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Certain Subclass of Analytic Functions Defined by Bell Distribution Series

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Abstract. The study of the geometric properties of analytic functions and their numerous applications in a variety of mathematical fields, including fractional calculus, probability distributions, and special functions, has drawn significant and impressive attention to Geometric Function Theory (GFT), one of the most prominent branches of complex analysis, in recent years. The focus of this article is to introduce a new subclass of analytic functions involving Bell Distribution series and obtain coefficient inequalities, neighborhood results and partial sums for this class.

1. Introduction

Let A denote the class of analytic functions F defined on the unit disk $U = \{\omega : |\omega| < 1\}$ with normalization $F(0) = 0$ and $F'(0) = 1$. Such a function has the Taylor series expansion about the origin in the form

$$F(\omega) = \omega + \sum_{n=2}^{\infty} \dot{a}_n \omega^n, \tag{1.1}$$

denoted by S, the subclass of A consisting of functions that are univalent in U.

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For $F \in A$ given by (1.1) and $g(\omega)$ given by

$$g(\omega) = \omega + \sum_{n=2}^{\infty} b_n \omega^n \tag{1.2}$$

their convolution (or Hadamard product), denoted by $(F * g)$, is defined as

$$(F * g)(\omega) = \omega + \sum_{n=2}^{\infty} \dot{a}_n b_n \omega^n = (g * F)(\omega), \ (\omega \in U). \tag{1.3}$$

Note that $F * g \in A$.

A function $F \in A$ is said to be in $\dot{k} - US(\ell)$, the class of $\dot{k}$−uniformly starlike functions of order $\ell, 0 \leq \ell < 1$, if satisfies the condition

$$\Re\left\{\frac{\omega F'(\omega)}{F(\omega)}\right\} > \dot{k}\left|\frac{\omega F'(\omega)}{F(\omega)} - 1\right| + \ell, \ (\dot{k} \geq 0) \tag{1.4}$$

and a function $F \in A$ is said to be in $\dot{k} - UC(\ell)$, the class of $\dot{k}$−uniformly convex functions of order $\ell, 0 \leq \ell < 1$, if satisfies the condition

$$\Re\left\{1 + \frac{\omega F''(\omega)}{F'(\omega)}\right\} > \dot{k}\left|\frac{\omega F''(\omega)}{F'(\omega)}\right| + \ell, \ (\dot{k} \geq 0). \tag{1.5}$$

Uniformly starlike and uniformly convex functions were first introduced by Goodman [16] and then studied by various authors.

In [24], Sakaguchi defined the class S_s of starlike functions with respect to symmetric points as follows:

Let $F \in A$. Then F is said to be starlike with respect to symmetric points in U if and only if

$$\Re\left\{\frac{2\omega F'(\omega)}{F(\omega) - F(-\omega)}\right\} > 0, \ (\omega \in U).$$

Recently, Owa et al. [20] defined the class $S_s(\hbar, t)$ as follows:

$$\Re\left\{\frac{(1-t)\omega F'(\omega)}{F(\omega) - F(t\omega)}\right\} > \hbar, \ (\omega \in U),$$

where $0 \leq \hbar < 1, |t| \leq 1, t \neq 1$. Note that $S_s(0, -1) = S_s$ and $S_s(\hbar, -1) = S_s(\hbar)$ is called Sakaguchi function of order $\hbar$.

The Bell distribution, sometimes referred to as the normal mixture distribution, is a probability distribution that is used in signal processing, statistical inference, and other scientific domains. A continuous probability distribution known as the Bell distribution is a

blend of the normal distributions. In a Bell distribution, roughly 0.68 percent of the data

0.95 lies between two standard deviations and one standard deviation from the mean, and

The value of 0.997 is three standard deviations away.

The probability density function of the Bell distribution is symmetric and bell-shaped; it resembles a normal distribution but has heavier tails. The distribution's degree of asymmetry is determined by the mixing parameter p, where $p = 0.5$ denotes a perfectly symmetric distribution.

Numerous disciplines, including finance, physics, engineering, and biology, use the Bell distribution. Among its applications are the modelling of biological system behaviour, noisy signal characteristics, and stock return distribution. In statistics, the Bell curve has numerous significant uses, including regression analysis confidence intervals, and hypothesis testing. It is also utilised in disciplines like psychology, economics, and finance to model the behaviour of intricate systems and generate predictions based on actual data.

A fundamental part of statistics and probability, random variable distributions serve to describe and model a wide range of real-world events [6]. They represent the distribution of probabilities over the values of the random variable. Some of the fundamental distributions, such as the Poisson, Pascal, logarithmic, binomial, and Borel distributions, have been used in geometric function theory (see [2–4]).

