(i+1)}(\mathsf{m}^2-5\mathsf{m}+6)}\right)\geq_{L^*} N'_{\flat 1,\flat_2}\left(\mathsf{g}(\mathsf{a},\mathsf{a},0,\cdots,0),\epsilon\right). \tag{2.14}$$

Utilizing condition (IFN3) in (2.14), we obtain

$$N_{\flat_1,\flat_2}\left(\frac{\mathsf{f}(2^{l+j}\mathsf{a})}{2^{3(l+j)}}-\frac{\mathsf{f}(2^j\mathsf{a})}{2^{3j}},\epsilon\right)\geq_{L^*} N'_{\flat_1,\flat_2}\left(\mathsf{g}(\mathsf{a},\mathsf{a},0,\cdots,0),\frac{\epsilon}{\sum_{i=j}^{l+j-1}\frac{\psi^i}{2^{3(i+1)}(\mathsf{m}^2-5\mathsf{m}+6)}}\right), \tag{2.15}$$

for every $j,l\geq 0$. Since $0<\psi<2^3$, it follows that the series

$$\sum_{i=0}^{l}\left(\frac{\psi}{2^3}\right)^i$$

is convergent. Therefore, the sequence

$$\left\{\frac{\mathsf{f}(2^l\mathsf{a})}{2^{3l}}\right\}_{l\in\mathbb{N}}$$

is Cauchy in the complete IFN-space $(T,N_{\flat_1,\flat_2},\Lambda)$. Hence, this sequence converges to some limit $Q_3(\mathsf{a})\in T$. Consequently, we may define $Q_3:G\to T$ by

$$N_{\flat_1,\flat_2}\left(Q_3(\mathsf{a})-\frac{\mathsf{f}(2^l\mathsf{a})}{2^{3l}}\right)\longrightarrow 1_{L^*}\quad\text{as}\quad l\to\infty.$$

By substituting $j = 0$ into inequality (2.15), we derive

$$N_{\flat_1,\flat_2}\left(\frac{\mathsf{f}(2^l\mathsf{a})}{2^{3l}} - \mathsf{f}(\mathsf{a}),\epsilon\right) \geq_{L^*} N'_{\flat_1,\flat_2}\left(\mathsf{g}(\mathsf{a},\mathsf{a},0,\cdots,0), \frac{\epsilon}{\sum_{i=0}^{l-1}\frac{\psi^i}{2^{3(i+1)}(\mathsf{m}^2-5\mathsf{m}+6)}}\right). \tag{2.16}$$

In (2.16), we obtain the limit as $l \to \infty$ and we apply

$$N_{\flat_1,\flat_2}\left(\mathsf{f}(\mathsf{a}) - Q_3(\mathsf{a}),\epsilon\right) \geq_{L^*} N'_{\flat_1,\flat_2}\Big(\mathsf{g}(\mathsf{a},\mathsf{a},0,\cdots,0),\epsilon(\mathsf{m}^2-5\mathsf{m}+6)(2^3-\psi)\Big).$$

Next, Our objective is to illustrate that (1.1) is fulfilled by Q_3. Switching $(\mathsf{a}_1,\mathsf{a}_2,\mathsf{a}_3,\cdots,\mathsf{a}_\mathsf{m})$ with $(2^l\mathsf{a}_1,2^l\mathsf{a}_2,2^l\mathsf{a}_3,\cdots,2^l\mathsf{a}_\mathsf{m})$ in (2.3), we obtain

$$N_{\flat_1,\flat_2}\left(\frac{1}{2^{3l}}D\mathsf{f}\left(2^l\mathsf{a}_1,2^l\mathsf{a}_2,2^l\mathsf{a}_3,\cdots,2^l\mathsf{a}_\mathsf{m}\right),\epsilon\right) \geq_{L^*} N'_{\flat_1,\flat_2}\left(g\left(2^l\mathsf{a}_1,2^l\mathsf{a}_2,2^l\mathsf{a}_3,\cdots,2^l\mathsf{a}_\mathsf{m}\right),2^{3l}\epsilon\right), \tag{2.17}$$

for every $\mathsf{a}_1,\mathsf{a}_2,\cdots,\mathsf{a}_\mathsf{m} \in G$ and every $\epsilon > 0$. Since

$$\lim_{l\to\infty} N'_{\flat_1,\flat_2}\left(\mathsf{g}(2^l\mathsf{a}_1,2^l\mathsf{a}_2,2^l\mathsf{a}_3,\cdots,2^l\mathsf{a}_m),\ 2^{3l}(\mathsf{m}^2-5\mathsf{m}+6)\epsilon\right) = 1_{L^*}.$$

From the functional equation (1.1), one can deduce that the function Q_3 fulfills the stipulated criterion, signifying that it represents a cubic function.

Next, to establish the uniqueness of Q_3. Consider one more cubic solution $Q'_3 : G \to T$ is also fulfills (1.1) and (2.5). Then, for any $\mathsf{a} \in G$ and $\epsilon > 0$, we obtain

$$\begin{aligned}
N_{\flat_1,\flat_2}\left(Q_3(\mathsf{a}) - Q'_3(\mathsf{a}),\epsilon\right) &= N_{\flat_1,\flat_2}\left(\frac{Q_3(2^l\mathsf{a})}{2^{3l}} - \frac{Q'_3(2^l\mathsf{a})}{2^{3l}},\epsilon\right)\\
&\geq_{L^*} \Lambda\left\{N_{\flat_1,\flat_2}\left(\frac{Q_3(2^l\mathsf{a})}{2^{3l}} - \frac{\mathsf{f}(2^l\mathsf{a})}{2^{3l}},\frac{\epsilon}{2}\right), N_{\flat_1,\flat_2}\left(\frac{\mathsf{f}(2^l\mathsf{a})}{2^{3l}} - \frac{Q'_3(2^l\mathsf{a})}{2^{3l}},\frac{\epsilon}{2}\right)\right\}\\
&\geq_{L^*} N'_{\flat_1,\flat_2}\left(g\left(2^l\mathsf{a},2^l\mathsf{a},0,\cdots,0\right),\frac{2^{3l}\epsilon(2^3-\psi)(\mathsf{m}^2-5\mathsf{m}+6)}{2}\right)\\
&\geq_{L^*} N'_{\flat_1,\flat_2}\left(g\left(\mathsf{a},\mathsf{a},0,\cdots,0\right),\frac{2^{3l}\epsilon(2^3-\psi)(\mathsf{m}^2-5\mathsf{m}+6)}{2\psi^l}\right).
\end{aligned}$$

Since

$$\lim_{l\to\infty}\frac{2^{3l}\epsilon(2^3-\psi)(\mathsf{m}^2-5\mathsf{m}+6)}{2\psi^l} = \infty,$$

it follows that

$$\lim_{l\to\infty} N'_{\flat_1,\flat_2}\left(g\left(\mathsf{a},\mathsf{a},0,\cdots,0\right),\frac{2^{3l}\epsilon(2^3-\psi)(\mathsf{m}^2-5\mathsf{m}+6)}{2\psi^l}\right) = 1_{L^*}.$$

Thus, we conclude that

$$N_{\flat_1,\flat_2}\left(Q_3(\mathsf{a}) - Q'_3(\mathsf{a}),\epsilon\right) = 1_{L^*}.$$

Therefore, $Q_3(\mathsf{a}) = Q'_3(\mathsf{a})$ for all $\mathsf{a} \in G$, establishing the uniqueness of the cubic function $Q_3(\mathsf{a})$. □

