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THE S-TRANSFORM ON HARDY SPACES AND ITS DUALS

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Abstract. In this paper, continuity and boundedness results for the continuous S-transform in BMO and Hardy spaces are obtained. Furthermore, the continuous S-transform is also studied on the weighted BMO_k and weighted Hardy spaces associated with a tempered weight function which was proposed by L. Hörmander in the study of the theory of partial differential equations.

1. Introduction

The S-transform is a time-frequency localization technique that has characteristics superior to both of the Fourier transform and the wavelet transform[12]. The n-dimensional continuous S-transform of a function f with respect to the window function ω is defined as [13]

$$(S_\omega f)(\tau,\xi) = \int_{\mathbb{R}^n} f(t)\ \omega(\tau - t,\xi)\ e^{-i2\pi\langle\xi,t\rangle}\ dt,\ \text{for}\ \tau,\xi\in\mathbb{R}^n, \tag{1.1}$$

provided the integral exists.

In signal analysis, at least in dimension $n = 1$, $\mathbb{R}^{2n}$ is called the time-frequency plane, and in physics $\mathbb{R}^{2n}$ is called the phase space[11]. Equation(1.1) can be rewritten as a convolution

$$(S_\omega f)(\tau,\xi) = \Big(f(\cdot)e^{-i2\pi\langle\xi,\cdot\rangle} * \omega(\cdot,\xi)\Big)(\tau). \tag{1.2}$$

Applying the convolution property for the Fourier transform in (1.2), we obtain

$$(S_\omega f)(\tau,\xi) = \mathscr{F}^{-1}\Big\{\hat{f}(\cdot+\xi)\ \hat{\omega}(\cdot,\xi)\Big\}(\tau), \tag{1.3}$$

where $\hat{f}(\eta) = (\mathscr{F}f)(\eta) = \int_{\mathbb{R}^n} f(t)\ e^{-i2\pi\langle\eta,t\rangle}dt$, is the Fourier transform of f.

2. THE S-TRANSFORM ON BMO SPACES

The bounded mean oscillation space $BMO(\mathbb{R}^n)$ was first introduced by F. John and L. Nirenberg in 1961 [3]. It is the dual space of the real Hardy space H^1 and serves in many ways as a substitute space for L^∞. The $BMO(\mathbb{R}^n)$ space has become extremely important in various areas of analysis including harmonic analysis, PDEs and function theory.

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Definition 2.1. The bounded mean oscillation space $BMO(\mathbb{R}^n)$ is defined as the space of all locally Lebesgue integrable functions defined on $\mathbb{R}^n$ such that

$$\| f \|_{BMO} = \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|}\int_B |f(x)-f_B|dx < \infty,$$

here the supremum is taken over the ball B in $\mathbb{R}^n$ of measure $|B|$ and f_B stands for the mean of f on B, namely

$$f_B := \frac{1}{|B|}\int_B f(x)dx \le \frac{1}{|B|}\int_B |f(x)|dx \le m < \infty. \tag{2.1}$$

Lemma 2.1. *Let $f \in L^1(\mathbb{R}^n)$, then $e^{-i2\pi<\xi,\cdot>}f(\cdot) \in L^1(\mathbb{R}^n)$ and*

$$\| e^{-i2\pi<\xi,\cdot>}f(\cdot) \|_{BMO} \le \| f \|_{BMO} + 2m$$

where m is a constant given in equation (2.1).

Proof.

$$\begin{aligned}
& \| e^{-i2\pi<\xi,\cdot>}f(\cdot) \|_{BMO} \\
= & \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|}\int_B \left| e^{-i2\pi<\xi,x>}f(x) - \frac{1}{|B|}\int_B e^{-i2\pi<\xi,t>}f(t)dt\right| dx \\
= & \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|}\int_B \left| e^{-i2\pi<\xi,x>}f(x) - \frac{e^{-i2\pi<\xi,x>}}{|B|}\int_B f(t)dt \right. \\
& \left. + \frac{e^{-i2\pi<\xi,x>}}{|B|}\int_B f(t)dt - \frac{1}{|B|}\int_B e^{-i2\pi<\xi,t>}f(t)dt\right| dx \\
\le & \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|}\int_B \left(\left| e^{-i2\pi<\xi,x>}\left(f(x) - \frac{1}{|B|}\int_B f(t)dt\right)\right| \right. \\
& \left. + \left|\frac{1}{|B|}\int_B f(t)dt\right| + \left|\frac{1}{|B|}\int_B e^{-i2\pi<\xi,t>}f(t)dt\right|\right) dx \\
\le & \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|}\int_B |f(x)-f_B|\,dx + \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|}\int_B |f_B|dx \\
& + \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|}\int_B \left(\frac{1}{|B|}\int_B |f(t)|dt\right) dx \\
\le & \| f \|_{BMO} + \frac{1}{|B|}m|B| + \frac{1}{|B|}m|B| \\
= & \| f \|_{BMO} + 2m.
\end{aligned}$$

