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TRIPLED COINCIDENCE POINTS OF MAPPINGS IN PARTIALLY ORDERED 0-COMPLETE PARTIAL METRIC SPACES

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Abstract. In this paper, we introduce the concept of a tripled coincidence point for a pair of nonlinear contractive mappings $F : X^3 \to X$ and $g : X \to X$ in partially ordered 0-complete partial metric spaces and obtain existence and uniqueness theorems. Our results generalize, extend, unify and complement recent tripled coincidence point theorems established by Marin Borcut, Vasile Berinde [M. Borcut, V. Berinde, Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces, Applied Mathematical and Computation 218 (2012) 5929-5935], Marin Borcut [M. Borcut, Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces, Applied Mathematical and Computation 218 (2012) 7339-7346], Hassen Aydi, Erdal Karapinar, Mihail Postolache [H. Aydi, E. Karapinar, M. Postolache, Tripled coincidence point theorems for weak ϕ–contractions in partially ordered metric spaces, Fixed Point Theory and Applications 2012, 2012:44, doi: 10.1186/1687-1812-2012-44] and Binayak S. Choudhury, Erdal Karapinar and Amaresh Kundu [B. Choudhury, E. Karapinar, A. Kundu, Tripled coincidence point theorems for nonlinear contractions in partially ordered metric spaces, International Journal of Mathematics and Mathematical Sciences,2012, in press]. Examples to support our new results are given.

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1. Introduction

Matthews [23, 24] introduced the concept of partial metric spaces as a part of the study of denotational semantics of dataflow networks. He showed that the Banach contraction principle can be generalized to the partial metric context. Based on the notion of partial metric spaces, several authors have proved fixed point theorems in partial metric spaces (see, for example, [2], [3], [4], [16], [17], [19], [23-26], [28]). See also the presentation by Bakutin et al. [7] where the motivation for introducing non-zero distance (i.e., the "distance" p where $p(x, x) = 0$ need not hold) is explained, which is also leading to interesting research in foundations of topology.

There are also many generalizations of Banach contractive condition. Remarks on some recent fixed point theorems can be found in [6] and [18]. In 2006, Bhaskar and Lakshmikantham [10] introduced and proved some coupled fixed point results in a partially ordered metric space. There are many authors obtained important coupled fixed point theorems (see [1], [14], [15], [18], [20], [21], [22], [29]).

Berinde and Borcut [8] introduced the concept of tripled fixed point for nonlinear mappings in partially ordered complete metric space. Aydi et al. [5] presented tripled coincidence theorem for weak φ-contractions in partially ordered metric space. There are some authors obtained important tripled fixed point theorems (see, for example [9], [12], [13]).

The aim of this paper is to continue the study of the tripled fixed points but now in partially ordered 0-complete partial metric spaces (see Bin Ahmad et al. [11] where authors proved fixed point theorems in 0-complete partial metric spaces). The following definitions and results will be needed in the sequel.

2. Notation and preliminaries

Definition 2.1. A partial metric on a nonempty set X is a function $p: X^2 \rightarrow R^+$ such that for all $x, y, z \in X$:
(**p**1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,
(**p**2) $p(x, x) \leq p(x, y)$,
(**p**3) $p(x, y) = p(y, x)$,

(**p4**) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X. Each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open p–balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Definition 2.2. A sequence $\{x_n\}$ in a partial metric space (X, p) converges to $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x_n, x)$;
(**i**) a sequence $\{x_n\}$ in a partial metric space (X, p) is called 0-Cauchy if $\lim_{n,m\to\infty} p(x_n, x_m) = 0$.
(**ii**) a partial metric space (X, p) is 0-complete if every 0-Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x)=0$. In this case, p is a 0-complete partial metric on X.
(iii) A mapping $f: X \to X$ is said to be continuous at $x_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_p(x_0, \delta)) \subset B_p(f(x_0), \varepsilon)$.
This definition of continuity is equivalent to the following statement:
(iv) A mapping $f: X \to X$ is said to be continuous at $x_0 \in X$ where (X, p) is a partial metric space if and only if $p(f(x_n), f(x_0)) \to p(f(x_0), f(x_0))$ whenever $p(x_n, x_0) \to p(x_0, x_0)$ as $n \to \infty$.

Remark 2.3. (1) A limit of a sequence in a partial metric space does not need to be unique. Moreover, the function $p(\cdot,\cdot)$ does not need to be continuous in the sense that $x_n \to x$ and $y_n \to y$ implies $p(x_n, y_n) \to p(x, y)$.
(2) [3] However, $p(x_n, y_n) \to p(x, y) = 0$ then $p(x_n, y) \to p(x, y)$ for all $y \in X$.

Definition 2.4. [8] Let $(X, \preceq)$ be a partially ordered set. The mapping $F: X^3 \to X$ is said to have the mixed monotone property if for any $x, y, z \in X$

$$x_1, x_2 \in X,\ x_1 \leq x_2 \Rightarrow F(x_1, y, z) \leq F(x_2, y, z),$$
$$y_1, y_2 \in X,\ y_1 \leq y_2 \Rightarrow F(x, y_1, z) \geq F(x, y_2, z),$$
$$z_1, z_2 \in X,\ z_1 \leq z_2 \Rightarrow F(x, y, z_1) \leq F(x, y, z_2).$$

An element $(x, y, z) \in X^3$ is called a tripled fixed point of F if

$$F(x,y,z)=x,\ F(y,x,y)=y \text{ and } F(z,y,x)=z.$$

Definition 2.5. [8] Let $(X,\preceq)$ be a partially ordered set, $F: X^3 \to X$ and g: $X \to X$ two mappings. The mapping F is said to have the mixed g-monotone property if for any $x, y, z \in X$

$$x_1, x_2 \in X,\ gx_1 \le gx_2 \Rightarrow F(x_1,y,z) \le F(x_2,y,z),$$
$$y_1, y_2 \in X,\ gy_1 \le gy_2 \Rightarrow F(x,y_1,z) \ge F(x,y_2,z),$$
$$z_1, z_2 \in X,\ gz_1 \le gz_2 \Rightarrow F(x,y,z_1) \le F(x,y,z_2).$$

An element (x,y,z) is called a tripled coincidence point of F and g if

$$F(x,y,z)=gx,\ F(y,x,y)=gy \text{ and } F(z,y,x)=gz.$$

while (gx, gy, gz) is said a tripled point of coincidence of mappings F and g. Moreover, (x,y,z) is called a tripled common fixed point of F and g if

$$F(x,y,z)=gx=x,\ F(y,x,y)=gy=y \text{ and } F(z,y,x)=gz=z.$$

At last, mappings F and g are called commutative if for all $x, y, z \in X$

$$g(F(x,y,z)) = F(gx, gy, gz).$$

Definition 2.6. [5] Let $(X,\preceq)$ be an ordered set and p be a partial metric on X. We say that $(X, p, \preceq)$ is regular if it has the following properties:

(i) if for non-decreasing sequence $\{x_n\}$ holds $p(x_n,x) \to p(x,x)$, then $x_n \preceq x$ for all n,
(i) if for non-increasing sequence $\{y_n\}$ holds $p(y_n,y) \to p(y,y)$, then $y_n \succeq y$ for all n.

Lemma 2.7. [18, 27] Let (X, p) be a 0-complete partial metric space and let $\{x_n\}$,$\{y_n\}$ and $\{z_n\}$ be sequences in X such that

$$\lim_{n\to\infty} p\,(x_n, x_{n+1}) = \lim_{n\to\infty} p\,(y_n, y_{n+1}) = \lim_{n\to\infty} p\,(z_n, z_{n+1}) = 0.$$

If at least one of sequences$\{x_n\}$,$\{y_n\}$ and $\{z_n\}$ is not a 0-Cauchy sequence in (X, p), then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $m(k) > n(k) > k$ and the following four sequences

$$p(x_{m(k)}, x_{n(k)}) + p(y_{m(k)}, y_{n(k)}) + p(z_{m(k)}, z_{n(k)}),$$
$$p(x_{m(k)}, x_{n(k)+1}) + p(y_{m(k)}, y_{n(k)+1}) + p(z_{m(k)}, z_{n(k)+1}),$$
$$p(x_{m(k)-1}, x_{n(k)}) + p(y_{m(k)-1}, y_{n(k)}) + p(z_{m(k)-1}, z_{n(k)}),$$
$$p(x_{m(k)-1}, x_{n(k)+1}) + p(y_{m(k)-1}, y_{n(k)+1}) + p(z_{m(k)-1}, z_{n(k)+1}),$$

tend to $\varepsilon+$ when $k \to \infty$.