The Bell distribution [10] was introduced by Castellares et al. in 2018 and is appropriate for count data that exhibit over-dispersion. Compared to the Bell numbers, the Bell distribution is superior [7,8].

$$\mathcal{P}(X=m)=\frac{\vartheta^m e^{e^{(-\vartheta^2)+1}}\mathcal{B}_m}{m!};\quad m=1,2,3\cdots, \tag{1.6}$$

where $\mathcal{B}_m=\frac{1}{e}\sum\limits_{j=0}^{\infty}\frac{j^m}{m!}$ are the Bell numbers, $m\geq 2$, and $\vartheta>0$.

Example of the Bell numbers are $\mathcal{B}_2=2,\ \mathcal{B}_3=5,\ \mathcal{B}_4=15$ and $\mathcal{B}_5=52$. Now, we introduce a new power series whose coefficients represent the probabilities of the Bell distribution

$$\mathcal{B}(\vartheta,\omega)=\omega+\sum_{n=2}^{\infty}\frac{\vartheta^{n-1}\,\mathcal{B}_n}{(n-1)!\,e^{\vartheta^2-1}}\omega^n,\quad (\omega\in U). \tag{1.7}$$

Next, we consider the linear operator $\mathbb{L}^\vartheta:\mathcal{A}\to\mathcal{A}$ defined by the convolution (or Hadamard product)

$$\mathbb{L}^\vartheta F(\omega)=\mathcal{B}(\vartheta,\omega)*F(\omega)=\omega+\sum_{n=2}^{\infty}\phi_n(\vartheta)\dot{a}_n\omega^n \tag{1.8}$$

where

$$\phi_n(\vartheta)=\frac{\vartheta^{n-1}\,\mathcal{B}_n}{(n-1)!\,e^{\vartheta^2-1}}.$$

Using the Hurwitz-Lerch zeta operator $\mathbb{L}^\vartheta F$, we now describe a new subclass of functions that belong to the class A.

Definition 1.1. *A function $F\in A\in \dot{k}-US_s(\vartheta,\ell,t)$ if for all $\omega\in U$*

$$\Re\left\{\frac{(1-t)\omega\left(\mathbb{L}^\vartheta F(\omega)\right)'}{\mathbb{L}^\vartheta F(\omega)-\mathbb{L}^\vartheta F(t\omega)}\right\}\geq \dot{k}\left|\frac{(1-t)\omega\left(\mathbb{L}^\vartheta F(\omega)\right)'}{\mathbb{L}^\vartheta F(\omega)-\mathbb{L}^\vartheta F(t\omega)}-1\right|+\ell,$$

for $\dot{k}\geq 0, |t|\leq 1, t\neq 1, 0\leq \ell<1$.

Furthermore, we say that a function $F \in \dot{k} - US_s(\vartheta, \ell, t)$ is in the subclass $\dot{k} - \widetilde{US}_s(\vartheta, \ell, t)$ if $F(\omega)$ is of the following form

$$F(\omega) = \omega - \sum_{n=2}^{\infty} \dot{a}_n \omega^n, \ (\dot{a}_n \geq 0, \ n \in \mathbb{N}, \ \omega \in U). \tag{1.9}$$

The aim of the present paper is to study the coefficient bounds, partial sums and certain neighborhood results of the class $\dot{k} - \widetilde{US}_s(\vartheta, \ell, t)$.

Firstly, we shall need the following lemmas [5].

Lemma 1.1. *Let w be a complex number. Then*

$$\Re(w) \geq \hbar \ \Leftrightarrow \ |w - (1 + \hbar)| \leq |w + (1 - \hbar)|.$$

Lemma 1.2. *Let w be a complex number and $\hbar, \ell$ be real numbers. Then*

$$\Re\ (w) > \hbar|w - 1| + \ell \ \Leftrightarrow \ \Re\{w(1 + \hbar e^{i\theta}) - \hbar e^{i\theta}\} > \ell, \ \ -\pi < \theta < \pi.$$

2. Coefficient bounds of the function class $\dot{k} - \widetilde{US}_s(\vartheta, \ell, t)$

Theorem 2.1. *The function F defined by (1.9) is in the class $\dot{k} - \widetilde{US}_s(\vartheta, \ell, t)$ if and only if*

$$\sum_{n=2}^{\infty} \phi_n(\vartheta)|n(\dot{k} + 1) - u_n(\dot{k} + \ell)|\dot{a}_n \leq 1 - \ell, \tag{2.1}$$

where $\dot{k} \geq 0, |t| \leq 1, t \neq 1, 0 \leq \ell < 1$ and $u_n = 1 + t + \cdots + t^{n-1}$.
The result is sharp for the function $F(\omega)$ given by

$$F(\omega) = \omega - \frac{1 - \ell}{\phi_n(\vartheta)|n(\dot{k} + 1) - u_n(\dot{k} + \ell)|}\omega^n.$$