□

Theorem 2.2. *Suppose $\omega(\cdot,\xi) \in L^1(\mathbb{R}^n)\bigcap L^2(\mathbb{R}^n)$, then, for any fixed $\xi \in \mathbb{R}^n_0 = \mathbb{R}^n \setminus \{\mathbf{0}\}$, the operator $S_\omega : BMO(\mathbb{R}^n) \to BMO(\mathbb{R}^n)$ is continuous. Furthermore, we have*

$$\| (S_\omega f)(\cdot,\xi) \|_{BMO} \le \| \omega(\cdot,\xi) \|_{L^1} (\| f \|_{BMO} + 2m)\,.$$

Proof. For any arbitrary ball B in $\mathbb{R}^n$, we have

$$\begin{aligned}
(S_\omega f)_B(\tau,\xi) &= \frac{1}{|B|}\int_B (S_\omega f)(\tau,\xi)d\tau \\
&= \frac{1}{|B|}\int_B \int_{\mathbb{R}^n} e^{-i2\pi<\xi,\tau-x>}f(\tau-x)\omega(x,\xi)dxd\tau,
\end{aligned}$$

and hence

$$
\begin{aligned}
& |(S_\omega f)(\tau,\xi) - (S_\omega f)_B(\tau,\xi)| \\
= & \left| \int_{\mathbb{R}^n} e^{-i2\pi<\xi,\tau-x>} f(\tau-x)\omega(x,\xi)dx \right. \\
& \left. -\frac{1}{|B|}\int_B\int_{\mathbb{R}^n} e^{-i2\pi<\xi,\alpha-x>} f(\alpha-x)\omega(x,\xi)dxd\alpha \right| \\
= & \left| \int_{\mathbb{R}^n} e^{-i2\pi<\xi,\tau-x>} f(\tau-x)\omega(x,\xi)dx \right. \\
& \left. -\int_{\mathbb{R}^n} \omega(x,\xi)\left(\frac{1}{|B|}\int_B e^{-i2\pi<\xi,\alpha-x>} f(\alpha-x)d\alpha\right)dx \right| \\
= & \left| \int_{\mathbb{R}^n} \omega(x,\xi)\Big(e^{-i2\pi<\xi,\tau-x>} f(\tau-x) \right. \\
& \left. -\frac{1}{|B|}\int_B e^{-i2\pi<\xi,\alpha-x>} f(\alpha-x)d\alpha\Big)dx \right| \\
\leq & \int_{\mathbb{R}^n} |\omega(x,\xi)| \left| e^{-i2\pi<\xi,\tau-x>} f(\tau-x) \right. \\
& \left. -\frac{1}{|B|}\int_B e^{-i2\pi<\xi,\alpha-x>} f(\alpha-x)d\alpha \right| dx.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \| (S_\omega f)(\cdot,\xi) \|_{BMO} \\
= & \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|}\int_B |(S_\omega f)(\tau,\xi) - (S_\omega f)_B(\tau,\xi)|d\tau \\
\leq & \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|}\int_B \left(\int_{\mathbb{R}^n} |\omega(x,\xi)| \left| e^{-i2\pi<\xi,\tau-x>} f(\tau-x) \right.\right. \\
& \left.\left. -\frac{1}{|B|}\int_B e^{-i2\pi<\xi,\alpha-x>} f(\alpha-x)d\alpha \right| dx\right)d\tau \\
= & \int_{\mathbb{R}^n} |\omega(x,\xi)| \left(\sup_{K\subset\mathbb{R}^n} \frac{1}{|K|}\int_K \left| e^{-i2\pi<\xi,y>} f(y) \right.\right. \\
& \left.\left. -\frac{1}{|K|}\int_K e^{-i2\pi<\xi,t>} f(t)dt \right| dy\right)dx \\
\leq & \| \omega(\cdot,\xi) \|_{L^1} \| e^{-i2\pi<\xi,\cdot>} f(\cdot) \|_{BMO},
\end{aligned}
$$

here $K = B - x$ for $x \in \mathbb{R}^n$.
By using above lemma we get,

$$\| (S_\omega f)(\cdot,\xi) \|_{BMO} \leq \| \omega(\cdot,\xi) \|_{L^1} (\| f \|_{BMO} + 2m).$$

