Int. J. Anal. Appl. (2026), 24:149

International Journal of Analysis and Applications

An Extended Form of $(\mathfrak{p}, \mathfrak{q})$-Hermite-Hadamard Inequalities via Convex Functions

Shiwani Singh[1], Shashi Kant Mishra[1], Pankaj Kumar[1], Abdelouahed Hamdi[2,*]

[1]*Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi, Uttar Pradesh, India*

[2]*Department of Mathematics and Statistics, College of Arts and Sciences, Qatar University, Doha, P.O. Box 2713, Qatar*

[*]*Corresponding author:* abhamdi@qu.edu.qa

Abstract. The aim of this paper is to demonstrate generalized estimations for the $(\mathfrak{p}, \mathfrak{q})$-Hermite-Hadamard inequalities for convex functions with two parameters. By using the same parameters, our findings are consistent with the previously established $(\mathfrak{p}, \mathfrak{q})$-Hermite-Hadamard inequalities. We introduce a new lemma to derive generalized post-quantum inequalities for convex functions and show that our results extend some previously established ones. Additionally, we provide mathematical examples for specific $(\mathfrak{p}, \mathfrak{q})$-functions to validate the newly obtained results.

1. Introduction

Convexity is generally known as one of the most fundamental principles of analysis that is frequently used in many fields of pure and applied sciences. In particular, where convexity makes it possible to construct necessary and sufficient global optimality conditions; where there is a direct connection between consumer theory in economics, information theory, and the field of inequalities. In recent decades, several extensions of the classical concept of convexity have been proposed and explored [1,2], such as generalized convexity, preinvexity, harmonic convexity, exponential convexity, (α,m)-convexity, m-convexity, log-convexity, h-convexity, and convexity with respect to function pairs, among others. These generalized forms of convexity have proved to be highly useful in diverse areas, including optimization theory, approximation theory, fractional mathematical modeling, and functional analysis [3–8].

In recent decades, the concept of convexity has played a central role in the generalization and extension of various inequalities. Numerous inequalities—including those of Hermite–Hadamard,

Received: Oct. 27, 2025.

2020 *Mathematics Subject Classification.* 26A33; 26A51; 33E12.

Key words and phrases. Hermite-Hadamard type inequality; convex function; post-quantum calculus; Hölder's inequality.

https://doi.org/10.28924/2291-8639-24-2026-149
ISSN: 2291-8639

Newton, Simpson, Opial, Ostrowski, Pachpatte, and Bullen types, among others—have been developed by researchers through the use of convex functions, with an emphasis on establishing new bounds for their left- and right-hand sides; see [9–24] and references cited therein for more details.

In mathematics, quantum calculus refers to a branch of calculus that is developed without relying on the notion of limits. In the beginning of 18th century, the first person to investigate quantum calculus was Euler [25] (1707-1783) who used the parameter '$\mathfrak{q}$' in Newton's study of infinite series. Jackson initiated the study of quantum calculus in the early twentieth century and developed the $\mathfrak{q}$-integral and $\mathfrak{q}$-derivative [26, 27]. When limit $\mathfrak{q}$ tends to 1, the $\mathfrak{q}$ calculus can be simplified to ordinary calculus and also the quantum calculus is subset of time scale calculus. A unified framework for studying dynamical equations in both discrete and continuous domains is provided by the time scale of calculus.

The idea of $\mathfrak{q}$-calculus has been used to study numerous well-known integral inequalities, including Ostrowski, Hermite-Hadamard, Hölder, trapezoidal, and Cauchy-Bunyakovsky-Schwarz inequalities. In 2015, Noor *et al.* [28] developed a quantum version of the classical integral identity and estimated Hermite-Hadamard inequalities for $\mathfrak{q}$-differentiable convex and quasi-convex functions. In addition, Alp *et al.* [29] derived extended quantum versions of the $\mathfrak{q}$-H–H inequality for convex functions using two distinct forms of quantum integrals.

The post-quantum calculus, commonly referred to as, $(\mathfrak{p}, \mathfrak{q})$-calculus, represents a generalization of $\mathfrak{q}$-calculus by introducing two independent parameters, $\mathfrak{p}$ and $\mathfrak{q}$. This framework plays a significant role in various areas of physics and mathematics. The study of $(\mathfrak{p}, \mathfrak{q})$-calculus was initiated by Chakrabarti and Jagannathan [30] in 1991, where they introduced the notions of the $(\mathfrak{p}, \mathfrak{q})$-integral and derivative on the interval $(0, \infty)$. Later, in 2016, Tunç and Göv [31] extended these concepts by defining the $(\mathfrak{p}, \mathfrak{q})_{\grave{\varrho}}$-integral and derivative over the finite intervals. More recently, Vivas-Cortez [32] $(\mathfrak{p}, \mathfrak{q})^{\grave{\varsigma}}$-integral and derivative, also formulated on finite intervals.

In recent years, $(\mathfrak{p}, \mathfrak{q})$-calculus has attracted considerable attention from researchers, particularly in the study of various integral inequalities, with numerous results available in [33–38], and the cited references. Lungboon *et al.* [39] presented a new $(\mathfrak{p}, \mathfrak{q})^{\grave{\varsigma}}$-integrable identity involving the first order $(\mathfrak{p}, \mathfrak{q})^{\grave{\varsigma}}$-derivative and, utilizing this identity, established several $(\mathfrak{p}, \mathfrak{q})^{\grave{\varsigma}}$-integrale inequalities of Hermite–Hadamard type for $(\mathfrak{p}, \mathfrak{q})^{\grave{\varsigma}}$-differentiable convex functions. In addition, You *et al.* [40] derived Hermite–Hadamard and Ostrowski type inequalities for s-convex functions through $(\mathfrak{p}, \mathfrak{q})$-calculus, along with some applications. Furthermore, Mishra and Singh [41] introduced a Hermite–Hadamard type inequality for s-preinvex functions within the framework of post-quantum calculus and formulated a new $(\mathfrak{p}, \mathfrak{q})$-integral identity, which was applied to obtain Ostrowski type inequalities for $(\mathfrak{p}, \mathfrak{q})$-differentiable functions.

This research is divided into five sections, including the introduction. Section 2 presents a concise overview of fundamental definitions and properties regarding quantum and post-quantum calculus. In section 3, we introduce parametrized post-quantum integral inequalities for convex function by using lemma and acheive some new type inequality into a single inequality. In section 4, Two examples are provided to examine the applicability of our main results. In section 5, we have given ideas for upcoming studies.

2. Preliminaries

In this section, we revisit some basic ideas and conclusion about convex function, $\mathfrak{q}$-calculus and $(\mathfrak{p},\mathfrak{q})$-calculus, respectively.

Definition 2.1. *[42] A function $\grave{\varphi}$ is said to be convex on an interval I if*

$$\grave{\varphi}(\underline{t}\grave{\varrho}+(1-\underline{t})\grave{\varsigma}) \le \underline{t}\grave{\varphi}(\grave{\varrho})+(1-\underline{t})\grave{\varphi}(\grave{\varsigma}), \tag{2.1}$$

holds for all $\grave{\varrho},\grave{\varsigma}\in I$ and $0\le \underline{t}\le 1$.

2.1. $\mathfrak{q}$**-Calculus and Some Inequalities.** For the sake of brevity, let $\mathfrak{q}\in(0,1)$ and we see the following notations (see, [43]).

$$[\Upsilon]_{\mathfrak{q}}=\frac{1-(\mathfrak{q})^{\Upsilon}}{1-\mathfrak{q}}=1+\mathfrak{q}+(\mathfrak{q})^2+......+(\mathfrak{q})^{\Upsilon-1}.$$

Definition 2.2. *[44] The left quantum derivative or $\mathfrak{q}_{\grave{\varrho}}$-derivative of $\grave{\varphi}:[\grave{\varrho},\grave{\varsigma}]\to\mathbb{R}$ at $\grave{\Lambda}\in[\grave{\varrho},\grave{\varsigma}]$ is characterized by the expression:*

$$_{\grave{\varrho}}D_{\mathfrak{q}}\grave{\varphi}(\grave{\Lambda})=\frac{\grave{\varphi}(\grave{\Lambda})-\grave{\varphi}(\mathfrak{q}\grave{\Lambda}+(1-\mathfrak{q})\grave{\varrho})}{(1-\mathfrak{q})(\grave{\Lambda}-\grave{\varrho})},\grave{\Lambda}\neq\grave{\varrho}. \tag{2.2}$$

For $\grave{\Lambda}=\grave{\varsigma}$, we state ${}^{\grave{\varsigma}}D_{\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})=\lim_{\grave{\Lambda}\to\grave{\varsigma}}{}^{\grave{\varsigma}}D_{\mathfrak{q}}\grave{\varphi}(\grave{\Lambda})$, if it exists and finite.

Definition 2.3. *[45] The right quantum derivative or $\mathfrak{q}^{\grave{\varsigma}}$-derivative of $\grave{\varphi}:[\grave{\varrho},\grave{\varsigma}]\to\mathbb{R}$ at $\grave{\Lambda}\in[\grave{\varrho},\grave{\varsigma}]$ is expressed as:*

$${}^{\grave{\varsigma}}D_{\mathfrak{q}}(\grave{\Lambda})=\frac{\grave{\varphi}(\mathfrak{q}\grave{\Lambda}+(1-\mathfrak{q})\grave{\varsigma})-\grave{\varphi}(\grave{\Lambda})}{(1-\mathfrak{q})(\grave{\varsigma}-\grave{\Lambda})},\grave{\Lambda}\neq\grave{\varsigma}. \tag{2.3}$$

For $\grave{\Lambda}=\grave{\varsigma}$, we state ${}^{\grave{\varsigma}}D_{\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})=\lim_{\grave{\Lambda}\to\grave{\varsigma}}{}^{\grave{\varsigma}}D_{\mathfrak{q}}\grave{\varphi}(\grave{\Lambda})$, if it exists and finite.

Definition 2.4. *[45] The left quantum integral or $\mathfrak{q}_{\grave{\varrho}}$-integral of $\grave{\varphi}:[\grave{\varrho},\grave{\varsigma}]\to\mathbb{R}$ at $\grave{\Lambda}\in[\grave{\varrho},\grave{\varsigma}]$ is defined as:*

$$\int_{\grave{\varrho}}^{\grave{\Lambda}}\grave{\varphi}(\underline{t})_{\grave{\varrho}}d_{\mathfrak{q}}\underline{t}=(1-\mathfrak{q})(\grave{\Lambda}-\grave{\varrho})\sum_{\Upsilon=0}^{\infty}\mathfrak{q}^{\Upsilon}\grave{\varphi}(\mathfrak{q}^{\Upsilon}\grave{\Lambda}+(1-\mathfrak{q}^{\Upsilon})\grave{\varrho})=(\grave{\varsigma}-\grave{\varrho})\int_0^1\grave{\varphi}((1-\underline{t})\grave{\varrho}+\underline{t}\grave{\varsigma}). \tag{2.4}$$

Definition 2.5. *[46] The right quantum integral or $\mathfrak{q}^{\grave{\varsigma}}$-integral of $\grave{\varphi}:[\grave{\varrho},\grave{\varsigma}]$ is defined as:*

$$\int_{\grave{\Lambda}}^{\grave{\varsigma}}\grave{\varphi}(\underline{t})^{\grave{\varsigma}}d_{\mathfrak{q}}\underline{t}=(1-\mathfrak{q})(\grave{\varsigma}-\grave{\Lambda})\sum_{\Upsilon=0}^{\infty}(\mathfrak{q}^{\Upsilon}\grave{\Lambda}+(1-\mathfrak{q}^{\Upsilon}))=(\grave{\varsigma}-\grave{\varrho})\int_0^1\grave{\varphi}(\underline{t}a+(1-\underline{t})\grave{\varsigma})d_{\mathfrak{q}}\underline{t}. \tag{2.5}$$

Lemma 2.1. *We have the equality for* $\mathfrak{q}_{\grave{\varrho}}$*-integrals*

$$\int_{\grave{\varrho}}^{\grave{\varsigma}} (\grave{\Lambda}-\grave{\varrho})^{\alpha}_{\grave{\varrho}} d_{\mathfrak{q}}\grave{\Lambda} = \frac{(\grave{\varsigma}-\grave{\varrho})^{\alpha+1}}{[\alpha+1]_{\mathfrak{q}}}, \tag{2.6}$$

for $\alpha \in \mathbb{R}\backslash\{-1\}$.