Proof. By Definition 1.1, we get

$$\Re\left\{\frac{(1 - t)\omega\left(\mathbb{L}^{\vartheta}F(\omega)\right)'}{\mathbb{L}^{\vartheta}F(\omega) - \mathbb{L}^{\vartheta}F(t\omega)}\right\} \geq \dot{k}\left|\frac{(1 - t)\omega\left(\mathbb{L}^{\vartheta}F(\omega)\right)'}{\mathbb{L}^{\vartheta}F(\omega) - \mathbb{L}^{\vartheta}F(t\omega)} - 1\right| + \ell.$$

Then by Lemma 1.2, we have

$$\Re\left\{\frac{(1 - t)\omega\left(\mathbb{L}^{\vartheta}F(\omega)\right)'}{\mathbb{L}^{\vartheta}F(\omega) - \mathbb{L}^{\vartheta}F(t\omega)}(1 + \dot{k}e^{i\theta}) - \dot{k}e^{i\theta}\right\} \geq \ell, \ -\pi < \theta < \pi$$

or equivalently

$$\Re\left\{\frac{(1 - t)\omega\left(\mathbb{L}^{\vartheta}F(\omega)\right)'(1 + \dot{k}e^{i\theta})}{\mathbb{L}^{\vartheta}F(\omega) - \mathbb{L}^{\vartheta}F(t\omega)} - \frac{\dot{k}e^{i\theta}\left[\mathbb{L}^{\vartheta}F(\omega) - \mathbb{L}^{\vartheta}F(t\omega)\right]}{\mathbb{L}^{\vartheta}F(\omega) - \mathbb{L}^{\vartheta}F(t\omega)}\right\} \geq \ell. \tag{2.2}$$

Let $F(\omega) = (1 - t)\omega\left(\mathbb{L}^{\vartheta}F(\omega)\right)'(1 + \dot{k}e^{i\theta}) - \dot{k}e^{i\theta}\left[\mathbb{L}^{\vartheta}F(\omega) - \mathbb{L}^{\vartheta}F(t\omega)\right]$
and $E(\omega) = \mathbb{L}^{\vartheta}F(\omega) - \mathbb{L}^{\vartheta}F(t\omega)$.
By Lemma 1.1, (2.2) is equivalent to

$$|F(\omega) + (1 - \ell)E(\omega)| \geq |F(\omega) - (1 + \ell)E(\omega)|, \text{ for } 0 \leq \ell < 1.$$

But

$$|F(\omega)+(1-\ell)E(\omega)| = \Big|(1-t)\Big\{(2-\ell)\omega - \sum_{n=2}^{\infty}\phi_n(\vartheta)(n+u_n(1-\ell))\dot{a}_n\omega^n - \dot{k}e^{i\theta}\sum_{n=2}^{\infty}\phi_n(\vartheta)(n-u_n)\dot{a}_n\omega^n\Big\}\Big|$$
$$\geq |1-t|\Big\{(2-\ell)|\omega| - \sum_{n=2}^{\infty}\phi_n(\vartheta)|n+u_n(1-\ell)|\dot{a}_n|\omega^n| - \dot{k}\sum_{n=2}^{\infty}\phi_n(\vartheta)|n-u_n|\dot{a}_n|\omega^n|\Big\}.$$

Also

$$|F(\omega)-(1+\ell)E(\omega)| = \Big|(1-t)\Big\{-\ell\omega - \sum_{n=2}^{\infty}\phi_n(\vartheta)(n-u_n(1+\ell))\dot{a}_n\omega^n - \dot{k}e^{i\theta}\sum_{n=2}^{\infty}\phi_n(\vartheta)(n-u_n)\dot{a}_n\omega^n\Big\}\Big|$$
$$\leq |1-t|\Big\{\ell|\omega| + \sum_{n=2}^{\infty}\phi_n(\vartheta)|n-u_n(1+\ell)|\dot{a}_n|\omega^n| + \dot{k}\sum_{n=2}^{\infty}\phi_n(\vartheta)|n-u_n|\dot{a}_n|\omega^n|\Big\}.$$

So

$$|F(\omega)+(1-\ell)E(\omega)| - |F(\omega)-(1+\ell)E(\omega)|$$
$$\geq |1-t|\Big\{2(1-\ell)|\omega| - \sum_{n=2}^{\infty}\phi_n(\vartheta)\Big[|n+u_n(1-\ell)| + |n-u_n(1+\ell)| + 2\dot{k}|n-u_n|\Big]\dot{a}_n|\omega^n|\Big\}$$
$$\geq 2(1-\ell)|\omega| - \sum_{n=2}^{\infty}2\phi_n(\vartheta)\Big|n(\dot{k}+1)-u_n(\dot{k}+\ell)\Big|\dot{a}_n|\omega^n| \geq 0$$

or

$$\sum_{n=2}^{\infty}\phi_n(\vartheta)\Big|n(\dot{k}+1)-u_n(\dot{k}+\ell)\Big|\dot{a}_n \leq 1-\ell.$$