Theorem 2.3. *Let G be a linear space and let $(Z, N'_{\mathfrak{b}_1,\mathfrak{b}_2}, \Lambda)$ be an IFN-space. Suppose $\mathsf{g} : G^{\mathsf{m}} \to Z$ is a function such that $0 < \frac{2^3}{\psi} < 1$ and*

$$N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\mathsf{g}\left(2^{-1}\mathsf{a}, 2^{-1}\mathsf{a}, 0, \cdots, 0\right), \epsilon\right) \geq_{L^*} N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\frac{1}{\psi}\,\mathsf{g}(\mathsf{a}, \mathsf{a}, 0, \cdots, 0), \epsilon\right), \tag{2.18}$$

and

$$\lim_{l\to\infty} N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\mathsf{g}\left(2^{-l}\mathsf{a}_1, 2^{-l}\mathsf{a}_2, \cdots, 2^{-l}\mathsf{a}_{\mathsf{m}}\right), 2^{-3l}\epsilon\right) = 1_{L^*},$$

for every $\mathsf{a}, \mathsf{a}_1, \mathsf{a}_2, \cdots, \mathsf{a}_{\mathsf{m}} \in G$. If a mapping $\mathsf{f} : G \to T$ satisfies (2.3)*, subsequently, the constraint*

$$N_{\mathfrak{b}_1,\mathfrak{b}_2}\left(Q_3(\mathsf{a}) - 2^{3l}\mathsf{f}\left(\tfrac{\mathsf{a}}{2^l}\right), \epsilon\right) \longrightarrow 1_{L^*} \quad as \quad l \to \infty$$

exists, and there is a unique cubic function $Q_3 : G \to T$ satisfying the functional equation (1.1)*. Moreover,*

$$N_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\mathsf{f}(\mathsf{a}) - Q_3(\mathsf{a}), \epsilon\right) \geq_{L^*} N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\mathsf{g}(\mathsf{a}, \mathsf{a}, 0, \cdots, 0), \epsilon(\psi - 2^3)(\mathsf{m}^2 - 5\mathsf{m} + 6)\right),$$

for any $\mathsf{a} \in G$ and each $\epsilon > 0$.

Proof. Fix $\mathsf{a} \in G$ and let $\epsilon > 0$ be arbitrary. By setting $(\mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_3, \cdots, \mathsf{a}_{\mathsf{m}}) = (\mathsf{a}, \mathsf{a}, 0, \cdots, 0)$ in (2.3), we obtain

$$N_{\mathfrak{b}_1,\mathfrak{b}_2}\Big((\mathsf{m}^2 - 5\mathsf{m} + 6)\mathsf{f}(2\mathsf{a}) - 2^3(\mathsf{m}^2 - 5\mathsf{m} + 6)\mathsf{f}(\mathsf{a}), \epsilon\Big) \geq_{L^*} N'_{\mathfrak{b}_1,\mathfrak{b}_2}\Big(\mathsf{g}(\mathsf{a}, \mathsf{a}, 0, \cdots, 0), \epsilon\Big). \tag{2.19}$$

Replacing a by $\frac{\mathsf{a}}{2}$ in (2.19), we get

$$N_{\mathfrak{b}_1,\mathfrak{b}_2}\Big((\mathsf{m}^2 - 5\mathsf{m} + 6)\mathsf{f}(\mathsf{a}) - 2^3(\mathsf{m}^2 - 5\mathsf{m} + 6)\mathsf{f}\Big(\frac{\mathsf{a}}{2}\Big), \epsilon\Big) \geq_{L^*} N'_{\mathfrak{b}_1,\mathfrak{b}_2}\Big(\mathsf{g}\Big(\frac{\mathsf{a}}{2}, \frac{\mathsf{a}}{2}, 0, \cdots, 0\Big), \epsilon\Big). \tag{2.20}$$

Next, substituting a by $\frac{\mathsf{a}}{2^l}$ in (2.20) and applying (IFN3), we obtain

$$N_{\mathfrak{b}_1,\mathfrak{b}_2}\Big((\mathsf{m}^2 - 5\mathsf{m} + 6)\mathsf{f}\Big(\frac{\mathsf{a}}{2^l}\Big) - 2^3(\mathsf{m}^2 - 5\mathsf{m} + 6)\mathsf{f}\Big(\frac{\mathsf{a}}{2^{l+1}}\Big), \epsilon\Big) \geq_{L^*} N'_{\mathfrak{b}_1,\mathfrak{b}_2}\Big(\mathsf{g}\Big(\frac{\mathsf{a}}{2^{l+1}}, \frac{\mathsf{a}}{2^{l+1}}, 0, \cdots, 0\Big)\Big). \tag{2.21}$$

Finally, by employing inequality (2.18) together with condition (IFN3) in (2.21), we arrive at

$$N_{\mathfrak{b}_1,\mathfrak{b}_2}\Big((\mathsf{m}^2 - 5\mathsf{m} + 6)\mathsf{f}\Big(\frac{\mathsf{a}}{2^l}\Big) - 2^3(\mathsf{m}^2 - 5\mathsf{m} + 6)\mathsf{f}\Big(\frac{\mathsf{a}}{2^{l+1}}\Big), \epsilon\Big) \geq_{L^*} N'_{\mathfrak{b}_1,\mathfrak{b}_2}\Big(\mathsf{g}(\mathsf{a}, \mathsf{a}, 0, 0, \cdots, 0),\ \psi^{l+1}\epsilon\Big).$$

By following the same reasoning as in Theorem 2.2, the remainder of the proof is then established. □

Corollary 2.1. *Suppose that $\mathsf{f} : G \to T$ fulfills*

$$N_{\mathfrak{b}_1,\mathfrak{b}_2}(D\mathsf{f}(\mathsf{a}_1, \mathsf{a}_2, \cdots, \mathsf{a}_{\mathsf{m}}), \epsilon) \ \geq_{L^*}\ N'_{\mathfrak{b}_1,\mathfrak{b}_2}(\theta, \epsilon)\,,$$

for all $\mathsf{a}_1, \mathsf{a}_2, \cdots, \mathsf{a}_{\mathsf{m}} \in G$ and every $\epsilon > 0$. Then there is only one cubic solution $Q_3 : G \to T$ satisfying

$$N_{\mathfrak{b}_1,\mathfrak{b}_2}(\mathsf{f}(\mathsf{a}) - Q_3(\mathsf{a}), \epsilon) \ \geq_{L^*}\ N'_{\mathfrak{b}_1,\mathfrak{b}_2}\Big(\theta,\ |2^3 - 1|\,\epsilon\Big),$$

where θ in $\mathbb{R}^+$.

Corollary 2.2. *If* $\mathsf{f}: G \to T$ *be a mapping which satisfies*

$$N_{\flat_1,\flat_2}(D\mathsf{f}(\mathsf{a}_1,\mathsf{a}_2,\ldots,\mathsf{a}_\mathsf{m}),\epsilon) \ \geq_{L^*} \ N'_{\flat_1,\flat_2}\left(\theta\sum_{i=1}^{\mathsf{m}}\|\mathsf{a}_i\|^{\xi},\ \epsilon\right)$$

for all $\mathsf{a}_1,\mathsf{a}_2,\cdots,\mathsf{a}_\mathsf{m} \in G$*,* $\epsilon > 0$*, then there exist a unique cubic mapping* $Q_3 : G \to T$ *satisfying*

$$N_{\flat_1,\flat_2}(\mathsf{f}(\mathsf{a}) - Q_3(\mathsf{a}),\ \epsilon) \ \geq_{L^*} \ N'_{\flat_1,\flat_2}\Big(2\theta\|\mathsf{a}\|^{\xi},\ \left|2^3 - 2^{\xi}\right|(\mathsf{m}^2 - 5\mathsf{m} + 6)\epsilon\Big),$$

where ξ *and* θ *are in* $\mathbb{R}^+$ *with* $\xi \in (0,3)\cup(3,+\infty)$*.*

Proof. If we take $\psi = 2^{\xi}$ and define $g(\mathsf{a}_1,\mathsf{a}_2,\mathsf{a}_3,\ldots,\mathsf{a}_\mathsf{m}) = \theta\sum_{1\leq i\leq \mathsf{m}}\|\mathsf{a}_i\|^{\xi}$, then the result follows as a consequence of Theorems 2.2 and 2.3. □