3. Main results

Firstly, let us consider the set of functions

$$\Phi = \left\{ \varphi : [0, +\infty) \to [0, +\infty) \,\middle|\, \varphi(t) < t \text{ and } \lim_{r \to t^+} \varphi(r) < t,\ t > 0 \right\}.$$

It is clear that for $\varphi \in \Phi$ holds $\varphi(0) = 0$ and $\lim_{n\to\infty} \varphi^n(t) = 0$ for $t > 0$. Indeed, since $\varphi^n(t)$ is a decreasing sequence for all $t > 0$, we have that $\lim_{n\to\infty} \varphi^n(t) = \delta \geq 0$. If $\delta > 0$, we get $\delta = \lim_{n\to\infty} \varphi^{n+1}(t) = \lim_{\varphi^n(t)\to\delta^+} \varphi(\varphi^n(t)) < \delta$, which is a contradiction.

Our first main result is the following theorem. This theorem extends some recently results from usual metric spaces to the case of partial metric spaces.

Theorem 3.1. Let $(X, \preceq)$ be a partially ordered set and suppose there is a partial metric p on X such that (X, p) is a partial metric space. Suppose $F: X^3 \to X$ and $g: X \to X$ are such that F has the mixed g-monotone property with $F(X^3) \subset g(X)$ and $g(X)$ is a 0-complete subspace of X. Assume there is a function $\varphi \in \Phi$ such that

$$p(F(x,y,z), F(u,v,w)) + p(F(y,x,y), F(v,u,v)) + p(F(z,y,x), F(w,v,u)) \\ \leq 3\varphi\left(\frac{p(gx,gu) + p(gy,gv) + p(gz,gw)}{3}\right) \tag{3.1}$$

for any $x, y, z, u, v, w \in X$ for which $gx \preceq gu$, $gy \succeq gv$ and $gz \preceq gw$. Suppose either F is continuous or $(X, p, \preceq)$ is regular. If there exist $x_0, y_0, z_0 \in X$ such that

$$gx_0 \preceq F(x_0, y_0, z_0), \quad gy_0 \succeq F(y_0, x_0, y_0) \text{ and } gz_0 \preceq F(z_0, y_0, x_0),$$

then there exist $x, y, z \in X$ such that

$$F(x,y,z) = gx, \ F(y,x,y) = gy, \text{ and } F(z,y,x) = gz,$$

that is., F and g have a tripled coincidence point.

Proof. Let us consider sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X such that

$gx_{n+1} = F(x_n, y_n, z_n)$, $gy_{n+1} = F(y_n, x_n, y_n)$ and $gz_{n+1} = F(z_n, y_n, x_n)$ for n=0,1,2,...

We will prove by induction that

$$gx_n \preceq gx_{n+1},\ gy_{n+1} \preceq gy_n \text{ and } gz_n \preceq gz_{n+1}. \tag{3.2}$$

Since $gx_0 \preceq F(x_0, y_0, z_0)$ and $gx_1 = F(x_0, y_0, z_0)$ we have $gx_0 \preceq gx_1$. This is true for n=0. We suppose that (3.2) is true for some n>0. Since F has the mixed

g-monotone property, by $gx_n \preceq gx_{n+1}$, $gy_{n+1} \preceq gy_n$ and $gz_n \preceq gz_{n+1}$, we have that

$$gx_{n+1} = F(x_n, y_n, z_n) \preceq F(x_{n+1}, y_n, z_n) \preceq F(x_{n+1}, y_n, z_{n+1}) \preceq F(x_{n+1}, y_{n+1}, z_{n+1}) = gx_{n+2}$$

and similarly $gy_{n+2} \preceq gy_{n+1}$ and $gz_{n+1} \preceq gz_{n+2}$. This means that (3.2) is true for any $n \in \mathbf{N}$.

Consider the two possible cases:

- $p(gx_{n+1}, gx_n)=0$, $p(gy_{n+1}, yx_n)=0$, $p(gz_{n+1}, gz_n)=0$ for some $n \in \mathbf{N}$.

In this case, $gx_n = F(x_n, y_n, z_n)$, $gy_n = F(y_n, x_n, y_n)$, $gz_n = F(z_n, y_n, x_n)$ and (x_n, y_n, z_n) is a tripled coincidence point of F and g.

- $p(gx_{n+1}, gx_n) > 0$, $p(gy_{n+1}, yx_n) > 0$, $p(gz_{n+1}, gz_n) > 0$ for every $n \in$N.

Applying (3.1) with $x=x_n$, $y=y_n$, $z=z_n$, $u=x_{n+1}$, $v=y_{n+1}$ and $w=z_{n+1}$ we have:

$$\begin{aligned} & p(F(x_n, y_n, z_n), F(x_{n+1}, y_{n+1}, z_{n+1})) + p(F(y_n, x_n, y_n), F(y_{n+1}, x_{n+1}, y_{n+1})) \\ & + p(F(z_n, y_n, x_n), F(z_{n+1}, y_{n+1}, x_{n+1})) \\ & \leq 3\varphi\left(\frac{p(gx_n, gx_{n+1}) + p(gy_n, gy_{n+1}) + p(gz_n, gz_{n+1})}{3}\right) \\ & < 3\,\frac{p(gx_n, gx_{n+1}) + p(gy_n, gy_{n+1}) + p(gz_n, gz_{n+1})}{3} \\ & = p(gx_n, gx_{n+1}) + p(gy_n, gy_{n+1}) + p(gz_n, gz_{n+1}) \end{aligned} \tag{3.3}$$

from which follows:

$$\begin{aligned} & p(gx_{n+1}, gx_{n+2}) + p(gy_{n+1}, gy_{n+2}) + p(gz_{n+1}, gz_{n+2}) \\ & < p(gx_n, gx_{n+1}) + p(gy_n, gy_{n+1}) + p(gz_n, gz_{n+1}) \end{aligned} \tag{3.4}$$

If we denote by $\delta_n=\frac{1}{3}(p(gx_n, gx_{n+1})+p(gy_n, gy_{n+1})+p(gz_n, gz_{n+1}))$, we can conclude that $\{\delta_n\}$ is monotone decreasing. Therefore, $\delta_n \to \delta^* \geq 0$ when $n \to \infty$.

We now prove that $\delta^*=0$. Assume, on the contrary, that $\delta^* > 0$. If we write (3.3) as $\delta_{n+1} \leq \varphi(\delta_n)$ and if we pass to the limit when $n \to \infty$, we obtain that

$$\delta^* \leq \lim_{n\to+\infty} \varphi(\delta_n) = \lim_{\delta_n \to \delta^{*+}} \varphi(\delta_n) < \delta^*,$$

which is a contradiction. Hence,

$$\lim_{n\to+\infty} \frac{1}{3}(p(gx_n, gx_{n+1})+p(gy_n, gy_{n+1})+p(gz_n, gz_{n+1})) = 0. \tag{3.5}$$

From (3.5) follows $\lim_{n\to\infty} p(gx_{n+1}, gx_n) = \lim_{n\to\infty} p(gy_{n+1}, gyx_n) = \lim_{n\to\infty} p(gz_{n+1}, gz_n) = 0$.