Conversely, suppose that (2.1) holds. Then we must show

$$\Re\left\{\frac{(1-t)\omega\left(\mathbb{L}^{\vartheta}F(\omega)\right)'(1+\dot{k}e^{i\theta}) - \dot{k}e^{i\theta}\left[\mathbb{L}^{\vartheta}F(\omega)-\mathbb{L}^{\vartheta}F(t\omega)\right]}{\mathbb{L}^{\vartheta}F(\omega)-\mathbb{L}^{\vartheta}F(t\omega)}\right\} \geq \ell.$$

Upon choosing the values of ω on the positive real axis where $0 \leq |\omega| = r < 1$, the above inequality reduces to

$$\Re\left\{\frac{(1-\ell)-\sum\limits_{n=2}^{\infty}\phi_n(\vartheta)[n(1+\dot{k}e^{i\theta})-u_n(\ell+\dot{k}e^{i\theta})]\dot{a}_n\omega^{n-1}}{1-\sum\limits_{n=2}^{\infty}\phi_n(\vartheta)u_n\dot{a}_n\omega^{n-1}}\right\}\geq 0.$$

Since $\Re(-e^{i\theta})\geq -|e^{i\theta}|=-1$, the above inequality reduces to

$$\Re\left\{\frac{(1-\ell)-\sum\limits_{n=2}^{\infty}\phi_n(\vartheta)[n(1+\dot{k})-u_n(\ell+\dot{k}]\dot{a}_nr^{n-1}}{1-\sum\limits_{n=2}^{\infty}\phi_n(\vartheta)u_n\dot{a}_nr^{n-1}}\right\}\geq 0.$$

Letting $r\to 1^-$, we have desired conclusion. ◻

Corollary 2.1. *If $F(\omega)\in \dot{k}-\widetilde{US}_s(\vartheta,\ell,t)$ then*

$$\dot{a}_n\leq \frac{1-\ell}{\phi_n(\vartheta)|n(\dot{k}+1)-u_n(\dot{k}+\ell)|}$$

where $\dot{k}\geq 0, |t|\leq 1, t\neq 1, 0\leq \ell<1$ and $u_n=1+t+\cdots+t^{n-1}$.

3. Neighborhood of the function class $\dot{k}-\widetilde{US}_s(\vartheta,\ell,t)$

We define the neighbourhood of a function $F\in T$. by following the earlier examinations (based upon the well-known concept of neighbourhoods of analytic functions) by Goodman [15], Ruscheweyh [23], and Santosh et al. [25].

Definition 3.1. *Let $\dot{k}\geq 0, |t|\leq 1, t\neq 1, 0\leq \ell<1, \hbar\geq 0$ and $u_n=1+t+\cdots+t^{n-1}$. We define the $\hbar-$neighborhood of a function $F\in T$ and denote by $N_\hbar(F)$ consisting of all functions $g(\omega)=\omega-\sum\limits_{n=2}^{\infty}b_n\omega^n\in S(b_n\geq 0, n\in\mathbb{N})$ satisfying*

$$\sum_{n=2}^{\infty}\frac{\phi_n(\vartheta)|n(\dot{k}+1)-u_n(\dot{k}+\ell)|}{1-\ell}|\dot{a}_n-b_n|\leq 1-\hbar.$$

Theorem 3.1. *Let $F(\omega)\in \dot{k}-\widetilde{US}_s(\vartheta,\ell,t)$ and $\Re(\ell)\neq 1$. For any complex number α with $|\alpha|<\hbar(\hbar\geq 0)$, if F satisfies the following condition:*

$$\frac{F(\omega)+\alpha\omega}{1+\alpha}\in \dot{k}-\widetilde{US}_s(\vartheta,\ell,t)$$

then $N_\hbar(F)\subset \dot{k}-\widetilde{US}_s(\vartheta,\ell,t)$.

Proof. It is obvious that $F\in \dot{k}-\widetilde{US}_s(\vartheta,\ell,t)$ if and only if

$$\left|\frac{(1-t)\omega\left(\mathbb{L}^{\vartheta}F(\omega)\right)'(1+\dot{k}e^{i\theta})-(\dot{k}e^{i\theta}+1+\ell)\left(\mathbb{L}^{\vartheta}F(\omega)-\mathbb{L}^{\vartheta}F(t\omega)\right)}{(1-t)\omega\left(\mathbb{L}^{\vartheta}F(\omega)\right)'(1+\dot{k}e^{i\theta})+(1-\dot{k}e^{i\theta}-\ell)\left(\mathbb{L}^{\vartheta}F(\omega)-\mathbb{L}^{\vartheta}F(t\omega)\right)}\right|<1,$$

$-\pi<\theta<\pi$.