Many studies in quantum analysis have recently been done.One of these is the following in 2018 Alp *et al.* [46] proved the q-Hermite-Hadamard inequality:-

Theorem 2.1. *Let* $\grave{\varphi} : [\grave{\varrho}, \grave{\varsigma}] \to \mathbb{R}$ *be a convex differential function on* $[\grave{\varrho}, \grave{\varsigma}]$ *and* $0 < \mathfrak{q} < 1$*. Then, we have*

$$\grave{\varphi}\left(\frac{\mathfrak{q}\grave{\varrho}+\grave{\varsigma}}{[2]_{\mathfrak{q}}}\right) \leq \frac{1}{\grave{\varsigma}-\grave{\varrho}} \int_{\grave{\varrho}}^{\grave{\varsigma}} \grave{\varphi}(\grave{\Lambda})_{\grave{\varrho}} d_{\mathfrak{q}}\grave{\Lambda} \leq \frac{\mathfrak{q}\grave{\varphi}(\grave{\varrho})+\grave{\varphi}(\grave{\varsigma})}{[2]_{\mathfrak{q}}}, \tag{2.7}$$

where, $[2]_{\mathfrak{q}} = 1+\mathfrak{q}$.

By using the $\mathfrak{q}^{\grave{\varsigma}}$-integral, Bermudo *et al.* [45] proved the following new $\mathfrak{q}$-Hermite-Hadamard type inequality.

Theorem 2.2. *If* $\grave{\varphi} : [\grave{\varrho}, \grave{\varsigma}] \to \mathbb{R}$ *is a convex differential function on* $[\grave{\varrho}, \grave{\varsigma}]$*. Then, we have the following* $\mathfrak{q}$*-Hermite-Hadamard inequalities.*

$$\grave{\varphi}\left(\frac{\grave{\varrho}+\mathfrak{q}\grave{\varsigma}}{[2]\mathfrak{q}}\right) \leq \frac{1}{\grave{\varsigma}-\grave{\varrho}} \int_{\grave{\varrho}}^{\grave{\varsigma}} \grave{\varphi}(\grave{\Lambda})^{\grave{\varsigma}} d_{\mathfrak{q}}\grave{\Lambda} \leq \frac{\grave{\varphi}(\grave{\varrho})+\mathfrak{q}\grave{\varphi}(\grave{\varsigma})}{[2]_{\mathfrak{q}}}. \tag{2.8}$$

For some studies in this regard, the reader is refer to [47,48].

In [29], Necmettin Alp *et al.* established the following generalized version of $\mathfrak{q}$-Hermite-Hadamard inequality.

Theorem 2.3. *Let* $\grave{\varphi} : [\grave{\varrho}, \grave{\varsigma}] \to \mathbb{R}$ *be a convex function on* $[\grave{\varrho}, \grave{\varsigma}]$ *with* $\grave{\varrho} < \grave{\varsigma}$*, then the following inequalities hold:*

$$\grave{\varphi}\left(\frac{\grave{\varrho}+\grave{\varsigma}}{2}\right) \leq \frac{1}{2\lambda(\grave{\varsigma}-\grave{\varrho})}\left(\int_{\grave{\varrho}}^{\lambda\grave{\varsigma}+(1-\lambda)\grave{\varrho}} \grave{\varphi}(\grave{\Lambda})_{\grave{\varrho}} d_{\mathfrak{q}}\grave{\Lambda} + \int_{\lambda\grave{\varrho}+(1-\lambda)\grave{\varsigma}}^{\grave{\varsigma}} \grave{\varphi}(\grave{\Lambda})^{\grave{\varsigma}} d_{\mathfrak{q}}\grave{\Lambda}\right) \leq \frac{\grave{\varphi}(\grave{\varrho})+\grave{\varphi}(\grave{\varsigma})}{2}. \tag{2.9}$$

2.2. $(\mathfrak{p},\mathfrak{q})$**-Calculus and Some Inequalities.** In this work, $[\grave{\varrho}, \grave{\varsigma}] \subset \mathbb{R}$ is an interval and $0 < \mathfrak{q} < \mathfrak{p} \leq 1$ are constants.

The $(\mathfrak{p},\mathfrak{q})$-number of 'Y' is expressed as:

$$[\mathrm{Y}]_{\mathfrak{p},\mathfrak{q}} = \frac{\mathfrak{p}^{\mathrm{Y}}-\mathfrak{q}^{\mathrm{Y}}}{\mathfrak{p}-\mathfrak{q}} = \mathfrak{p}^{\mathrm{Y}-1}+\mathfrak{p}^{\mathrm{Y}-2}\mathfrak{q}+......+\mathfrak{p}\mathfrak{q}^{\mathrm{Y}-2}+\mathfrak{q}^{\mathrm{Y}-1},\ \mathrm{Y} \in N \tag{2.10}$$

By setting, $\mathfrak{q}$=1 in (2.10), the expression simplifies to:

$$[\mathrm{Y}]_{\mathfrak{q}} = \frac{1-\mathfrak{q}^{\mathrm{Y}}}{1-\mathfrak{q}} = 1+\mathfrak{q}+\mathfrak{q}^2+......+\mathfrak{q}^{\mathrm{Y}-1},\ \mathrm{Y} \in N;$$

Which is also known as the $\mathfrak{q}$-number of 'Y'; for further information, see [49].

Definition 2.6. *[31] The* $(\mathfrak{p},\mathfrak{q})_{\grave{\varrho}}$*-derivative of function* $\grave{\varphi}:[\grave{\varrho},\grave{\varsigma}]\to\mathbb{R}$ *is given as follows:*

$$ {}_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\Lambda})=\frac{\grave{\varphi}(\mathfrak{p}\grave{\Lambda}+(1-\mathfrak{p})\grave{\varrho})-\grave{\varphi}(\mathfrak{q}\grave{\Lambda}+(1-\mathfrak{q})\grave{\varrho})}{(\mathfrak{p}-\mathfrak{q})(\grave{\Lambda}-\grave{\varrho})},\ \grave{\Lambda}\neq\grave{\varrho}; \tag{2.11}$$

with $0<\mathfrak{q}<\mathfrak{p}\leq 1$. *For* $\grave{\Lambda}=\grave{\varrho}$*, we state* ${}_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varrho})=\lim_{\grave{\Lambda}\to\grave{\varrho}}{}_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\Lambda})$*, if it exists and finite.*

Definition 2.7. *[45] The* $(\mathfrak{p},\mathfrak{q})^{\grave{\varsigma}}$*-derivative of function* $\grave{\varphi}:[\grave{\varrho},\grave{\varsigma}]\to\mathbb{R}$ *is given as follows:*

$$ {}^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\Lambda})=\frac{\grave{\varphi}(\mathfrak{q}\grave{\Lambda}+(1-\mathfrak{q})\grave{\varsigma})-\grave{\varphi}(\mathfrak{p}\grave{\Lambda}+(1-\mathfrak{p})\grave{\varsigma})}{(\mathfrak{p}-\mathfrak{q})(\grave{\Lambda}-\grave{\varsigma})},\ \grave{\Lambda}\neq\grave{\varsigma}; \tag{2.12}$$

with $0<\mathfrak{q}<\mathfrak{p}\leq 1$. *For* $\grave{\Lambda}=\grave{\varsigma}$*, we state* ${}^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})=\lim_{\grave{\Lambda}\to\grave{\varsigma}}{}^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\Lambda})$*, if it exists and finite.*

Remark 2.1. *It follows that if we take* $\mathfrak{q}=1$ *in (2.6) and (2.7), then these equations simplify to (2.2) and (2.3), respectively.*

Definition 2.8. *[31] The defnition* $(\mathfrak{p},\mathfrak{q})_{\grave{\varrho}}$*-integral of function* $\grave{\varphi}:[\grave{\varrho},\grave{\varsigma}]\to\mathbb{R}$ *on* $[\grave{\varrho},\grave{\varsigma}]$ *is stated as follows:*

$$\int_{\grave{\varrho}}^{\grave{\Lambda}}\grave{\varphi}(t)\,{}_{\grave{\varrho}}d_{\mathfrak{p},\mathfrak{q}}t=(\mathfrak{p}-\mathfrak{q})(\grave{\Lambda}-\grave{\varrho})\sum_{\Upsilon=0}^{\infty}\frac{\mathfrak{q}^{\Upsilon}}{\mathfrak{p}^{\Upsilon+1}}\grave{\varphi}\left(\frac{\mathfrak{q}^{\Upsilon}}{\mathfrak{p}^{\Upsilon+1}}\grave{\Lambda}+\left(1-\frac{\mathfrak{q}^{\Upsilon}}{\mathfrak{p}^{\Upsilon+1}}\right)\grave{\varrho}\right), \tag{2.13}$$

with $0<\mathfrak{q}<\mathfrak{p}\leq 1$.

Definition 2.9. *[32] The defnition* $(\mathfrak{p},\mathfrak{q})^{\grave{\varsigma}}$*-integral of function* $\grave{\varphi}:[\grave{\varrho},\grave{\varsigma}]\to\mathbb{R}$ *on* $[\grave{\varrho},\grave{\varsigma}]$ *is stated as follows:*

$$\int_{\grave{\Lambda}}^{\grave{\varsigma}}\grave{\varphi}(t)\,{}^{\grave{\varsigma}}d_{\mathfrak{p},\mathfrak{q}}t=(\mathfrak{p}-\mathfrak{q})(\grave{\varsigma}-\grave{\Lambda})\sum_{\Upsilon=0}^{\infty}\frac{\mathfrak{q}^{\Upsilon}}{\mathfrak{p}^{\Upsilon+1}}\grave{\varphi}\left(\frac{\mathfrak{q}^{\Upsilon}}{\mathfrak{p}^{\Upsilon+1}}\grave{\Lambda}+\left(1-\frac{\mathfrak{q}^{\Upsilon}}{\mathfrak{p}^{\Upsilon+1}}\right)\grave{\varsigma}\right), \tag{2.14}$$

with $0<\mathfrak{q}<\mathfrak{p}\leq 1$.

Remark 2.2. *Taking* $\mathfrak{p}=1$ *in equations (2.13) and (2.14) simplifies them to (2.4) and (2.5) , respectively.*

Remark 2.3. *In (2.13), if we take* $\grave{\varrho}=0$ *and* $\grave{\Lambda}=\grave{\varsigma}=1$ *in (2.13), we get*

$$\int_0^1\grave{\varphi}(\underline{t})\,{}_0d_{\mathfrak{p},\mathfrak{p}}\underline{t}=(\mathfrak{p}-\mathfrak{q})\sum_{\Upsilon=0}^{\infty}\frac{\mathfrak{q}^{\Upsilon}}{\mathfrak{p}^{\Upsilon+1}}\grave{\varphi}\left(\frac{\mathfrak{q}^{\Upsilon}}{\mathfrak{p}^{\Upsilon+1}}\right).$$

Similarly, by taking $\grave{\Lambda}=\grave{\varrho}=0$ *is (2.14), it follows that*

$$\int_0^1\grave{\varphi}(\underline{t})\,{}^1d_{\mathfrak{p},\mathfrak{q}}\underline{t}=(\mathfrak{p}-\mathfrak{q})\sum_{\Upsilon=0}^{\infty}\frac{\mathfrak{q}^{\Upsilon}}{\mathfrak{p}^{\Upsilon+1}}\grave{\varphi}\left(1-\frac{\mathfrak{q}^{\Upsilon}}{\mathfrak{p}^{\Upsilon+1}}\right).$$

Lemma 2.2. *[32] The following equalities apply:*

$$\int_{\grave{\varrho}}^{\grave{\varsigma}}(\grave{\varsigma}-\grave{\Lambda})^{\alpha}\,{}^{\grave{\varsigma}}d_{\mathfrak{p},\mathfrak{q}}\grave{\Lambda}=\frac{(\grave{\varsigma}-\grave{\varrho})^{\alpha+1}}{[\alpha+1]_{\mathfrak{p},\mathfrak{q}}},$$

$$\int_{\grave{\varrho}}^{b}(\grave{\Lambda}-\grave{\varrho})^{\alpha}\,{}_{\grave{\varrho}}d_{\mathfrak{p},\mathfrak{q}}\grave{\Lambda}=\frac{(b-\grave{\varrho})^{\alpha+1}}{[\alpha+1]_{\mathfrak{p},\mathfrak{q}}},$$

where $\alpha\in\mathbb{R}\backslash\{-1\}$.