Corollary 2.3. *If* $\mathsf{f}: G \to T$ *be a mapping which fulfills*

$$N_{\flat_1,\flat_2}\left(D\mathsf{f}(\mathsf{a}_1,\mathsf{a}_2,\mathsf{a}_3,\cdots,\mathsf{a}_\mathsf{m}),\epsilon\right) \geq_{L^*} N'_{\flat_1,\flat_2}\left(\theta\sum_{i=1}^{\mathsf{m}}\|\mathsf{a}_i\|^{m\xi} + \gamma\prod_{i=1}^{\mathsf{m}}\|\mathsf{a}_i\|^{\tau},\epsilon\right)$$

for any $\mathsf{a}_1,\mathsf{a}_2,\cdots,\mathsf{a}_\mathsf{m} \in G$ *and each* $\epsilon > 0$*, then there exists a unique cubic mapping* $Q_3 : G \to T$ *fulfills*

$$N_{\flat_1,\flat_2}\left(\mathsf{f}(\mathsf{a}) - Q_3(\mathsf{a}),\epsilon\right) \geq_{L^*} N'_{\flat_1,\flat_2}\left(2\theta\|a\|^{\mathsf{m}\xi}, |2^3 - 2^{\mathsf{m}\xi}|(\mathsf{m}^2 - 5\mathsf{m} + 6)\epsilon\right)$$

for each $\mathsf{a} \in G$ *and any* $\epsilon > 0$*, where* $\gamma,\tau,\theta,\xi \in \mathbb{R}^+$ *with* $\mathsf{m}\tau, \mathsf{m}\xi \in (0,3)\cup(3,+\infty)$*.*

Corollary 2.4. *If* $\mathsf{f}: G \to T$ *be a mapping which satisfies*

$$N_{\flat_1,\flat_2}\left(D\mathsf{f}(\mathsf{a}_1,\mathsf{a}_2,\cdots,\mathsf{a}_\mathsf{m}),\epsilon\right) \geq_{L^*} N'_{\flat_1,\flat_2}\left(\gamma\prod_{i=1}^{m}\|\mathsf{a}_i\|^{\tau},\epsilon\right)$$

for any $\mathsf{a}_1,\mathsf{a}_2,\cdots,\mathsf{a}_\mathsf{m} \in G$*,* $\epsilon > 0$*, then the solution* f *is cubic, where* γ *and* τ *in* $\mathbb{R}^+$ *with* $0 < \tau\mathsf{m} \neq 3$*.*

2.1.2. *Fixed Point Technique.* Before proceeding, let us consider a constant ρ_i defined by

$$\rho_i = \begin{cases} 2, & \text{if } i = 0, \\ \dfrac{1}{2}, & \text{if } i = 1. \end{cases}$$

Furthermore, let Φ denote the set

$$\Phi = \{\, t_1 \mid t_1 : G \to T,\ t_1(0) = 0\,\}.$$

Theorem 2.4. *Let* $\mathsf{f}: G \to T$ *be a mapping for which there exists a function* $\mathsf{g} : G^{\mathsf{m}} \to Z$ *such that*

$$\lim_{l\to\infty} N'_{\flat_1,\flat_2}\Big(\mathsf{g}(2^l\mathsf{a}_1, 2^l\mathsf{a}_2,\ldots,2^l\mathsf{a}_\mathsf{m}),\ 2^{3l}\epsilon\Big) = 1_{L^*}, \tag{2.22}$$

and which also satisfies inequality (2.3)*. Suppose that there is* $L = L(i)$ *satisfying*

$$\mu(\mathsf{a}) = \frac{1}{(\mathsf{m}^2 - 5\mathsf{m} + 6)}\,\mathsf{g}\Big(\tfrac{\mathsf{a}}{2},\tfrac{\mathsf{a}}{2},0,\ldots,0\Big)$$

has the property

$$N'_{\flat_1,\flat_2}\left(L\frac{1}{\rho_i^3}\,\mu(\rho_i\mathsf{a}),\epsilon\right) = N'_{\flat_1,\flat_2}(\mu(\mathsf{a}),\epsilon)\,. \tag{2.23}$$

Then there exists a unique cubic solution $Q_3 : G \to T$ satisfying (1.1)*, and moreover,*

$$N_{\flat_1,\flat_2}(\mathsf{f}(\mathsf{a}) - Q_3(\mathsf{a}), \epsilon) \geq_{L^*} N'_{\flat_1,\flat_2}\left(\frac{L^{1-i}}{1-L}\mu(\mathsf{a}), \epsilon\right) \tag{2.24}$$

for any $\mathsf{a} \in G$ *and each* $\epsilon > 0$.

Proof. Let ψ denote a generalized metric defined on the set Φ:

$$\psi(r_1, r_2) = \inf\left\{j \in (0,\infty) | N_{\flat_1,\flat_2}(r_1(\mathsf{a}) - r_2(\mathsf{a}), \epsilon) \geq_{L^*} N'_{\flat_1,\flat_2}(j\mu(\mathsf{a}), \epsilon), \mathsf{a} \in G, \epsilon > 0\right\}.$$

Therefore, (Φ, ψ) is complete.

Define a function $\Upsilon : \Phi \to \Phi$ by $\Upsilon t_1(\mathsf{a}) = \frac{1}{\rho_i^3} t_1(\rho_i \mathsf{a})$, for every $\mathsf{a} \in G$. For $t_1, t_2 \in \Phi$, we have

$$\begin{aligned}
\psi(t_1, t_2) &\leq j \\
\Rightarrow N_{\flat_1,\flat_2}(t_1(\mathsf{a}) - t_2(\mathsf{a}), \epsilon) &\geq_{L^*} N'_{\flat_1,\flat_2}(j\mu(\mathsf{a}), \epsilon) \\
\Rightarrow N_{\flat_1,\flat_2}\left(\frac{t_1(\rho_i \mathsf{a})}{\rho_i^3} - \frac{t_2(\rho_i \mathsf{a})}{\rho_i^3}, \epsilon\right) &\geq_{L^*} N'_{\flat_1,\flat_2}\left(\frac{j\mu(\rho_i \mathsf{a})}{\rho_i^3}, \epsilon\right) \\
\Rightarrow N_{\flat_1,\flat_2}(\Upsilon t_1(\mathsf{a}) - \Upsilon t_2(\mathsf{a}), \epsilon) &\geq_{L^*} N_{\flat_1,\flat_2}(j\, L\mu(\mathsf{a}), \epsilon) \\
\Rightarrow \psi(\Upsilon t_1(\mathsf{a}), \Upsilon t_2(\mathsf{a})) &\leq j\, L \\
\Rightarrow \psi(\Upsilon t_1, \Upsilon t_2) &\leq L\psi(t_1, t_2).
\end{aligned}$$

Consequently, Φ is a strictly contractive domain for the function Υ with L being the Lipschitz constant. Replacing $(\mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_3, \cdots, \mathsf{a}_\mathsf{m})$ by $(\mathsf{a}, \mathsf{a}, 0, \cdots, 0)$ in (2.3), we have

$$N_{\flat_1,\flat_2}\left(\mathsf{f}(2\mathsf{a}) - 2^3\mathsf{f}(\mathsf{a}), \epsilon\right) \geq_{L^*} N'_{\flat_1,\flat_2}(\mathsf{g}(\mathsf{a}, \mathsf{a}, 0, \cdots, 0), \epsilon). \tag{2.25}$$