□

3. THE S-TRANSFORM ON WEIGHTED BMO SPACES.

Definition 3.1. A positive function k defined on $\mathbb{R}^n$ is called a tempered weight function[2] if there exists positive constants C and N such that

$$k(\xi+\eta) \leq (1 + C|\xi|)^N k(\eta) \text{ for all } \xi,\eta \in \mathbb{R}^n. \tag{3.1}$$

Definition 3.2. For $1\leq p \leq \infty$, the weighted Lebesgue space $L^p_k(\mathbb{R}^n)$ is defined as the space of all measurable functions f on $\mathbb{R}^n$ such that

$$\| f \|_{L^p_k}= \left(\int_{\mathbb{R}^n} |f(x)|^p k(x)dx\right)^{\frac{1}{p}} < \infty.$$

Definition 3.3. The weighted bounded mean oscillation space $\mathrm{BMO}_k(\mathbb{R}^n)$ is defined as the space of all weighted Lebesgue integrable (locally) functions defined on $\mathbb{R}^n$ such that

$$\| f \|_{BMO_k}= \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|_k}\int_B |f(x)-f_B|k(x)dx < \infty,$$

where the supremum is taken over the ball B in $\mathbb{R}^n$ and $|B|_k=\int_B k(x)dx$.

Lemma 3.1. *Let $f\in L^1_k(\mathbb{R}^n)$, then $e^{-i2\pi<\xi,\cdot>}f(\cdot)\in L^1_k(\mathbb{R}^n)$ and*

$$\| e^{-i2\pi<\xi,\cdot>}f(\cdot) \|_{BMO_k}\leq\| f \|_{BMO_k}+2m,$$

where m is a constant defined in equation (2.1).

Proof.

$$\begin{aligned}
& \| e^{-i2\pi<\xi,\cdot>}f(\cdot) \|_{BMO_k}\\
= & \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|_k}\int_B \left|e^{-i2\pi<\xi,x>}f(x)-\frac{1}{|B|}\int_B e^{-i2\pi<\xi,t>}f(t)dt\right|k(x)dx\\
= & \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|_k}\int_B \left|e^{-i2\pi<\xi,x>}f(x)-\frac{e^{-i2\pi<\xi,x>}}{|B|}\int_B f(t)dt\right.\\
& \left.+\frac{e^{-i2\pi<\xi,x>}}{|B|}\int_B f(t)dt-\frac{1}{|B|}\int_B e^{-i2\pi<\xi,t>}f(t)dt\right|k(x)dx\\
\leq & \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|_k}\int_B \left(\left|e^{-i2\pi<\xi,x>}\left(f(x)-\frac{1}{|B|}\int_B f(t)dt\right)\right|\right.\\
& \left.+\left|\frac{1}{|B|}\int_B f(t)dt\right|+\left|\frac{1}{|B|}\int_B e^{-i2\pi<\xi,t>}f(t)dt\right|\right)k(x)dx\\
\leq & \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|_k}\int_B |f(x)-f_B|\,k(x)dx+\sup_{B\subset\mathbb{R}^n} \frac{1}{|B|_k}\int_B |f_B|k(x)dx\\
& +\sup_{B\subset\mathbb{R}^n} \frac{1}{|B|_k}\int_B\left(\frac{1}{|B|}\int_B|f(t)|dt\right)k(x)dx\\
\leq & \| f \|_{BMO_k}+\frac{1}{|B|_k}m\int_B k(x)dx+\frac{1}{|B|_k}m\int_B k(x)dx\\
= & \| f \|_{BMO_k}+\frac{1}{|B|_k}m|B|_k+\frac{1}{|B|_k}m|B|_k\\
= & \| f \|_{BMO_k}+2m.
\end{aligned}$$