Kunt *et. al* [50] demonstrated the Hermite-Hadamard type inequality for convex functions using the $(\mathfrak{p},\mathfrak{q})_{\grave{\varrho}}$ integral.

Theorem 2.4. *For a convex mapping $\grave{\varphi}:[\grave{\varrho},\grave{\varsigma}]\to\mathbb{R}$ which is differential on $[\grave{\varrho},\grave{\varsigma}]$, the following inequalities hold for $[\mathfrak{p},\mathfrak{q}]_{\grave{\varrho}}$-integral:*

$$\grave{\varphi}\left(\frac{\mathfrak{q}\grave{\varrho}+\mathfrak{p}\grave{\varsigma}}{[2]_{\mathfrak{p},\mathfrak{q}}}\right)\leq\frac{1}{\mathfrak{p}(\grave{\varsigma}-\grave{\varrho})}\int_{\grave{\varrho}}^{\mathfrak{p}\grave{\varsigma}+(1-\mathfrak{p})\grave{\varrho}}\grave{\varphi}(\grave{\Lambda})_{\grave{\varrho}}d_{\mathfrak{p},\mathfrak{q}}\grave{\Lambda}\leq\frac{\mathfrak{q}\grave{\varphi}(\grave{\varrho})+\mathfrak{p}\grave{\varphi}(\grave{\varsigma})}{[2]_{\mathfrak{p},\mathfrak{q}}},\tag{2.15}$$

where $0<q<p\leq 1$.

Vivas-Cortez *et. al* [32] demonstrated the Hermite-Hadamard type inequality for convex functions using the $(\mathfrak{p},\mathfrak{q})^{\grave{\varsigma}}$ integral.-

Theorem 2.5. *Consider a convex mapping $\grave{\varphi}:[\grave{\varrho},\grave{\varsigma}]\to\mathbb{R}$ which is differentiable on $[\grave{\varrho},\grave{\varsigma}]$ the following inequalities hold for $(\mathfrak{p},\mathfrak{q})^{\grave{\varsigma}}$-integral:*

$$\grave{\varphi}\left(\frac{\mathfrak{p}\grave{\varrho}+\mathfrak{q}\grave{\varsigma}}{[2]_{\mathfrak{p},\mathfrak{q}}}\right)\leq\frac{1}{\mathfrak{p}(\grave{\varsigma}-\grave{\varrho})}\int_{\mathfrak{p}\grave{\varrho}+(1-\mathfrak{p})\grave{\varsigma}}^{\grave{\varsigma}}\grave{\varphi}(\grave{\Lambda})^{\grave{\varsigma}}d_{\mathfrak{p},\mathfrak{q}}\grave{\Lambda}\leq\frac{\mathfrak{p}\grave{\varphi}(\grave{\varrho})+\mathfrak{q}\grave{\varphi}(\grave{\varsigma})}{[2]_{\mathfrak{p},\mathfrak{q}}}.\tag{2.16}$$

where, $0<\mathfrak{q}<\mathfrak{p}\leq 1$.

Theorem 2.6. *[32] Consider a convex mapping $\grave{\varphi}:[\grave{\varrho},\grave{\varsigma}]\to\mathbb{R}$, the following inequality holds:*

$$\grave{\varphi}\left(\frac{\grave{\varrho}+\grave{\varsigma}}{2}\right)\leq\frac{1}{2\mathfrak{q}(\grave{\varsigma}-\grave{\varrho})}\left[\int_{\grave{\varrho}}^{\mathfrak{p}\grave{\varsigma}+(1-\mathfrak{p})\grave{\varrho}}\grave{\varphi}(\grave{\Lambda})_{\grave{\varrho}}d_{\mathfrak{p},\mathfrak{q}}\grave{\Lambda}+\int_{\mathfrak{p}\grave{\varrho}+(1-\mathfrak{p})\grave{\varsigma}}^{\grave{\varsigma}}\grave{\varphi}(\grave{\Lambda})^{\grave{\varsigma}}d_{\mathfrak{p},\mathfrak{q}}\grave{\Lambda}\right]\leq\frac{\grave{\varphi}(\grave{\varrho})+\grave{\varphi}(\grave{\varsigma})}{2}.\tag{2.17}$$

where $0<\mathfrak{q}<\mathfrak{p}\leq 1$.

3. Main Result

In this section, we present certain generalizations of the $(\mathfrak{p},\mathfrak{q})$-Hermite–Hadamard type inequalities. This section begins with a lemma that establishes the inequalities.

Lemma 3.1. *Consider a function $\grave{\varphi}:[\grave{\varrho},\grave{\varsigma}]\to\mathbb{R}$ which is $(\mathfrak{p},\mathfrak{q})$-differentiable. Provided that ${}_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}$ and ${}^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}$ are continuous and $(\mathfrak{p},\mathfrak{q})$-integrable over $[\grave{\varrho},\grave{\varsigma}]$, then we arrive at the following equality:*

$$\begin{aligned}&\frac{(\grave{\varsigma}-\grave{\varrho})}{2}\int_0^1\underline{t}[\mathfrak{q}\,{}^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\underline{t}\grave{\varrho}+(1-\underline{t})\grave{\varsigma})-\mathfrak{p}\,{}_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}((1-\underline{t})\grave{\varrho}+\underline{t}\grave{\varsigma})]d_{\mathfrak{p},\mathfrak{q}}\underline{t}\\&=\frac{1}{2(\grave{\varsigma}-\grave{\varrho})}\left[\frac{1}{\mathfrak{q}}\int_{\grave{\varrho}}^{\mathfrak{p}\grave{\varsigma}+(1-\mathfrak{p})\grave{\varrho}}\grave{\varphi}(\underline{t})_{\grave{\varrho}}d_{\mathfrak{p},\mathfrak{q}}\underline{t}+\frac{1}{\mathfrak{p}}\int_{\mathfrak{p}\grave{\varrho}+(1-\mathfrak{p})\grave{\varsigma}}^{\grave{\varsigma}}\grave{\varphi}(\underline{t})^{\grave{\varsigma}}d_{\mathfrak{p},\mathfrak{q}}\underline{t}\right]-\left[\frac{\grave{\varphi}(\grave{\varrho})+\grave{\varphi}(\grave{\varsigma})}{2}\right].\end{aligned}\tag{3.1}$$

Proof. Assume that

$$\begin{aligned}&\frac{(\grave{\varsigma}-\grave{\varrho})}{2}\left(\int_0^1\mathfrak{q}\underline{\mathrm{t}}\,{}^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}((1-\underline{\mathrm{t}})\grave{\varsigma}+\underline{\mathrm{t}}\grave{\varrho})d_{\mathfrak{p},\mathfrak{q}}\underline{\mathrm{t}}-\int_0^1\mathfrak{p}\underline{\mathrm{t}}\,{}_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}((1-\underline{\mathrm{t}})\grave{\varrho}+\underline{\mathrm{t}}\grave{\varsigma})d_{\mathfrak{p},\mathfrak{q}}\underline{\mathrm{t}}\right)\\&=\frac{(\grave{\varsigma}-\grave{\varrho})}{2}(I_1-I_2).\end{aligned}\tag{3.2}$$

By definition 2.7, we have

$$I_1 = \int_0^1 \mathfrak{q}\underline{\mathfrak{t}}_{\grave{\varrho}} D_{\mathfrak{p},\mathfrak{q}} \grave{\varphi}(\underline{\mathfrak{t}}\grave{\varrho} + (1-\underline{\mathfrak{t}})\grave{\varsigma}) d_{\mathfrak{p},\mathfrak{q}}\underline{\mathfrak{t}} \tag{3.3}$$

$$= \int_0^1 \mathfrak{q}\underline{\mathfrak{t}} \frac{\grave{\varphi}(\mathfrak{q}\underline{\mathfrak{t}}\grave{\varrho} + (1-\mathfrak{q}\underline{\mathfrak{t}})\grave{\varsigma}) - \grave{\varphi}(\mathfrak{p}\underline{\mathfrak{t}}\grave{\varrho} + (1-\mathfrak{p}\underline{\mathfrak{t}})\grave{\varsigma})}{(\mathfrak{p}-\mathfrak{q})(\grave{\varsigma}-\grave{\varrho})\underline{\mathfrak{t}}}$$

$$= \frac{\mathfrak{q}(\mathfrak{p}-\mathfrak{q})}{(\mathfrak{p}-\mathfrak{q})(\grave{\varsigma}-\grave{\varrho})}\left[\sum_{Y=0}^{\infty} \frac{\mathfrak{q}^Y}{\mathfrak{p}^{Y+1}} \grave{\varphi}\left(\frac{\mathfrak{q}^{Y+1}}{\mathfrak{p}^{Y+1}}\grave{\varrho} + \left(1-\frac{\mathfrak{q}^{Y+1}}{\mathfrak{p}^{Y+1}}\right)\grave{\varsigma}\right) - \sum_{Y=0}^{\infty} \frac{\mathfrak{q}^Y}{\mathfrak{p}^{Y+1}} \grave{\varphi}\left(\frac{\mathfrak{q}^{Y}}{\mathfrak{p}^{Y}}\grave{\varrho} + \left(1-\frac{\mathfrak{q}^{Y}}{\mathfrak{p}^{Y}}\right)\grave{\varsigma}\right)\right]$$

$$= \frac{\mathfrak{q}}{(\grave{\varsigma}-\grave{\varrho})}\left[\frac{1}{\mathfrak{q}}\sum_{Y=0}^{\infty} \frac{\mathfrak{q}^{Y+1}}{\mathfrak{p}^{Y+1}} \grave{\varphi}\left(\frac{\mathfrak{q}^{Y+1}}{\mathfrak{p}^{Y+1}}\grave{\varrho} + \left(1-\frac{\mathfrak{q}^{Y+1}}{\mathfrak{p}^{Y+1}}\right)\grave{\varsigma}\right) - \frac{1}{\mathfrak{p}}\sum_{Y=0}^{\infty} \frac{\mathfrak{q}^Y}{\mathfrak{p}^{Y}} \grave{\varphi}\left(\frac{\mathfrak{q}^{Y}}{\mathfrak{p}^{Y}}\grave{\varrho} + \left(1-\frac{\mathfrak{q}^{Y}}{\mathfrak{p}^{Y}}\right)\grave{\varsigma}\right)\right]$$

$$= \frac{\mathfrak{q}}{(\grave{\varsigma}-\grave{\varrho})}\left[\left(\frac{1}{\mathfrak{q}}-\frac{1}{\mathfrak{p}}\right)\sum_{Y=0}^{\infty} \frac{\mathfrak{q}^Y}{\mathfrak{p}^{Y}} \grave{\varphi}\left(\frac{\mathfrak{q}^{Y}}{\mathfrak{p}^{Y}}\grave{\varrho} + \left(1-\frac{\mathfrak{q}^{Y}}{\mathfrak{p}^{Y}}\right)\grave{\varsigma}\right) - \frac{1}{\mathfrak{q}}\grave{\varphi}(\grave{\varrho})\right]$$

$$= \frac{(\mathfrak{p}-\mathfrak{q})(\grave{\varsigma}-\grave{\varrho})}{\mathfrak{p}(\grave{\varsigma}-\grave{\varrho})^2}\sum_{Y=0}^{\infty} \frac{\mathfrak{q}^Y}{\mathfrak{p}^{Y}} \grave{\varphi}\left(\frac{\mathfrak{q}^{Y}}{\mathfrak{p}^{Y}}\grave{\varrho} + \left(1-\frac{\mathfrak{q}^{Y}}{\mathfrak{p}^{Y}}\right)\grave{\varsigma}\right) - \frac{1}{(b-\grave{\varrho})}\grave{\varphi}(\grave{\varrho})$$

$$= \frac{1}{\mathfrak{p}(\grave{\varsigma}-\grave{\varrho})^2}\int_{\mathfrak{p}\grave{\varrho}+(1-\mathfrak{p})\grave{\varsigma}}^{\grave{\varsigma}} \grave{\varphi}(\underline{\mathfrak{t}})^{\grave{\varsigma}} d_{\mathfrak{p},\mathfrak{q}}\underline{\mathfrak{t}} - \frac{1}{\grave{\varsigma}-\grave{\varrho}}\grave{\varphi}(\grave{\varrho}).$$