We next prove that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are 0-Cauchy sequence in the space (X, p). Suppose that at least one of sequences $\{gx_n\}$, $\{gy_n\}$, $\{gz_n\}$ is not a 0-Cauchy sequence. Then using Lemma 2.7. we get that there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $m(k) > n(k) > k$ and the following four sequences

$$p(gx_{m(k)}, gx_{n(k)}) + p(gy_{m(k)}, gy_{n(k)}) + p(gz_{m(k)}, gz_{n(k)}),$$
$$p(gx_{m(k)}, gx_{n(k)+1}) + p(gy_{m(k)}, gy_{n(k)+1}) + p(gz_{m(k)}, gz_{n(k)+1}),$$
$$p(gx_{m(k)-1}, gx_{n(k)}) + p(gy_{m(k)-1}, gy_{n(k)}) + p(gz_{m(k)-1}, gz_{n(k)}),$$
$$p(gx_{m(k)-1}, gx_{n(k)+1}) + p(gy_{m(k)-1}, gy_{n(k)+1}) + p(gz_{m(k)-1}, gz_{n(k)+1}),$$

all tend to $\varepsilon+$ when $k \to \infty$. Applying condition (3.1) to elements $x=x_{m(k)-1}$, $y=y_{m(k)-1}$, $z=z_{m(k)-1}$, $u=x_{n(k)}$, $v=y_{n(k)}$ and $w=z_{n(k)}$ we get that

$$p(F(x_{m(k)-1}, y_{m(k)-1}, z_{m(k)-1}), F(x_{n(k)}, y_{n(k)}, z_{n(k)})) +$$
$$p(F(y_{m(k)-1}, x_{m(k)-1}, y_{m(k)-1}), F(y_{n(k)}, x_{n(k)}, y_{n(k)})) +$$
$$p(F(z_{m(k)-1}, y_{m(k)-1}, x_{m(k)-1}), F(z_{n(k)}, y_{n(k)}, x_{n(k)}))$$
$$\leq 3\varphi\left(\frac{p(gx_{m(k)-1}, gx_{n(k)}) + p(gy_{m(k)-1}, gy_{n(k)}) + p(gz_{m(k)-1}, gz_{n(k)})}{3}\right)$$

or

$$p(gx_{m(k)-1}, gx_{n(k)}) + p(gy_{m(k)-1}, gy_{n(k)}) + p(gz_{m(k)-1}, gz_{n(k)})$$
$$\leq 3\varphi\left(\frac{p(gx_{m(k)-1}, gx_{n(k)}) + p(gy_{m(k)-1}, gy_{n(k)}) + p(gz_{m(k)-1}, gz_{n(k)})}{3}\right).$$

If we pass to the limit when $n \to \infty$, we obtain

$$\varepsilon \leq 3\lim_{k\to\infty} \varphi\left(\frac{p(gx_{m(k)-1}, gx_{n(k)}) + p(gy_{m(k)-1}, gy_{n(k)}) + p(gz_{m(k)-1}, gz_{n(k)})}{3}\right)$$
$$= 3\lim_{\frac{A_k}{3}\to\frac{\varepsilon}{3}} \varphi\left(\frac{A_k}{3}\right) < 3\frac{\varepsilon}{3} = \varepsilon,$$

which is a contradiction. (We denote $p(gx_{m(k)-1}, gx_{n(k)}) + p(gy_{m(k)-1}, gy_{n(k)}) + p(gz_{m(k)-1}, gz_{n(k)})$ with A_k).

This shows that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are 0-Cauchy sequences in the space (X, p). Since gX is a 0-complete, there exist $x, y, z \in X$ such that

$\lim_{n\to\infty} p(gx_n, gx) = p(gx, gx) = 0$, $\lim_{n\to\infty} p(gy_n, gy) = p(gy, gy) = 0$, and $\lim_{n\to\infty} p(gz_n, gz) = p(gz, gz) = 0$.

Suppose that F is continuous. We have

$$p(gx, F(x, y, z)) \leq p(gx, gx_{n+1}) + p(gx_{n+1}, F(x, y, z)). \tag{3.6}$$

It holds that $p(gx, gx_{n+1}) \to p(gx, gx) = 0$ when $n \to \infty$ and

$p(gx_{n+1}, F(x,y,z)) = p(F(x_n, y_n, z_n), F(x,y,z)) \to p(F(x,y,z), F(x,y,z))$ when $n \to \infty$.

When $n \to \infty$ from (3.6) follows:

$$p(gx, F(x,y,z)) \le 0 + p(F(x,y,z), F(x,y,z)) = p(F(x,y,z), F(x,y,z)).$$

According to the property (**p**2) of the partial metric space, we have

$$p(gx, F(x,y,z)) = p(F(x,y,z), F(x,y,z)).$$

Similarly, we have

$$p(gy, F(y,x,y)) = p(F(y,x,y), F(y,x,y))$$

and

$$p(gz, F(z,y,x)) = p(F(z,y,x), F(z,y,x)).$$

Since $x \preceq x$, $y \succeq y$ and $z \preceq z$, according to condition (3.1) with $x=u$, $y=v$, $z=w$ we obtain

$$p(F(x,y,z), F(x,y,z)) + p(F(y,x,y), F(y,x,y)) + p(F(z,y,x), F(z,y,x)) = 0$$

from which follows

$$p(F(x,y,z), F(x,y,z)) = 0,\ p(F(y,x,y), F(y,x,y)) = 0 \text{ and } p(F(z,y,x), F(z,y,x)) = 0$$

or, equivalently

$$p(gx, F(x,y,z)) = 0,\ p(gy, F(y,x,y)) = 0 \text{ and } p(gz, F(z,y,x)) = 0. \qquad (3.7)$$

From (3.7) follows

$$gx = F(x,y,z)\,,\; gy = F(y,x,y) \text{ and } gz = F(z,y,x)\,. \tag{3.8}$$

Hence, in this case, we have proved that F and g have a tripled coincidence point.

Suppose that $(X,\ p, \preceq)$ is regular. Then, since (gx_n, gy_n, gz_n) is comparable with (gx, gy, gz) we have that according to (3.1)

$$p(F(x_n, y_n, z_n), F(x,y,z)) + p(F(y_n, x_n, y_n), F(y,x,y)) + p(F(z_n, y_n, x_n), F(z,y,x))$$

$$\leq 3\varphi\left(\frac{p(gx_n, gx) + p(gy_n, gy) + p(gz_n, gz)}{3}\right) \tag{3.9}$$

or

$$p(gx_{n+1}, F(x,y,z)) + p(gy_{n+1}, F(y,x,y)) + p(gz_{n+1}, F(z,y,x))$$

$$\leq 3\varphi\left(\frac{p(gx_n, gx) + p(gy_n, gy) + p(gz_n, gz)}{3}\right) \tag{3.10}$$

Now, taking limit as $n \to \infty$ from (3.10) follows:

$$p(gx, F(x,y,z)) + p(gy, F(y,x,y)) + p(gz, F(z,y,x)) = 0. \tag{3.11}$$

As $n \to +\infty$ we have $p(gx_n, gx) \to p(gx, gx) = 0$, $p(gy_n, gy) \to p(gy, gy) = 0$ and $p(gz_n, gz) \to p(gz, gz) = 0$. We also have, as $n \to +\infty$, $p(gx_{n+1}, F(x,y,z)) \to p(gx, F(x,y,z))$, $p(gy_{n+1}, F(y,x,y)) \to p(gy, F(y, x,y)$ and $p(gz_{n+1}, F(z,y,x)) \to p(gz, F(z,y,x))$ (see Remark 2.3 (2)]).

From (3.11) follows

$$p(gx, F(x,y,z)) = 0,\ p(gy, F(y,x,y)) = 0 \text{ and } p(gz, F(z,y,x)) = 0$$

or

$$gx = F(x,y,z),\ gy = F(y,x,y) \text{ and } gz = F(z,y,x).$$

In this case we have also proved that F and g have a tripled coincidence point.

□

When we consider a partial metric instead of a standard metric then the Theorem 3.1 is the result of Aydi et al. [5] but the method used in the proof of this theorem is completely different from that used by Aydi et al. [5]. Also, in Theorem 3.1. we consider a weaker assumption for the function g.

