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EXISTENCE RESULT FOR NONLINEAR INITIAL VALUE PROBLEMS INVOLVING THE DIFFERENCE OF TWO MONOTONE FUNCTIONS

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Abstract. In this paper, monotone iterative technique for nonlinear initial value problems involving the difference of two functions is developed. As an application of this technique, existence of solution of nonlinear initial value problems involving the difference of two functions is obtained.

1. INTRODUCTION

In the last few decades many authors pointed out that fractional derivatives and fractional integrals are very suitable for the description of properties of various real materials, e.g. polymers. It has been shown that new fractional - order models are more adequate than integer - order models. The advantages of fractional derivatives become apparent in modeling mechanical and electrical properties of real materials, and in many other fields, like theory of fractals [10]. Many dynamical models are described by fractional differential equations. Analytical as well as numerical methods are available for studying fractional differential equations such as power series method, compositional method, transform method and Adomain methods etc. (see details in [4, 14, 21] and references therein).

The method of lower and upper solutions has been effectively used for proving the existence results for a class of variety of nonlinear problems. Monotone iterative technique coupled with method of lower and upper solutions is an effective mechanism that offers constructive procedure to obtain existence results in a closed set [5]. The basic theory of fractional differential equation with Riemann-Liouville fractional derivative is developed in [2, 7, 9]. In 2008, Lakshmikantham and Vatsala obtained the local and global existence of solution of Riemann-Liouville fractional differential equation and uniqueness of solution in [6, 8]. Recently, McRae [11] developed monotone method for Riemann-Liouville fractional differential equation with initial conditions and studied the qualitative properties of solutions of initial value problem. Recently, Nanware et.al. developed monotone method for system of Caputo fractional differential equations with periodic boundary conditions when the function is quasimonotone nondecreasing and mixed quasimonotone[3, 19], Riemann-Liouville fractional differential equations with integral boundary conditions when the function on the right is sum of nondecreasing and nonincreasing functions [15] and system of Riemann-Liouville fractional differential equations with

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integral boundary conditions when the function is quasimonotone nondecreasing [12, 16, 20]. Monotone method is successfully applied to obtain existence and uniqueness of solutions of these problems [12, 13, 17, 18].

Monotone iterative technique for the following initial value problem

$$u' = f(t,u) - g(t,u), \quad u(0) = u_0,$$

where f and g are in $C([J \times \mathbb{R}, \mathbb{R}])$ and nondecreasing in u, uniformly in t, is developed by Bhaskar and McRae [1]. In this paper, monotone iterative technique is developed for nonlinear initial value problems involving the difference of two monotone functions with Riemann-Liouville fractional derivative and successfully applied this technique to obtain existence of solution of the problem.

The paper is organized in the following manner:

Basic definitions and results are considered in the second section. Monotone iterative technique is developed in the third section and the technique is successively employed to prove existence result. In the last section some remarks are given.

2. DEFINITIONS AND BASIC RESULTS

The Riemann-Liouville fractional derivative of order q, $(0 < q < 1)$ [21] is defined as

$$[{}_0D_t^q]u(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t \frac{u(s)}{(t-s)^q} ds. \tag{2.1}$$

Consider the following Riemann-Liouville fractional differential equation

$$[{}_0D_t^q]u(t) = f(t,u(t)) - g(t,u(t)), \quad t \in J = [0,T] \tag{2.2}$$

with initial condition

$$u(0) = u_0 \tag{2.3}$$

where $f, g \in C(J \times \mathbb{R}, \mathbb{R})$ are both nondecreasing in $u(t)$, uniformly in t. This is called a nonlinear initial value problem (IVP).