Similarly, by definition 2.6, we get

$$I_2 = \int_0^1 \mathfrak{p}\underline{\mathfrak{t}}_{\grave{\varrho}} D_{\mathfrak{p},\mathfrak{q}} \grave{\varphi}((1-\underline{\mathfrak{t}})\grave{\varrho} + \underline{\mathfrak{t}}\grave{\varsigma}) d_{\mathfrak{p},\mathfrak{q}}\underline{\mathfrak{t}}$$

$$= \int_0^1 \mathfrak{p}\underline{\mathfrak{t}} \frac{\grave{\varphi}(\mathfrak{p}((1-\underline{\mathfrak{t}})\grave{\varrho} + \underline{\mathfrak{t}}\grave{\varsigma}) + (1-\mathfrak{p})\grave{\varrho}) - \grave{\varphi}(\mathfrak{q}((1-\underline{\mathfrak{t}})\grave{\varrho} + \underline{\mathfrak{t}}\grave{\varsigma}) + (1-\mathfrak{q})\grave{\varrho})}{(\mathfrak{p}-\mathfrak{q})((1-\underline{\mathfrak{t}})\grave{\varrho} + \underline{\mathfrak{t}}\grave{\varsigma} - \grave{\varrho})}$$

$$= \mathfrak{p}\int_0^1 \frac{\grave{\varphi}(\grave{\varrho}(1-\mathfrak{p}\underline{\mathfrak{t}}) + \mathfrak{p}\underline{\mathfrak{t}}\grave{\varsigma}) - \grave{\varphi}(\grave{\varrho}(1-\mathfrak{q}\underline{\mathfrak{t}}) + \mathfrak{q}\underline{\mathfrak{t}}\grave{\varsigma})}{(\mathfrak{p}-\mathfrak{q})(\grave{\varsigma}-\grave{\varrho})} d_{\mathfrak{p},\mathfrak{q}}\underline{\mathfrak{t}}$$

$$= \frac{\mathfrak{p}(\mathfrak{p}-\mathfrak{q})}{(\mathfrak{p}-\mathfrak{q})(\grave{\varsigma}-\grave{\varrho})}\left[\sum_{Y=0}^{\infty} \frac{\mathfrak{q}^Y}{\mathfrak{p}^{Y+1}} \grave{\varphi}\left(\grave{\varrho}\left(1-\frac{\mathfrak{q}^{Y}}{\mathfrak{p}^{Y}}\right) + \frac{\mathfrak{q}^{Y}}{\mathfrak{p}^{Y}}\grave{\varsigma}\right) - \sum_{Y=0}^{\infty} \frac{\mathfrak{q}^Y}{\mathfrak{p}^{Y+1}} \grave{\varphi}\left(\grave{\varrho}\left(1-\frac{\mathfrak{q}^{Y+1}}{\mathfrak{p}^{Y+1}}\right) + \frac{\mathfrak{q}^{Y+1}}{\mathfrak{p}^{Y+1}}\grave{\varsigma}\right)\right]$$

$$= \frac{\mathfrak{p}}{(\grave{\varsigma}-\grave{\varrho})}\left[\frac{1}{\mathfrak{p}}\sum_{Y=0}^{\infty} \frac{\mathfrak{q}^Y}{\mathfrak{p}^{Y}} \grave{\varphi}\left(\grave{\varrho}\left(1-\frac{\mathfrak{q}^{Y}}{\mathfrak{p}^{Y}}\right) + \frac{\mathfrak{q}^{Y}}{\mathfrak{p}^{Y}}\grave{\varsigma}\right) - \frac{1}{\mathfrak{q}}\sum_{Y=0}^{\infty} \frac{\mathfrak{q}^{Y+1}}{\mathfrak{p}^{Y+1}} \grave{\varphi}\left(\grave{\varrho}\left(1-\frac{\mathfrak{q}^{Y+1}}{\mathfrak{p}^{Y+1}}\right) + \frac{\mathfrak{q}^{Y+1}}{\mathfrak{p}^{Y+1}}\grave{\varsigma}\right)\right]$$

$$= \frac{\mathfrak{p}}{(\grave{\varsigma}-\grave{\varrho})}\left[\left(\frac{1}{\mathfrak{p}}-\frac{1}{\mathfrak{q}}\right)\sum_{Y=0}^{\infty} \frac{\mathfrak{q}^Y}{\mathfrak{p}^{Y}} \grave{\varphi}\left(\grave{\varrho}\left(1-\frac{\mathfrak{q}^{Y}}{\mathfrak{p}^{Y}}\right) + \frac{\mathfrak{q}^{Y}}{\mathfrak{p}^{Y}}\grave{\varsigma}\right) + \frac{1}{\mathfrak{p}}\grave{\varphi}(\grave{\varrho})\right]$$

$$= \frac{\mathfrak{p}}{(\grave{\varsigma}-\grave{\varrho})}\left[-\left(\frac{\mathfrak{p}-\mathfrak{q}}{\mathfrak{p}\mathfrak{q}}\right)\sum_{Y=0}^{\infty} \frac{\mathfrak{q}^Y}{\mathfrak{p}^{Y}} \grave{\varphi}\left(\grave{\varrho}\left(1-\frac{\mathfrak{q}^{Y}}{\mathfrak{p}^{Y}}\right) + \frac{\mathfrak{q}^{Y}}{\mathfrak{p}^{Y}}\grave{\varsigma}\right) + \frac{1}{\mathfrak{p}}\grave{\varphi}(b)\right]$$

$$= \frac{1}{\grave{\varsigma}-\grave{\varrho}}\grave{\varphi}(\grave{\varsigma}) - \frac{1}{\mathfrak{q}(\grave{\varsigma}-\grave{\varrho})^2}\int_{\grave{\varrho}}^{\mathfrak{p}\grave{\varsigma}+(1-\mathfrak{p})\grave{\varrho}} \grave{\varphi}(t)_{\grave{\varrho}} d_{\mathfrak{p},\mathfrak{q}}t.$$

Then it follows that

$$\frac{(\grave{\varsigma}-\grave{\varrho})}{2}(I_1 - I_2)$$

$$= \frac{1}{\mathfrak{p}(\grave{\varsigma}-\grave{\varrho})^2}\int_{\mathfrak{p}\grave{\varrho}+(1-\mathfrak{p})\grave{\varsigma}}^{\grave{\varsigma}}\grave{\varphi}(\underline{\mathrm{t}})^{\grave{\varsigma}}d_{\mathfrak{p},\mathfrak{q}}\underline{\mathrm{t}} - \frac{1}{\grave{\varsigma}-\grave{\varrho}}\grave{\varphi}(\grave{\varrho}) - \frac{1}{\grave{\varsigma}-\grave{\varrho}}\grave{\varphi}(\grave{\varsigma}) + \frac{1}{\mathfrak{q}(\grave{\varsigma}-\grave{\varrho})^2}\int_{\grave{\varrho}}^{\mathfrak{p}\grave{\varsigma}+(1-\mathfrak{p})\grave{\varrho}}\grave{\varphi}(\underline{\mathrm{t}})_{\grave{\varrho}}d_{\mathfrak{p},\mathfrak{q}}\underline{\mathrm{t}}$$
$$= \frac{1}{2(\grave{\varsigma}-\grave{\varrho})}\left[\frac{1}{\mathfrak{q}}\int_{\grave{\varrho}}^{\mathfrak{p}\grave{\varsigma}+(1-\mathfrak{p})\grave{\varrho}}\grave{\varphi}(\underline{\mathrm{t}})_{\grave{\varrho}}d_{\mathfrak{p},\mathfrak{q}}\underline{\mathrm{t}} + \frac{1}{\mathfrak{p}}\int_{\mathfrak{p}\grave{\varrho}+(1-\mathfrak{p})\grave{\varsigma}}^{\grave{\varsigma}}\grave{\varphi}(\underline{\mathrm{t}})^{\grave{\varsigma}}d_{\mathfrak{p},\mathfrak{q}}\underline{\mathrm{t}}\right] - \left[\frac{\grave{\varphi}(\grave{\varrho})+\grave{\varphi}(\grave{\varsigma})}{2}\right].$$

Which completes the proof.

□

Remark 3.1. *We have presented the lemma that generalizes the corresponding results given in [29] for q-Hermite-Hadamard inequality for convex functions.*

Theorem 3.1. *We work under the conditions stated in Lemma 3.1. If the $|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}|$ and $|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}|$ are convex on $[\grave{\varrho},\grave{\varsigma}]$, then the following inequality holds:*

$$\left|\frac{1}{2(\grave{\varsigma}-\grave{\varrho})}\left(\frac{1}{\mathfrak{q}}\int_{\grave{\varrho}}^{\mathfrak{p}\grave{\varsigma}+(1-\mathfrak{p})\grave{\varrho}}\grave{\varphi}(\underline{t})_{\grave{\varrho}}d_{\mathfrak{p},\mathfrak{q}}\underline{t} + \frac{1}{\mathfrak{p}}\int_{\mathfrak{p}\grave{\varrho}+(1-\mathfrak{p})\grave{\varsigma}}^{\grave{\varsigma}}\grave{\varphi}(\underline{t})^{\grave{\varsigma}}d_{\mathfrak{p},\mathfrak{q}}\underline{t}\right) - \left[\frac{\grave{\varphi}(\grave{\varrho})+\grave{\varphi}(\grave{\varsigma})}{2}\right]\right|$$
$$\le \frac{\grave{\varsigma}-\grave{\varrho}}{2[2]_{\mathfrak{p},\mathfrak{q}}[3]_{\mathfrak{p},\mathfrak{q}}}\Big[([3]_{\mathfrak{p},\mathfrak{q}}-[2]_{\mathfrak{p},\mathfrak{q}})[\mathfrak{q}|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})| + \mathfrak{p}|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varrho})|] \tag{3.4}$$
$$+ [2]_{\mathfrak{p},\mathfrak{q}}[\mathfrak{q}|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varrho})| + \mathfrak{p}|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})|]\Big].$$