For any complex number s with $|s| = 1$, we have

$$\frac{(1-t)\omega\left(\mathbb{L}^{\vartheta}F(\omega)\right)'(1+\dot{k}e^{i\theta}) - (\dot{k}e^{i\theta}+1+\ell)\left(\mathbb{L}^{\vartheta}F(\omega) - \mathbb{L}^{\vartheta}F(t\omega)\right)}{(1-t)\omega\left(\mathbb{L}^{\vartheta}F(\omega)\right)'(1+\dot{k}e^{i\theta}) + (1-\dot{k}e^{i\theta}-\ell)\left(\mathbb{L}^{\vartheta}F(\omega) - \mathbb{L}^{\vartheta}F(t\omega)\right)} \neq s.$$

In other words, we must have

$$(1-s)(1-t)\omega\left(\mathbb{L}^{\vartheta}F(\omega)\right)'(1+\dot{k}e^{i\theta}) - (\dot{k}e^{i\theta}+1+\ell+s(-1+\dot{k}e^{i\theta}+\ell))$$
$$\times\left(\mathbb{L}^{\vartheta}F(\omega) - \mathbb{L}^{\vartheta}F(t\omega)\right) \neq 0$$

which is equivalent to

$$\omega - \sum_{n=2}^{\infty} \frac{\phi_n(\vartheta)\left((n-u_n)(1+\dot{k}e^{i\theta} - s\dot{k}e^{i\theta}) - s(n+u_n) - u_n\ell(1-s)\right)}{\ell(s-1)-2s}\omega^n \neq 0.$$

However, $F \in \dot{k} - \widetilde{US}_s(\vartheta, \ell, t)$ if and only $\frac{(F*h)}{\omega} \neq 0,\ \omega \in U - \{0\}$, where $h(\omega) = \omega - \sum_{n=2}^{\infty} c_n\omega^n$ and

$$c_n = \frac{\phi_n(\vartheta)\left((n-u_n)(1+\dot{k}e^{i\theta} - s\dot{k}e^{i\theta}) - s(n+u_n) - u_n\ell(1-s)\right)}{\ell(s-1)-2s}.$$

We note that

$$|c_n| \leq \frac{\phi_n(\vartheta)\left|n(1+\dot{k}) - u_n(\dot{k}+\ell)\right|}{1-\ell}$$

since $\frac{F(\omega)+\alpha\omega}{1+\alpha} \in \dot{k} - \widetilde{US}_s(\vartheta, \ell, t)$, therefore $\omega^{-1}\left(\frac{F(\omega)+\alpha\omega}{1+\alpha} * h(\omega)\right) \neq 0$, which is equivalent to

$$\frac{(F*h)(\omega)}{(1+\alpha)\omega} + \frac{\alpha}{1+\alpha} \neq 0. \tag{3.1}$$

Now suppose that $\left|\frac{(F*h)(\omega)}{\omega}\right| < \hbar$. Then by (3.1), we must have

$$\left|\frac{(F*h)(\omega)}{(1+\alpha)\omega} + \frac{\alpha}{1+\alpha}\right| \geq \frac{|\alpha|}{|1+\alpha|} - \frac{1}{|1+\alpha|}\left|\frac{(F*h)(\omega)}{\omega}\right|$$
$$> \frac{|\alpha| - \hbar}{|1+\alpha|} \geq 0,$$

this is a contradiction by $|\alpha| < \hbar$ and however, we have $\left|\frac{(F*h)(\omega)}{\omega}\right| \geq \hbar$.

If $g(\omega) = \omega - \sum_{n=2}^{\infty} b_n\omega^n \in N_{\hbar}(F)$ then

$$\hbar - \left|\frac{(g*h)(\omega)}{\omega}\right| \leq \left|\frac{((F-g)*h)(\omega)}{\omega}\right| \leq \sum_{n=2}^{\infty} |\dot{a}_n - b_n||c_n||\omega^n|$$
$$< \sum_{n=2}^{\infty} \frac{\phi_n(\vartheta)|n(1+\dot{k}) - u_n(\dot{k}+\ell)|}{1-\ell}|\dot{a}_n - b_n|$$
$$\leq \hbar.$$

□

4. Partial sums of the function class $\dot{k}-\widetilde{US}_s(\vartheta,\ell,t)$

In this section, applying methods used by Silverman [26] and Silvia [27], we investigate the ratio of a function of the form (1.9) to its sequence of partial sums $F_m(\omega)=\omega+\sum_{n=2}^{m}\dot{a}_n\omega^n$.