Corollary 3.2. Let $(X, \preceq)$ be a partially ordered set and suppose there is a partial metric p on X such that (X, p) is a 0-complete partial metric space. Suppose $F: X^3 \to X$ and $g: X \to X$ are such that F has the mixed g-monotone property with $F(X^3) \subset g(X)$ and $g(X)$ is a 0-complete subspace of X. Assume there exists $\alpha \in [0,1)$ such that

$$\begin{aligned} p(F(x,y,z), F(u,v,w)) + p(F(y,x,y), F(v,u,v)) + p(F(z,y,x), F(w,v,u)) \\ \leq \alpha\,(p(gx,gu) + p(gy,gv) + p(gz,gw)) \end{aligned} \tag{3.12}$$

for any $x,y,z,u,v,w \in X$ for which $gx \preceq gu$, $gy \succeq gv$ and $gz \preceq gw$. Suppose either F is continuous or $(X, p, \preceq)$ is regular. If there exist $x_0, y_0, z_0 \in X$ such that

$$gx_0 \preceq F(x_0, y_0, z_0),\quad gy_0 \succeq F(y_0, x_0, y_0) \text{ and } gz_0 \preceq F(z_0, y_0, x_0),$$

then there exist $x, y, z \in X$ such that

$$F(x,y,z)=gx,\ F(y,x,y)=gy,\ \text{and}\ F(z,y,x)=gz\ ,$$

that is., F and g have a tripled coincidence point.

Proof: Follows by taking $\varphi(t)=\alpha t$ in Theorem 3.1.

□

Corollary 3.3 Let $(X,\preceq)$ be a partially ordered set and suppose there is a partial metric p on X such that (X,p) is a 0-complete partial metric space. Suppose $F:X^3\to X$ has the mixed monotone property. Assume there is a function $\varphi\in\Phi$ such that

$$p(F(x,y,z),F(u,v,w))+p(F(y,x,y),F(v,u,v))+p(F(z,y,x),F(w,v,u))$$
$$\leq 3\varphi\left(\frac{p(x,u)+p(y,v)+p(z,w)}{3}\right)$$

for any $x,y,z,u,v,w\in X$ for which $x\preceq u$, $y\succeq v$ and $z\preceq w$. Suppose either F is continuous or $(X,p,\preceq)$ is regular. If there exist $x_0,y_0,z_0\in X$ such that

$$x_0\preceq F(x_0,y_0,z_0)\ ,\quad y_0\succeq F(y_0,x_0,y_0)\ \text{and}\ z_0\preceq F(z_0,y_0,x_0)\ ,$$

then there exist $x,y,z\in X$ such that

$$F(x,y,z)=x,\ F(y,x,y)=y,\ \text{and}\ F(z,y,x)=z\ ,$$

that is., F has a tripled fixed point.

Proof. Follows from Theorem 3.1 by taking $g=i_X$ (the identity map).

□

Our second result is the following theorem which improve the main result from [13] in several direction.

Theorem 3.4. Let $(X, \preceq)$ be a partially ordered set and suppose there is a partial metric p on X such that (X, p) is a partial metric space. Suppose $F : X^3 \to X$ and $g : X \to X$ are such that F has the mixed g-monotone property with $F(X^3) \subset g(X)$ and $g(X)$ is a 0-complete subspace of X. Assume there is a function $\varphi \in \Phi$ such that

$$p(F(x,y,z), F(u,v,w)) \leq \varphi\left(\max\{p(gx,gu), p(gy,gv), p(gz,gw)\}\right) \quad (3.13)$$

for any $x,y,z,u,v,w \in X$ for which $gx \preceq gu$, $gy \succeq gv$ and $gz \preceq gw$. Suppose either F is continuous or $(X, p, \preceq)$ is regular. If there exist $x_0, y_0, z_0 \in X$ such that

$$gx_0 \preceq F(x_0, y_0, z_0), \quad gy_0 \succeq F(y_0, x_0, y_0) \text{ and } gz_0 \preceq F(z_0, y_0, x_0),$$

then there exist $x, y, z \in X$ such that

$$F(x,y,z) = gx, \; F(y,x,y) = gy, \text{ and } F(z,y,x) = gz,$$

that is., F and g have a tripled coincidence point.

Proof. Let us consider sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X such that

$gx_{n+1} = F(x_n, y_n, z_n)$, $gy_{n+1} = F(y_n, x_n, y_n)$ and $gz_{n+1} = F(z_n, y_n, x_n)$ for n=0,1,2,...

Proceeding exactly as in Theorem 3.1 we have that $gx_n \preceq gx_{n+1}$, $gy_{n+1} \preceq gy_n$ and $gz_n \preceq gz_{n+1}$.

If $p(gx_{n+1}, gx_n)=0$, $p(gy_{n+1}, yx_n)=0$ and $p(gz_{n+1}, gz_n)=0$ for some $n \in \mathbf{N}$, then (x_n, y_n, z_n) is a tripled coincidence point of F and g.

So, we will consider the case when $p(gx_{n+1}, gx_n)>0$, $p(gy_{n+1}, yx_n)>0$, $p(gz_{n+1}, gz_n)>0$ for every $n \in \mathbf{N}$. Applying (3.13) with $x=x_n$, $y=y_n$, $z=z_n$, $u=x_{n+1}$, $v=y_{n+1}$ and $w=z_{n+1}$ we have:

$$p(F(x_n, y_n, z_n), F(x_{n+1}, y_{n+1}, z_{n+1})) \leq \varphi\left(\max\{p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1}), p(gz_n, gz_{n+1})\}\right)$$

that is.,

$$p(gx_{n+1}, gx_{n+2}) \leq \varphi\left(\max\{p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1}), p(gz_n, gz_{n+1})\}\right).$$

Similarly, we have

$$p(gy_{n+1}, gy_{n+2}) \leq \varphi\left(\max\{p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1}), p(gz_n, gz_{n+1})\}\right)$$

and

$$p(gz_{n+1}, gz_{n+2}) \leq \varphi\left(\max\{p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1}), p(gz_n, gz_{n+1})\}\right).$$

It follows:

$$\max\{p(gx_{n+1}, gx_{n+2}), p(gy_{n+1}, gy_{n+2}), p(gz_{n+1}, gz_{n+2})\} \tag{3.14}$$

$$\leq \varphi\left(\max\{p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1}), p(gz_n, gz_{n+1})\}\right)$$

If we denote by $\delta_n = \max\{p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1}), p(gz_n, gz_{n+1})\}$, we can conclude that $\{\delta_n\}$ is monotone decreasing. Therefore, $\delta_n \to \delta^* \geq 0$ when $n \to \infty$.

We now prove that $\delta^*=0$. Assume, on the contrary, that $\delta^* > 0$. If we write (3.14) as $\delta_{n+1} < \varphi(\delta_n)$ and if we pass to the limit when $n \to \infty$, we obtain that $\delta^* \leq \varphi(\delta^*) < \delta^*$, which is a contradiction. Hence,

$$\lim_{n\to+\infty} \max\{p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1}), p(gz_n, gz_{n+1})\} = 0,$$

and it follows $\lim_{n\to\infty} p(gx_{n+1},gx_n) = \lim_{n\to\infty} p(gy_{n+1},gyx_n) = \lim_{n\to\infty} p(gz_{n+1},gz_n) = 0$.