Using (IFN3) in (2.25), we have

$$N_{\flat_1,\flat_2}\left(\frac{\mathsf{f}(2\mathsf{a})}{2^3} - \mathsf{f}(\mathsf{a}), \epsilon\right) \geq_{L^*} N'_{\flat_1,\flat_2}\left(\left(\frac{1}{2^3}\right)(\mathsf{m}^2 - 5\mathsf{m} + 6)\mathsf{g}(\mathsf{a}, \mathsf{a}, 0, \cdots, 0), \epsilon\right).$$

Using (2.23) for the case $i = 0$, that

$$\begin{aligned}
N_{\flat_1,\flat_2}\left(\frac{\mathsf{f}(2\mathsf{a})}{2^3} - \mathsf{f}(\mathsf{a}), \epsilon\right) &\geq_{L^*} N'_{\flat_1,\flat_2}(L\mu(\mathsf{a}), \epsilon) \\
\Rightarrow \psi(\Upsilon \mathsf{f}, \mathsf{f}) &\leq L = L^1 = L^{1-i}.
\end{aligned} \tag{2.26}$$

Setting $\mathsf{a} = \frac{\mathsf{a}}{2}$ in (2.25), we obtain

$$N_{\flat_1,\flat_2}\left(\mathsf{f}(\mathsf{a}) - 2^3\mathsf{f}\left(\tfrac{\mathsf{a}}{2}\right), 2^3\epsilon\right) \geq_{L^*} N'_{\flat_1,\flat_2}\left(g\left(\tfrac{\mathsf{a}}{2}, \tfrac{\mathsf{a}}{2}, 0, \cdots, 0\right), 2^3(\mathsf{m}^2 - 5\mathsf{m} + 6)\epsilon\right) \tag{2.27}$$

for all $\mathsf{a} \in G$, $\epsilon > 0$. Now, applying (2.23) for $i = 1$, we have

$$\begin{aligned}
N_{\flat_1,\flat_2}\left(\mathsf{f}(\mathsf{a}) - 2^3\mathsf{f}\left(\tfrac{\mathsf{a}}{2}\right), \epsilon\right) &\geq_{L^*} N'_{\flat_1,\flat_2}(\mu(\mathsf{a}), \epsilon) \\
\Rightarrow \psi(\mathsf{f}, \Upsilon \mathsf{f}) &\leq 1 = L^0 = L^{1-i}.
\end{aligned} \tag{2.28}$$

From (2.26) and (2.28), it follows that

$$\psi(\mathsf{f}, \Upsilon\mathsf{f}) \leq L^{1-i} < \infty.$$

Consequently, based on the fixed-point approach, there is a fixed point Q_3 of Υ in Φ satisfying

$$\lim_{l\to\infty} N_{\flat_1,\flat_2}\left(\frac{\mathsf{f}(\rho_i^l \mathsf{a})}{\rho_i^{3l}} - Q_3(\mathsf{a}), \epsilon\right) \to 1_{L^*},\ \forall\ \mathsf{a} \in G,\ \epsilon > 0.$$

By replacing $(\mathsf{a}_1, \mathsf{a}_2, \cdots, \mathsf{a}_\mathsf{m})$ with $(\rho_i\mathsf{a}_1, \rho_i\mathsf{a}_2, \cdots, \rho_i\mathsf{a}_\mathsf{m})$ in (2.3), we obtain

$$N_{\flat_1,\flat_2}\left(\tfrac{1}{\rho_i^3} D\mathsf{f}(\rho_i\mathsf{a}_1, \rho_i\mathsf{a}_2, \cdots, \rho_i\mathsf{a}_\mathsf{m}), \epsilon\right) \geq_{L^*} N'_{\flat_1,\flat_2}\left(\psi(\rho_i\mathsf{a}_1, \rho_i\mathsf{a}_2, \cdots, \rho_i\mathsf{a}_\mathsf{m}), \rho_i^3\epsilon\right)$$

for every $\mathsf{a}_1, \mathsf{a}_2, \cdots, \mathsf{a}_\mathsf{m} \in G$ and every $\epsilon > 0$.

Using the argument from Theorem 2.2, it follows that the function Q_3 fulfills the functional equation (1.1). Furthermore, as Q_3 is the only one fixed point of Υ according to Theorem 2.1, we obtain

$$\Delta = \{\mathsf{f} \in \Phi \mid \psi(\mathsf{f}, Q_3) < \infty\}.$$

Hence, the function Q_3 serves as the unique solution satisfying

$$N_{\flat_1,\flat_2}(Q_3(\mathsf{a}) - \mathsf{f}(\mathsf{a}), \epsilon) \geq_{L^*} N'_{\flat_1,\flat_2}(j\,\mu(\mathsf{a}), \epsilon),$$

for $t > 0$. Finally, by applying Theorem 2.1, we obtain

$$\begin{aligned} \psi(\mathsf{f}, Q_3) &\leq \frac{1}{1-L}\,\psi(\mathsf{f}, \Upsilon\mathsf{f}) \\ &\leq \frac{L^{1-i}}{1-L}, \\ N_{\flat_1,\flat_2}(\mathsf{f}(\mathsf{a}) - Q_3(\mathsf{a}), \epsilon) &\geq_{L^*} N'_{\flat_1,\flat_2}\left(L^{1-i}(1-L)^{-1}\mu(\mathsf{a}), \epsilon\right). \end{aligned}$$

□

Corollary 2.5. *Assume that a mapping* $\mathsf{f} : G \to T$ *satisfies*

$$N_{\flat_1,\flat_2}(D\mathsf{f}(\mathsf{a}_1, \mathsf{a}_2, \cdots, \mathsf{a}_\mathsf{m}), \epsilon) \geq_{L^*} \begin{cases} N'_{\flat_1,\flat_2}(\theta, \epsilon), \\ N'_{\flat_1,\flat_2}\left(\theta \sum_{j=1}^{\mathsf{m}} \|\mathsf{a}_j\|^{\xi}, \epsilon\right), \\ N'_{\flat_1,\flat_2}\left(\theta\left(\prod_{j=1}^{\mathsf{m}} \|\mathsf{a}_j\|^{\xi} + \sum_{j=1}^{\mathsf{m}} \|\mathsf{a}_j\|^{\mathsf{m}\xi}\right), \epsilon\right), \end{cases}$$

for all $\mathsf{a}_1, \mathsf{a}_2, \cdots, \mathsf{a}_\mathsf{m} \in G$ *and* $\epsilon > 0$*. Then there is only one cubic solution* $Q_3 : G \to T$ *fulfilling*

$$N_{\flat_1,\flat_2}(\mathsf{f}(\mathsf{a}) - Q_3(\mathsf{a}), \epsilon) \geq_{L^*} \begin{cases} N'_{\flat_1,\flat_2}\left(\theta, |2^3 - 1|\,\epsilon\right), \\ N'_{\flat_1,\flat_2}\left(2\theta\|\mathsf{a}\|^{\mathsf{m}}, (\mathsf{m}^2 - 5\mathsf{m} + 6)\,|2^3 - 2^{\xi}|\,\epsilon\right), & \xi < 3 \text{ or } \xi > 3, \\ N'_{\flat_1,\flat_2}\left(2\theta\|\mathsf{a}\|^{\mathsf{m}}, (\mathsf{m}^2 - 5\mathsf{m} + 6)\,|2^3 - 2^{\mathsf{m}\xi}|\,\epsilon\right), & \xi < \frac{3}{\mathsf{m}} \text{ or } \xi > \frac{3}{\mathsf{m}}, \end{cases}$$

for any $\mathsf{a} \in G$, $\epsilon > 0$*, where* $\xi, \theta \in \mathbb{R}^+$ *and* $\theta > 0$.