Theorem 4.1. *If* $F\in\dot{k}-\widetilde{US}_s(\vartheta,\ell,t)$ *then*

$$\Re\left\{\frac{F(\omega)}{F_m(\omega)}\right\}\geq\left(\frac{\chi_{m+1}\phi_{m+1}(\vartheta)-1+\ell}{\chi_{m+1}\phi_{m+1}(\vartheta)}\right) \tag{4.1}$$

where

$$\chi_n=\chi_n(\dot{k},\ell,u_n)\phi_n(\vartheta)\geq\begin{cases}1-\ell, & \text{if } n=2,3,\cdots,m;\\ \chi_{m+1}\phi_{m+1}(\vartheta), & \text{if } n=m+1,m+2,\cdots\end{cases} \tag{4.2}$$

and

$$\chi_n=\chi_n(\dot{k},\ell,u_n)=n(1+\dot{k})-u_n(\dot{k}+\ell).$$

The result in (4.1) *is sharp with the following given by*

$$F(\omega)=\omega+\frac{1-\ell}{\chi_{m+1}\phi_{m+1}(\vartheta)}\omega^{m+1}. \tag{4.3}$$

Proof. Define the function w, we may write

$$\begin{aligned}\frac{1+w(\omega)}{1-w(\omega)}&=\frac{\chi_{m+1}\phi_{m+1}(\vartheta)}{1-\ell}\left\{\frac{F(\omega)}{F_m(\omega)}-\frac{\chi_{m+1}\phi_{m+1}(\vartheta)-1+\ell}{\chi_{m+1}\phi_{m+1}(\vartheta)}\right\}\\&=\left\{\frac{1+\sum_{n=2}^{m}\dot{a}_n\omega^{n-1}+\left(\frac{\chi_{m+1}\phi_{m+1}(\vartheta)}{1-\ell}\right)\sum_{n=m+1}^{\infty}\dot{a}_n\omega^{n-1}}{1+\sum_{n=2}^{m}\dot{a}_n\omega^{n-1}}\right\}.\end{aligned} \tag{4.4}$$

It suffices to show that $|w(\omega)|\leq 1$. Now, from (4.4), we can obtain

$$w(\omega)=\frac{\left(\frac{\chi_{m+1}\phi_{m+1}(\vartheta)}{1-\ell}\right)\sum_{n=m+1}^{\infty}\dot{a}_n\omega^{n-1}}{2+2\sum_{n=2}^{m}\dot{a}_n\omega^{n-1}+\left(\frac{\chi_{m+1}\phi_{m+1}(\vartheta)}{1-\ell}\right)\sum_{n=m+1}^{\infty}\dot{a}_n\omega^{n-1}}.$$

Hence we obtain

$$|w(\omega)|\leq\frac{\left(\frac{\chi_{m+1}\phi_{m+1}(\vartheta)}{1-\ell}\right)\sum_{n=m+1}^{\infty}|\dot{a}_n|}{2-2\sum_{n=2}^{m}|\dot{a}_n|-\left(\frac{\chi_{m+1}\phi_{m+1}(\vartheta)}{1-\ell}\right)\sum_{n=m+1}^{\infty}|\dot{a}_n|}.$$

Now $|w(\omega)|\leq 1$ if

$$2\left(\frac{\chi_{m+1}\phi_{m+1}(\vartheta)}{1-\ell}\right)\sum_{n=m+1}^{\infty}|\dot{a}_n|\leq 2-2\sum_{n=2}^{m}|\dot{a}_n|,$$

or, equivalently

$$\sum_{n=2}^{m}|\dot{a}_n|+\left(\frac{\chi_{m+1}\phi_{m+1}(\vartheta)}{1-\ell}\right)\sum_{n=m+1}^{\infty}|\dot{a}_n|\leq 1.$$

From the condition (2.1), it is sufficient to show that

$$\sum_{n=2}^{m}|\dot{a}_n|+\left(\frac{\chi_{m+1}\phi_{m+1}(\vartheta)}{1-\ell}\right)\sum_{n=m+1}^{\infty}|\dot{a}_n|\leq\sum_{n=2}^{\infty}\frac{\chi_n\phi_n(\vartheta)}{1-\ell}|\dot{a}_n|$$

which is equivalent to

$$\sum_{n=2}^{m}\left(\frac{\chi_n\phi_n(\vartheta)-1+\ell}{1-\ell}\right)|\dot{a}_n|+\sum_{n=m+1}^{\infty}\left(\frac{\chi_n\phi_n(\vartheta)-\chi_{n+1}\phi_{n+1}(\vartheta)}{1-\ell}\right)|\dot{a}_n|\geq 0. \tag{4.5}$$