Definition 2.1. *A pair of functions $v(t)$ and $w(t)$ in $C_p(J, \mathbb{R})$ are called ordered lower and upper solutions of the nonlinear IVP* (2.2) − (2.3) *if*

$$[{}_0D_t^q]v(t) \le f(t,v(t)) - g(t,v(t)), \qquad v(0) \le u_0$$
$$[{}_0D_t^q]w(t) \ge f(t,w(t)) - g(t,w(t)), \qquad w(0) \ge u_0$$

Definition 2.2. *The functions $v(t)$ and $w(t)$ in $C_p(J, \mathbb{R})$ are called coupled lower and upper solutions of the nonlinear IVP* (2.2) − (2.3) *if*

$$[{}_0D_t^q]v(t) \le f(t,v(t)) - g(t,w(t)), \qquad v(0) \le u_0$$
$$[{}_0D_t^q]w(t) \ge f(t,w(t)) - g(t,v(t)), \qquad w(0) \ge u_0$$

Lemma 2.1. [2] *Let $m \in C_p(J, \mathbb{R})$ and for any $t_1 \in (0, T]$ we have $m(t_1) = 0$ and $m(t) < 0$ for $0 \le t < t_1$. Then it follows that $D^q m(t_1) \ge 0$.*

Lemma 2.2. [6] *Let $\{u_\epsilon(t)\}$ be a family of continuous functions on J, for each $\epsilon > 0$ where $D^q u_\epsilon(t) = f(t, u_\epsilon(t))$, $\quad u_\epsilon(t_0) = u_\epsilon(t)(t-t_0)^{1-q}\}_{t=t_0}$ and $|f(t, u_\epsilon(t))| \le M$ for $t_0 \le t \le T$. Then the family $\{u_\epsilon(t)\}$ is equicontinuous on $[t_0, T]$.*

Theorem 2.1. [11] *Let $v, w \in C_p(J, \mathbb{R}), f \in C([t_0, T] \times \mathbb{R}, \mathbb{R})$ and*

: *(i)* $D^q v(t) \leq f(t, v(t))$
and
: *(ii)*$D^q w(t) \geq f(t, w(t))$,

$t_0 < t \leq T$. *Assume* $f(t,u)$ *satisfies the Lipschitz condition*

$$f(t,x) - f(t,y) \leq L(x-y), \quad x \geq y, L > 0.$$

Then $v^0 < w^0$, *where* $v^0 = v(t)(t-t_0)^{1-q}|_{t=t_0}$ *and* $w^0 = w(t)(t-t_0)^{1-q}|_{t=t_0}$, *implies* $v(t) \leq w(t), t \in [t_0, T]$.

3. MAIN RESULTS

In this section we develop monotone iterative technique for nonlinear IVP (2.2) − (2.3). As an application of the technique we prove the existence of solution of nonlinear initial value problem (2.2) − (2.3).

Theorem 3.1. *Assume that:*

(i): *$f(t,u(t))$ and $g(t,u(t))$ in $C[J \times \mathbb{R}, \mathbb{R}]$ are nondecreasing in u(t),*
(ii): *$v_0(t)$ and $w_0(t)$ in $C(J, \mathbb{R})$ are coupled lower and upper solutions of IVP* (2.2) − (2.3) *such that* $v_0(t) \leq w_0(t), t \in J = [0, T]$.

Then there exist monotone sequences $\{v_n(t)\}$ *and* $\{w_n(t)\}$ *in* $C(J, \mathbb{R})$ *such that*

$$\{v_n(t)\} \to v(t) \quad and \quad \{w_n(t)\} \to w(t) \quad as \quad n \to \infty,$$

uniformly and monotonically on J and the functions $v(t)$ and $w(t)$ are the coupled minimal and maximal solutions of nonlinear IVP (2.2) − (2.3) *respectively.*

Proof : Consider the following coupled linear system of fractional differential equations with initial conditions (LIVP)

$$\begin{aligned} v_{n+1}(t) &= f(t, v_n) - g(t, w_n), \qquad v_{n+1}(0) = u_0 \\ [{}_0D_t^q]w_{n+1}(t) &= f(t, w_n) - g(t, v_n), \qquad w_{n+1}(0) = u_0 \end{aligned} \tag{3.1}$$