Proof. Letting

$$g(\mathsf{a}_1,\mathsf{a}_2,\cdots,\mathsf{a}_\mathsf{m}) = \begin{cases} \theta, \\ \theta\sum_{j=1}^{\mathsf{m}} \|\mathsf{a}_j\|^{\xi}, \\ \theta\left(\prod_{j=1}^{\mathsf{m}} \|\mathsf{a}_j\|^{\xi} + \sum_{j=1}^{\mathsf{m}} \|\mathsf{a}_j\|^{\mathsf{m}\xi}\right). \end{cases}$$

Then,

$$\begin{aligned} N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(g\left(\rho_i^l\mathsf{a}_1,\rho_i^l\mathsf{a}_2,\cdots,\rho_i^l\mathsf{a}_\mathsf{m}\right),\rho_i^{3l}\epsilon\right) &= \begin{cases} N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\theta,(\rho_i)^{3l}\epsilon\right), \\ N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\theta\sum_{j=1}^{\mathsf{m}}\|\mathsf{a}_j\|^{\xi},\left(\rho_i^{1-\xi}\right)^{3l}\epsilon\right), \\ N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\theta\left(\prod_{j=1}^{\mathsf{m}}\|\mathsf{a}_j\|^{\xi}+\sum_{j=1}^{\mathsf{m}}\|\mathsf{a}_j\|^{\mathsf{m}\xi}\right),\left(\rho_i^{1-\mathsf{m}\xi}\right)^{3l}\epsilon\right), \end{cases} \\ &= \begin{cases} \to 1_{L^*} \text{ as } l\to\infty, \\ \to 1_{L^*} \text{ as } l\to\infty, \\ \to 1_{L^*} \text{ as } l\to\infty. \end{cases} \end{aligned}$$

Therefore, (2.22) holds. Moreover, the function

$$\mu(\mathsf{a}) = g\left(\frac{\mathsf{a}}{2},\frac{\mathsf{a}}{2},0,\ldots,0\right)$$

satisfies the property

$$N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(L\frac{1}{\rho_i^3}\mu(\rho_i\mathsf{a}),\epsilon\right) \geq_{L^*} N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\mu(\mathsf{a}),\epsilon\right),\ \mathsf{a}\in G,\epsilon>0.$$

Hence,

$$\begin{aligned} N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\mu(\mathsf{a}),\epsilon\right) &= N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(g\left(\frac{\mathsf{a}}{2},\frac{\mathsf{a}}{2},0,\cdots,0\right),\epsilon\right) \\ &= \begin{cases} N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\theta,\epsilon\right), \\ N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\frac{2\theta}{2^{\xi}}\|\mathsf{a}\|^{\xi},\epsilon\right), \\ N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\frac{2\theta}{2^{m\xi}}\|\mathsf{a}\|^{\mathsf{m}\xi},\epsilon\right). \end{cases} \end{aligned}$$

Now,

$$\begin{aligned} N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\frac{1}{\rho_i^3}\mu(\rho_i\mathsf{a}),\epsilon\right) &= \begin{cases} N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\frac{\theta}{\rho_i^3},\epsilon\right), \\ N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\frac{2\theta}{2^{\xi}\rho_i^3(\mathsf{m}^2-5\mathsf{m}+6)}\|\rho_i\mathsf{a}\|^{\xi},\epsilon\right), \\ N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\frac{2\theta}{2^{\mathsf{m}\xi}\rho_i^3(\mathsf{m}^2-5\mathsf{m}+6)}\|\rho_i\mathsf{a}\|^{\mathsf{m}\xi},\epsilon\right), \end{cases} \\ &= \begin{cases} N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\rho_i^{-3}\mu(\mathsf{a}),\epsilon\right), \\ N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\rho_i^{\xi-3}\mu(\mathsf{a}),\epsilon\right), \\ N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\rho_i^{\mathsf{m}\xi-3}\mu(\mathsf{a}),\epsilon\right). \end{cases} \end{aligned}$$

We are able to verify the following situations for conditions of ρ_i on the basis of inequality (2.23) for our verification.

Case:1 Let $L = 2^{-3}$ if $i = 0$.

$$N_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\mathsf{f}(\mathsf{a}) - Q_3(\mathsf{a}), \epsilon\right) \geq_{L^*} N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\frac{2^{-3}}{1-2^{-3}}\mu(\mathsf{a}), \epsilon\right) = N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\theta, 7\,\epsilon\right).$$

Case:2 Let $L = 2^{3}$ if $i = 1$.

$$N_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\mathsf{f}(\mathsf{a}) - Q_3(\mathsf{a}), \epsilon\right) \geq_{L^*} N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\frac{1}{1-2^{3}}\mu(\mathsf{a}), \epsilon\right) = N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\theta, -7\,\epsilon\right).$$

Case:3 Let $L = 2^{\xi-3}$ for $\xi < 3$ if $i = 0$.

$$N_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\mathsf{f}(\mathsf{a}) - Q_3(\mathsf{a}), \epsilon\right) \geq_{L^*} N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\frac{2^{\xi-3}}{1-2^{\xi-3}}\mu(\mathsf{a}), \epsilon\right) = N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(2\theta||\mathsf{a}||^{\xi}, (\mathsf{m}^2 - 5\mathsf{m} + 6)(2^3 - 2^{\xi})\epsilon\right).$$

Case:4 Let $L = 2^{3-\xi}$ for $\xi > 3$ if $i = 1$.

$$N_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\mathsf{f}(\mathsf{a}) - Q_3(\mathsf{a}), \epsilon\right) \geq_{L^*} N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\frac{1}{1-3^{3-\xi}}\mu(\mathsf{a}), \epsilon\right) = N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(2\theta||\mathsf{a}||^{\xi}, (\mathsf{m}^2 - 5\mathsf{m} + 6)(2^{\xi} - 2^3)\epsilon\right).$$

Case:5 Let $L = 2^{\mathsf{m}\xi-3}$ for $\xi < \frac{3}{\mathsf{m}}$ if $i = 0$.

$$N_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\mathsf{f}(\mathsf{a}) - Q_3(\mathsf{a}), \epsilon\right) \geq_{L^*} N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\frac{2^{\mathsf{m}\xi-3}}{1-2^{\mathsf{m}\xi-3}}\mu(\mathsf{a}), \epsilon\right) = N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(2\theta||\mathsf{a}||^{\mathsf{m}\xi}, (\mathsf{m}^2 - 5\mathsf{m} + 6)(2^3 - 2^{\mathsf{m}\xi})\epsilon\right).$$

Case:6 Let $L = 2^{3-\mathsf{m}\xi}$ for $\xi > \frac{3}{\mathsf{m}}$ if $i = 1$.

$$N_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\mathsf{f}(\mathsf{a}) - Q_3(\mathsf{a}), \epsilon\right) \geq_{L^*} N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(\frac{1}{1-2^{3-\mathsf{m}\xi}}\mu(\mathsf{a}), \epsilon\right) = N'_{\mathfrak{b}_1,\mathfrak{b}_2}\left(2\theta||\mathsf{a}||^{\mathsf{m}\xi}, (\mathsf{m}^2 - 5\mathsf{m} + 6)(2^{\mathsf{m}\xi} - 2^3)\epsilon\right).$$

□

2.2. **Analysis of Hyers-Ulam Stability in 2-Banach Spaces.**

Definition 2.2. *[15] Let G be a real linear space of dimension greater than 1, and let $|\cdot,\cdot| : G^2 \to \mathbb{R}$ be a mapping satisfying the following conditions*

(a) $||p_1, p_2|| = 0$ *iff p_1 and p_2 are linearly dependent.*
(b) $||p_1, p_2|| = ||p_2, p_1||$,
(c) $||\omega p_1, p_2|| = |\omega|\, ||p_1, p_2||$,
(d) $||p_1, p_2 + p_3|| \leq ||p_1, p_2|| + ||p_1, p_3||$,

for every $p_1, p_2, p_3 \in G$ and $\omega \in \mathbb{R}$.