Proof. With help of Lemma 3.1, we get

$$\left|\frac{1}{2(\grave{\varsigma}-\grave{\varrho})}\left(\frac{1}{\mathfrak{q}}\int_{\grave{\varrho}}^{\mathfrak{p}\grave{\varsigma}+(1-\mathfrak{p})\grave{\varrho}}\grave{\varphi}(\underline{\mathrm{t}})_{\grave{\varrho}}d_{\mathfrak{p},\mathfrak{q}}\underline{\mathrm{t}} + \frac{1}{\mathfrak{p}}\int_{\mathfrak{p}\grave{\varrho}+(1-\mathfrak{p})\grave{\varsigma}}^{\grave{\varsigma}}\grave{\varphi}(\underline{\mathrm{t}})^{\grave{\varsigma}}d_{\mathfrak{p},\mathfrak{q}}\underline{\mathrm{t}}\right) - \left[\frac{\grave{\varphi}(\grave{\varrho})+\grave{\varphi}(\grave{\varsigma})}{2}\right]\right|$$
$$\le \frac{(\grave{\varsigma}-\grave{\varrho})}{2}\int_0^1 \mathfrak{q}\underline{\mathrm{t}}|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}((1-\underline{\mathrm{t}})\grave{\varsigma}+\underline{\mathrm{t}}\grave{\varrho})d_{\mathfrak{p},\mathfrak{q}}\underline{\mathrm{t}}| + \frac{(\grave{\varsigma}-\grave{\varrho})}{2}\int_0^1 \mathfrak{p}\underline{\mathrm{t}}|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}((1-\underline{\mathrm{t}})\grave{\varrho}+\underline{\mathrm{t}}\grave{\varsigma})d_{\mathfrak{p},\mathfrak{q}}\underline{\mathrm{t}}|. \tag{3.5}$$

By using the convexity of $|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}|$ and $|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}|$, we have

$$\left|\frac{1}{2(\grave{\varsigma}-\grave{\varrho})}\left(\frac{1}{\mathfrak{q}}\int_{\grave{\varrho}}^{\mathfrak{p}\grave{\varsigma}+(1-\mathfrak{p})\grave{\varrho}}\grave{\varphi}(\underline{\mathrm{t}})_{\grave{\varrho}}d_{\mathfrak{p},\mathfrak{q}}\underline{\mathrm{t}} + \frac{1}{\mathfrak{p}}\int_{\mathfrak{p}\grave{\varrho}+(1-\mathfrak{p})\grave{\varsigma}}^{\grave{\varsigma}}\grave{\varphi}(\underline{\mathrm{t}})^{\grave{\varsigma}}d_{\mathfrak{p},\mathfrak{q}}\underline{\mathrm{t}}\right) - \left[\frac{\grave{\varphi}(\grave{\varrho})+\grave{\varphi}(\grave{\varsigma})}{2}\right]\right|$$
$$\le \frac{(\grave{\varsigma}-\grave{\varrho})}{2}\int_0^1 \mathfrak{q}\underline{\mathrm{t}}\Big((1-\underline{\mathrm{t}})|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})| + \underline{\mathrm{t}}|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varrho})|\Big)d_{\mathfrak{p},\mathfrak{q}}\underline{\mathrm{t}}$$
$$+ \frac{(\grave{\varsigma}-\grave{\varrho})}{2}\int_0^1 \mathfrak{p}\underline{\mathrm{t}}\Big((1-\underline{\mathrm{t}})|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varrho})| + \underline{\mathrm{t}}|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\underline{\mathrm{t}}|\Big)d_{\mathfrak{p},\mathfrak{q}}\underline{\mathrm{t}}$$
$$= \frac{(\grave{\varsigma}-\grave{\varrho})}{2}\Big[\mathfrak{q}|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})| + \mathfrak{p}|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varrho})|\Big]\int_0^1(\underline{\mathrm{t}}-\underline{\mathrm{t}}^2)d_{\mathfrak{p},\mathfrak{q}}\underline{\mathrm{t}}$$
$$+ \Big[\mathfrak{q}|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varrho})| + \mathfrak{p}|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})|\Big]\int_0^1\underline{\mathrm{t}}^2 d_{\mathfrak{p},\mathfrak{q}}\underline{\mathrm{t}}$$
$$= \frac{(\grave{\varsigma}-\grave{\varrho})}{2}\Big[\mathfrak{q}|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})| + \mathfrak{p}|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varrho})|\Big]\left(\frac{1}{[2]_{\mathfrak{p},\mathfrak{q}}} - \frac{1}{[3]_{\mathfrak{p},\mathfrak{q}}}\right)$$
$$+ \Big[\mathfrak{q}|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varrho}) + \mathfrak{p}|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})\Big]\frac{1}{[3]_{\mathfrak{p},\mathfrak{q}}}$$

$$= \frac{(\grave{\varsigma}-\grave{\varrho})}{2[2]_{\mathfrak{p},\mathfrak{q}}[3]_{\mathfrak{p},\mathfrak{q}}}\Big[\Big([3]_{\mathfrak{p},\mathfrak{q}}-[2]_{\mathfrak{p},\mathfrak{q}}\Big)\Big[\mathfrak{q}|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})+\mathfrak{p}|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varrho})\Big] + [2]_{\mathfrak{p},\mathfrak{q}}\Big[\mathfrak{q}|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varrho})|+\mathfrak{p}|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})|\Big]\Big].$$

Thus, the proof is completed. □

Remark 3.2. *By taking $\mathfrak{p}=1$ in Theorem 3.1, we arrive at the following inequality:*

$$\left|\frac{1}{2(\grave{\varsigma}-\grave{\varrho})}\Big(\frac{1}{\mathfrak{q}}\int_{\grave{\varrho}}^{\grave{\varsigma}}\grave{\varphi}(\underline{t})\,_{\grave{\varrho}}d_{\mathfrak{q}}\underline{t}+\int_{\grave{\varrho}}^{\grave{\varsigma}}\grave{\varphi}(\underline{t})\,^{\grave{\varsigma}}d_{\mathfrak{q}}\underline{t}\Big)-\Big[\frac{\grave{\varphi}(\grave{\varrho})+\grave{\varphi}(\grave{\varsigma})}{2}\Big]\right|$$
$$\leq \frac{\grave{\varsigma}-\grave{\varrho}}{2[2]_{\mathfrak{q}}[3]_{\mathfrak{q}}}\Big[([3]_{\mathfrak{q}}-[2]_{\mathfrak{q}})[\mathfrak{q}|^{\grave{\varsigma}}D_{\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})|+|_{\grave{\varrho}}D_{\mathfrak{q}}\grave{\varphi}(\grave{\varrho})|]+[2]_{\mathfrak{q}}[\mathfrak{q}|^{\grave{\varsigma}}D_{\mathfrak{q}}\grave{\varphi}(\grave{\varrho})|+|_{\grave{\varrho}}D_{\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})|]\Big].$$

Theorem 3.2. *We work under the conditions stated in Lemma 3.1. If the functions $|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}|^s$ and $|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}|^s$, $s>1$ are convex, then the following inequality holds:*

$$\left|\frac{1}{2(\grave{\varsigma}-\grave{\varrho})}\Big(\frac{1}{\mathfrak{q}}\int_{\grave{\varrho}}^{\mathfrak{p}\grave{\varsigma}+(1-\mathfrak{p})\grave{\varrho}}\grave{\varphi}(\underline{t})\,_{\grave{\varrho}}d_{\mathfrak{p},\mathfrak{q}}\underline{t}+\frac{1}{\mathfrak{p}}\int_{\mathfrak{p}\grave{\varrho}+(1-\mathfrak{p})\grave{\varsigma}}^{\grave{\varsigma}}\grave{\varphi}(\underline{t})\,^{\grave{\varsigma}}d_{\mathfrak{p},\mathfrak{q}}\underline{t}\Big)-\Big[\frac{\grave{\varphi}(\grave{\varrho})+\grave{\varphi}(\grave{\varsigma})}{2}\Big]\right|$$
$$\leq \frac{\grave{\varsigma}-\grave{\varrho}}{2}\Big(\frac{1}{[r+1]_{\mathfrak{p},\mathfrak{q}}}\Big)^{\frac{1}{r}}\Bigg[\mathfrak{q}\Big(\Big(\frac{[2]_{\mathfrak{p},\mathfrak{q}}-1}{[2]_{\mathfrak{p},\mathfrak{q}}}\Big)|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})|^s+\frac{1}{[2]_{\mathfrak{p},\mathfrak{q}}}|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varrho})|^s\Big)^{\frac{1}{s}}$$
$$+\mathfrak{p}\Big(\Big(\frac{[2]_{\mathfrak{p},\mathfrak{q}}-1}{[2]_{\mathfrak{p},\mathfrak{q}}}\Big)|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varrho})|^s+\frac{1}{[2]_{\mathfrak{p},\mathfrak{q}}}|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})|^s\Big)^{\frac{1}{s}}\Bigg], \tag{3.6}$$

where $s^{-1}+r^{-1}=1$.

Proof. Applying Hölder's inequality to (3.5), it follows that:

$$\left|\frac{1}{2(\grave{\varsigma}-\grave{\varrho})}\Big(\frac{1}{\mathfrak{q}}\int_{\grave{\varrho}}^{\mathfrak{p}\grave{\varsigma}+(1-\mathfrak{p})\grave{\varrho}}\grave{\varphi}(\underline{t})\,_{\grave{\varrho}}d_{\mathfrak{p},\mathfrak{q}}\underline{t}+\frac{1}{\mathfrak{p}}\int_{\mathfrak{p}\grave{\varrho}+(1-\mathfrak{p})\grave{\varsigma}}^{\grave{\varsigma}}\grave{\varphi}(\underline{t})\,^{\grave{\varsigma}}d_{\mathfrak{p},\mathfrak{q}}\underline{t}\Big)-\Big[\frac{\grave{\varphi}(\grave{\varrho})+\grave{\varphi}(\grave{\varsigma})}{2}\Big]\right|$$
$$\leq \frac{(\grave{\varsigma}-\grave{\varrho})}{2}\Bigg[\Big(\int_0^1(\mathfrak{q}\underline{t})^r d_{\mathfrak{p},\mathfrak{q}}\underline{t}\Big)^{\frac{1}{r}}\Big(\int_0^1|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}((1-\underline{t})\grave{\varsigma}+\underline{t}\grave{\varrho})|^s d_{\mathfrak{p},\mathfrak{q}}\underline{t}\Big)^{\frac{1}{s}}$$
$$+\Big(\int_0^1(\mathfrak{p}\underline{t})^r d_{\mathfrak{p},\mathfrak{q}}\underline{t}\Big)^{\frac{1}{r}}\Big(\int_0^1|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}((1-\underline{t})\grave{\varrho}+\underline{t}\grave{\varsigma})|^s d_{\mathfrak{p},\mathfrak{q}}\underline{t}\Big)^{\frac{1}{s}}\Bigg].$$