Since the functions $f(t,u)$ and $g(t,u)$ are continuous on $J \times \mathbb{R}$, the solutions of LIVP (3.1) exist on J. We claim that $v_0(t) \leq v_1(t) \leq w_1(t) \leq w_0(t)$ on J.
For this, set $p(t) = v_1(t) - v_0(t)$ then we have

$$\begin{aligned} [{}_0D_t^q]p(t) &= [{}_0D_t^q]v_1(t) - [{}_0D_t^q]v_0(t) \\ &\geq f(t, v_0) - g(t, w_0) - f(t, v_0) + g(t, w_0) \\ [{}_0D_t^q]p(t) &\geq 0 \\ p(0) &= 0 \end{aligned}$$

By applying Theorem 2.1, we get $v_0(t) \leq v_1(t)$. Similarly, we can prove $w_1(t) \leq w_0(t)$ on J.
Also, we prove that $v_1(t) \leq w_1(t)$ on J. Set $p(t) = w_1(t) - v_1(t)$. Then we have

$$\begin{aligned} [{}_0D_t^q]p(t) &= [{}_0D_t^q]w_1(t) - [{}_0D_t^q]v_1(t) \\ &\geq f(t, w_0) - g(t, v_0) - f(t, v_0) + g(t, w_0) \\ [{}_0D_t^q]p(t) &\geq 0 \\ p(0) &\geq 0 \end{aligned}$$

Thus,by applying Theorem 2.1, we get $v_1(t) \leq w_1(t)$.
Assume that for some $k > 1$, $\quad v_{k-1}(t) \leq v_k(t) \leq w_k(t) \leq w_{k-1}(t)$. We claim that

$v_k(t) \leq v_{k+1}(t) \leq w_{k+1}(t) \leq w_k(t)$ on J. To prove this, set $p(t) = v_k(t) - v_{k+1}(t)$. Since $f(t,u)$ and $g(t,u)$ are nondecreasing in u, we get

$$\begin{aligned}
[{}_0D_t^q]p(t) &= [{}_0D_t^q]v_k(t) - [{}_0D_t^q]v_{k+1}(t)\\
&\leq f(t, v_{k-1}) - g(t, w_{k-1}) - f(t, v_k) + g(t, w_k)\\
[{}_0D_t^q]p(t) &\leq 0\\
p(0) &= 0
\end{aligned}$$

By applying Theorem 2.1, we have $v_k \leq v_{k+1}$ on J. By induction, it follows that $v_k \leq v_{k+1}$ for all $k \geq 1, t \in J$.
Similarly we prove $w_{k+1}(t) \leq w_k(t)$ on J. Next we prove $v_{k+1}(t) \leq w_{k+1}(t)$. Consider $p(t) = w_{k+1}(t) - v_{k+1}(t)$. Since $f(t,u)$ and $g(t,u)$ are nondecreasing in u, we have

$$\begin{aligned}
[{}_0D_t^q]p(t) &= [{}_0D_t^q]w_{k+1}(t) - [{}_0D_t^q]v_{k+1}(t)\\
&\geq f(t, w_k) - g(t, v_k) - f(t, v_k) + g(t, w_k)\\
[{}_0D_t^q]p(t) &\geq 0\\
p(0) &= 0
\end{aligned}$$

Hence, by applying Theorem 2.1, we get $v_{k+1} \leq w_{k+1}$ on J. By induction, we get $v_{k+1} \geq w_{k+1}$ for all $k \geq 1, t \in J$.
Thus we have sequences v_n and w_n on J such that

$$v_0 \leq v_1 \leq v_2 \leq ... \leq v_n \leq w_n \leq w_{n-1} \leq ... \leq w_2 \leq w_1 \leq w_0.$$

Clearly the sequences $\{v_n\}$ and $\{w_n\}$ are nondecreasing and bounded below and nondecreasing and bounded above respectively. By Lemma 2.2 it follows that the sequences $\{v_n\}$ and $\{w_n\}$ are equicontinuous and uniformly bounded. Applying Ascoli-Arzela theorem, there exist convergent subsequences $\{v_{n_k}\}$ and $\{w_{n_k}\}$ converging to v and w uniformly and monotonically on J respectively. Then we have

$$\{v_n(t)\} \to v(t) \quad \text{and} \quad \{w_n(t)\} \to w(t) \quad \text{as} \quad n \to \infty.$$