Hence, $||\cdot,\cdot||$ is called a 2-norm on G, and the pair $(G, ||\cdot,\cdot||)$ is known as a linear 2-normed space. An example of a 2-normed space is $\mathbb{R}^2$ equipped with the 2-norm defined by the area of the triangle determined by the points 0, p_1, and p_2.

Lemma 2.1. *[15] Let $(G, ||\cdot,\cdot||)$ be a linear 2-normed space. If $p_1 \in G$ and $||p_1, p_2|| = 0$ for every $p_2 \in G$, then $p_1 = 0$.*

Lemma 2.2. *[15] Consider a sequence $\{(p_1)_j\}$ in the linear 2-normed space G that converges.*

$$\lim_{j\to\infty}\|(p_1)_j, p_2\| = \|\lim_{j\to\infty}(p_1)_j, p_2\|$$

for every $p_2 \in G$.

Our focus here is on G in its normed linear space and T in its 2-Banach space.

Theorem 2.5. *Let $\mathsf{g} : G \to [0, +\infty)$ be a mapping such that*

$$\lim_{i\to\infty}\frac{1}{2^{3i}}\,\mathsf{g}\Big(2^i\mathsf{a}_1, 2^i\mathsf{a}_2, \cdots, 2^i\mathsf{a}_\mathsf{m}, s\Big) = 0 \tag{2.29}$$

for all $\mathsf{a}_1, \mathsf{a}_2, \cdots, \mathsf{a}_\mathsf{m}, s \in G$. Suppose that Any mapping $\mathsf{f} : G \to T$ with $\mathsf{f}(0) = 0$ is a mapping, and fulfills

$$\Big\|D\mathsf{f}(\mathsf{a}_1, \mathsf{a}_2, \cdots, \mathsf{a}_\mathsf{m}), s\Big\| \le \mathsf{g}(\mathsf{a}_1, \mathsf{a}_2, \cdots, \mathsf{a}_\mathsf{m}, s), \tag{2.30}$$

with the condition

$$\hat{\mathsf{g}}(\mathsf{a}, s) := \sum_{j=0}^{\infty}\frac{1}{2^{3j}}\,\mathsf{g}\Big(2^j\mathsf{a}, 2^j\mathsf{a}, 0, \cdots, 0, s\Big) < \infty, \tag{2.31}$$

for every $\mathsf{a}, s \in G$. Then there exists a unique cubic mapping $Q_3 : G \to T$ such that

$$\Big\|\mathsf{f}(\mathsf{a}) - Q_3(\mathsf{a}), s\Big\| \le \frac{1}{2^3(\mathsf{m}^2 - 5\mathsf{m} + 6)}\,\hat{\mathsf{g}}(\mathsf{a}, s), \tag{2.32}$$

for all $\mathsf{a}, s \in G$.

Proof. Setting $(\mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_3, \cdots, \mathsf{a}_\mathsf{m})$ by $(\mathsf{a}, \mathsf{a}, 0, \cdots, 0)$ in (2.30), we arrive

$$\Big\|(\mathsf{m}^2 - 5\mathsf{m} + 6)\mathsf{f}(2\mathsf{a}) - 2^3(\mathsf{m}^2 - 5\mathsf{m} + 6)\mathsf{f}(\mathsf{a}), s\Big\| \le \mathsf{g}(\mathsf{a}, \mathsf{a}, 0, \cdots, 0, s) \tag{2.33}$$

for every $\mathsf{a}, s \in G$. Switching a by $2^n\mathsf{a}$ in (2.33) and dividing both sides by 2^{n-3}, we attain

$$\Big\|\frac{1}{2^{3(n+1)}}\mathsf{f}(2^{n+1}\mathsf{a}) - \frac{1}{2^{3n}}\mathsf{f}(2^n\mathsf{a}), s\Big\| \le \frac{1}{2^{n+1}(\mathsf{m}^2 - 5\mathsf{m} + 6)}\mathsf{g}(2^i\mathsf{a}, 2^i\mathsf{a}, 0, \cdots, 0, s)$$

for each $\mathsf{a}, s \in G$ and any positive integer i. Hence,

$$\begin{aligned} \Big\|\frac{1}{2^{3(i+1)}}\mathsf{f}(2^{i+1}\mathsf{a}) - \frac{1}{2^{3\mathsf{m}}}\mathsf{f}(2^\mathsf{m}\mathsf{a}), s\Big\| &\le \sum_{j=\mathsf{m}}^{i}\Big\|\frac{1}{2^{3(j+1)}}\mathsf{f}(2^{j-1}\mathsf{a}) - \frac{1}{2^{3j}}\mathsf{f}(2^j\mathsf{a}), s\Big\| \\ &\le \frac{1}{(\mathsf{m}^2 - 5\mathsf{m} + 6)}\sum_{j=\mathsf{m}}^{i}\frac{1}{2^{3j}}\mathsf{g}(2^j\mathsf{a}, 2^j\mathsf{a}, 0, \cdots, 0, s) \end{aligned} \tag{2.34}$$

For all $\mathsf{a}, s \in G$ and for any positive integers i and m with $\mathsf{m} \le i$, it follows from (2.3) and (2.7) that the sequence

$$\Big\{\frac{1}{2^{3i}}\mathsf{f}(2^i\mathsf{a})\Big\}$$

This expression forms a Cauchy sequence in the space T for every element a in the set G. Since T is complete, this sequence converges in T for every $\mathsf{a} \in G$. Therefore, we can define $Q_3 : G \to T$ as follows:

$$Q_3(\mathsf{a}) := \lim_{i\to\infty} \frac{1}{2^{3i}} \mathsf{f}(2^i\mathsf{a}) \tag{2.35}$$

for every $\mathsf{a} \in G$. Then,

$$\lim_{i\to\infty} \left\| \frac{1}{2^{3i}} \mathsf{f}(2^i\mathsf{a}) - Q_3(\mathsf{a}), s \right\| = 0$$

when a and s are elements of G. We get equation (2.32) by placing $m = 0$ and taking a limit in (2.34) as $i \to \infty$. Following that, our goal is to prove that Q_3 is a cubic functioning. We may deduce from Lemma 2.2 and the inequality sets (2.29), (2.30), and (2.35) that

$$\begin{aligned}\left\| D\mathsf{f}(\mathsf{a}_1, \mathsf{a}_2, \cdots, \mathsf{a}_\mathsf{m}), s \right\| &= \lim_{i\to\infty} \left\| D\mathsf{f}(2^i\mathsf{a}_1, 2^i\mathsf{a}_2, \cdots, 2^i\mathsf{a}_\mathsf{m}), s \right\| \\ &\le \lim_{i\to\infty} \frac{1}{2^{3i}} \mathsf{g}(2^i\mathsf{a}_1, 2^i\mathsf{a}_2, \cdots, 2^i\mathsf{a}_\mathsf{m}, s) = 0\end{aligned}$$

for every $\mathsf{a}_1, \mathsf{a}_2, \cdots, \mathsf{a}_\mathsf{m}, s \in G$. By Lemma 2.1,

$$DQ_3(\mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_3, \cdots, \mathsf{a}_\mathsf{m}) = 0$$

for every $\mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_3, \cdots, \mathsf{a}_\mathsf{m} \in G$. The cubic function is represented by the mapping $Q_3 : G \to T$.