Since, the functions $|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}|^s$ and $|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}|^s$ are convex, we have

$$\left|\frac{1}{2(\grave{\varsigma}-\grave{\varrho})}\Big(\frac{1}{\mathfrak{q}}\int_{\grave{\varrho}}^{\mathfrak{p}\grave{\varsigma}+(1-\grave{\varsigma})\grave{\varrho}}\grave{\varphi}(\underline{t})\,_{\grave{\varrho}}d_{\mathfrak{p},\mathfrak{q}}\underline{t}+\frac{1}{\mathfrak{p}}\int_{\mathfrak{p}\grave{\varrho}+(1-\mathfrak{p})\grave{\varsigma}}^{\grave{\varsigma}}\grave{\varphi}(\underline{t})\,^{\grave{\varsigma}}d_{\mathfrak{p},\mathfrak{q}}\underline{t}\Big)-\Big[\frac{\grave{\varphi}(\grave{\varrho})+\grave{\varphi}(\grave{\varsigma})}{2}\Big]\right|$$
$$\leq \frac{(\grave{\varsigma}-\grave{\varrho})}{2}\Bigg[\Big(\int_0^1(\mathfrak{q}\underline{t})^r d_{\mathfrak{p},\mathfrak{q}}\underline{t}\Big)^{\frac{1}{r}}\Big(\int_0^1[(1-\underline{t})|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})|^s+\underline{t}|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varrho})|^s]d_{\mathfrak{p},\mathfrak{q}}\underline{t}\Big)^{\frac{1}{s}}\Bigg]$$
$$+\frac{(\grave{\varsigma}-\grave{\varrho})}{2}\Bigg[\Big(\int_0^1(\mathfrak{p}\underline{t})^r d_{\mathfrak{p},\mathfrak{q}}\underline{t}\Big)^{\frac{1}{r}}\Big(\int_0^1[(1-\underline{t})|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varrho})|^s+\underline{t}|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})|^s]d_{\mathfrak{p},\mathfrak{q}}\underline{t}\Big)^{\frac{1}{s}}\Bigg]$$

$$= \frac{\mathfrak{q}(\grave{\varsigma}-\grave{\varrho})}{2}\left(\frac{1}{[r+1]_{\mathfrak{p},\mathfrak{q}}}\right)^{\frac{1}{r}}\left[\left(\frac{[2]_{\mathfrak{p},\mathfrak{q}}-1}{[2]_{\mathfrak{p},\mathfrak{q}}}\right)|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})|^{s}+\frac{1}{[2]_{\mathfrak{p},\mathfrak{q}}}|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varrho})|^{s}\right]^{\frac{1}{s}}$$

$$+ \frac{\mathfrak{p}(\grave{\varsigma}-\grave{\varrho})}{2}\left(\frac{1}{[r+1]_{\mathfrak{p},\mathfrak{q}}}\right)^{\frac{1}{r}}\left[\left(\frac{[2]_{\mathfrak{p},\mathfrak{q}}-1}{[2]_{\mathfrak{p},\mathfrak{q}}}\right)|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varrho})|^{s}+\frac{1}{[2]_{\mathfrak{p},\mathfrak{q}}}|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})|^{s}\right]^{\frac{1}{s}}$$

$$= \frac{(\grave{\varsigma}-\grave{\varrho})}{2}\left(\frac{1}{[r+1]_{\mathfrak{p},\mathfrak{q}}}\right)^{\frac{1}{r}}\left[\mathfrak{q}\left(\left(\frac{[2]_{\mathfrak{p},\mathfrak{q}}-1}{[2]_{\mathfrak{p},\mathfrak{q}}}\right)|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})|^{s}+\frac{1}{[2]_{\mathfrak{p},\mathfrak{q}}}|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varrho})|^{s}\right)^{\frac{1}{s}}\right.$$

$$\left.+\mathfrak{p}\left(\left(\frac{[2]_{\mathfrak{p},\mathfrak{q}}-1}{[2]_{\mathfrak{p},\mathfrak{q}}}\right)|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varrho})|^{s}+\frac{1}{[2]_{\mathfrak{p},\mathfrak{q}}}|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})|^{s}\right)^{\frac{1}{s}}\right].$$

Hence, the proof is completed. □

Remark 3.3. *By taking* $\mathfrak{p}=1$ *in Theorem 3.2, we arrive at the following inequality:*

$$\left|\frac{1}{2(\grave{\varsigma}-\grave{\varrho})}\left(\frac{1}{\mathfrak{q}}\int_{a}^{\grave{\varsigma}}\grave{\varphi}(\underline{t})_{\grave{\varrho}}d_{\mathfrak{q}}\underline{t}+\int_{\grave{\varrho}}^{\grave{\varsigma}}\grave{\varphi}(\underline{t})^{\grave{\varsigma}}d_{\mathfrak{q}}\underline{t}\right)-\left[\frac{\grave{\varphi}(\grave{\varrho})+\grave{\varphi}(\grave{\varsigma})}{2}\right]\right|$$

$$\leq \frac{\grave{\varsigma}-\grave{\varrho}}{2}\left(\frac{1}{[r+1]_{\mathfrak{q}}}\right)^{\frac{1}{r}}\left[\mathfrak{q}\left(\left(\frac{[2]_{\mathfrak{q}}-1}{[2]_{\mathfrak{q}}}\right)|^{\grave{\varsigma}}D_{\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})|^{s}+\frac{1}{[2]_{\mathfrak{q}}}|^{\grave{\varsigma}}D_{\mathfrak{q}}\grave{\varphi}(\grave{\varrho})|^{s}\right)^{\frac{1}{s}}\right.$$

$$\left.+\left(\left(\frac{[2]_{\mathfrak{q}}-1}{[2]_{\mathfrak{q}}}\right)|_{\grave{\varrho}}D_{\mathfrak{q}}\grave{\varphi}(\grave{\varrho})|^{s}+\frac{1}{[2]_{\mathfrak{q}}}|_{a}D_{\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})|^{s}\right)^{\frac{1}{s}}\right].$$

Theorem 3.3. *We work under the conditions stated in Lemma 3.1. If the functions* $|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}|^{s}$ *and* $|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}|^{s}$*,* $s\geq 1$ *are convex, then the following inequality holds:*

$$\left|\frac{1}{2(\grave{\varsigma}-\grave{\varrho})}\left(\frac{1}{\mathfrak{q}}\int_{\grave{\varrho}}^{\mathfrak{p}\grave{\varsigma}+(1-\mathfrak{p})\grave{\varrho}}\grave{\varphi}(\underline{t})_{\grave{\varrho}}d_{\mathfrak{p},\mathfrak{q}}\underline{t}+\frac{1}{\mathfrak{p}}\int_{\mathfrak{p}\grave{\varrho}+(1-\mathfrak{p})\grave{\varsigma}}^{\grave{\varsigma}}\grave{\varphi}(\underline{t})^{\grave{\varsigma}}d_{\mathfrak{p},\mathfrak{q}}\underline{t}\right)-\left[\frac{\grave{\varphi}(\grave{\varrho})+\grave{\varphi}(\grave{\varsigma})}{2}\right]\right|$$

$$\leq \frac{(\grave{\varsigma}-\grave{\varrho})}{2[2]_{\mathfrak{p},\mathfrak{q}}}\left[\mathfrak{q}\left(\frac{([3]_{\mathfrak{p},\mathfrak{q}}-[2]_{\mathfrak{p},\mathfrak{q}})|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})|^{s}+[2]_{\mathfrak{p},\mathfrak{q}}|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varrho})|^{s}}{[3]_{\mathfrak{p},\mathfrak{q}}}\right)^{\frac{1}{s}}\right.$$

$$\left.+\mathfrak{p}\left(\frac{([3]_{\mathfrak{p},\mathfrak{q}}-[2]_{\mathfrak{p},\mathfrak{q}})|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varrho})|^{s}+[2]_{\mathfrak{p},\mathfrak{q}}|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})|^{s}}{[3]_{\mathfrak{p},\mathfrak{q}}}\right)^{\frac{1}{s}}\right].$$

Proof. Applying the power mean inequality to (3.5), it follows that:

$$\left|\frac{1}{2(\grave{\varsigma}-\grave{\varrho})}\left(\frac{1}{\mathfrak{q}}\int_{\grave{\varrho}}^{\mathfrak{p}\grave{\varsigma}+(1-\mathfrak{p})\grave{\varrho}}\grave{\varphi}(\underline{\mathrm{t}})_{\grave{\varrho}}d_{\mathfrak{p},\mathfrak{q}}\underline{\mathrm{t}}+\frac{1}{\mathfrak{p}}\int_{\mathfrak{p}\grave{\varrho}+(1-\mathfrak{p})\grave{\varsigma}}^{\grave{\varsigma}}\grave{\varphi}(\underline{\mathrm{t}})^{\grave{\varsigma}}d_{\mathfrak{p},\mathfrak{q}}\underline{\mathrm{t}}\right)-\left[\frac{\grave{\varphi}(\grave{\varrho})+\grave{\varphi}(\grave{\varsigma})}{2}\right]\right|$$

$$\leq \frac{(\grave{\varsigma}-\grave{\varrho})}{2}\left[\left(\int_{0}^{1}\mathfrak{q}\underline{\mathrm{t}}d_{\mathfrak{p},\mathfrak{q}}\underline{\mathrm{t}}\right)^{1-\frac{1}{s}}\left(\int_{0}^{1}\mathfrak{q}\underline{\mathrm{t}}|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}((1-\underline{\mathrm{t}})\grave{\varsigma}+\underline{\mathrm{t}}\grave{\varrho})|^{s}d_{\mathfrak{p},\mathfrak{q}}\underline{\mathrm{t}}\right)^{\frac{1}{s}}\right.$$

$$\left.+\left(\int_{0}^{1}\mathfrak{p}\underline{\mathrm{t}}d_{\mathfrak{p},\mathfrak{q}}\underline{\mathrm{t}}\right)^{1-\frac{1}{s}}\left(\int_{0}^{1}p\underline{\mathrm{t}}|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}((1-\underline{\mathrm{t}})\grave{\varrho}+\underline{\mathrm{t}}\grave{\varsigma})|^{s}d_{\mathfrak{p},\mathfrak{q}}\underline{\mathrm{t}}\right)^{\frac{1}{s}}\right].$$

By using convexity of the functions $|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}|^s$ and $|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}|^s$, we have

$$\begin{aligned}
&\left|\frac{1}{2(\grave{\varsigma}-\grave{\varrho})}\left(\frac{1}{\mathfrak{q}}\int_{\grave{\varrho}}^{\mathfrak{p}\grave{\varsigma}+(1-\mathfrak{p})\grave{\varrho}}\grave{\varphi}(\underline{\mathrm{t}})\,_{\grave{\varrho}}d_{\mathfrak{p},\mathfrak{q}}\underline{\mathrm{t}}+\frac{1}{\mathfrak{p}}\int_{\mathfrak{p}\grave{\varrho}+(1-\mathfrak{p})\grave{\varsigma}}^{\grave{\varsigma}}\grave{\varphi}(\underline{\mathrm{t}})\,^{\grave{\varsigma}}d_{\mathfrak{p},\mathfrak{q}}\underline{\mathrm{t}}\right)-\left[\frac{\grave{\varphi}(\grave{\varrho})+\grave{\varphi}(\grave{\varsigma})}{2}\right]\right|\\
&\leq\frac{\grave{\varsigma}-\grave{\varrho}}{2}\Bigg[\left(\int_0^1\mathfrak{q}\underline{\mathrm{t}}d_{\mathfrak{p},\mathfrak{q}}\underline{\mathrm{t}}\right)^{1-\frac{1}{s}}\left(\int_0^1\mathfrak{q}\underline{\mathrm{t}}[(1-\underline{\mathrm{t}})|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})|^s+\underline{\mathrm{t}}|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varrho})|^s]d_{\mathfrak{p},\mathfrak{q}}\underline{\mathrm{t}}\right)^{\frac{1}{s}}\\
&\quad+\left(\int_0^1\mathfrak{p}\underline{\mathrm{t}}d_{\mathfrak{p},\mathfrak{q}}\underline{\mathrm{t}}\right)^{1-\frac{1}{s}}\left(\int_0^1\mathfrak{p}\underline{\mathrm{t}}[(1-\underline{\mathrm{t}})|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varrho})|^s+\underline{\mathrm{t}}|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})|^s]d_{\mathfrak{p},\mathfrak{q}}\underline{\mathrm{t}}\right)^{\frac{1}{s}}\Bigg]\\
&=\frac{\grave{\varsigma}-\grave{\varrho}}{2}\left(\frac{\mathfrak{q}}{[2]_{\mathfrak{p},\mathfrak{q}}}\right)^{1-\frac{1}{s}}\left[\left(\frac{\mathfrak{q}}{[2]_{\mathfrak{p},\mathfrak{q}}}-\frac{\mathfrak{q}}{[3]_{\mathfrak{p},\mathfrak{q}}}\right)|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})|^s+\frac{\mathfrak{q}}{[3]_{\mathfrak{p},\mathfrak{q}}}|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varrho})|^s\right]^{\frac{1}{s}}\\
&\quad+\frac{\grave{\varsigma}-\grave{\varrho}}{2}\left(\frac{\mathfrak{p}}{[2]_{\mathfrak{p},\mathfrak{q}}}\right)^{1-\frac{1}{s}}\left[\left(\frac{\mathfrak{p}}{[2]_{\mathfrak{p},\mathfrak{q}}}-\frac{\mathfrak{p}}{[3]_{\mathfrak{p},\mathfrak{q}}}\right)|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varrho})|^s+\frac{\mathfrak{p}}{[3]_{\mathfrak{p},\mathfrak{q}}}|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})|^s\right]^{\frac{1}{s}}\\
&=\frac{(\grave{\varsigma}-\grave{\varrho})}{2[2]_{\mathfrak{p},\mathfrak{q}}}\Bigg[\mathfrak{q}\left(\frac{([3]_{\mathfrak{p},\mathfrak{q}}-[2]_{\mathfrak{p},\mathfrak{q}})|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})|^s+[2]_{\mathfrak{p},\mathfrak{q}}|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varrho})|^s}{[3]_{\mathfrak{p},\mathfrak{q}}}\right)^{\frac{1}{s}}\\
&\quad+\mathfrak{p}\left(\frac{([3]_{\mathfrak{p},\mathfrak{q}}-[2]_{\mathfrak{p},\mathfrak{q}})|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varrho})|^s+[2]_{\mathfrak{p},\mathfrak{q}}|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})|^s}{[3]_{\mathfrak{p},\mathfrak{q}}}\right)^{\frac{1}{s}}\Bigg].
\end{aligned}$$