□

Theorem 3.2. *Suppose ω is a window function such that for any fixed $\xi\in\mathbb{R}^n_0$*

$$\int_{\mathbb{R}^n}|\omega(x,\xi)|\,(1+C|x|)^N dx\leq A<\infty, \tag{3.2}$$

where A,C and N are positive constants. Then the operator $S_\omega : BMO_k(\mathbb{R}^n) \to BMO_k(\mathbb{R}^n)$ *is continuous. Furthermore, we have*

$$\| (S_\omega f)(\cdot,\xi) \|_{BMO_k} \leq A\,(\| f \|_{BMO_k} + 2m)$$

where m *is a constant given in equation (2.1).*

Proof. By using the techniques of Theorem 2.2, for any arbitrary ball B in $\mathbb{R}^n$, we have

$$\begin{aligned}
\| (S_\omega f)(\cdot,\xi) \|_{BMO_k} &= \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|_k}\int_B |(S_\omega f)(\tau,\xi) - (S_\omega f)_B(\tau,\xi)| k(\tau)d\tau \\
&\leq \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|_k}\int_B \left(\int_{\mathbb{R}^n} |\omega(x,\xi)| \left| e^{-i2\pi<\xi,\tau-x>} f(\tau-x) \right.\right.\\
&\quad \left.\left. - \frac{1}{|B|}\int_B e^{-i2\pi<\xi,\alpha-x>} f(\alpha-x)d\alpha \right| dx\right) k(\tau)d\tau \\
&\leq \sup_{K\subset\mathbb{R}^n} \frac{1}{|K|_k}\int_K \left(\int_{\mathbb{R}^n} |\omega(x,\xi)| \left| e^{-i2\pi<\xi,y>} f(y) \right.\right.\\
&\quad \left.\left. - \frac{1}{|K|}\int_K e^{-i2\pi<\xi,t>} f(t)dt \right| dx\right) (1+C|x|)^N k(y)dy \\
&= \int_{\mathbb{R}^n} |\omega(x,\xi)|\,(1+C|x|)^N \left(\sup_{K\subset\mathbb{R}^n} \frac{1}{|K|_k}\int_K \left| e^{-i2\pi<\xi,y>} f(y) \right.\right.\\
&\quad \left.\left. - \frac{1}{|K|}\int_K e^{-i2\pi<\xi,t>} f(t)dt \right| k(y)dy\right) dx \\
&\leq A \,\| e^{-i2\pi<\xi,\cdot>} f(\cdot) \|_{BMO_k},
\end{aligned}$$

here $K = B - x$ for $x \in \mathbb{R}^n$.
By using above lemma we get

$$\| (S_\omega f)(\cdot,\xi) \|_{BMO_k} \leq A\,(\| f \|_{BMO_k} + 2m)\,.$$

□

4. THE S-TRANSFORM ON HARDY SPACES.

Definition 4.1. The Hardy space is defined as the space of all functions $f \in L^1(\mathbb{R}^n)$ such that

$$\| f \|_{H^1} = \int_{\mathbb{R}^n} \sup_{t>0} |\,(f * \phi_t)\,(x)|dx < \infty,$$

where ϕ is any test function with $\int \phi \neq 0$ and $\phi_t(x) = t^{-n}\phi(x/t);\ t>0, x\in\mathbb{R}^n$.

Theorem 4.1. *Let* $f \in L^1(\mathbb{R}^n)$ *such that*

$$\sup_{t>0}\left|\int_{\mathbb{R}^n} f(x-y)\phi_t(y)dy\right| = \sup_{t>0}\int_{\mathbb{R}^n} |f(x-y)\phi_t(y)|\,dy < \infty. \tag{4.1}$$

Then for any fixed $\xi \in \mathbb{R}_0^n$*, the operator* $S_\omega : H^1(\mathbb{R}^n) \to H^1(\mathbb{R}^n)$ *is continuous. Furthermore, we have*

$$\| (S_\omega f)(\cdot,\xi) \|_{H^1} \leq 3 \,\| \omega(\cdot,\xi) \|_{L^1} \| f \|_{H^1}\ .$$