We next prove that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are 0-Cauchy sequences in the space (X, p). If at least one of sequences $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ is not a 0-Cauchy sequence, it means that $\max\{p(gx_n, gx_m), p(gy_n, gy_m), p(gz_n, gz_m)\}$ does not tend to 0 when $n, m \to \infty$. It means that there exist two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $m(k) > n(k) > k$ and the following four sequences

$$\max\{p(gx_{n(k)}, gx_{m(k)}), p(gy_{n(k)}, gy_{m(k)}), p(gz_{n(k)}, gz_{m(k)})\},$$
$$\max\{p(gx_{n(k)+1}, gx_{m(k)}), p(gy_{n(k)+1}, gy_{m(k)}), p(gz_{n(k)+1}, gz_{m(k)})\},$$
$$\max\{p(gx_{n(k)}, gx_{m(k)-1}), p(gy_{n(k)}, gy_{m(k)-1}), p(gz_{n(k)}, gz_{m(k)-1})\},$$
$$\max\{p(gx_{n(k)+1}, gx_{m(k)-1}), p(gy_{n(k)+1}, gy_{m(k)-1}), p(gz_{n(k)+1}, gz_{m(k)-1})\},$$

all tend to $\varepsilon+$ when $k \to \infty$. The proof is identical as in [18] and [27].

Applying condition (3.13) to elements $x=x_{n(k)}$, $y=y_{n(k)}$, $z=z_{n(k)}$, $u=x_{m(k)-1}$, $v=y_{m(k)-1}$ and $w=z_{m(k)-1}$ we get that:

$$\begin{aligned} p(gx_{n(k)+1}, gx_{m(k)}) &= p(F(x_{n(k)}, y_{n(k)}, z_{n(k)}), F(x_{m(k)-1}, y_{m(k)-1}, z_{m(k)-1})) \\ &\leq \varphi\left(\max\left\{p(gx_{n(k)}, gx_{m(k)-1}), p(gy_{n(k)}, gy_{m(k)-1}), p(gz_{n(k)}, gz_{m(k)-1})\right\}\right). \end{aligned} \tag{3.15}$$

Similarly, we have

$$\begin{aligned} p(gy_{n(k)+1}, gy_{m(k)}) &= p(F(y_{n(k)}, x_{n(k)}, y_{n(k)}), F(y_{m(k)-1}, x_{m(k)-1}, y_{m(k)-1})) \\ &\leq \varphi\left(\max\left\{p(gy_{n(k)}, gy_{m(k)-1}), p(gx_{n(k)}, gx_{m(k)-1})\right\}\right) \\ &\leq \varphi\left(\max\left\{p(gx_{n(k)}, gx_{m(k)-1}), p(gy_{n(k)}, gy_{m(k)-1}), p(gz_{n(k)}, gz_{m(k)-1})\right\}\right) \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} p(gz_{n(k)+1}, gz_{m(k)}) &= p(F(z_{n(k)}, y_{n(k)}, x_{n(k)}), F(z_{m(k)-1}, y_{m(k)-1}, x_{m(k)-1})) \\ &\leq \varphi\left(\max\left\{p(gz_{n(k)}, gz_{m(k)-1}), p(gy_{n(k)}, gy_{m(k)-1}), p(gx_{n(k)}, gx_{m(k)-1})\right\}\right). \end{aligned} \tag{3.17}$$

From (3.15), (3.16) and (3.17) we have

$$\max\left\{p(gx_{n(k)+1},gx_{m(k)}),\,p(gy_{n(k)+1},gy_{m(k)}),\,p(gz_{n(k)+1},gz_{m(k)})\right\}$$
$$\leq \max\left\{\varphi\left(\max\left\{p(gx_{n(k)},gx_{m(k)-1}),\,p(gy_{n(k)},gy_{m(k)-1}),\,p(gz_{n(k)},gz_{m(k)-1})\right\}\right)\right\}$$
$$=\varphi\left(\max\left\{p(gx_{n(k)},gx_{m(k)-1}),\,p(gy_{n(k)},gy_{m(k)-1}),\,p(gz_{n(k)},gz_{m(k)-1})\right\}\right)$$

If we pass to the limit when $k\to\infty$ we get $\varepsilon\leq\varphi(\varepsilon)<\varepsilon$, which is a contradiction since $\varepsilon>0$.

This shows that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are 0-Cauchy sequences in the space (X,p). Since gX is a 0-complete, there exist $x,y,z\in X$ such that

$\lim_{n\to\infty}p(gx_n,gx)=p(gx,gx)=0$, $\lim_{n\to\infty}p(gy_n,gy)=p(gy,gy)=0$, and $\lim_{n\to\infty}p(gz_n,gz)=p(gz,gz)=0$.

Suppose that F is continuous. As in the proof of the Theorem 3.1 we have

$$p(gx,F(x,y,z))=p(F(x,y,z),F(x,y,z)),$$
$$p(gy,F(y,x,y))=p(F(y,x,y),F(y,x,y)),$$

and

$$p(gz,F(z,y,x))=p(F(z,y,x),F(z,y,x)).$$

Applying condition (3.13) with $x=u$, $y=v$, $z=w$ we have that $p(F(x,y,z),F(x,y,z))=0$. Similarly, we have $p(F(y,x,y),F(y,x,y))=0$ and $p(F(z,y,x),F(z,y,x))=0$. It follows $gx=F(x,y,z)$, $gy=F(y,x,y)$ and $gz=F(z,y,x)$.

Suppose that $(X,p,\preceq)$ is regular. Then, since (gx_n,gy_n,gz_n) is comparable with (gx,gy,gz) we have that according to (3.13)

$$p(F(x_n, y_n, z_n), F(x, y, z)) \leq \varphi\left(\max\{p(gx_n, gx), p(gy_n, gy), p(gz_n, gz)\}\right)$$

$$p(F(y_n, x_n, y_n), F(y, x, y)) \leq \varphi\left(\max\{p(gy_n, gy), p(gx_n, gx), p(gy_n, gy)\}\right)$$

$$p(F(z_n, y_n, x_n), F(z, y, x)) \leq \varphi\left(\max\{p(gz_n, gz), p(gy_n, gy), p(gx_n, gx)\}\right)$$

or

$$p(gx_{n+1}, F(x, y, z)) + p(gy_{n+1}, F(y, x, y)) + p(gz_{n+1}, F(z, y, x)) \leq 3\varphi\left(\max\{p(gx_n, gx), p(gy_n, gy), p(gz_n, gz)\}\right) \tag{3.18}$$

Now, taking limit as $n \to \infty$ from (3.18) follows:

$$p(gx, F(x, y, z)) + p(gy, F(y, x, y)) + p(gz, F(z, y, x)) = 0. \tag{3.19}$$

From (3.19) follows $gx = F(x, y, z)$, $gy = F(y, x, y)$ and $gz = F(z, y, x)$.

In both cases we have obtained that F and g have a tripled coincidence point.

□

Corollary 3.5. Let $(X, \preceq)$ be a partially ordered set and suppose there is a partial metric p on X such that (X, p) is a 0-complete partial metric space. Suppose $F : X^3 \to X$ and $g : X \to X$ are such that F has the mixed g-monotone property with $F(X^3) \subset g(X)$ and $g(X)$ is a 0-complete subspace of X and

$$p(F(x, y, z), F(u, v, w)) \leq jp(gx, gu) + kp(gy, gv) + lp(gz, gw)$$

for any $x, y, z, u, v, w \in X$ for which $gx \preceq gu$, $gy \succeq gv$ and $gz \preceq gw$ and $j + k + l < 1$. Suppose either F is continuous or $(X, p, \preceq)$ is regular. If there exist $x_0, y_0, z_0 \in X$ such that

$$gx_0 \preceq F(x_0, y_0, z_0), \quad gy_0 \succeq F(y_0, x_0, y_0) \text{ and } gz_0 \preceq F(z_0, y_0, x_0),$$

then there exist $x, y, z \in X$ such that

$$F(x, y, z) = gx, \ F(y, x, y) = gy, \text{ and } F(z, y, x) = gz \ ,$$

that is., F and g have a tripled coincidence point.

Proof.
Since
$jp(gx, gu) + kp(gy, gv) + lp(gz, gw) \leq \lambda \max\{p(gx, gu), p(gy, gv), p(gz, gw)\}$, for $\lambda = j + k + l$, then the proof follows from Theorem 3.4 taking $\varphi(t) = \lambda t$.
□

The following corollary is the result of Berinde and Borcut in [8] when we consider a partial metric instead of a standard metric.