Using corresponding fractional Volterra integral equations

$$\begin{aligned}
v_{n+1}(t) &= u_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{T} (t-s)^{q-1} \Big\{ f(s, v_n(s)) - g(s, w_n(s)) \Big\} ds\\
w_{n+1}(t) &= u_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{T} (t-s)^{q-1} \Big\{ f(s, w_n(s)) - g(s, v_n(s)) \Big\} ds
\end{aligned} \tag{3.2}$$

it follows that $v(t)$ and $w(t)$ are solutions of (3.1).
Next we claim that $v(t)$ and $w(t)$ are the coupled minimal and maximal solutions of LIVP (3.1). For this, let $u(t))$ be any solution of nonlinear IVP (2.2) – (2.3) different from $v(t)$ and $w(t)$, so that there exists k such that $v_k(t) \leq u(t) \leq w_k(t)$ on J and set $p(t) = u(t) - v_{k+1}(t)$ so that

$$\begin{aligned}
[{}_0D_t^q]p(t) &= [{}_0D_t^q]u(t) - [{}_0D_t^q]v_{k+1}(t)\\
&\geq f(t, u) - g(t, u) - f(t, v_k) + g(t, w_k)\\
[{}_0D_t^q]p(t) &\geq 0\\
p(0) &= 0.
\end{aligned}$$

By applying Theorem 2.1, we have $v_{k+1}(t) \leq u(t)$ on J. Since $v_0(t) \leq u(t)$ on J, by induction it follows that $v_k(t) \leq u(t)$ for all k.

Similarly we prove $u(t) \leq w_k(t)$ for all k on J. Thus $v_k(t) \leq u_k(t) \leq w_k(t)$ on $[0,T]$. In limiting case, we have $v(t) \leq u(t) \leq w(t)$ on $[0,T]$. This completes the proof.

4. REMARKS

(1) If $f(t,u)$ and $g(t,u)$ in $C(J\times\mathbb{R})$ and if there exists positive constants M, N such that $f(t,u)+Mu$ and $g(t,u)+Nu$ are both nondecreasing, for $t \in J$ and $v_0 \leq u \leq w_0$ then we may write

$$\begin{aligned} G(t,u) &= f(t,u) - g(t,u) \\ &= |f(t,u) + (M+N)u| - |g(t,u) + (M+N)u| \\ &= f_1(t,u) - g_1(t,u). \end{aligned}$$

Clearly f_1 and g_1 are both monotone nondecreasing functions and Theorem 3.1 can be applied.

(2) If $g = 0$ and as in Theorem 2.1, f satisfies for some $M > 0$,

$$f(t,u_1) - f(t,u_2) \geq -M(u_1 - u_2)$$

whenever $u_1 \geq u_2$. Define $f_1 = f(t,u) + Mu$. Then f_1 is nondecreasing in u and we may write

$$[{}_0D_t^q]u(t) = f(t,u), \qquad u(0) = u_0$$

as

$$[{}_0D_t^q]u(t) = f_1(t,u) - Mu, \qquad u(0) = u_0$$

and with appropriate modifications we can apply Theorem 3.1 to

$$[{}_0D_t^q]u(t) = f_1(t,u) - Mu, \qquad u(0) = u_0.$$

Thus we obtain new result.

(3) If $f = 0$ and g satisfies for some $M > 0$,

$$g(t,u_1) - g(t,u_2) \geq -M(u_1 - u_2),$$

whenever $u_1 \geq u_2$, we define $g_1(t,u) = g(t,u) + Mu$. Then g_1 is nondecreasing in u and we write

$$[{}_0D_t^q]u(t) = -g(t,u), \qquad u(0) = u_0$$

as

$$[{}_0D_t^q]u(t) = Mu - g_1(t,u), \qquad u(0) = u_0.$$

Theorem 2.1 may be applied to obtain the coupled minimal and maximal solutions of the original problem.

(4) Theorem 3.1 can easily be modified to include the IVP of the form

$$[{}_0D_t^q]u(t) + Ku(t) = f(t,u) - g(t,u), \qquad u(0) = 0.$$

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