Proof. Since

$$
\begin{aligned}
&((S_\omega f)(\cdot,\xi) * \phi_t)(\tau) \\
= \quad & \left(\left(\int_{\mathbb{R}^n} e^{-i2\pi<\xi,\,\cdot-x>} f(\cdot - x)\omega(x,\xi)dx\right) * \phi_t\right)(\tau) \\
= \quad & \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} e^{-i2\pi<\xi,\tau-x-y>} f(\tau-x-y)\omega(x,\xi)\ dx\right)\phi_t(y)\ dy \\
= \quad & \int_{\mathbb{R}^n}\omega(x,\xi)\left(\int_{\mathbb{R}^n} e^{-i2\pi<\xi,\tau-x-y>} f(\tau-x-y)\phi_t(y)dy\right)dx.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\| (S_\omega f)(\cdot,\xi) \|_{H^1} \\
= \quad & \int_{\mathbb{R}^n}\sup_{t>0}\left|((S_\omega f)(\cdot,\xi) * \phi_t)(\tau)\right| d\tau \\
= \quad & \int_{\mathbb{R}^n}\sup_{t>0}\left|\int_{\mathbb{R}^n}\omega(x,\xi)\left(\int_{\mathbb{R}^n} e^{-i2\pi<\xi,\tau-x-y>} f(\tau-x-y)\phi_t(y)dy\right)dx\right| d\tau \\
\leq \quad & \int_{\mathbb{R}^n}|\omega(x,\xi)|\left(\int_{\mathbb{R}^n}\sup_{t>0}\left|\int_{\mathbb{R}^n} e^{-i2\pi<\xi,\tau-x-y>} f(\tau-x-y)\phi_t(y)dy\right| d\tau\right)dx \\
= \quad & \int_{\mathbb{R}^n}|\omega(x,\xi)|\left(\int_{\mathbb{R}^n}\sup_{t>0}\left|\int_{\mathbb{R}^n} e^{-i2\pi<\xi,\eta-y>} f(\eta-y)\phi_t(y)dy\right| d\eta\right)dx.
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\sup_{t>0}\left|\int_{\mathbb{R}^n} e^{-i2\pi<\xi,\eta-y>} f(\eta-y)\phi_t(y)dy\right| d\eta \\
&= \int_{\mathbb{R}^n}\sup_{t>0}\Bigg|\int_{\mathbb{R}^n} e^{-i2\pi<\xi,\eta-y>} f(\eta-y)\phi_t(y)dy \\
&\quad - \int_{\mathbb{R}^n} f(\eta-y)\phi_t(y)dy + \int_{\mathbb{R}^n} f(\eta-y)\phi_t(y)dy\Bigg| d\eta \\
&= \int_{\mathbb{R}^n}\sup_{t>0}\Bigg|\int_{\mathbb{R}^n}(e^{-i2\pi<\xi,\eta-y>}-1) f(\eta-y)\phi_t(y)dy \\
&\quad + \int_{\mathbb{R}^n} f(\eta-y)\phi_t(y)dy\Bigg| d\eta \\
&\leq \int_{\mathbb{R}^n}\sup_{t>0}\int_{\mathbb{R}^n}\left|(e^{-i2\pi<\xi,\eta-y>}-1)\right| |f(\eta-y)\phi_t(y)|\, dy d\eta \\
&\quad + \int_{\mathbb{R}^n}\sup_{t>0}\left|\int_{\mathbb{R}^n} f(\eta-y)\phi_t(y)dy\right| d\eta \\
&\leq 2\int_{\mathbb{R}^n}\sup_{t>0}\int_{\mathbb{R}^n}|f(\eta-y)\phi_t(y)|\, dy d\eta + \| f \|_{H^1} \\
&= 2\| f \|_{H^1} + \| f \|_{H^1} \\
&= 3\| f \|_{H^1}\ .
\end{aligned}
$$

Therefore,

$$
\| (S_\omega f)(\cdot,\xi) \|_{H^1} \leq \int_{\mathbb{R}^n}|\omega(x,\xi)|\, 3 \| f \|_{H^1}\ dx = 3 \| \omega(\cdot,\xi) \|_{L^1} \| f \|_{H^1}\ .
$$

□

5. THE S-TRANSFORM ON WEIGHTED HARDY SPACES.

Definition 5.1. The weighted Hardy space is defined as the space of all functions $f \in L_k^1(\mathbb{R}^n)$ such that

$$\| f \|_{H_k^1} = \int_{\mathbb{R}^n} \sup_{t>0} |(f * \phi_t)(x)| \, k(x) dx < \infty.$$