Corollary 3.6. Let $(X, \preceq)$ be a partially ordered set and suppose there is a partial metric p on X such that (X, p) is a 0-complete partial metric space. Suppose $F : X^3 \to X$ has the mixed monotone property and

$$p(F(x, y, z), F(u, v, w)) \leq jp(x, u) + kp(y, v) + lp(z, w)$$

for any $x, y, z, u, v, w \in X$ for which $x \preceq u$, $y \succeq v$ and $z \preceq w$ and $j + k + l < 1$. Suppose either F is continuous or $(X, p, \preceq)$ is regular. If there exist $x_0, y_0, z_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0, z_0), \ \ y_0 \succeq F(y_0, x_0, y_0) \text{ and } z_0 \preceq F(z_0, y_0, x_0),$$

then there exist $x, y, z \in X$ such that

$$F(x, y, z) = x, \ F(y, x, y) = y, \text{ and } F(z, y, x) = z \ ,$$

that is, F has a tripled fixed point.

Proof. Follows from corollary 3.5 by taking $g = i_X$ (the identity map).

□

In several papers, the authors also considered some additional conditions to ensure the uniqueness of the coupled fixed point, of the coupled coincidence or of the tripled fixed point, respectively. So, we state and prove the corresponding result regarding the uniqueness of tripled coincidence points in the context of partially ordered 0-complete partial metric spaces.

Theorem 3.7. In addition to hypotheses of Theorem 3.1 (resp. Theorem 3.4.) assume that for any two elements (x, y, z), $(x^*, y^*, z^*) \in X^3$ there exists $(u, v, w) \in X^3$ such that $(F(u, v, w), F(v, u, v), F(w, v, u))$ is comparable to both $(F(x, y, z), F(y, x, y), F(z, y, x))$ and $(F(x^*, y^*, z^*), F(y^*, x^*, y^*), F(z^*, y^*, x^*))$. Then g and F have a unique tripled coincidence point.

Proof. Theorem 3.1 implies that there exists a tripled coincidence point $(x, y, z) \in X^3$, that is

$$gx = F(x, y, z),\ gy = F(y, x, y),\ gz = F(z, y, x).$$

Suppose that there exists another tripled coincidence point $(x^*, y^*, z^*) \in X^3$ and hence

$$gx^* = F(x^*, y^*, z^*),\ ,\ gy^* = F(y^*, x^*, y^*),\ gz^* = F(z^*, y^*, x^*).$$

We will prove that $gx = gx^*$, $gy = gy^*$ and $gz = gz^*$.

From given condition, we get there exists $(u, v, w) \in X^3$ such that $(F(u, v, w), F(v, u, v), F(w, v, u))$ is comparable to both $(F(x, y, z), F(y, x, y), F(z, y, x))$ and $(F(x^*, y^*, z^*), F(y^*, x^*, y^*), F(z^*, y^*, x^*))$.

Put $u_0 = u$, $v_0 = v$, $w_0 = w$ and analogously to the proof of Theorem 3.1, choose sequences $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ satisfying

$$gu_{n+1} = F(u_n, v_n, w_n),\ gv_{n+1} = F(v_n, u_n, v_n) \text{ and } gw_{n+1} = F(w_n, v_n, u_n) \text{ for } n = 0, 1, 2, \ldots$$

Starting from $x_0 = x$, $y_0 = y$, $z_0 = z$ and $x_0^* = x^*$, $y_0^* = y^*$, $z_0^* = z^*$, choose sequences $\{x_n\},\{y_n\},\{z_n\}$ and $\{x_n^*\},\{y_n^*\},\{z_n^*\}$ satisfying

$$gx_{n+1} = F(x_n, y_n, z_n),\ gy_{n+1} = F(y_n, x_n, y_n),\ gz_{n+1} = F(z_n, y_n, x_n)$$

and $gx_{n+1}^* = F(x_n^*, y_n^*, z_n^*)$, $gy_{n+1}^* = F(y_n^*, x_n^*, y_n^*)$, $gz_{n+1}^* = F(z_n^*, y_n^*, x_n^*)$ for n=0,1,2,...

Taking into account properties of coincidence points, it is easy to see that it can be done so that $x_n = x$, $y_n = y$, $z_n = z$ and $x_n^* = x^*$, $y_n^* = y^*$, $z_n^* = z^*$, i.e.,

$$gx_n = F(x, y, z),\ gy_n = F(y, x, y),\ gz_n = F(z, y, x)$$

and

$$gx_n^* = F(x^*, y^*, z^*),\ ,gy_n^* = F(y^*, x^*, y^*),\ gz_n^* = F(z^*, y^*, x^*).$$

for all $n \in \mathbf{N}$.

Since $(F(x, y, z), F(y, x, y), F(z, y, x)) = (gx_1, gy_1, gz_1) = (gx, gy, gz)$ and $(F(u, v, w), F(v, u, v), F(w, v, u)) = (gu_1, gv_1, gw_1)$ are comparable, then $gx \preceq gu_1$, $gy \succeq gv_1$ and $gz \preceq gw_1$. It is clear that (gx, gy, gz) and (gu_n, gv_n, gw_n) are also comparable, that is $gx \preceq gu_n$, $gy \succeq gv_n$ and $gz \preceq gw_n$ for all $n \geq 1$. Then, from (3.1) and using the proof of Theorem 3.1 we have:

$$\begin{aligned}
&\frac{p(gx, gu_{n+1}) + p(gy, gv_{n+1}) + p(gz, gw_{n+1})}{3} \\
&= \frac{p(F(x,y,z), F(u_n, v_n, w_n)) + p(F(y,x,y), F(v_n, u_n, v_n)) + p(F(z,y,x), F(w_n, v_n, u_n))}{3} \\
&\leq \varphi\left(\frac{p(gx, gu_n) + p(gy, gv_n) + p(gz, gw_n)}{3}\right) \\
&\leq \varphi^2\left(\frac{p(gx, gu_{n-1}) + p(gy, gv_{n-1}) + p(gz, gw_{n-1})}{3}\right) \\
&\leq \dots \\
&\leq \varphi^{n+1}\left(\frac{p(gx, gu_0) + p(gy, gv_0) + p(gz, gw_0)}{3}\right)
\end{aligned} \tag{3.20}$$

By letting $n \to \infty$ in relation (3.20) we obtain

$$\lim_{n\to\infty} \frac{p(gx, gu_{n+1}) + p(gy, gv_{n+1}) + p(gz, gw_{n+1})}{3} = 0,$$

that is.,

$$\lim_{n\to\infty} p(gx, gu_{n+1}) = 0,\ \lim_{n\to\infty} p(gy, gv_{n+1}) = 0,\ \lim_{n\to\infty} p(gz, gw_{n+1}) = 0. \quad (3.21)$$

Similarly, one can prove that:

$$\lim_{n\to\infty} p(gx^*, gu_{n+1}) = 0,\ \lim_{n\to\infty} p(gy^*, gv_{n+1}) = 0,$$
$$\lim_{n\to\infty} p(gz^*, gw_{n+1}) = 0. \quad (3.22)$$

Now by (**p**4), (3.21) and (3.22) we have:

$$p(gx, gx^*) \le p(gx, gu_{n+1}) + p(gu_{n+1}, gx^*) - p(gu_{n+1}, gu_{n+1})$$
$$\le p(gx, gu_{n+1}) + p(gu_{n+1}, gx^*) \to 0 \text{ as } n \to \infty,$$

$$p(gy, gy^*) \le p(gy, gv_{n+1}) + p(gv_{n+1}, gy^*) - p(gv_{n+1}, gv_{n+1})$$
$$\le p(gy, gv_{n+1}) + p(gv_{n+1}, gx^*) \to 0 \text{ as } n \to \infty,$$

$$p(gz, gz^*) \le p(gz, gw_{n+1}) + p(gw_{n+1}, gz^*) - p(gw_{n+1}, gw_{n+1})$$
$$\le p(gz, gw_{n+1}) + p(gw_{n+1}, gz^*) \to 0 \text{ as } n \to \infty,$$

from which it follows that $gx = gx^*$, $gy = gy^*$ and $gz = gz^*$.