To see that the function gives by (4.3) given the sharp result, we observe that for $\omega=re^{\frac{i\pi}{n}}$

$$\begin{aligned}\frac{F(\omega)}{F_m(\omega)}=&1+\frac{1-\ell}{\chi_{m+1}\phi_{m+1}(\vartheta)}\omega^n\to 1-\frac{1-\ell}{\chi_{m+1}\phi_{m+1}(\vartheta)}\\ =&\frac{\chi_{m+1}\phi_{m+1}(\vartheta)-1+\ell}{\chi_{m+1}\phi_{m+1}(\vartheta)},\text{ when } r\to 1^-.\end{aligned}$$

□

Theorem 4.2. *If* $F\in\dot{k}-\widetilde{US}_s(\vartheta,\ell,t)$ *then*

$$\Re\left\{\frac{F_m(\omega)}{F(\omega)}\right\}\geq\frac{\chi_{m+1}\phi_{m+1}(\vartheta)}{\chi_{m+1}\phi_{m+1}(\vartheta)+1-\ell},\quad(\omega\in U) \tag{4.6}$$

where $\chi_{m+1}\phi_{m+1}(\vartheta)\geq 1-\ell$ *and*

$$\chi_n\phi_n(\vartheta)\geq\begin{cases}1-\ell, & \text{if } n=2,3,\cdots,m;\\ \chi_{m+1}\phi_{m+1}(\vartheta), & \text{if } n=m+1,m+2,\cdots.\end{cases} \tag{4.7}$$

The result (4.6) *is sharp with the function given by* (4.3).

Proof. We write

$$\begin{aligned}\frac{1+w(\omega)}{1-w(\omega)}=&\frac{\chi_{m+1}\phi_{m+1}(\vartheta)+1-\ell}{1-\ell}\left\{\frac{F_m(\omega)}{F(\omega)}-\frac{\chi_{m+1}\phi_{m+1}(\vartheta)}{\chi_{m+1}\phi_{m+1}(\vartheta)+1-\ell}\right\}\\ =&\left\{\frac{1+\sum_{n=2}^{m}\dot{a}_n\omega^{n-2}-\left(\frac{\chi_{m+1}\phi_{m+1}(\vartheta)}{1-\ell}\right)\sum_{n=m+1}^{\infty}\dot{a}_n\omega^{n-1}}{1+\sum_{n=2}^{\infty}\dot{a}_n\omega^{n-1}}\right\}\end{aligned}$$

$$|w(\omega)|\leq\frac{\left(\frac{\chi_{m+1}\phi_{m+1}(\vartheta)+1-\ell}{1-\ell}\right)\sum_{n=m+1}^{\infty}|\dot{a}_n|}{2-2\sum_{n=2}^{m}|\dot{a}_n|-\left(\frac{\chi_{m+1}\phi_{m+1}(\vartheta)+1-\ell}{1-\ell}\right)\sum_{n=m+1}^{\infty}|\dot{a}_n|}\leq 1.$$

This last inequality is equivalent to

$$\sum_{n=2}^{m}|\dot{a}_n|+\sum_{n=m+1}^{\infty}\left(\frac{\chi_{m+1}\phi_{m+1}(\vartheta)}{1-\ell}\right)|\dot{a}_n|\leq 1.$$

Making use of (2.1) to get (4.5). Lastly, for the extremal function $F(\omega)$ provided by (4.3), equality is maintained in (4.6). □

Theorem 4.3. *If $F \in \dot{k} - \widetilde{US}_s(\vartheta, \ell, t)$ then*

$$\Re\left\{\frac{F'(\omega)}{F'_m(\omega)}\right\} \geq \left(\frac{\chi_{m+1}\phi_{m+1}(\vartheta) - (1-\ell)(m+1)}{\chi_{m+1}\phi_{m+1}(\vartheta)}\right), \quad (\omega \in U) \tag{4.8}$$

$$\text{and } \Re\left\{\frac{F'_m(\omega)}{F'(\omega)}\right\} \geq \left(\frac{\chi_{m+1}\phi_{m+1}(\vartheta)}{\chi_{m+1}\phi_{m+1}(\vartheta) + (1-\ell)(m-1)}\right), \quad (\omega \in U) \tag{4.9}$$

where $\chi_{m+1}\phi_{m+1}(\vartheta) \geq (m+1)(1-\ell)$ *and*

$$\chi_n\phi_n(\vartheta) \geq \begin{cases} n(1-\ell), & \text{if } n = 1, 2, 3, \cdots, m; \\ n\left(\frac{\chi_{m+1}\phi_{m+1}(\vartheta)}{m+1}\right), & \text{if } n = m+1, m+2, \cdots. \end{cases}$$