Theorem 5.1. *Suppose ω is a window function and satisfies the condition (3.2). Let $f \in L^1(\mathbb{R}^n)$ and satisfies the condition (4.1). Then, for any fixed $\xi \in \mathbb{R}_0^n$, the operator $S_\omega : H_k^1(\mathbb{R}^n) \to H_k^1(\mathbb{R}^n)$ is continuous. Furthermore, we have*

$$\| (S_\omega f)(\cdot, \xi) \|_{H_k^1} \leq 3A \| f \|_{H_k^1} .$$

Proof. Since

$$\begin{aligned}
&\| (S_\omega f)(\cdot, \xi) \|_{H_k^1} \\
&= \int_{\mathbb{R}^n} \sup_{t>0} | ((S_\omega f)(\cdot, \xi) * \phi_t)(\tau) | k(\tau) d\tau \\
&= \int_{\mathbb{R}^n} \sup_{t>0} \left| \int_{\mathbb{R}^n} \omega(x, \xi) \left(\int_{\mathbb{R}^n} e^{-i2\pi<\xi, \tau - x - y>} f(\tau - x - y) \phi_t(y) dy \right) dx \right| k(\tau) d\tau \\
&\leq \int_{\mathbb{R}^n} |\omega(x, \xi)| \left(\int_{\mathbb{R}^n} \sup_{t>0} \left| \int_{\mathbb{R}^n} e^{-i2\pi<\xi, \tau - x - y>} f(\tau - x - y) \phi_t(y) dy \right| k(\tau) d\tau \right) dx \\
&\leq \int_{\mathbb{R}^n} |\omega(x, \xi)| \left(\int_{\mathbb{R}^n} \sup_{t>0} \left| \int_{\mathbb{R}^n} e^{-i2\pi<\xi, \eta - y>} f(\eta - y) \phi_t(y) dy \right| (1 + C|x|)^N k(\eta) d\eta \right) dx \\
&= \int_{\mathbb{R}^n} |\omega(x, \xi)| (1 + C|x|)^N \left(\int_{\mathbb{R}^n} \sup_{t>0} \left| \int_{\mathbb{R}^n} e^{-i2\pi<\xi, \eta - y>} f(\eta - y) \phi_t(y) dy \right| k(\eta) d\eta \right) dx.
\end{aligned}$$

And

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sup_{t>0} \left| \int_{\mathbb{R}^n} e^{-i2\pi<\xi, \eta - y>} f(\eta - y) \phi_t(y) dy \right| k(\eta) d\eta \\
&= \int_{\mathbb{R}^n} \sup_{t>0} \Bigg| \int_{\mathbb{R}^n} e^{-i2\pi<\xi, \eta - y>} f(\eta - y) \phi_t(y) dy \\
&\quad - \int_{\mathbb{R}^n} f(\eta - y) \phi_t(y) dy + \int_{\mathbb{R}^n} f(\eta - y) \phi_t(y) dy \Bigg| k(\eta) d\eta \\
&\leq \int_{\mathbb{R}^n} \sup_{t>0} \left| \int_{\mathbb{R}^n} (e^{-i2\pi<\xi, \eta - y>} - 1) f(\eta - y) \phi_t(y) dy \right| k(\eta) d\eta \\
&\quad + \int_{\mathbb{R}^n} \sup_{t>0} \left| \int_{\mathbb{R}^n} f(\eta - y) \phi_t(y) dy \right| k(\eta) d\eta \\
&\leq 2 \int_{\mathbb{R}^n} \sup_{t>0} \int_{\mathbb{R}^n} |f(\eta - y) \phi_t(y)| \, dy \, k(\eta) \, d\eta \\
&\quad + \int_{\mathbb{R}^n} \sup_{t>0} \left| \int_{\mathbb{R}^n} f(\eta - y) \phi_t(y) dy \right| k(\eta) d\eta \\
&= 2 \| f \|_{H_k^1} + \| f \|_{H_k^1} \\
&= 3 \| f \|_{H_k^1} .
\end{aligned}$$

Therefore, using equation (3.2), we have

$$\| (S_\omega f)(\cdot, \xi) \|_{H_k^1} \leq \int_{\mathbb{R}^n} |\omega(x, \xi)| (1 + C|x|)^N 3 \| f \|_{H_k^1} dx \leq 3A \| f \|_{H_k^1} .$$

This completes the proof. □

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