□

The proof of the uniqueness result in the case of Theorem 3.4. is similar.

Remark 3.8. Taking $\varphi(t) = \alpha t$ (resp. $g = i_X$) in Theorem 3.7. we obtain the uniqueness result of tripled fixed points in Corollary 3.2 (resp. Corollary 3.3).

The following example shows that Theorem 3.1 is more general than Theorem 2.2 from [5].

Example 3.9. Let $(X, p) = (\mathbf{Q} \cap [0,+\infty), p)$ be endowed with usual order, where $\mathbf{Q}$ denotes the set of rational numbers and p is given by $p(x, y) = \max\{x, y\}$. A partial metric space (X, p) is a 0-complete partial metric space. Suppose that $g: X \to X$ and $F: X^3 \to X$ are such that $gx = x$ and

$$F(x,y,z) = \begin{cases} \dfrac{x-y+z}{4}, & x \geq y \\ 0, & x < y \end{cases}$$

for all $x, y, z \in X$ and $\varphi(t) = \frac{3t}{4}$, for all $t \in [0,\infty)$. From (1) follows:

L=

$$\max\left\{\frac{x-y+z}{4}, \frac{u-v+w}{4}\right\} + \max\left\{\frac{y-x+y}{4}, \frac{v-u+v}{4}\right\} + \max\left\{\frac{z-y+x}{4}, \frac{w-v+u}{4}\right\}$$

=

$$\frac{u-v+w}{4} + \frac{y-x+y}{4} + \frac{w-v+u}{4} = \frac{2u+2y+2w-2v-x}{4},$$

$$R = \frac{3}{4}\left(\max\{x,u\} + \max\{y,v\} + \max\{z,w\}\right) = 3\frac{u+y+w}{4}.$$

We have that $L = \frac{2u+2y+2w-2v-x}{4} < \frac{2u+2y+2w}{4} < R$ and it follows that F and g have a tripled coincidence point. It is easy to show that (0,0,0) is a tripled coincidence point.

On the other hand, consider the same problem in the standard metric $d(x,y) = |x-y|$ and take $x=u$ and $z=w$. Then, from (1) follows

$$L = \left|\frac{u-y+w}{4} - \frac{u-v+w}{4}\right| + \left|\frac{y-u+y}{4} - \frac{v-u+v}{4}\right| + \left|\frac{w-y+u}{4} - \frac{w-v+u}{4}\right| = |y-v|$$

$$R = \frac{3}{4}\left(|x-u| + |y-v| + |z-w|\right) = 3\frac{|y-v|}{4}.$$

Hence L ≤ R does not hold and the existence of a tripled coincidence point can not be obtained.

4. A new type tripled coincidence point

In the sequel of this paper we introduce a new type tripled coincidence point which is maybe more natural than ones in Berinde and Borcut [8].

Definition 4.1. Let $(X,\preceq)$ be a partially ordered set, $F: X^3 \to X$ and $g: X \to X$ two mappings. Suppose that the mapping F has the mixed g-monotone property. An element (x, y, z) is called a new type tripled coincidence point of F and g if

$$F(x, y, z) = gx,\ F(y, x, y) = gy \text{ and } F(z, y, z) = gz .$$

while (gx, gy, gz) is said a new type tripled point of coincidence of mappings F and g. Moreover, (x, y, z) is called a new type tripled common fixed point of F and g if

$$F(x, y, z) = gx = x,\ F(y, x, y) = gy = y \text{ and } F(z, y, z) = gz = z .$$

Now, for this new tripled case, we announce the following two results:

Theorem 4.2. Let $(X, \preceq)$ be a partially ordered set and suppose there is a partial metric p on X such that (X, p) is a partial metric space. Suppose $F : X^3 \to X$ and $g : X \to X$ are such that F has the mixed g-monotone property with $F(X^3) \subset g(X)$ and $g(X)$ is a 0-complete subspace of X. Assume there is a function $\varphi \in \Phi$ such that

$$p(F(x, y, z), F(u, v, w)) + p(F(y, x, y), F(v, u, v)) + p(F(z, y, z), F(w, v, w)) \le 3\varphi\left(\frac{p(gx, gu) + p(gy, gv) + p(gz, gw)}{3}\right) \tag{4.1}$$

for any $x,y,z,u,v,w \in X$ for which $gx \preceq gu$, $gy \succeq gv$ and $gz \preceq gw$. Suppose either F is continuous or $(X, p, \preceq)$ is regular. If there exist $x_0, y_0, z_0 \in X$ such that

$$gx_0 \preceq F(x_0, y_0, z_0)\,,\quad gy_0 \succeq F(y_0, x_0, y_0) \text{ and } gz_0 \preceq F(z_0, y_0, z_0),$$

then there exist $x, y, z \in X$ such that

$$F(x,y,z) = gx,\ F(y,x,y) = gy,\ \text{and } F(z,y,z) = gz\,,$$

that is., F and g have a new type tripled coincidence point.

Proof. The proof is very similiar to the proof of the Theorem 3.1, so we will omit it. Only here, we consider sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X such that
$gx_{n+1} = F(x_n, y_n, z_n)$, $gy_{n+1} = F(y_n, x_n, y_n)$ and $gz_{n+1} = F(z_n, y_n, z_n)$ for n=0,1,2,...
□

Theorem 4.3. Let $(X, \preceq)$ be a partially ordered set and suppose there is a partial metric p on X such that (X, p) is a partial metric space. Suppose $F : X^3 \to X$ and $g : X \to X$ are such that F has the mixed g-monotone property with $F(X^3) \subset g(X)$ and $g(X)$ is a 0-complete subspace of X. Assume there is a function $\varphi \in \Phi$ such that

$$p(F(x,y,z), F(u,v,w)) \le \varphi\left(\max\{p(gx,gu), p(gy,gv), p(gz,gw)\}\right) \tag{4.2}$$

for any $x,y,z,u,v,w \in X$ for which $gx \preceq gu$, $gy \succeq gv$ and $gz \preceq gw$. Suppose either F is continuous or $(X, p, \preceq)$ is regular. If there exist $x_0, y_0, z_0 \in X$ such that

$$gx_0 \preceq F(x_0, y_0, z_0)\,,\quad gy_0 \succeq F(y_0, x_0, y_0) \text{ and } gz_0 \preceq F(z_0, y_0, z_0),$$

then there exist $x, y, z \in X$ such that

$$F(x,y,z) = gx,\ F(y,x,y) = gy,\ \text{and}\ F(z,y,z) = gz\ ,$$

that is., F and g have a new type tripled coincidence point.

Proof. The proof is similiar to the proof of the Theorem 3.4. Here, we consider sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X such that

$gx_{n+1} = F(x_n, y_n, z_n)$, $gy_{n+1} = F(y_n, x_n, y_n)$ and $gz_{n+1} = F(z_n, y_n, z_n)$ for n=0,1,2,...

It holds that $gx_n \preceq gx_{n+1}$, $gy_{n+1} \preceq gy_n$ and $gz_n \preceq gz_{n+1}$. Applying (4.2) with $x=x_n$, $y=z_n$, $z=x_n$, $u=x_{n+1}$, $v=z_{n+1}$ and $w=x_{n+1}$ we have:

$$\begin{aligned} & p(F(x_n, y_n, z_n), F(x_{n+1}, y_{n+1}, z_{n+1})) \\ & \le \varphi\left(\max\{p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1}), p(gz_n, gz_{n+1})\}\right) \end{aligned}$$

that is.,

$$p(gx_{n+1}, gx_{n+2}) \le \varphi\left(\max\{p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1}), p(gz_n, gz_{n+1})\}\right)$$

Similarly, we have

$$\begin{aligned} p(gy_{n+1}, gy_{n+2}) & \le \varphi\left(\max\{p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1})\}\right) \\ & \le \varphi\left(\max\{p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1}), p(gz_n, gz_{n+1})\}\right) \end{aligned}$$

and

$$\begin{aligned} p(gz_{n+1}, gz_{n+2}) & \le \varphi\left(\max\{p(gy_n, gy_{n+1}), p(gz_n, gz_{n+1})\}\right) \\ & \le \varphi\left(\max\{p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1}), p(gz_n, gz_{n+1})\}\right) \end{aligned}$$

It follows:

$$\max\{p(gx_{n+1},gx_{n+2}),p(gy_{n+1},gy_{n+2}),p(gz_{n+1},gz_{n+2})\} \tag{4.3}$$

$$\leq \varphi\left(\max\{p(gx_n,gx_{n+1}),p(gy_n,gy_{n+1}),p(gz_n,gz_{n+1})\}\right)$$

As in the proof of the Theorem 3.4. it follows $\lim_{n\to\infty} p(gx_{n+1},gx_n) = \lim_{n\to\infty} p(gy_{n+1},gyx_n) = \lim_{n\to\infty} p(gz_{n+1},gz_n) = 0$.