With respect to the function provided by (4.3), *the results are sharp.*

Proof. We write

$$\frac{1+w(\omega)}{1-w(\omega)} = \left(\frac{\chi_{m+1}\phi_{m+1}(\vartheta)}{(m+1)(1-\ell)}\right)\left\{\frac{F'(\omega)}{F'_m(\omega)} - \left(\frac{\chi_{m+1}\phi_{m+1}(\vartheta) - (1-\ell)(m+1)}{\chi_{m+1}\phi_{m+1}(\vartheta)}\right)\right\}$$

where

$$w(\omega) = \frac{\left(\frac{\chi_{m+1}\phi_{m+1}(\vartheta)}{(m+1)(1-\ell)}\right)\sum_{n=m+1}^{\infty} n\dot{a}_n\omega^{n-1}}{2 + 2\sum_{n=2}^{m} n\dot{a}_n\omega^{n-1} + \left(\frac{\chi_{m+1}\phi_{m+1}(\vartheta)}{(m+1)(1-\ell)}\right)\sum_{n=m+1}^{\infty} n\dot{a}_n\omega^{n-1}}.$$

Now $|w(\omega)| \leq 1$ if and only if

$$\sum_{n=2}^{m} n|\dot{a}_n| + \left(\frac{\chi_{m+1}\phi_{m+1}(\vartheta)}{(m+1)(1-\ell)}\right)\sum_{n=m+1}^{\infty} n|\dot{a}_n| \leq 1.$$

Given (2.1), it suffices to demonstrate that

$$\sum_{n=2}^{m} n|\dot{a}_n| + \left(\frac{\chi_{m+1}\phi_{m+1}(\vartheta)}{(m+1)(1-\ell)}\right)\sum_{n=m+1}^{\infty} n|\dot{a}_n| \leq \sum_{n=2}^{\infty}\frac{\chi_n\phi_n(\vartheta)}{1-\ell}|\dot{a}_n|$$

which is equivalent to

$$\sum_{n=2}^{m}\frac{\chi_n\phi_n(\vartheta) - (1-\ell)n}{1-\ell}|\dot{a}_n| + \sum_{n=m+1}^{\infty}\frac{(m+1)\chi_n\phi_n(\vartheta) - n\chi_{m+1}\phi_{m+1}(\vartheta)}{(m+1)(1-\ell)}|\dot{a}_n| \geq 0.$$

To prove the result (4.9), define the function $w(\omega)$

$$\frac{1+w(\omega)}{1-w(\omega)} = \left(\frac{(m+1)(1-\ell) + \chi_{m+1}\phi_{m+1}(\vartheta)}{(m+1)(1-\ell)}\right) \times \left\{\frac{F'_m(\omega)}{F'(\omega)} - \left(\frac{\chi_{m+1}\phi_{m+1}(\vartheta)}{\chi_{m+1}\phi_{m+1}(\vartheta) + (m+1)(1-\ell)}\right)\right\}$$

where

$$w(\omega) = \frac{-\left(1 + \frac{\chi_{m+1}\phi_{m+1}(\vartheta)}{(m+1)(1-\ell)}\right)\sum_{n=m+1}^{\infty} n\dot{a}_n\omega^{n-1}}{2 + 2\sum_{n=2}^{m} n\dot{a}_n\omega^{n-1} + \left(1 - \frac{\chi_{m+1}\phi_{m+1}(\vartheta)}{(m+1)(1-\ell)}\right)\sum_{n=m+1}^{\infty} n\dot{a}_n\omega^{n-1}}.$$

Now $|w(\omega)| \leq 1$ if and only if

$$\sum_{n=2}^{m} n|\dot{a}_n| + \left(\frac{\chi_{m+1}\phi_{m+1}(\vartheta)}{(m+1)(1-\ell)}\right)\sum_{n=m+1}^{\infty} n|\dot{a}_n| \leq 1. \tag{4.10}$$

It is adequate for evidence to show that the condition above bounded the left hand side of (4.10).

$$\sum_{n=2}^{\infty} \frac{\chi_n\phi_n(\vartheta)}{1-\ell}|\dot{a}_n|$$

which is equivalent to

$$\sum_{n=2}^{m}\left(\frac{\chi_n\phi_n(\vartheta)}{1-\ell} - n\right)|\dot{a}_n| + \sum_{n=m+1}^{\infty}\left(\frac{\chi_n\phi_n(\vartheta)}{1-\ell} - \frac{\chi_{m+1}\phi_{m+1}(\vartheta)}{(m+1)(1-\ell)}\right)n|\dot{a}_n| \geq 0.$$

□

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