Thus, the proof is completed. □

Remark 3.4. *By taking $\mathfrak{p}=1$ in Theorem 3.3, we arrive at the following inequality:*

$$\begin{aligned}
&\left|\frac{1}{2(\grave{\varsigma}-\grave{\varrho})}\left(\frac{1}{\mathfrak{q}}\int_{\grave{\varrho}}^{\grave{\varsigma}}\grave{\varphi}(\underline{t})\,_{\grave{\varrho}}d_{\mathfrak{q}}\underline{t}+\int_{\grave{\varrho}}^{\grave{\varsigma}}\grave{\varphi}(\underline{t})\,^{\grave{\varsigma}}d_{\mathfrak{q}}\underline{t}\right)-\left[\frac{\grave{\varphi}(\grave{\varrho})+\grave{\varphi}(\grave{\varsigma})}{2}\right]\right|\\
&\leq\frac{(\grave{\varsigma}-\grave{\varrho})}{2[2]_{\mathfrak{q}}}\Bigg[\mathfrak{q}\left(\frac{([3]_{\mathfrak{q}}-[2]_{\mathfrak{q}})|^{\grave{\varsigma}}D_{\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})|^s+[2]_{\mathfrak{q}}|^{\grave{\varsigma}}D_{\mathfrak{q}}\grave{\varphi}(\grave{\varrho})|^s}{[3]_{\mathfrak{q}}}\right)^{\frac{1}{s}}\\
&\quad+\left(\frac{([3]_{\mathfrak{q}}-[2]_{\mathfrak{q}})|_{\grave{\varrho}}D_{\mathfrak{q}}\grave{\varphi}(\grave{\varrho})|^s+[2]_{\mathfrak{q}}|_{\grave{\varrho}}D_{\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})|^s}{[3]_{\mathfrak{q}}}\right)^{\frac{1}{s}}\Bigg].
\end{aligned}$$

4. Examples

In order to explore our theorem, we provide two examples.

Example 4.1. *For a convex function $\grave{\varphi}(\underline{t})=\underline{t}^2$ and $\grave{\varrho}=0$, $\grave{\varsigma}=1$, $\mathfrak{q}$=1/2 and $\mathfrak{p}$=3/4. The left side of (3.4) becomes*

$$\begin{aligned}
&\left|\frac{1}{2(\grave{\varsigma}-\grave{\varrho})}\left(\frac{1}{\mathfrak{q}}\int_{\grave{\varrho}}^{\mathfrak{p}\grave{\varsigma}+(1-\mathfrak{p})\grave{\varrho}}\grave{\varphi}(\underline{t})\,_{\grave{\varrho}}d_{\mathfrak{p},\mathfrak{q}}\underline{t}+\frac{1}{\mathfrak{p}}\int_{\mathfrak{p}\grave{\varrho}+(1-\mathfrak{p})\grave{\varsigma}}^{\grave{\varsigma}}\grave{\varphi}(\underline{t})\,^{\grave{\varsigma}}d_{\mathfrak{p},\mathfrak{q}}\underline{t}\right)-\left[\frac{\grave{\varphi}(\grave{\varrho})+\grave{\varphi}(\grave{\varsigma})}{2}\right]\right|\\
&=\left|\frac{1}{2(1-0)}\left(2\int_0^{\frac{3}{4}}\underline{t}^2\,_{\grave{\varrho}}d_{\mathfrak{p},\mathfrak{q}}\underline{t}+\frac{4}{3}\int_{\frac{1}{4}}^1\underline{t}^2\,^{\grave{\varsigma}}d_{\mathfrak{p},\mathfrak{q}}\underline{t}\right)-\left[\frac{\grave{\varphi}(0)+\grave{\varphi}(1)}{2}\right]\right|\\
&\approx\left|\frac{1}{2}(2\times 0.36+\frac{4}{3}\times 0.2736)-0.5\right|\approx 0.0532
\end{aligned}$$

and the right side of (3.4) becomes

$$\frac{\grave{\varsigma}-\grave{\varrho}}{2[2]_{\mathfrak{p},\mathfrak{q}}[3]_{\mathfrak{p},\mathfrak{q}}}\Bigg[([3]_{\mathfrak{p},\mathfrak{q}}-[2]_{\mathfrak{p},\mathfrak{q}})[\mathfrak{q}|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})|+\mathfrak{p}|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varrho})|]+[2]_{\mathfrak{p},\mathfrak{q}}[\mathfrak{q}|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varrho})|+\mathfrak{p}|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})|]\Bigg]$$
$$=\frac{1-0}{2[2]_{\mathfrak{p},\mathfrak{q}}[3]_{\mathfrak{p},\mathfrak{q}}}\Bigg[([3]_{\mathfrak{p},\mathfrak{q}}-[2]_{\mathfrak{p},\mathfrak{q}})[\frac{1}{2}|2|+\frac{3}{4}|0|]+[2]_{\mathfrak{p},\mathfrak{q}}[\frac{1}{2}|0.75|+\frac{3}{4}|1.25|]\Bigg]\approx 0.5207$$

It is clear that

$$0.0532\leq 0.5207$$

which confirms the validity of inequality (3.4).

Example 4.2. *For a convex function $\grave{\varphi}(\underline{t})=\underline{t}^2$ and $\grave{\varrho}=0$, $\grave{\varsigma}=1$, $\mathfrak{q}$=1/2, $\mathfrak{p}$=3/4, s=2 and r=2. The left side of (3.6) becomes*

$$\left|\frac{1}{2(\grave{\varsigma}-\grave{\varrho})}\left(\frac{1}{\mathfrak{q}}\int_{\grave{\varrho}}^{\mathfrak{p}\grave{\varsigma}+(1-\mathfrak{p})\grave{\varrho}}\grave{\varphi}(\underline{t})_{\grave{\varrho}}d_{\mathfrak{p},\mathfrak{q}}\underline{t}+\frac{1}{\mathfrak{p}}\int_{\mathfrak{p}\grave{\varrho}+(1-\mathfrak{p})\grave{\varsigma}}^{\grave{\varsigma}}\grave{\varphi}(\underline{t})^{\grave{\varsigma}}d_{\mathfrak{p},\mathfrak{q}}\underline{t}\right)-\left[\frac{\grave{\varphi}(\grave{\varrho})+\grave{\varphi}(\grave{\varsigma})}{2}\right]\right|$$
$$=\left|\frac{1}{2(1-0)}\left(2\int_0^{\frac{3}{4}}\underline{t}^2{}_{\grave{\varrho}}d_{\mathfrak{p},\mathfrak{q}}\underline{t}+\frac{4}{3}\int_{\frac{1}{4}}^{1}\underline{t}^{2\grave{\varsigma}}d_{\mathfrak{p},\mathfrak{q}}\underline{t}\right)-\left[\frac{\grave{\varphi}(0)+\grave{\varphi}(1)}{2}\right]\right|$$
$$\approx\left|\frac{1}{2}(2\times 0.36+\frac{4}{3}\times 0.2736)-0.5\right|\approx 0.0532$$

and the right side of (3.6) becomes

$$\leq\frac{\grave{\varsigma}-\grave{\varrho}}{2}\left(\frac{1}{[r+1]_{\mathfrak{p},\mathfrak{q}}}\right)^{\frac{1}{r}}\Bigg[\mathfrak{q}\left(\left(\frac{[2]_{\mathfrak{p},\mathfrak{q}}-1}{[2]_{\mathfrak{p},\mathfrak{q}}}\right)|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})|^s+\frac{1}{[2]_{\mathfrak{p},\mathfrak{q}}}|^{\grave{\varsigma}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varrho})|^s\right)^{\frac{1}{s}}$$
$$+\mathfrak{p}\left(\left(\frac{[2]_{\mathfrak{p},\mathfrak{q}}-1}{[2]_{\mathfrak{p},\mathfrak{q}}}\right)|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varrho})|^s+\frac{1}{[2]_{\mathfrak{p},\mathfrak{q}}}|_{\grave{\varrho}}D_{\mathfrak{p},\mathfrak{q}}\grave{\varphi}(\grave{\varsigma})|^s\right)^{\frac{1}{s}}\Bigg]$$
$$=\frac{1}{2}\left(\frac{1}{[3]_{\mathfrak{p},\mathfrak{q}}}\right)^{\frac{1}{2}}\left[\frac{1}{2}\left(\left(\frac{[2]_{\mathfrak{p},\mathfrak{q}}-1}{[2]_{\mathfrak{p},\mathfrak{q}}}\right)(2)^2+\frac{1}{[2]_{\mathfrak{p},\mathfrak{q}}}(0.75)^2\right)^{\frac{1}{2}}+\frac{3}{4}\left(\left(\frac{[2]_{\mathfrak{p},\mathfrak{q}}-1}{[2]_{\mathfrak{p},\mathfrak{q}}}\right)(0)^2+\frac{1}{[2]_{\mathfrak{p},\mathfrak{q}}}(1.25)^2\right)^{\frac{1}{2}}\right]$$
$$\approx 0.657$$

It is clear that

$$0.0532\leq 0.657$$

which confirms the validity of inequality (3.6).

5. Conclusion and Future Directions

In this study, we demonstrate generalized post-quantum estimations for the $(\mathfrak{p},\mathfrak{q})$-Hermite-Hadamard inequality for convex functions involving two kinds of post-quantum integrals. The outcomes of this study are generalizable to $(\mathfrak{p},\mathfrak{q})$-fractional inequalities, similar inequalities, or outcomes for various types of convexity. We expect that the approaches and results of this work may encourage continued research on inequalities.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] T.H. Zhao, M.K. Wang, Y.M. Chu, Monotonicity and Convexity Involving Generalized Elliptic Integral of the First Kind, RACSAM, Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat. 115 (2021), 46. https://doi.org/10.1007/s13398-020-00992-3.

[2] T.H. Zhao, L. Shi, Y.M. Chu, Convexity and Concavity of the Modified Bessel Functions of the First Kind with Respect to Hölder Means, RACSAM, Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat. 114 (2020), 96. https://doi.org/10.1007/s13398-020-00825-3.

[3] M.M. Matar, M.I. Abbas, J. Alzabut, M.K.A. Kaabar, S. Etemad, et al., Investigation of the P-Laplacian Nonperiodic Nonlinear Boundary Value Problem via Generalized Caputo Fractional Derivatives, Adv. Differ. Equ. 2021 (2021), 68. https://doi.org/10.1186/s13662-021-03228-9.

[4] A. Omame, M. Abbas, A.H. Abdel-Aty, Assessing the Impact of SARS-Cov-2 Infection on the Dynamics of Dengue and HIV via Fractional Derivatives, Chaos Solitons Fractals 162 (2022), 112427. https://doi.org/10.1016/j.chaos.2022.112427.