We next prove that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are 0-Cauchy sequences in the space (X,p) in the same way as in the proof of the Theorem 3.4. Applying condition (4.2) to elements $x=x_{n(k)}$, $y=z_{n(k)}$, $z=x_{n(k)}$, $u=x_{m(k)-1}$, $v=z_{m(k)-1}$ and $w=x_{m(k)-1}$ we have:

$$\begin{aligned} p(gx_{n(k)+1},gx_{m(k)}) &= p(F(x_{n(k)},y_{n(k)},z_{n(k)}),F(x_{m(k)-1},y_{m(k)-1},z_{m(k)-1})) \\ &\leq \varphi\left(\max\left\{p(gx_{n(k)},gx_{m(k)-1}),p(gy_{n(k)},gy_{m(k)-1}),p(gz_{n(k)},gz_{m(k)-1})\right\}\right) \end{aligned}$$

Similarly, we have

$$p(gy_{n(k)+1},gy_{m(k)}) \leq \varphi\left(\max\left\{p(gx_{n(k)},gx_{m(k)-1}),p(gy_{n(k)},gy_{m(k)-1}),p(gz_{n(k)},gz_{m(k)-1})\right\}\right)$$

and

$$p(gz_{n(k)+1},gz_{m(k)}) \leq \varphi\left(\max\left\{p(gx_{n(k)},gx_{m(k)-1}),p(gy_{n(k)},gy_{m(k)-1}),p(gz_{n(k)},gz_{m(k)-1})\right\}\right).$$

From the previous we have:

$$\max\left\{p(gx_{n(k)+1}, gx_{m(k)}), p(gy_{n(k)+1}, gy_{m(k)}), p(gz_{n(k)+1}, gz_{m(k)})\right\}$$
$$\le \max\left\{\varphi\left(\max\left\{p(gx_{n(k)}, gx_{m(k)-1}), p(gy_{n(k)}, gy_{m(k)-1}), p(gz_{n(k)}, gz_{m(k)-1})\right\}\right)\right\}$$
$$= \varphi\left(\max\left\{p(gx_{n(k)}, gx_{m(k)-1}), p(gy_{n(k)}, gy_{m(k)-1}), p(gz_{n(k)}, gz_{m(k)-1})\right\}\right)$$

If we pass to the limit when $k \to \infty$ we get $\varepsilon \le \varphi(\varepsilon) < \varepsilon$, which is a contradiction since $\varepsilon > 0$. This shows that $\{gx_n\}, \{gy_n\}$ and $\{gz_n\}$ are 0-Cauchy sequences in the space (X, p). Since gX is a 0-complete, there exist $x, y, z \in X$ such that

$\lim_{n\to\infty} p(gx_n, gx) = p(gx, gx) = 0$, $\lim_{n\to\infty} p(gy_n, gy) = p(gy, gy) = 0$, and $\lim_{n\to\infty} p(gz_n, gz) = p(gz, gz) = 0$.

Suppose that F is continuous. As in the proof of the Theorem 3.1 we have

$$p(gx, F(x, y, z)) = p(F(x, y, z), F(x, y, z)),$$
$$p(gy, F(y, x, y)) = p(F(y, x, y), F(y, x, y)),$$

and

$$p(gz, F(z, y, z)) = p(F(z, y, z), F(z, y, z)).$$

Since $gx \preceq gx$, $gz \succeq gz$ and $gx \preceq gx$, then applying condition (4.2) with $u=x$, $v=z$, $w=x$ we obtain that $p(F(x, y, z), F(x, y, z)) = 0$.

Similarly, we have $p(F(y, x, y), F(y, x, y)) = 0$ and $p(F(z, y, z), F(z, y, z)) = 0$. It follows $gx = F(x, y, z)$, $gy = F(y, x, y)$ and $gz = F(z, y, z)$.

Suppose that $(X, p, \preceq)$ is regular. Then, since (gx_n, gy_n, gz_n) is comparable with (gx, gy, gz) we have that according to (4.2)

$$p(F(x_n, y_n, z_n), F(x, y, z)) \le \varphi\left(\max\left\{p(gx_n, gx), p(gy_n, gy), p(gz_n, gz)\right\}\right)$$
$$p(F(y_n, x_n, y_n), F(y, x, y)) \le \varphi\left(\max\left\{p(gy_n, gy), p(gx_n, gx), p(gy_n, gy)\right\}\right)$$
$$p(F(z_n, y_n, z_n), F(z, y, z)) \le \varphi\left(\max\left\{p(gz_n, gz), p(gy_n, gy), p(gz_n, gz)\right\}\right)$$

or

$$p(gx_{n+1}, F(x,y,z)) + p(gy_{n+1}, F(y,x,y)) + p(gz_{n+1}, F(z,y,z)) \leq 3\varphi\left(\max\{p(gx_n, gx), p(gy_n, gy), p(gz_n, gz)\}\right) \quad (4.4)$$

Now, taking limit as $n \to \infty$ from (4.4) follows:

$$p(gx, F(x,y,z)) + p(gy, F(y,x,y)) + p(gz, F(z,y,z)) = 0. \quad (4.5)$$

From (4.5) follows $gx = F(x,y,z)$, $gy = F(y,x,y)$ and $gz = F(z,y,z)$.

In both cases we have obtained that F and g have a new type tripled coincidence point.

□

The following example supports our new result.

Example 4.4 Let $X = \mathbf{R}$ with $p(x,y) = d(x,y) = |x - y|$ and usual order and let $g(x) = x$, $F(x,y,z) = \frac{4x - 4y + 3z + 1}{48}$ for all $x, y, z \in X$. Further, take $\varphi(t) = \frac{7t}{16}$, $t \in [0, +\infty)$. All conditions of Theorem 4.1 are satisfied. Indeed, for $x \preceq u$, $y \succeq v$ and $z \preceq w$ we have

$$p(F(x,y,z), F(u,v,w)) = \frac{1}{48}\left|4(x-u) - 4(y-v) + 3(z-w)\right| \leq \frac{4}{48}(|x-u| + |y-v| + |z-w|)$$

and similar

$$p(F(y,x,y), F(v,u,v)) = \frac{7}{48}(|x-u| + |y-v|) \leq \frac{7}{48}(|x-u| + |y-v| + |z-w|)$$

and

$$p(F(z,y,z),F(w,v,w)) = \frac{7}{48}(|y-v|+|z-w|)$$
$$\leq \frac{7}{48}(|x-u|+|y-v|+|z-w|).$$

Hence,

$$p(F(x,y,z),F(u,v,w)) + p(F(y,x,y),F(v,u,v)) + p(F(z,y,z),F(w,v,w))$$
$$\leq 3\frac{7}{16}\frac{p(x,u)+p(y,v)+p(z,w)}{3} = 3\varphi\left(\frac{p(x,u)+p(y,v)+p(z,w)}{3}\right).$$

It follows that F and g have a new type tripled coincidence point. Here, $(\frac{1}{45},\frac{1}{45},\frac{1}{45})$ is a new type tripled coincidence point.

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