An additional cubic solution $Q_3' : G \to T$ satisfying (2.32) is assumed to exist in order to prove that the function Q_3 is unique. Following that

$$\begin{aligned}\left\| Q_3(\mathsf{a}) - Q_3'(\mathsf{a}), s \right\| &= \lim_{i\to\infty} \frac{1}{2^{3i}} \left\| Q_3(2^i\mathsf{a}) - \mathsf{f}(2^i\mathsf{a}) + \mathsf{f}(2^i\mathsf{a}) - Q_3'(2^i\mathsf{a}), s \right\| \\ &\le \frac{1}{(\mathsf{m}^2 - 5\mathsf{m} + 6)} \lim_{i\to\infty} \frac{1}{2^{3i}} \hat{g}(2^i\mathsf{a}, s) = 0\end{aligned}$$

for all $\mathsf{a}, s \in G$. By Lemma 2.1, $Q_3(\mathsf{a}) - Q_3'(\mathsf{a}) = 0$ for every $\mathsf{a} \in G$. Hence, $Q_3 = Q_3'$. □

Remark 2.1. *A theorem analogous to 2.5 can be formulated, where the sequence is given by*

$$Q_3(\mathsf{a}) := \lim_{i\to\infty} 2^{3i}\, \mathsf{f}\Big(\frac{\mathsf{a}}{2^i}\Big),$$

under appropriate conditions imposed on g.

Corollary 2.6. *Consider a function ω that satisfies the equation $\omega(0) = 0$ and maps $\omega : [0, \infty) \to [0, \infty)$ as well as*

(i) *$\omega(p) < p$ for each $p > 3$.*
(ii) *$\omega(pq) \le \omega(p)\omega(q)$.*

If a mapping $f : G \to T$ satisfies

$$\left\| D\mathsf{f}(\mathsf{a}_1, \mathsf{a}_2, \cdots, \mathsf{a}_\mathsf{m}), s \right\| \le \sum_{i=1}^{m} \omega(\|\mathsf{a}_i\|) + \omega(\|s\|)$$

for any $\mathsf{a}_1, \mathsf{a}_2, \cdots, \mathsf{a}_\mathsf{m}, s \in G$, then there exists a unique cubic mapping $Q_3 : G \to T$ satisfy

$$\left\| \mathsf{f}(\mathsf{a}) - Q_3(\mathsf{a}), s \right\| \le \left[\frac{\omega(\|\mathsf{a}\|)}{2^3 - \omega(2)} + \omega(\|s\|) \right] \tag{2.36}$$

for every $\mathsf{a}, s \in G$.

Proof. Setting

$$\mathsf{g}(\mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_3, \cdots, \mathsf{a}_\mathsf{m}, s) = \sum_{1 \le i \le \mathsf{m}} \omega(\|\mathsf{a}_i\|) + \omega(\|s\|)$$

for each $\mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_3, \cdots, \mathsf{a}_\mathsf{m}, s \in G$. If we look at (i) we can see that

$$\omega(2^{3i}) \le (\omega(2))^{3i}$$

and

$$\mathsf{g}(2^i\mathsf{a}_1, 2^i\mathsf{a}_2, \cdots, 2^i\mathsf{a}_\mathsf{m}, s) \le (\omega(2))^{3i} \left(\sum_{1 \le i \le \mathsf{m}} \omega(\|\mathsf{a}_i\|) \right) + \omega(\|s\|).$$

By applying Theorem 2.5, we obtain (2.36). □

Corollary 2.7. *Let* $q < 3$ *and let* $H : [0, \infty) \times [0, \infty) \to [0, \infty)$ *be a homogeneous function. Suppose that a function* $\mathsf{f} : G \to T$ *fulfills*

$$\left\| D\mathsf{f}(\mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_3, \ldots, \mathsf{a}_\mathsf{m}), s \right\| \le H\Big(\|\mathsf{a}_1\|, \|\mathsf{a}_2\|, \|\mathsf{a}_3\|, \ldots, \|\mathsf{a}_\mathsf{m}\|\Big) + \|s\|$$

for every $\mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_3, \ldots, \mathsf{a}_\mathsf{m}, s \in G$. *Then, there exists a unique cubic function* $Q_3 : G \to T$ *satisfying*

$$\left\| \mathsf{f}(\mathsf{a}) - Q_3(\mathsf{a}), s \right\| \le \frac{H(\|\mathsf{a}\|, \|\mathsf{a}\|, 0, \ldots, 0) + \|s\|}{2 - q} \tag{2.37}$$

for each $\mathsf{a}, s \in G$ *and for all* $q \in \mathbb{R}^+$.

Proof. Setting

$$\mathsf{g}(\mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_3, \cdots, \mathsf{a}_\mathsf{m}, s) = H\left(\|\mathsf{a}_1\|, \|\mathsf{a}_2\|, \|\mathsf{a}_3\|, \cdots, \|\mathsf{a}_\mathsf{m}\|\right) + \|s\|$$

for any $\mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_3, \cdots, \mathsf{a}_\mathsf{m}, s \in G$. By satisfying Theorem 2.5, we reach (2.37). □

Corollary 2.8. *Let* $q < 3$ *and consider* $H : [0, \infty) \times [0, \infty) \to [0, \infty)$ *a function of degree* q *is homogeneous. Suppose that a function* $\mathsf{f} : G \to T$ *fulfills*

$$\left\| D\mathsf{f}(\mathsf{a}_1, \mathsf{a}_2, \ldots, \mathsf{a}_m), s \right\| \le H\Big(\|\mathsf{a}_1\|, \|\mathsf{a}_2\|, \|\mathsf{a}_3\|, \ldots, \|\mathsf{a}_m\|\Big) \|s\|,$$

for every $\mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_3, \ldots, \mathsf{a}_m, s \in G$. *Then there exists a unique cubic mapping* $Q_3 : G \to T$ *fulfilling*

$$\left\| \mathsf{f}(\mathsf{a}) - Q_3(\mathsf{a}), s \right\| \le \frac{H(\|\mathsf{a}\|, \|\mathsf{a}\|, 0, \ldots, 0) \|s\|}{3 - 3^q},$$

for any $\mathsf{a}, s \in G$ *and* $q \in \mathbb{R}^+$.

Proof. Arrangement

$$\mathsf{g}(\mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_3, \cdots, \mathsf{a}_\mathsf{m}, s) = H\left(\|\mathsf{a}_1\|, \|\mathsf{a}_2\|, \|\mathsf{a}_3\|, \cdots, \|\mathsf{a}_\mathsf{m}\|\right) \|s\| \tag{2.38}$$

for every $\mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_3, \cdots, \mathsf{a}_\mathsf{m}, s \in G$. By fulfilling Theorem 2.5, we achieve (2.38). □

Corollary 2.9. *Let $p < 3$ and let $\mathsf{f} : G \to T$ be a function fulfilling*

$$\left\| D\mathsf{f}(\mathsf{a}_1, \mathsf{a}_2, \ldots, \mathsf{a}_\mathsf{m}), s \right\| \leq \sum_{i=1}^{\mathsf{m}} \|\mathsf{a}_i\|^p + \|s\|,$$

for all $\mathsf{a}_1, \mathsf{a}_2, \ldots, \mathsf{a}_\mathsf{m}, s \in G$. Then there is only one cubic function $Q_3 : G \to T$ satisfying

$$\left\| \mathsf{f}(\mathsf{a}) - Q_3(\mathsf{a}), s \right\| \leq \frac{2\|\mathsf{a}\|^p + \|s\|}{2 - p},$$

for each $\mathsf{a}, s \in G$ and $p \in \mathbb{R}^+$.