[5] H. Mohammadi, S. Kumar, S. Rezapour, S. Etemad, A Theoretical Study of the Caputo–Fabrizio Fractional Modeling for Hearing Loss Due to Mumps Virus with Optimal Control, Chaos Solitons Fractals 144 (2021), 110668. https://doi.org/10.1016/j.chaos.2021.110668.

[6] P. Kumar, V.S. Erturk, H. Abboubakar, K.S. Nisar, Prediction Studies of the Epidemic Peak of Coronavirus Disease in Brazil via New Generalised Caputo Type Fractional Derivatives, Alex. Eng. J. 60 (2021), 3189–3204. https://doi.org/10.1016/j.aej.2021.01.032.

[7] K.N. Nabi, H. Abboubakar, P. Kumar, Forecasting of COVID-19 Pandemic: From Integer Derivatives to Fractional Derivatives, Chaos Solitons Fractals 141 (2020), 110283. https://doi.org/10.1016/j.chaos.2020.110283.

[8] T. Eiter, K. Hopf, R. Lasarzik, Weak-Strong Uniqueness and Energy-Variational Solutions for a Class of Viscoelastoplastic Fluid Models, Adv. Nonlinear Anal. 12 (2022), 20220274. https://doi.org/10.1515/anona-2022-0274.

[9] O. Omotoyinbo, A. Mogbodemu, Some New Hermite-Hadamard Integral Inequalities for Convex Functions, Int. J. Sci. Innov. Technol. 1 (2014), 1–12.

[10] M. Tariq, J.N. Jamshed Nasir, S.K. Sahoo, A.A. Mallah, A Note on Some Ostrowski Type Inequalities via Generalized Exponentially Convexity, J. Math. Anal. Model. 2 (2021), 1–15. https://doi.org/10.48185/jmam.v2i2.216.

[11] M.A. Ali, C.S. Goodrich, H. Budak, Some New Parameterized Newton-Type Inequalities for Differentiable Functions via Fractional Integrals, J. Inequal. Appl. 2023 (2023), 49. https://doi.org/10.1186/s13660-023-02953-x.

[12] A. Moumen, H. Boulares, B. Meftah, R. Shafqat, T. Alraqad, et al., Multiplicatively Simpson Type Inequalities via Fractional Integral, Symmetry 15 (2023), 460. https://doi.org/10.3390/sym15020460.

[13] A.J. Barr, P. Caban, J. Rembieliński, Bell-Type Inequalities for Systems of Relativistic Vector Bosons, Quantum 7 (2023), 1070. https://doi.org/10.22331/q-2023-07-27-1070.

[14] J. Bisht, N. Sharma, S.K. Mishra, A. Hamdi, Some New Integral Inequalities for Higher-Order Strongly Exponentially Convex Functions, J. Inequal. Appl. 2023 (2023), 41. https://doi.org/10.1186/s13660-023-02952-y.

[15] K.K. Lai, S.K. Mishra, V. Singh, Ostrowski Type Inequalities via Some Exponentially S-Preinvex Functions on Time Scales with Applications, Symmetry 15 (2023), 410. https://doi.org/10.3390/sym15020410.

[16] N. Sharma, S.K. Singh, S.K. Mishra, A. Hamdi, Hermite–Hadamard–Type Inequalities for Interval-Valued Preinvex Functions via Riemann–Liouville Fractional Integrals, J. Inequal. Appl. 2021 (2021), 98. https://doi.org/10.1186/s13660-021-02623-w.

[17] J. Bisht, R. Mishra, A. Hamdi, Hermite-Hadamard and Fejér-Type Inequalities for Generalized η-Convex Stochastic Processes, Commun. Stat. - Theory Methods 53 (2023), 5299–5310. https://doi.org/10.1080/03610926.2023.2218506.

[18] K.K. Lai, J. Bisht, N. Sharma, S.K. Mishra, Hermite-Hadamard Type Integral Inequalities for the Class of Strongly Convex Functions on Time Scales, J. Math. Inequal. 16 (2022), 975–991. https://doi.org/10.7153/jmi-2022-16-66.

[19] S. Singh, S.K. Mishra, V. Singh, P. Pankaj, H. Budak, Hermite-Hadamard Type Inequalities for Preinvex Functions with Applications, Korean J. Math. 33 (2025), 65–74. https://doi.org/10.11568/kjm.2025.33.1.65.

[20] T. Zhou, Z. Yuan, T. Du, On the Fractional Integral Inclusions Having Exponential Kernels for Interval-Valued Convex Functions, Math. Sci. 17 (2021), 107–120. https://doi.org/10.1007/s40096-021-00445-x.

[21] W. Sun, On Generalization of Some Inequalities for Generalized Harmonically Convex Functions via Local Fractional Integrals, Quaest. Math. 42 (2018), 1159–1183. https://doi.org/10.2989/16073606.2018.1509242.

[22] W. Sun, Some New Inequalities for Generalized h-Convex Functions Involving Local Fractional Integral Operators with Mittag-Leffler Kernel, Math. Methods Appl. Sci. 44 (2020), 4985–4998. https://doi.org/10.1002/mma.7081.

[23] W. Sun, Q. Liu, Hadamard Type Local Fractional Integral Inequalities for Generalized Harmonically Convex Functions and Applications, Math. Methods Appl. Sci. 43 (2020), 5776–5787. https://doi.org/10.1002/mma.6319.

[24] W. Sun, Hermite–hadamard Type Local Fractional Integral Inequalities With Mittag-Leffler Kernel for Generalized Preinvex Functions, Fractals 29 (2021), 2150253. https://doi.org/10.1142/S0218348X21502534.

[25] T. Ernst, A Comprehensive Treatment of q-Calculus, Springer, 2021. https://doi.org/10.1007/978-3-0348-0431-8.

[26] F.H. Jackson, On q-Definite Integrals, Quart. J. Pure Appl. Math. 41 (1910), 193–203.

[27] F.H. Jackson, Q-Difference Equations, Am. J. Math. 32 (1910), 305–314. https://doi.org/10.2307/2370183.

[28] M.A. Noor, K.I. Noor, M.U. Awan, Some Quantum Estimates for Hermite–Hadamard Inequalities, Appl. Math. Comput. 251 (2015), 675–679. https://doi.org/10.1016/j.amc.2014.11.090.

[29] N. Alp, H. Budak, M.Z. Sarikaya, M.A. Ali, On New Refinements and Generalizations of q–hermite–hadamard Inequalities for Convex Functions, Rocky Mt. J. Math. 54 (2024), 361–374. https://doi.org/10.1216/rmj.2024.54.361.

[30] R. Chakrabarti, R. Jagannathan, A (p, q)-Oscillator Realization of Two-Parameter Quantum Algebras, J. Phys.: Math. Gen. 24 (1991), L711–L718. https://doi.org/10.1088/0305-4470/24/13/002.

[31] M. Tunç, E. Göv, Some Integral Inequalities via (p,q)-Calculus on Finite Intervals, Filomat 35 (2021), 1421–1430. https://doi.org/10.2298/FIL2105421T.

[32] M. Vivas-Cortez, M.A. Ali, H. Budak, H. Kalsoom, P. Agarwal, Some New Hermite–Hadamard and Related Inequalities for Convex Functions via (p,q)-Integral, Entropy 23 (2021), 828. https://doi.org/10.3390/e23070828.

[33] J. Soontharanon, T. Sitthiwirattham, Existence Results of Nonlocal Robin Boundary Value Problems for Fractional (p,q)-Integrodifference Equations, Adv. Differ. Equ. 2020 (2020), 342. https://doi.org/10.1186/s13662-020-02806-7.

[34] J. Prabseang, K. Nonlaopon, J. Tariboon, (p, q)–Hermite–Hadamard Inequalities for Double Integral and (p, q)-Differentiable Convex Functions, Axioms 8 (2019), 68. https://doi.org/10.3390/axioms8020068.

[35] U. Duran, M. Acikgoz, A. Esi, S. Araci, A Note on the (p,q)-Hermite Polynomials, Appl. Math. Inf. Sci. 12 (2018), 227–231. https://doi.org/10.18576/amis/120122.

[36] S. Thongjob, K. Nonlaopon, S.K. Ntouyas, Some (p, q)-Hardy Type Inequalities for (p, q)-Integrable Functions, AIMS Math. 6 (2021), 77–89. https://doi.org/10.3934/math.2021006.

[37] M. Nasiruzzaman, A. Mukheimer, M. Mursaleen, Some Opial-Type Integral Inequalities via (p,q)-Calculus, J. Inequal. Appl. 2019 (2019), 295. https://doi.org/10.1186/s13660-019-2247-8.

[38] M. Tariq, S.K. Ntouyas, A.A. Shaikh, A Comprehensive Review of the Hermite–Hadamard Inequality Pertaining to Quantum Calculus, Foundations 3 (2023), 340–379. https://doi.org/10.3390/foundations3020026.

[39] W. Luangboon, K. Nonlaopon, J. Tariboon, S.K. Ntouyas, H. Budak, Some (p, q)-Integral Inequalities of Hermite–Hadamard Inequalities for (p, q)-Differentiable Convex Functions, Mathematics 10 (2022), 826. https://doi.org/10.3390/math10050826.

[40] X.X. You, M.A. Ali, H. Kalsoom, J. Soontharanon, T. Sitthiwirattham, On Some New Hermite-Hadamard and Ostrowski Type Inequalities for s-Convex Functions in (p, q)-Calculus with Applications, Open Math. 20 (2022), 707–723. https://doi.org/10.1515/math-2022-0037.

[41] S.K. Mishra, V. Singh, Some New Hermite-Hadamard and Ostrowski Type Inequalities for S-Preinvex Functions in (p,q)-Calculus with Applications, Bull. Iran. Math. Soc. 49 (2023), 33. https://doi.org/10.1007/s41980-023-00776-6.

[42] J. Hadamard, Essai sur l'Étude des Fonctions Données par Leur Développement de Taylor, J. Math. Pures Appl. 8 (1892), 101–186. http://www.numdam.org/item?id=JMPA_1892_4_8__101_0.

[43] V. Kac, P. Cheung, Quantum Calculus, Springer, 2002. https://doi.org/10.1007/978-1-4613-0071-7.

[44] J. Tariboon, S.K. Ntouyas, Quantum Calculus on Finite Intervals and Applications to Impulsive Difference Equations, Adv. Differ. Equ. 2013 (2013), 282. https://doi.org/10.1186/1687-1847-2013-282.

[45] S. Bermudo, P. Kórus, J.E. Nápoles Valdés, On Q–Hermite–Hadamard Inequalities for General Convex Functions, Acta Math. Hung. 162 (2020), 364–374. https://doi.org/10.1007/s10474-020-01025-6.

[46] N. Alp, M.Z. Sarıkaya, M. Kunt, İ. İşcan, Q -Hermite Hadamard Inequalities and Quantum Estimates for Midpoint Type Inequalities via Convex and Quasi-Convex Functions, J. King Saud Univ. - Sci. 30 (2018), 193–203. https://doi.org/10.1016/j.jksus.2016.09.007.

[47] M. KUNT, A. BAİDAR, Z. ŞANLI, Some Quantum Integral Inequalities Based on Left-Right Quantum Integrals, Turk. J. Sci. Technol. 17 (2022), 343–356. https://doi.org/10.55525/tjst.1112582.

[48] A. BAİDAR, M. KUNT, Quantum Analog of Some Trapezoid and Midpoint Type Inequalities for Convex Functions, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. 71 (2022), 456–480. https://doi.org/10.31801/cfsuasmas.1009988.

[49] V. Kac, P. Cheung, Quantum Calculus, Springer, 2001. https://doi.org/10.1007/978-1-4613-0071-7.

[50] M. Kunt, I. İşcan, N. Alp, M.Z. Sarikaya, (p, q)-Hermite–Hadamard Inequalities and (p, q)-Estimates for Midpoint Type Inequalities via Convex and Quasi-Convex Functions, RACSAM, Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat. 112 (2017), 969–992. https://doi.org/10.1007/s13398-017-0402-y.