3. Conclusion

In this study, we proved the Hyers-Ulam stability of a cubic functional equation in the contexts of 2-Banach spaces and Intuitionistic Fuzzy Normed spaces. We obtained adequate conditions guaranteeing the existence and uniqueness of cubic mappings approximating the specified functional equation by using both fixed-point and direct analytical approaches. Our findings show that both additive and multiplicative structures may be used to quantify the stability through combinations of norm powers.

The dual methodology of this study is significant because it uses direct methods to get constructive estimations and fixed-point methods to give a systematic framework for stability analysis. This duality expands the results' application and strengthens their robustness. Additionally, the given examples demonstrate the usefulness and practical significance of the theoretical conclusions.

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References

[1] J. Aczel, J. Dhombres, Functional Equations in Several Variables, Cambridge University Press, 1989. https://doi.org/10.1017/cbo9781139086578.

[2] T. Aoki, On the Stability of the Linear Transformation in Banach Spaces., J. Math. Soc. Jpn. 2 (1950), 64–66. https://doi.org/10.2969/jmsj/00210064.

[3] Y. Almalki, B. Radhakrishnan, U. Jayaraman, K. Tamilvanan, Some Common Fixed Point Results in Modular Ultrametric Space Using Various Contractions and Their Application to Well-Posedness, Mathematics 11 (2023), 4077. https://doi.org/10.3390/math11194077.

[4] A.R. Baias, D. Popa, M.T. Rassias, Set-Valued Solutions of an Equation of Jensen Type, Quaest. Math. 46 (2022), 1237–1244. https://doi.org/10.2989/16073606.2022.2072249.

[5] A. Bodaghi, C. Park, J.M. Rassias, Fundamental Stabilities of the Nonic Functional Equation in Intuitionistic Fuzzy Normed Spaces, Commun. Korean Math. Soc. 31 (2016), 729–743. https://doi.org/10.4134/ckms.c150147.

[6] D.G. Bourgin, Classes of Transformations and Bordering Transformations, Bull. Am. Math. Soc. 57 (1951), 223–237. https://doi.org/10.1090/s0002-9904-1951-09511-7.

[7] G.Z. Eskandani, P. Gavruta, On the Stability Problems in Quasi-Banach Spaces, Nonlinear Funct. Anal. Appl. 15 (2010), 569–579.

[8] P. Gavruta, A Generalization of the Hyers-Ulam-Rassias Stability of Approximately Additive Mappings, J. Math. Anal. Appl. 184 (1994), 431–436. https://doi.org/10.1006/jmaa.1994.1211.

[9] P. Gavruta, An Answer to a Question of Th. M. Rassias and J. Tabor on Mixed Stability of Mappings, Bul. St. Univ. Politehn. Timisoara, Ser. Mat. Fiz. 42 (1997), 1–6.

[10] P. Gavruta, On a Problem of G. Isac and Th. M. Rassias Concerning the Stability of Mappings, J. Math. Anal. Appl. 261 (2001), 543–553. https://doi.org/10.1006/jmaa.2001.7539.

[11] D. H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.

[12] L. Maligranda, A Result of Tosio Aoki About a Generalization of Hyers–Ulam Stability of Additive Functions – A Question of Priority, Aequ. Math. 75 (2008), 289–296. https://doi.org/10.1007/s00010-007-2892-8.

[13] P. Nakmahachalasint, On the Generalized Ulam-Gavruta-Rassias Stability of Mixed-Type Linear and Euler-Lagrange-Rassias Functional Equations, Int. J. Math. Math. Sci. 2007 (2007), 63239. https://doi.org/10.1155/2007/63239.

[14] S. Paokanta, C. Park, N. Jun-on, R. Suparatulatorn, An Additive-Cubic Functional Equation in a Banach Space, J. Math. Comput. Sci. 33 (2024), 264–274. https://doi.org/10.22436/jmcs.033.03.05.

[15] W. Park, Approximate Additive Mappings in 2-Banach Spaces and Related Topics, J. Math. Anal. Appl. 376 (2011), 193–202. https://doi.org/10.1016/j.jmaa.2010.10.004.

[16] B. Radhakrishnan, U. Jayaraman, S.O. Hilali, M. Kameswari, M. Alhagyan, et al., Coincidence Point Results for Self-Mapping with Extended Rational Contraction in Partially Ordered Ultrametric Spaces Using p-Adic Distance, J. Math. 2024 (2024), 8846076. https://doi.org/10.1155/2024/8846076.

[17] B. Radhakrishnan, U. Jayaraman, Fixed Point Results in Partially Ordered Ultrametric Space via p-Adic Distance, IAENG Int. J. Appl. Math. 53 (2023), 1–7.

[18] T.M. Rassias, On the Stability of the Linear Mapping in Banach Spaces, Proc. Am. Math. Soc. 72 (1978), 297–300. https://doi.org/10.1090/s0002-9939-1978-0507327-1.

[19] J.M. Rassias, On Approximation of Approximately Linear Mappings by Linear Mappings, J. Funct. Anal. 46 (1982), 126–130. https://doi.org/10.1016/0022-1236(82)90048-9.

[20] T.M. Rassias, Functional Equations and Inequalities, Springer, 2000. https://doi.org/10.1007/978-94-011-4341-7.

[21] J.M. Rassias, K.W. Jun, H. Kim, Approximate (m,n)-Cauchy-Jensen Additive Mappings in C^*-Algebras, Acta Math. Sin. Engl. Ser. 27 (2011), 1907–1922. https://doi.org/10.1007/s10114-011-0179-4.

[22] J.M. Rassias, H. Kim, Generalized Hyers–Ulam Stability for General Additive Functional Equations in Quasi-β-Normed Spaces, J. Math. Anal. Appl. 356 (2009), 302–309. https://doi.org/10.1016/j.jmaa.2009.03.005.

[23] R. Saadati, C. Park, Non-Archimedean L-Fuzzy Normed Spaces and Stability of Functional Equations, Comput. Math. Appl. 60 (2010), 2488–2496. https://doi.org/10.1016/j.camwa.2010.08.055.

[24] R. Saadati, J.H. Park, On the Intuitionistic Fuzzy Topological Spaces, Chaos Solitons Fractals 27 (2006), 331–344. https://doi.org/10.1016/j.chaos.2005.03.019.

[25] G. Senthil, A.P. Pushpalatha, N. Vijaya, V.S. Devi, M. Balamurugan, et al., Ulam Stability of Quadratic Mapping Connected with Homomorphisms and Derivations in Non-Archimedean Banach Algebras, Int. J. Anal. Appl. 23 (2025), 119. https://doi.org/10.28924/2291-8639-23-2025-119.

[26] G. Senthil, S. Sudharsan, V.B. Priya, S. Annadurai, G. Ganapathy, et al., Hyers-Ulam Stability of n-Dimensional Additive Functional Equation in Modular Spaces Using Fixed Point Method, Int. J. Anal. Appl. 23 (2025), 148. https://doi.org/10.28924/2291-8639-23-2025-148.

[27] G. Senthil, N. Vijaya, S. Gayathri, P. Vijayalakshmi, M. Balamurugan, et al., Hyers-Ulam Stability of Quartic Functional Equation in IFN-Spaces and 2-Banach Spaces by Classical Methods, Int. J. Anal. Appl. 23 (2025), 68. https://doi.org/10.28924/2291-8639-23-2025-68.

[28] S.M. Ulam, Problems in Modern Mathematics (Science Editions), Wiley, 1964.